Describing $\beta\beta$ Decay Processes in the In-Medium SRG Framework

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National Superconducting Cyclotron Laboratory
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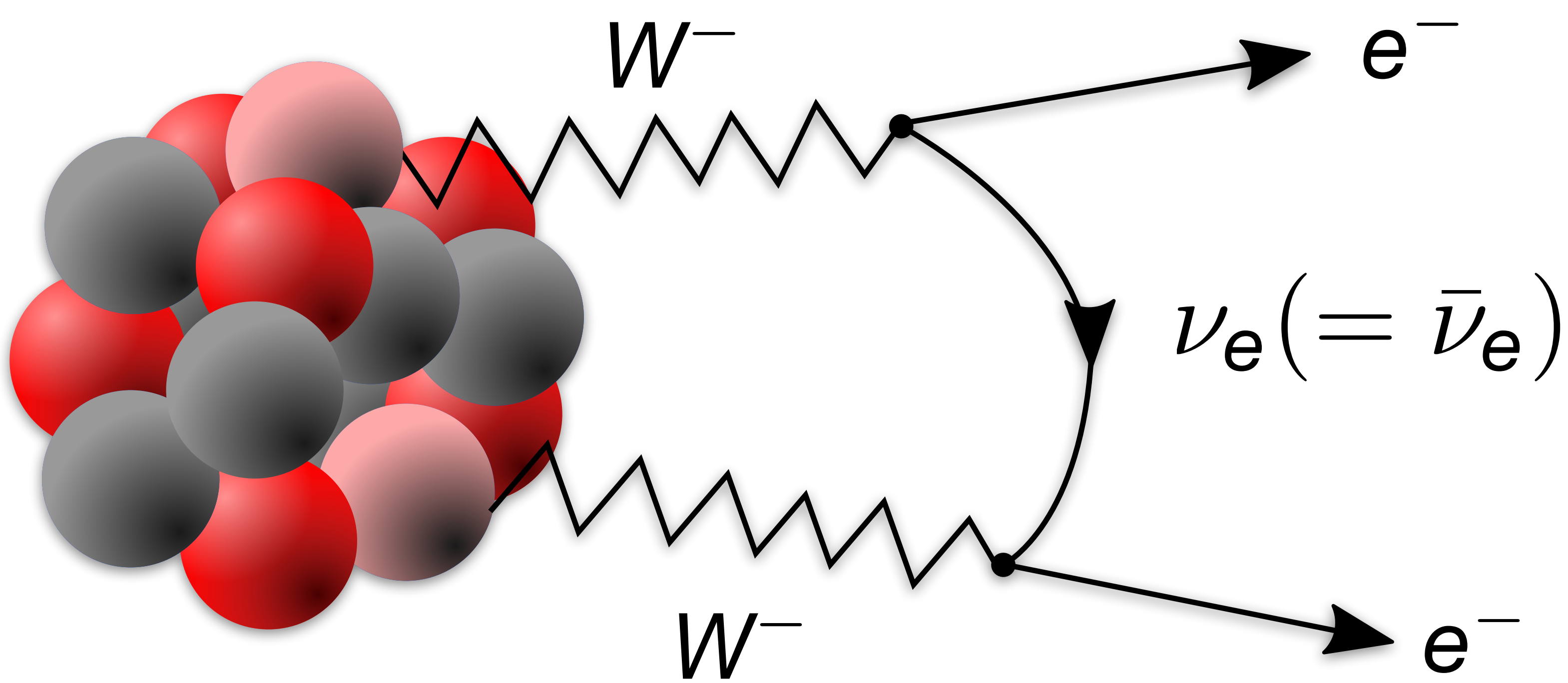
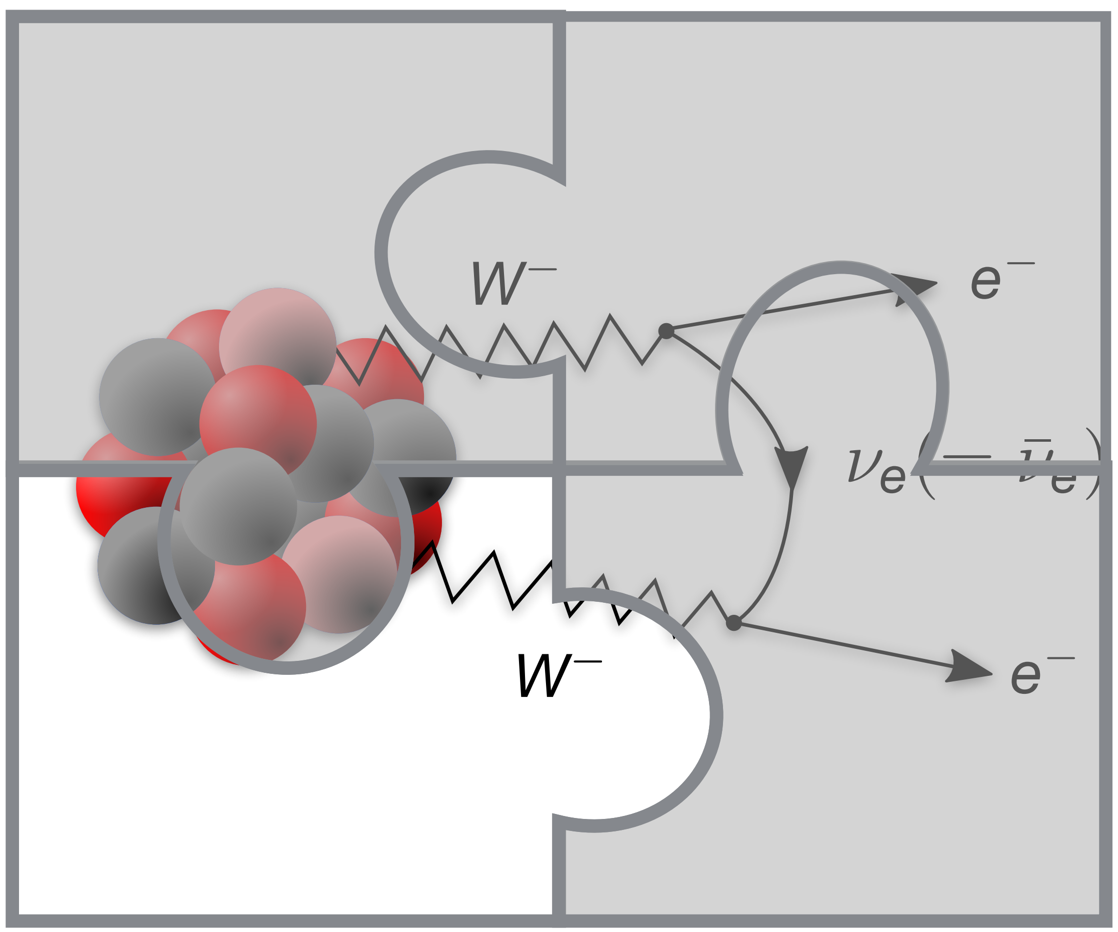
Michigan State University

Neutrinoless Double Beta Decay



Neutrinoless Double Beta Decay



- **interactions and transition operators** from Chiral EFT, **including currents**
  (cf. talks by S. Pastore and A. Schwenk)

- tune **resolution scale** of the Hamiltonian / Hilbert space
  (cf. talk by P. Navrátil, also M. Horoi)

- **(MR-)IM-SRG:** calculate ground (and excited) states or derive Shell Model interaction
  (also see talk by R. Stroberg)

- evaluate **1B, 2B** (, 3B,…) **transition operator**

The Similarity Renormalization Group

Review:
S. Bogner, R. Furnstahl, and A. Schwenk, Prog. Part. Nucl. Phys. 65 (2010), 94

E. Anderson, S. Bogner, R. Furnstahl, and R. Perry, Phys. Rev. C82 (2011), 054001
E. Jurgenson, P. Navratil, and R. Furnstahl, Phys. Rev. C83 (2011), 034301
R. Roth, S. Reinhardt, and H. H., Phys. Rev. C77 (2008), 064003
H. H. and R. Roth, Phys. Rev. C75 (2007), 051001

Similarity Renormalization Group

Basic Idea

continuous unitary transformation of the Hamiltonian to band-diagonal form w.r.t. a given “uncorrelated” many-body basis

- flow equation for Hamiltonian $H(s) = U(s)HU^\dagger(s)$:

$$\frac{d}{ds}H(s) = \left[\eta(s), H(s)\right] , \quad \eta(s) = \frac{dU(s)}{ds}U^\dagger(s) = -\eta^\dagger(s)$$

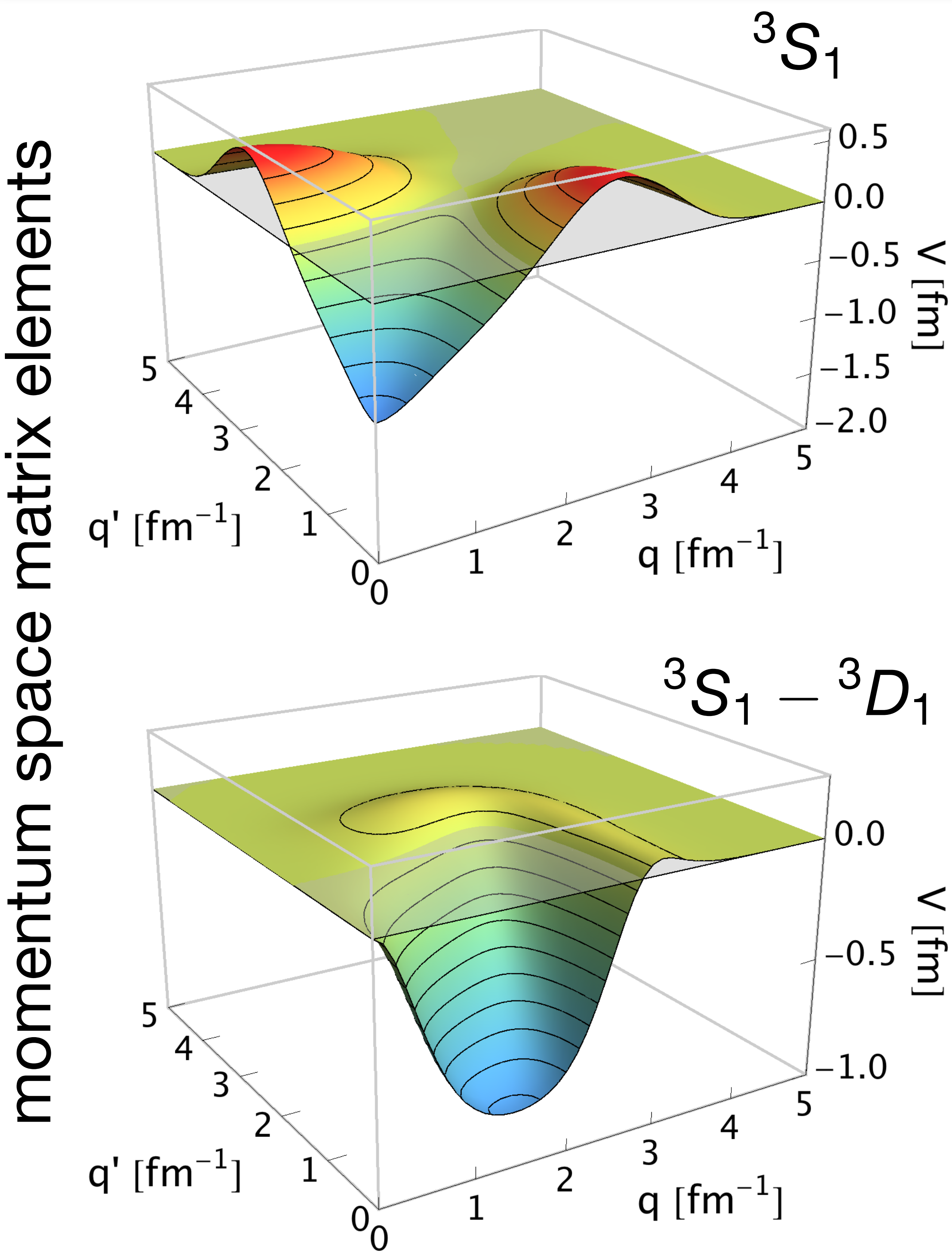
- choose $\eta(s)$ to achieve desired behavior, e.g.,

$$\eta(s) = \left[H_d(s), H_{od}(s)\right]$$

to suppress (suitably defined) off-diagonal Hamiltonian

- consistent evolution for all observables of interest

SRG in Two-Body Space



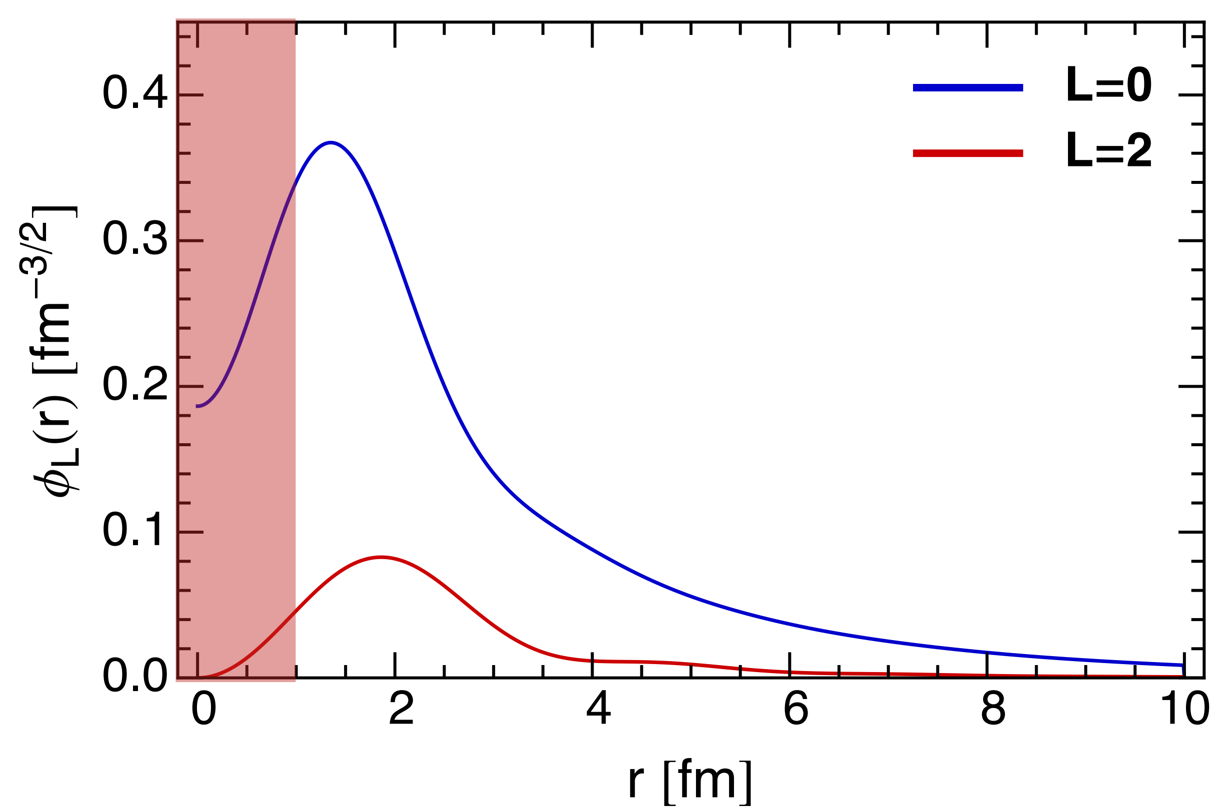
chiral NN

Entem & Machleidt, N3LO

$$\eta(\lambda) = 2\mu\left[T_{\text{rel}}, H(\lambda)\right]$$

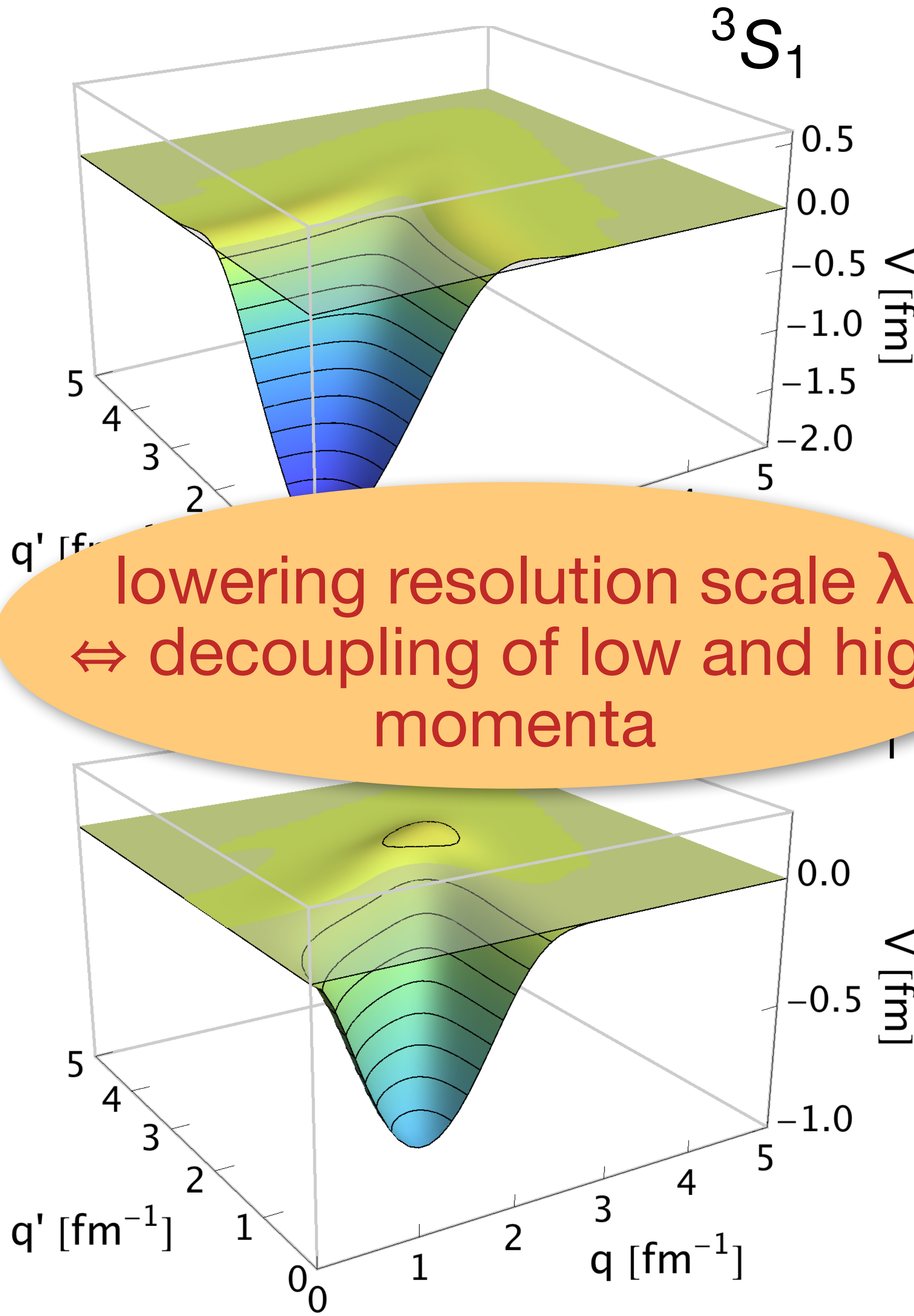
$$\lambda = s^{-1/4}$$

deuteron wave function



SRG in Two-Body Space

momentum space matrix elements



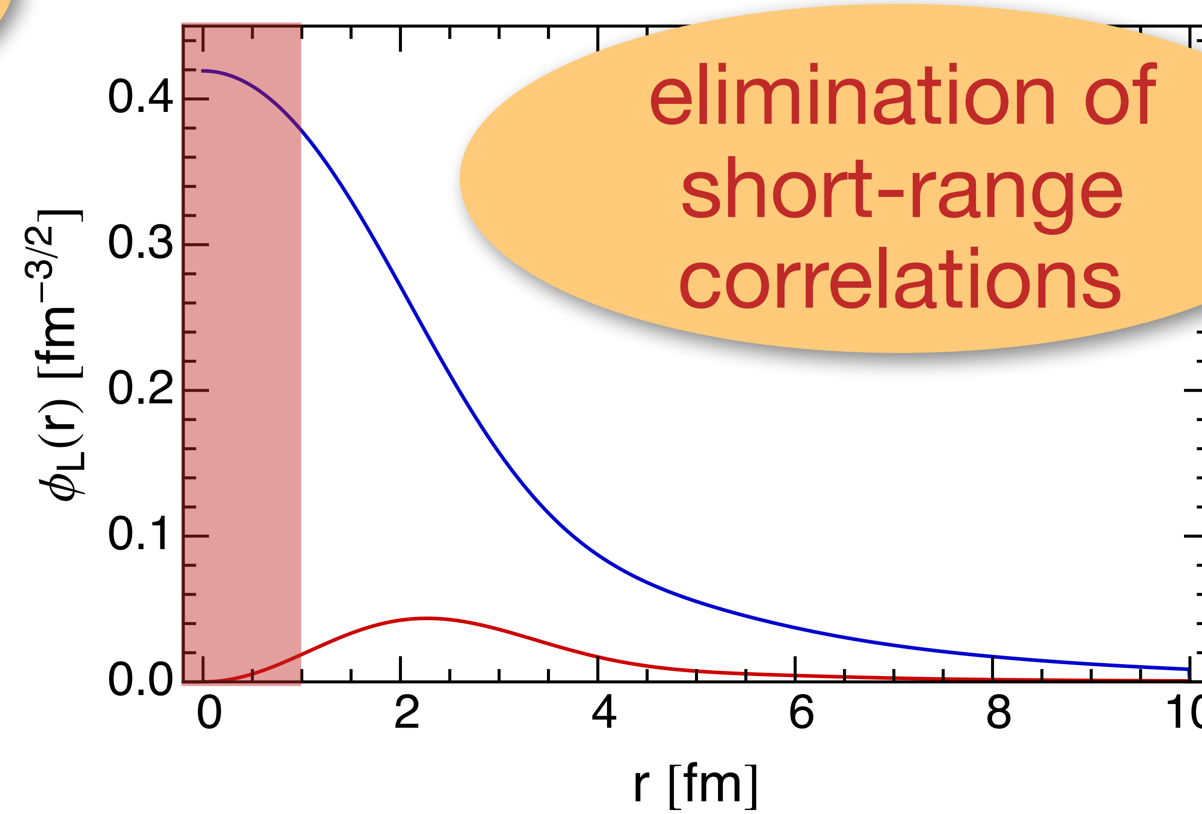
lowering resolution scale λ
⇔ decoupling of low and high momenta

$\lambda = 1.8\ \mathrm{fm}^{-1}$

$$\eta(\lambda) = 2\mu\left[T_{\mathrm{rel}}, H(\lambda)\right]$$

$$\lambda = s^{-1/4}$$

deuteron wave function



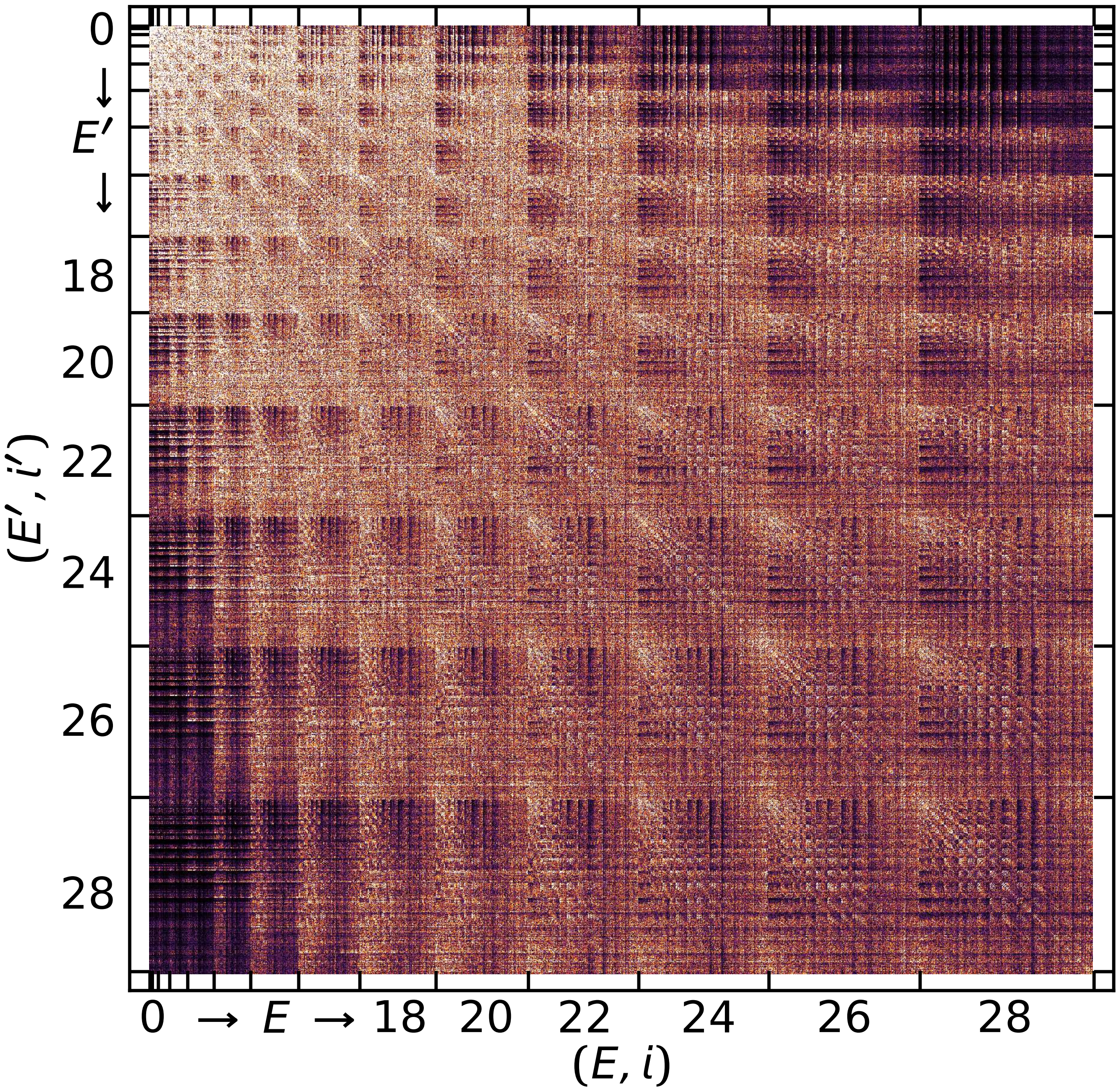
elimination of short-range correlations

SRG in Three-Body Space

[figures by R. Roth, A. Calci, J. Langhammer]

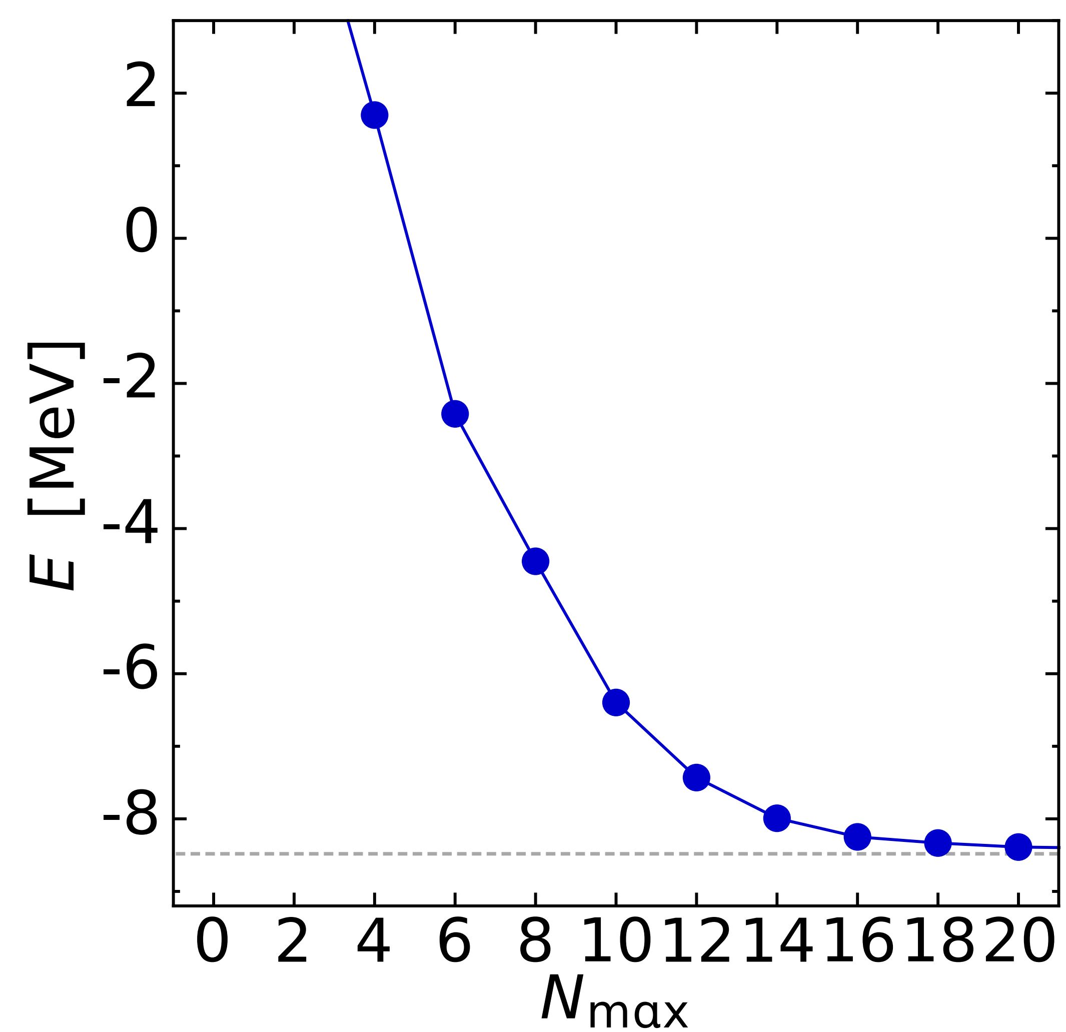
3B Jacobi-HO Matrix Elements

$J^\pi = \frac{1}{2}^+, T = \frac{1}{2}, \hbar\Omega = 28$ MeV



chiral NN + 3N
$N^3LO + N^2LO$ (^3H fit)

^3H ground-state (NCSM)



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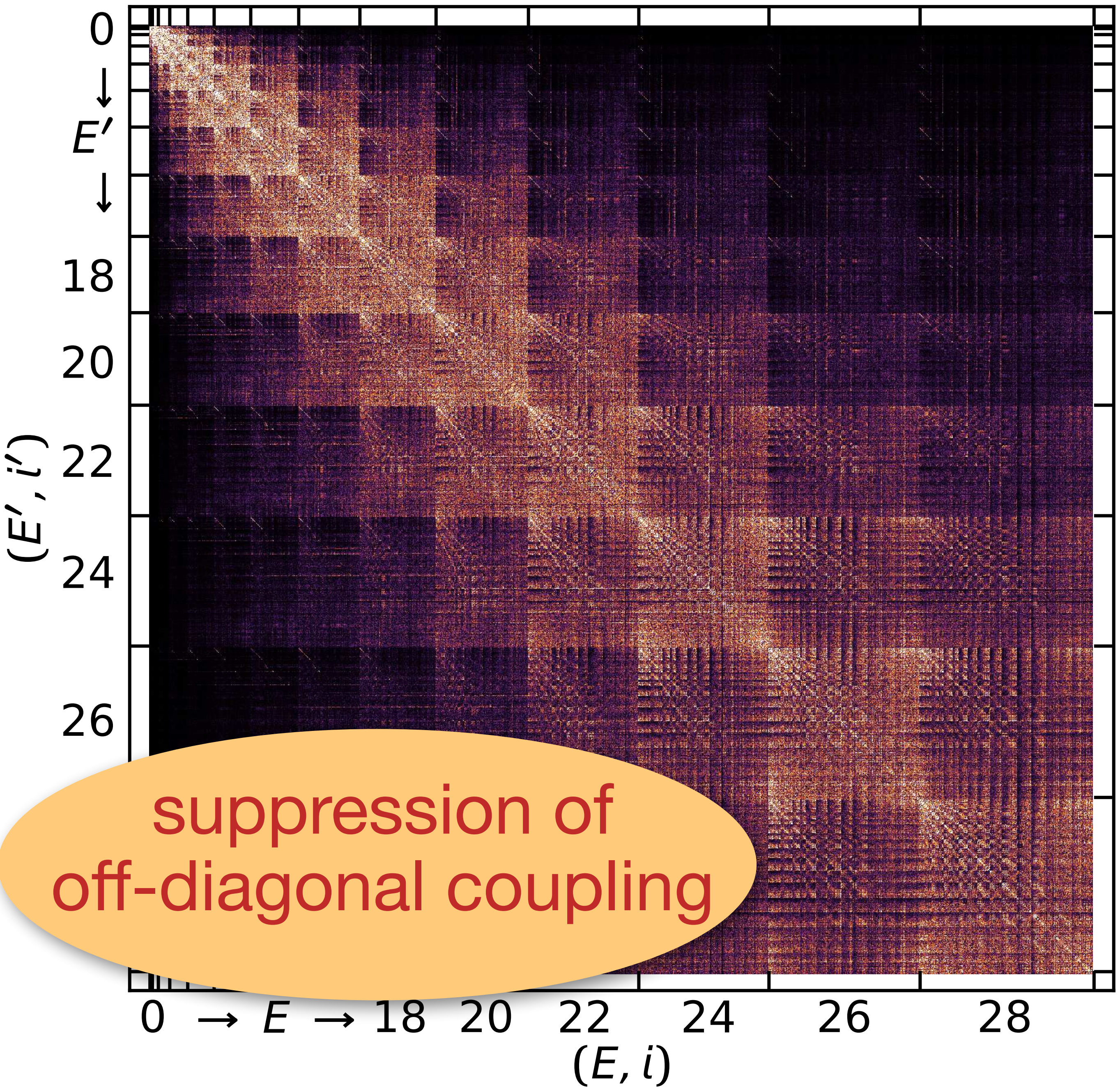
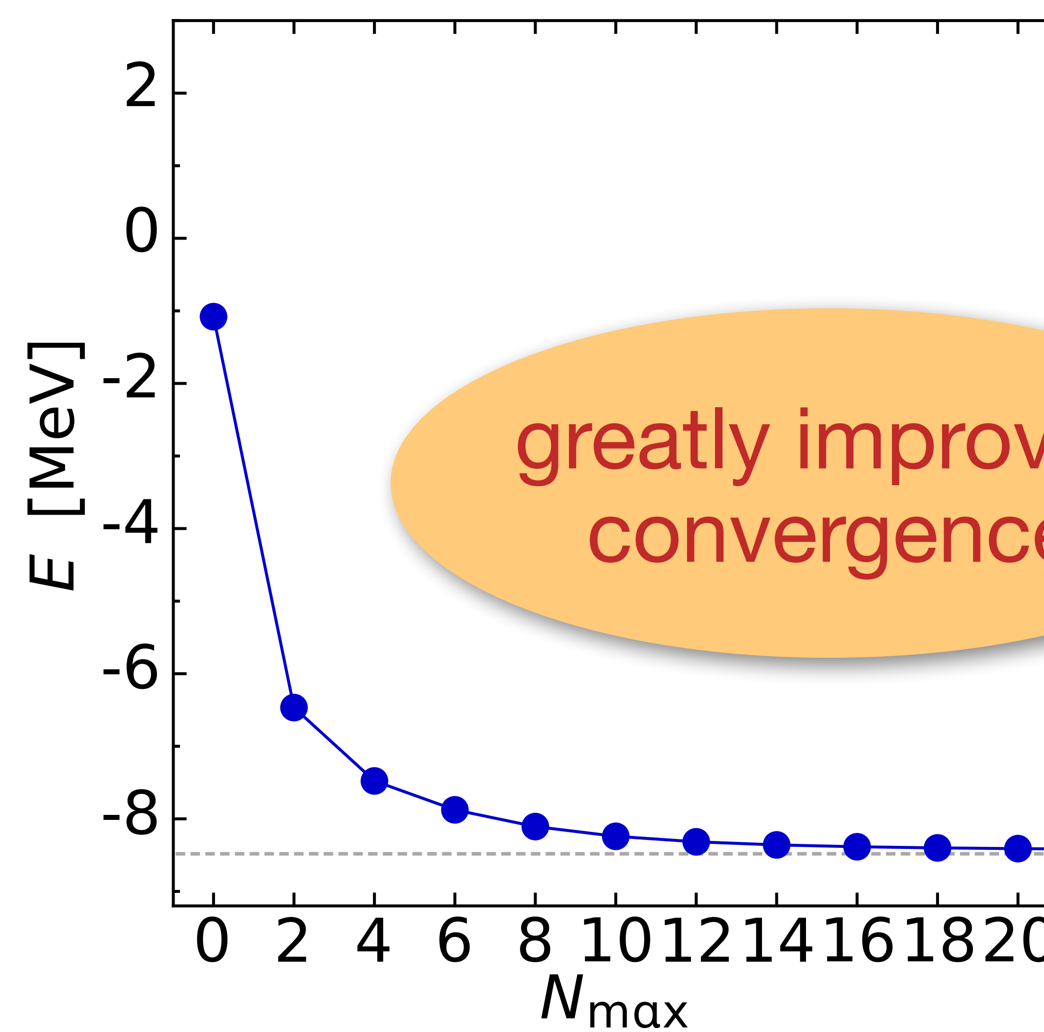


suppression of off-diagonal coupling

$\lambda = 1.33\ \mathrm{fm}^{-1}$

^{3}H ground-state (NCSM)



greatly improved convergence

(Multi-Reference) In-Medium SRG

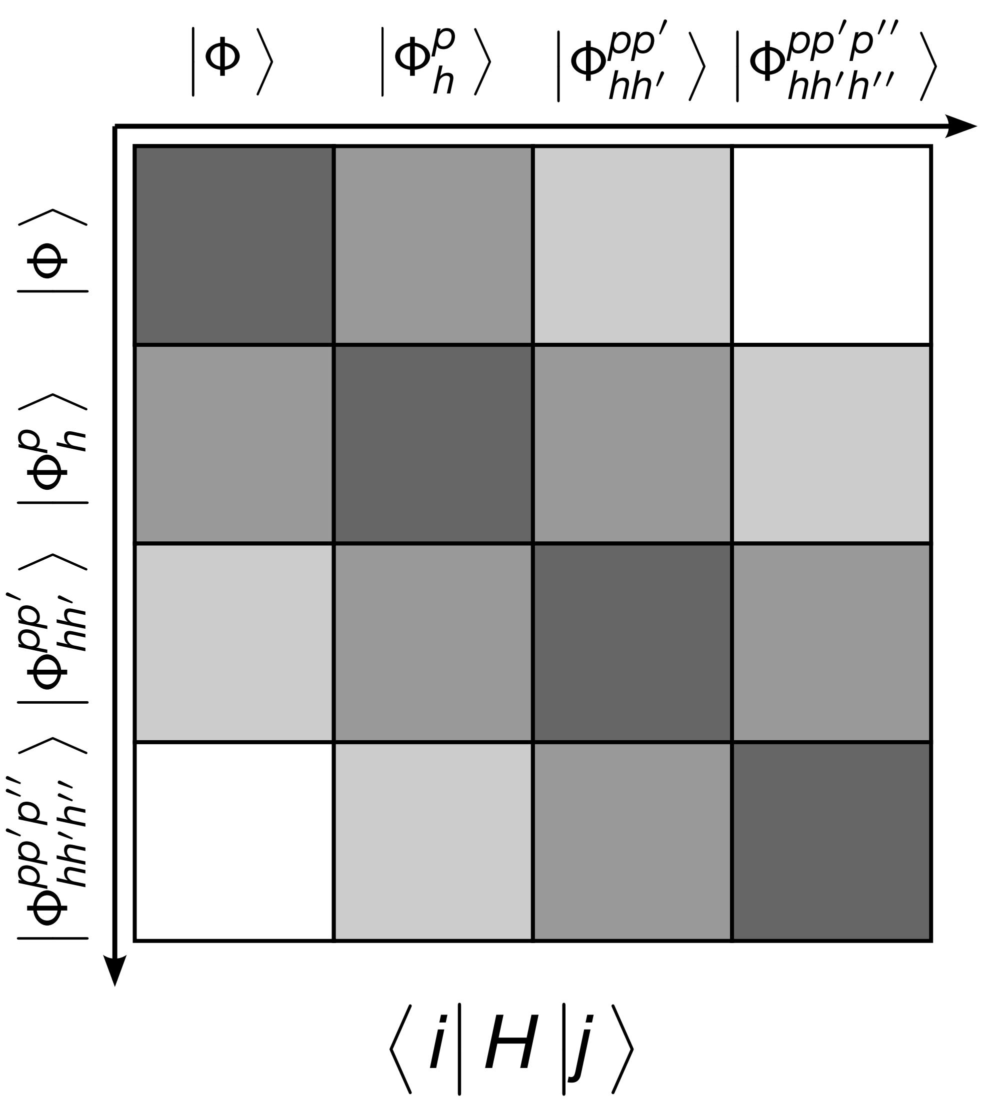
H. H., in preparation

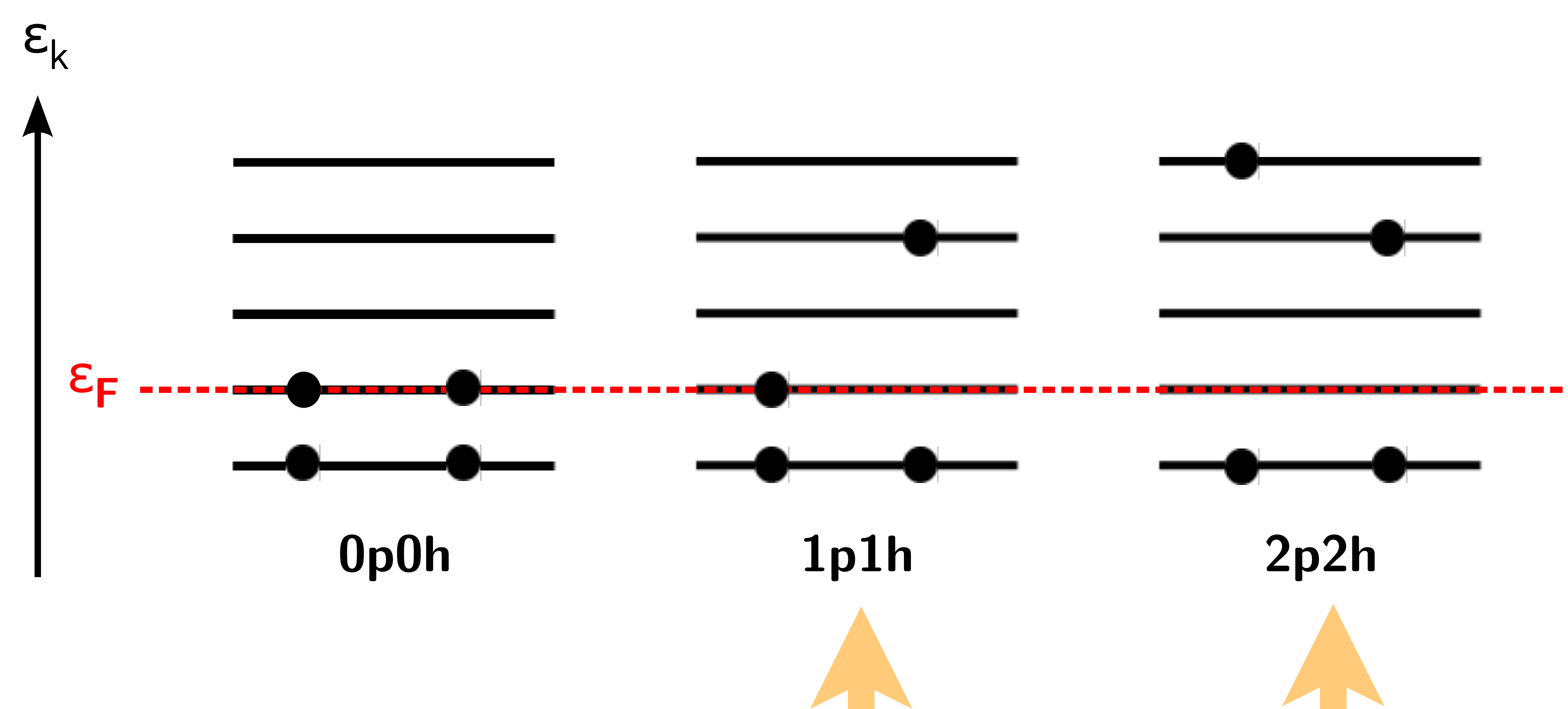
H. H., S. K. Bogner, T. D. Morris, A. Schwenk, and K. Tuskiyama, Phys. Rept. **621**, 165 (2016)

H. H., S. Bogner, T. Morris, S. Binder, A. Calci, J. Langhammer, R. Roth, Phys. Rev. C **90**, 041302 (2014)

H. H., S. Binder, A. Calci, J. Langhammer, and R. Roth, Phys. Rev. Lett **110**, 242501 (2013)

Ground-State Decoupling



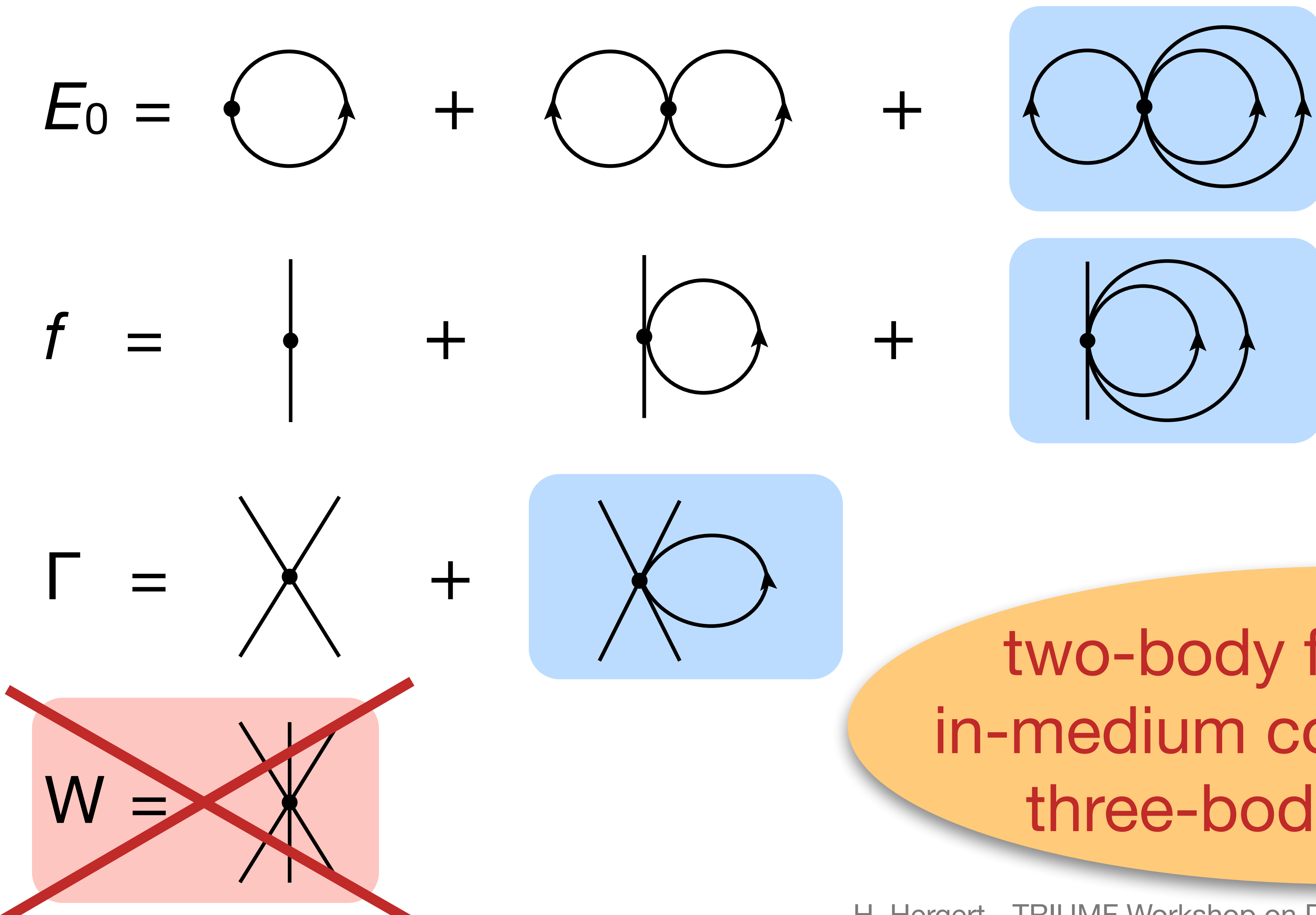


excitations **relative** to reference state: **normal-ordering**

Normal-Ordered Hamiltonian

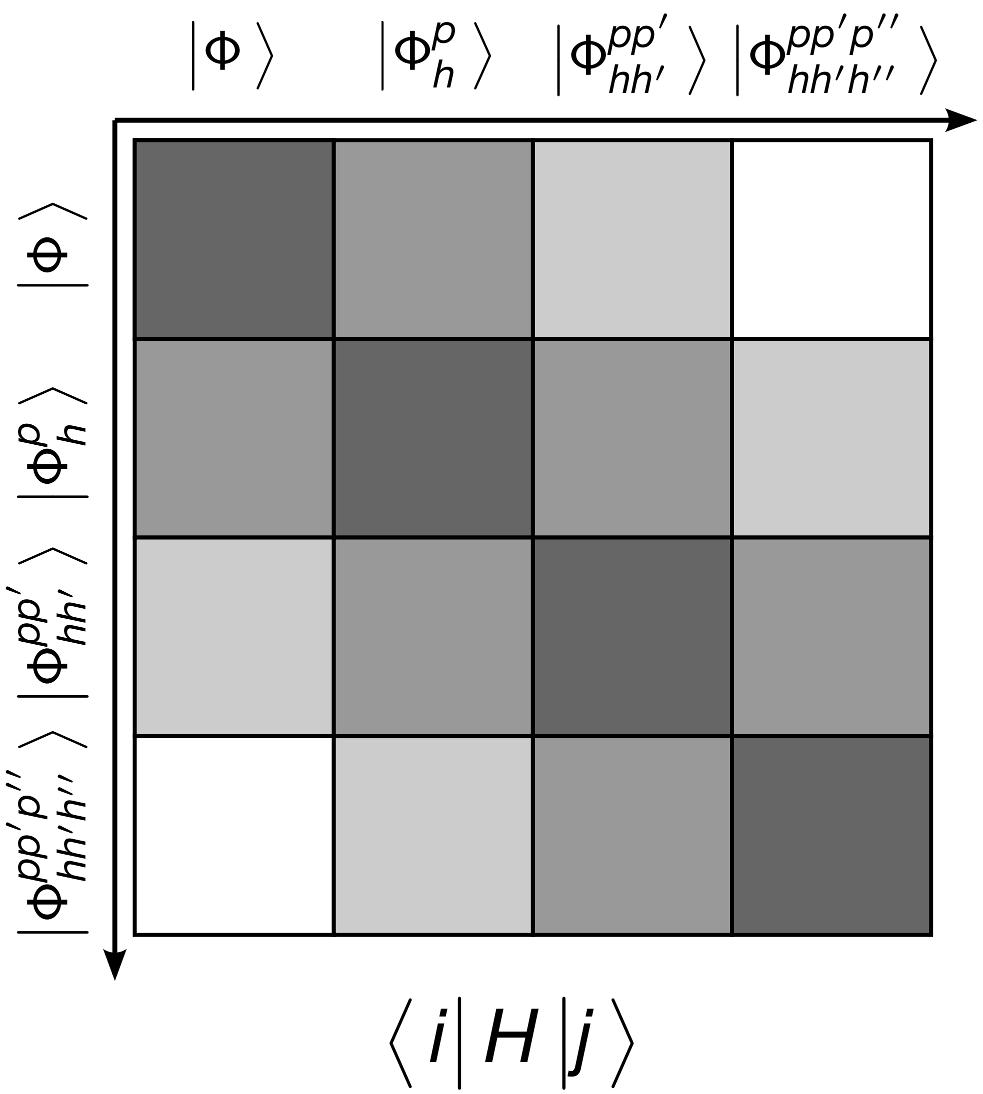
Normal-Ordered Hamiltonian

$$H = E_0 + \sum_{kl} f^k_l : A^k_l : + \frac{1}{4} \sum_{klmn} \Gamma^{kl}_{mn} : A^{kl}_{mn} : + \frac{1}{36} \sum_{ijklmn} W^{ijk}_{lmn} : A^{ijk}_{lmn} :$$



two-body formalism with in-medium contributions from three-body interactions

Single-Reference Case



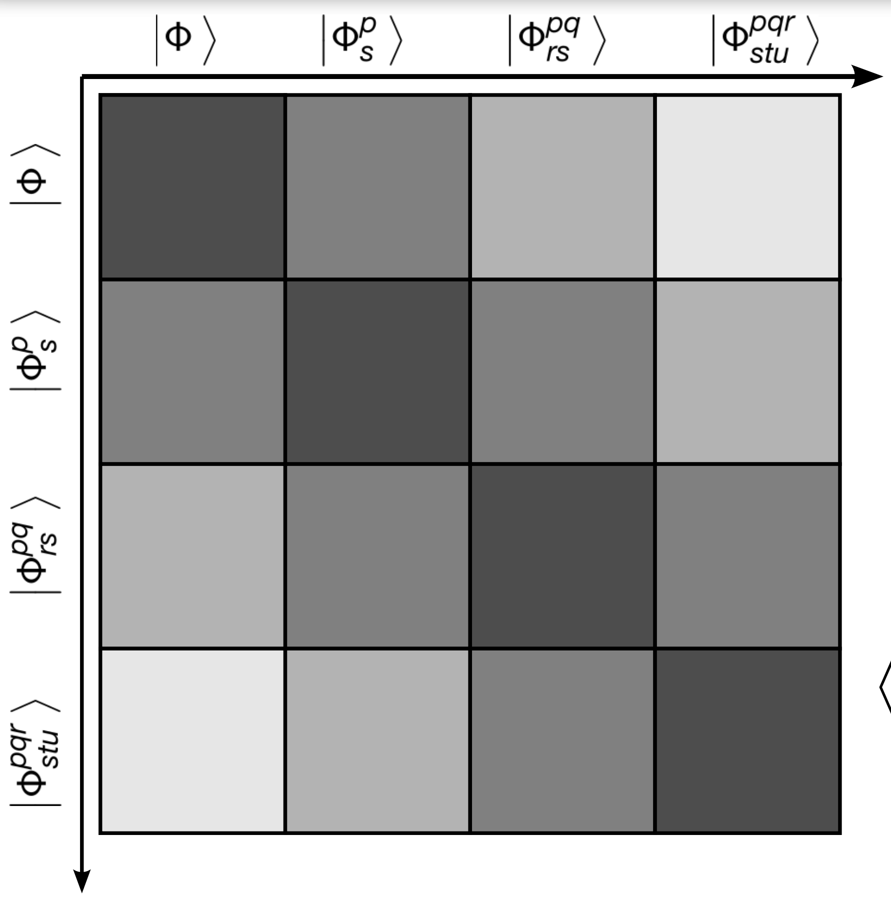
$$A^{i_1 \dots i_N}_{j_1 \dots j_N} \equiv a^{\dagger}_{i_1} \dots a^{\dagger}_{i_N} a_{j_N} \dots a_{j_1}$$

$$\langle {}^{p}_{h} | H | \Psi \rangle = \sum_{kl} f^{k}_{l} \langle \Psi | : A^{h}_{p} :: A^{k}_{l} : | \Psi \rangle = -n_h \bar{n}_p f^{p}_{h}$$

$$\langle {}^{pp'}_{hh'} | H | \Psi \rangle = \sum_{klmn} \Gamma^{kl}_{mn} \langle \Psi | : A^{hh'}_{pp'} :: A^{kl}_{mn} : | \Psi \rangle \sim \Gamma^{pp'}_{hh'}$$

- reference state: **Slater determinant**
- normal-ordered operators **depend on occupation numbers (one-body density)**

Multi-Reference Case



$$\left\langle {}^{p}_{s} \middle| H \middle| \Phi \right\rangle \sim \bar{n}_p n_s f^{p}_{s}, \sum_{kl} f^{k}_{l} \lambda^{sk}_{pl}, \sum_{klmn} \Gamma^{kl}_{mn} \lambda^{skl}_{pmn}, \ldots$$

$$\left\langle {}^{pq}_{st} \middle| H \middle| \Phi \right\rangle \sim \bar{n}_p \bar{n}_q n_s n_t \Gamma^{pq}_{st}, \sum_{kl} \Gamma^{pk}_{sl} \lambda^{tk}_{ql}, \sum_{kl} f^{k}_{l} \lambda^{stk}_{pql}, \sum_{klmn} \Gamma^{kl}_{mn} \lambda^{stkl}_{pqmn}, \ldots$$

$$\left\langle {}^{pqr}_{stu} \middle| H \middle| \Phi \right\rangle \sim \ldots$$

- reference state: **arbitrary**
- normal-ordered operators depend on up to **irreducible n-body density matrices** of the reference state

$$\rho^{kl}_{mn} = \lambda^{kl}_{mn} + \lambda^{k}_{m}\lambda^{l}_{n} - \lambda^{k}_{n}\lambda^{l}_{m}$$
$$\rho^{ijk}_{lmn} = \lambda^{ijk}_{lmn} + \lambda^{i}_{l}\lambda^{jk}_{mn} + \lambda^{i}_{l}\lambda^{j}_{m}\lambda^{k}_{n} + \text{permutations}$$
$$\ldots$$

MR-IM-SRG References States

available

future

- **number-projected Hartree-Fock Bogoliubov** vacua:

$$\left|\Phi_{ZN}\right\rangle = \frac{1}{(2\pi)^2}\int d\phi_p \int d\phi_n\, e^{i\phi_p(\hat{Z}-Z)} e^{i\phi_n(\hat{N}-N)}\left|\Phi\right\rangle$$

- small-scale (e.g., $0\hbar\Omega$, $2\hbar\Omega$) **No-Core Shell Model**:

$$\left|\Phi\right\rangle = \sum_{N=0}^{N_{\max}} \sum_{i=1}^{\dim(N)} C_i^{(N)} \left|\Phi_i^{(N)}\right\rangle$$

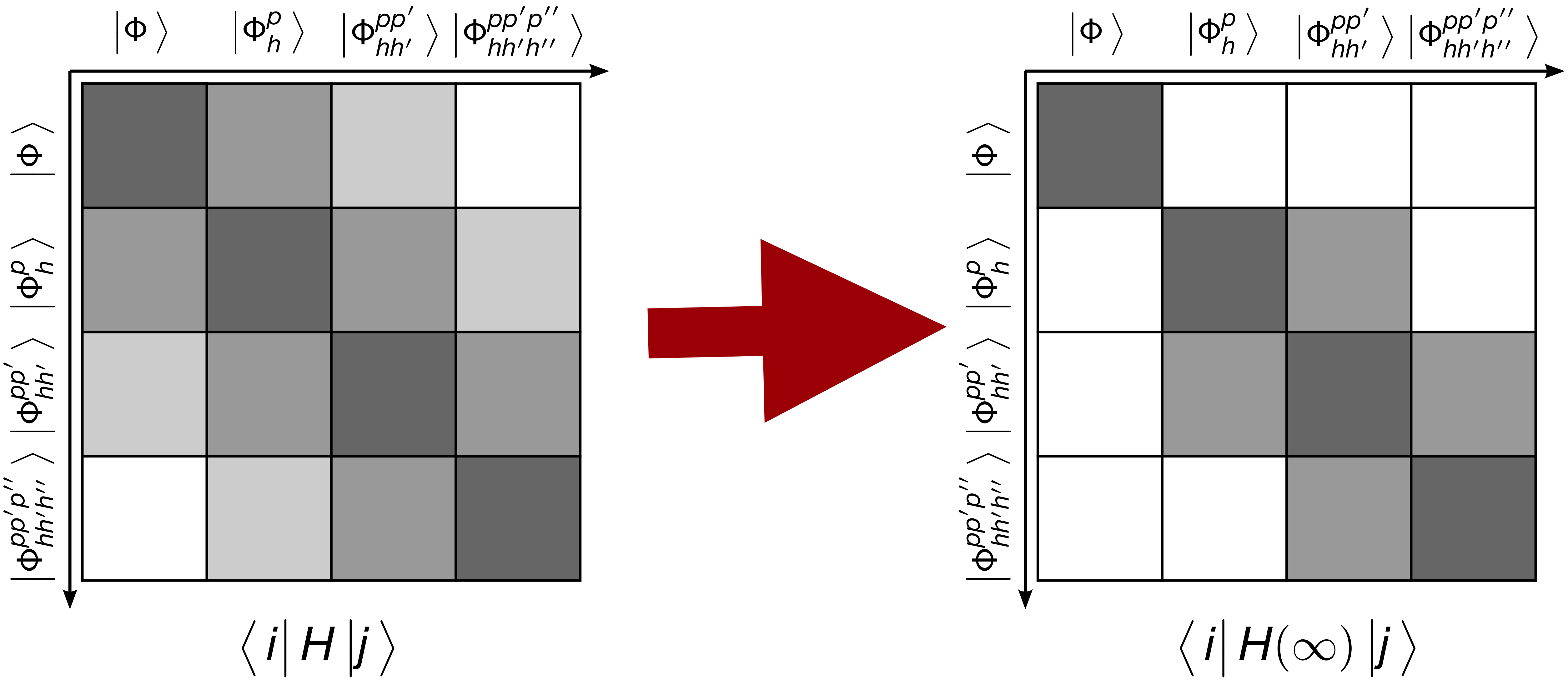
- **Generator Coordinate Method** (w/projections):

$$\left|\Phi\right\rangle = \int dq\, f(q)\, P_{J=0M=0} P_Z P_N \left|q\right\rangle$$

- Density Matrix Renormalization Group, Tensor Network States, …

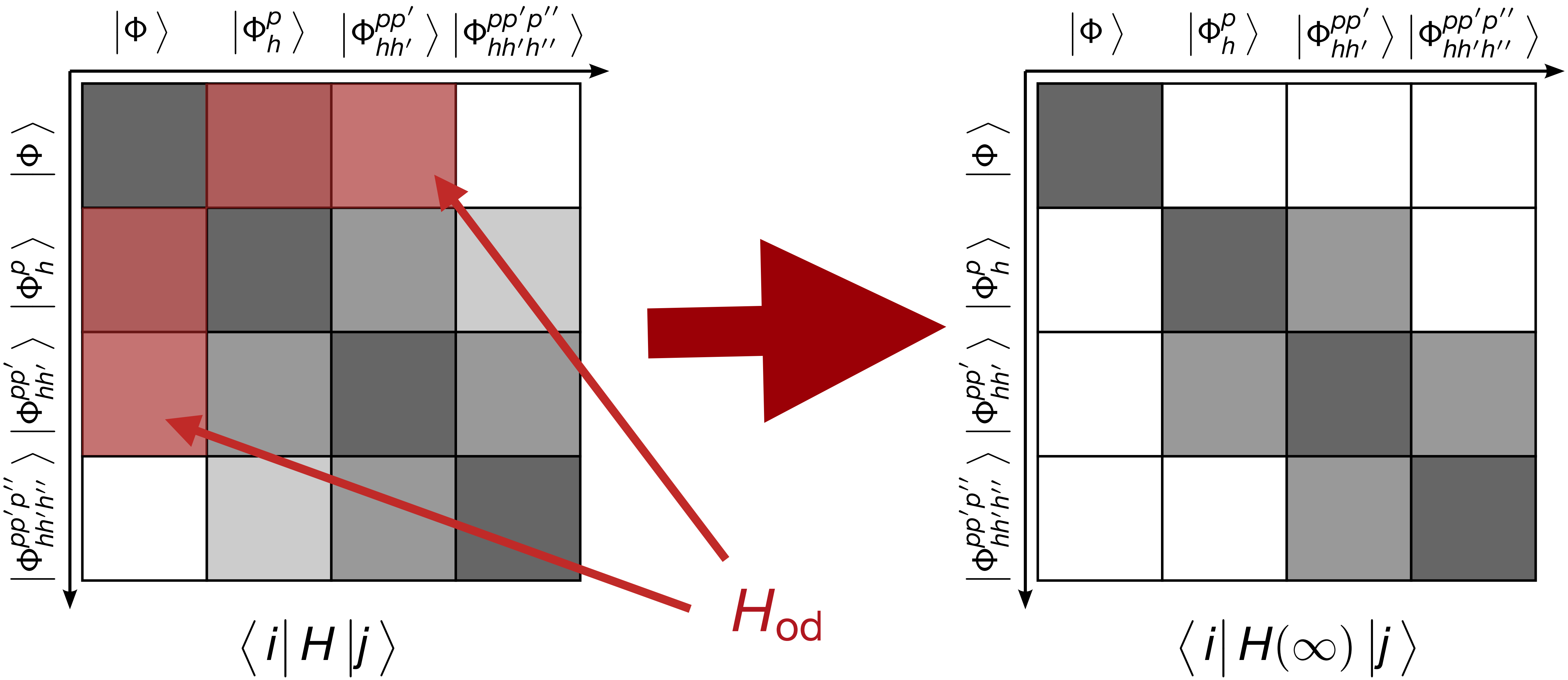
complement particle-hole type correlations

Decoupling in A-Body Space



aim: decouple reference state $|\Phi\rangle$ from excitations

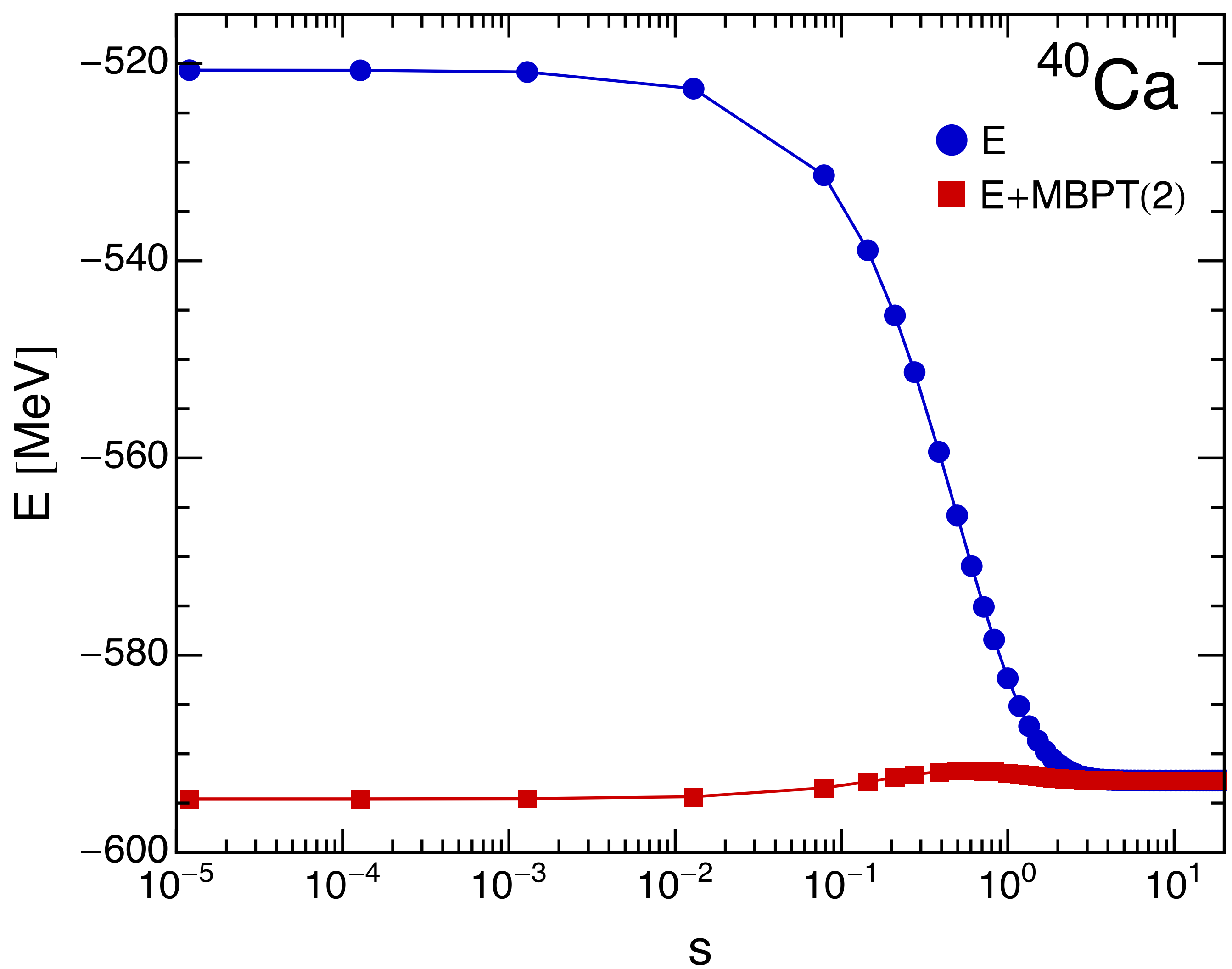
Flow Equation

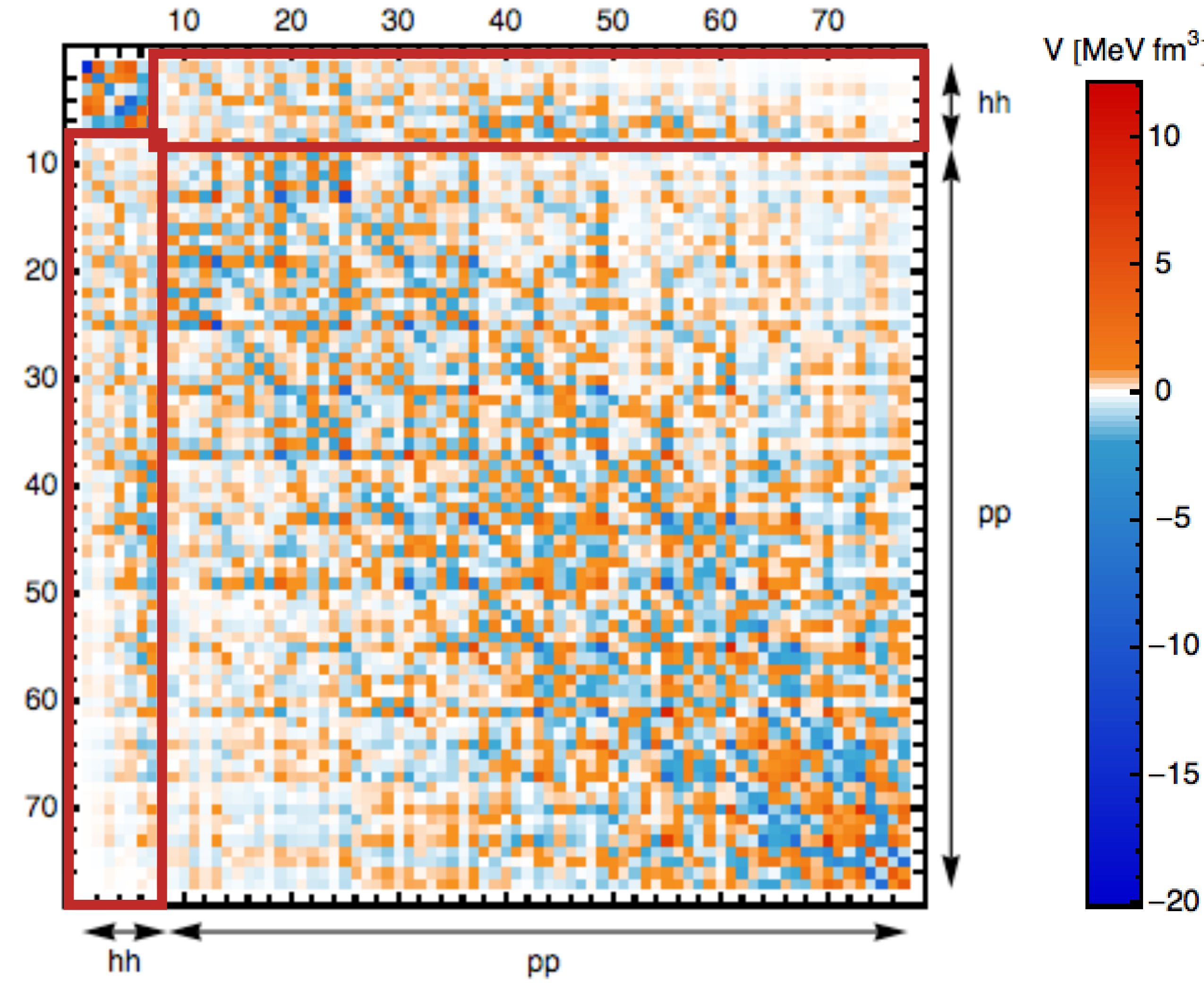


$$\frac{d}{ds}H(s) = \left[\eta(s), H(s)\right], \quad \text{e.g.,} \quad \eta(s) \equiv \left[H_d(s), H_{od}(s)\right]$$

more efficient: solve flow using Magnus methods, **T. D. Morris** et al., PRC **92**, 034331 (2015)

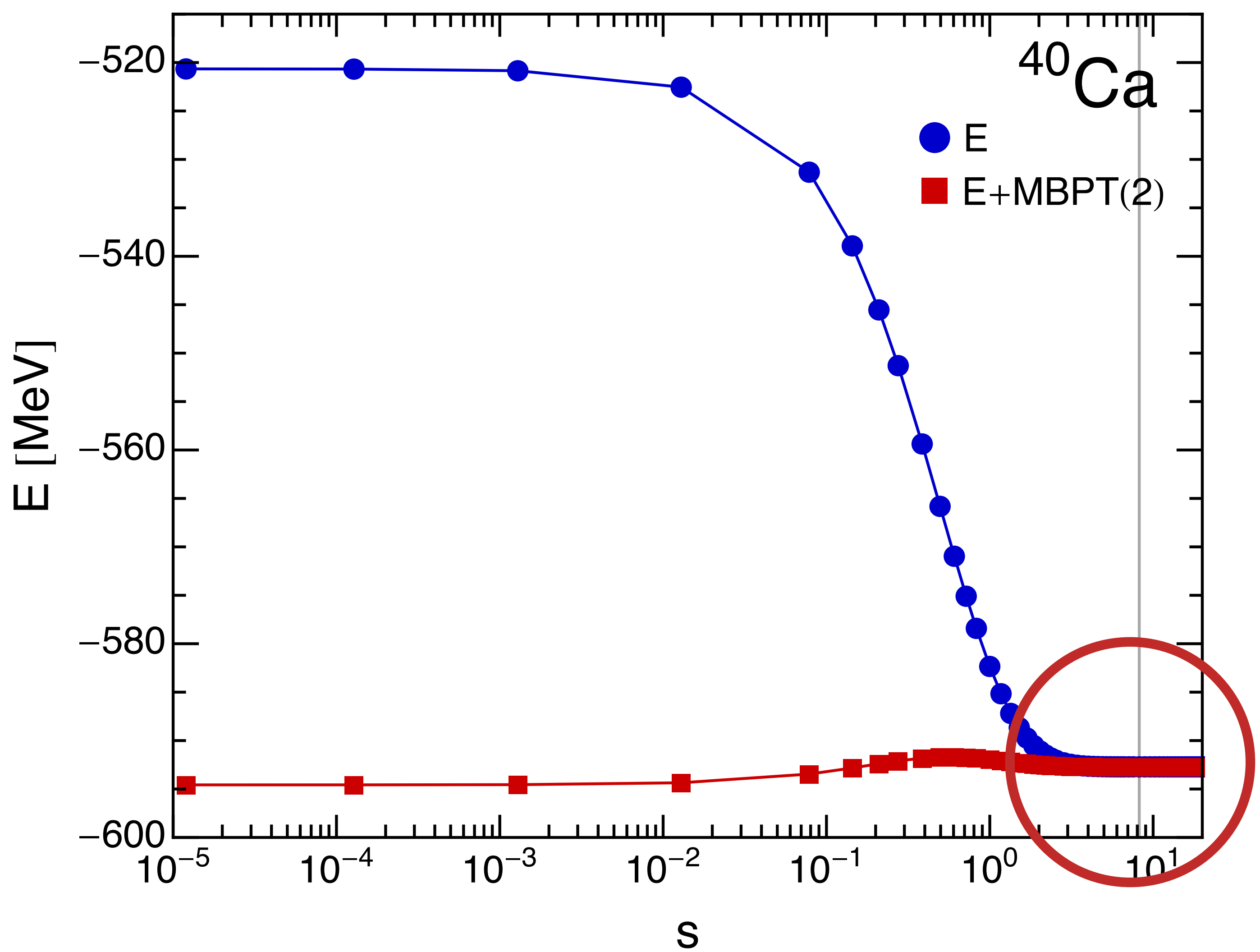
Decoupling

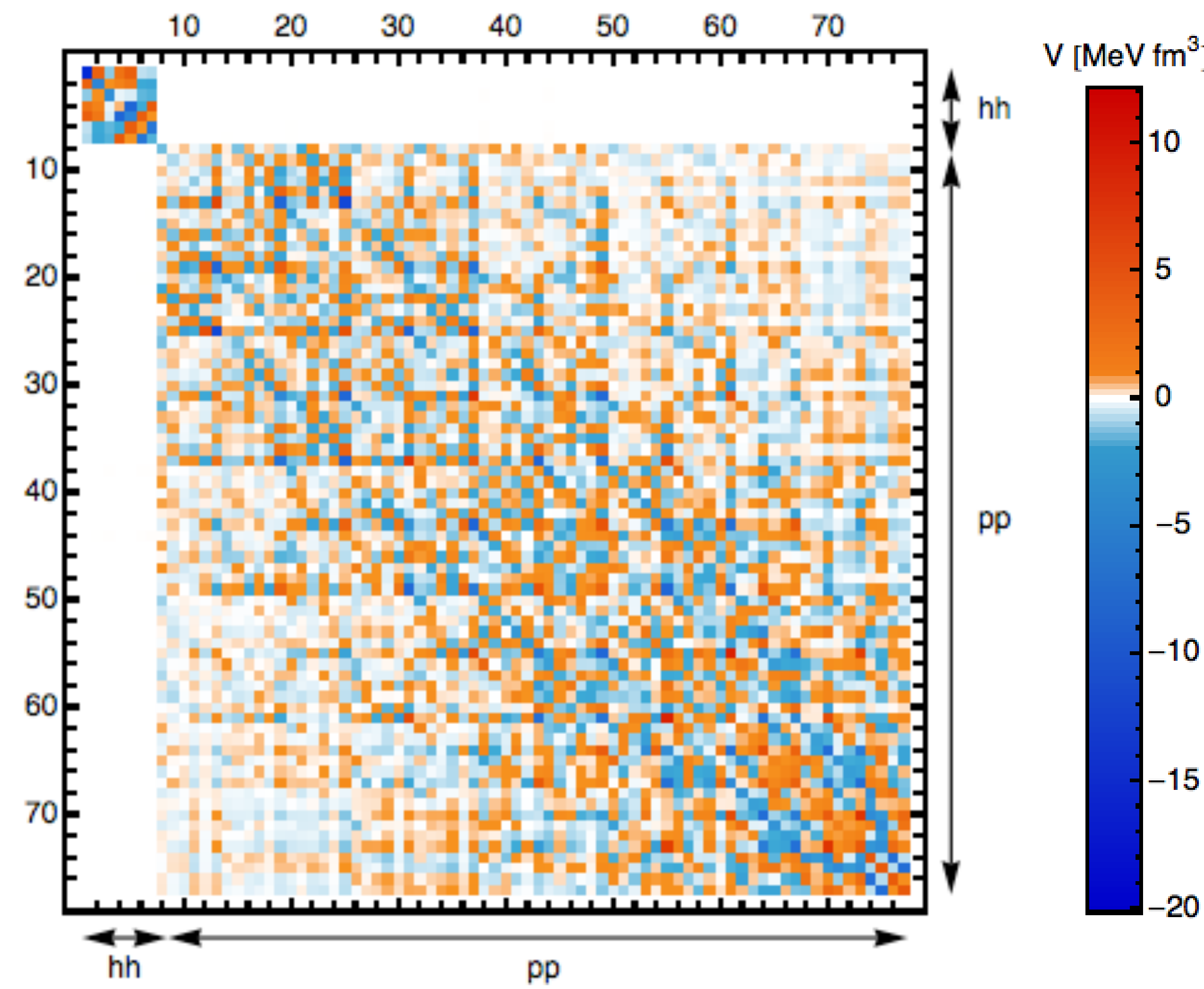




N3LO, $\lambda = 2.0\ \text{fm}^{-1}, e_{\text{Max}} = 8$

Decoupling





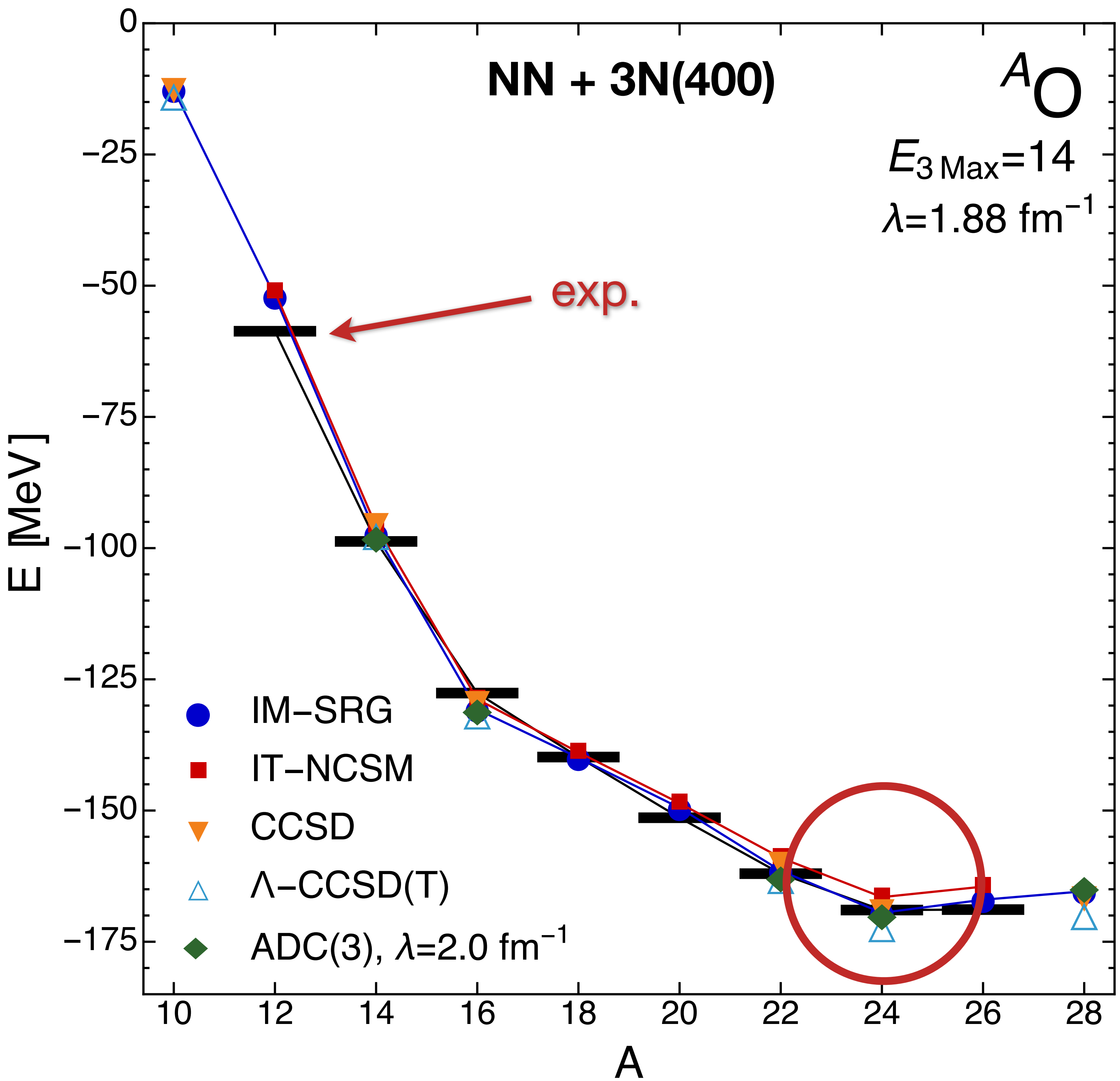
N3LO, $\lambda = 2.0$ fm^{-1}, $e_{\text{Max}} = 8$

non-perturbative resummation of MBPT series (correlations)

off-diagonal couplings are rapidly driven to zero

Oxygen Isotopes

HH et al., PRL **110**, 242501 (2013), ADC(3): A. Cipollone et al., PRL **111**, 242501 (2013)

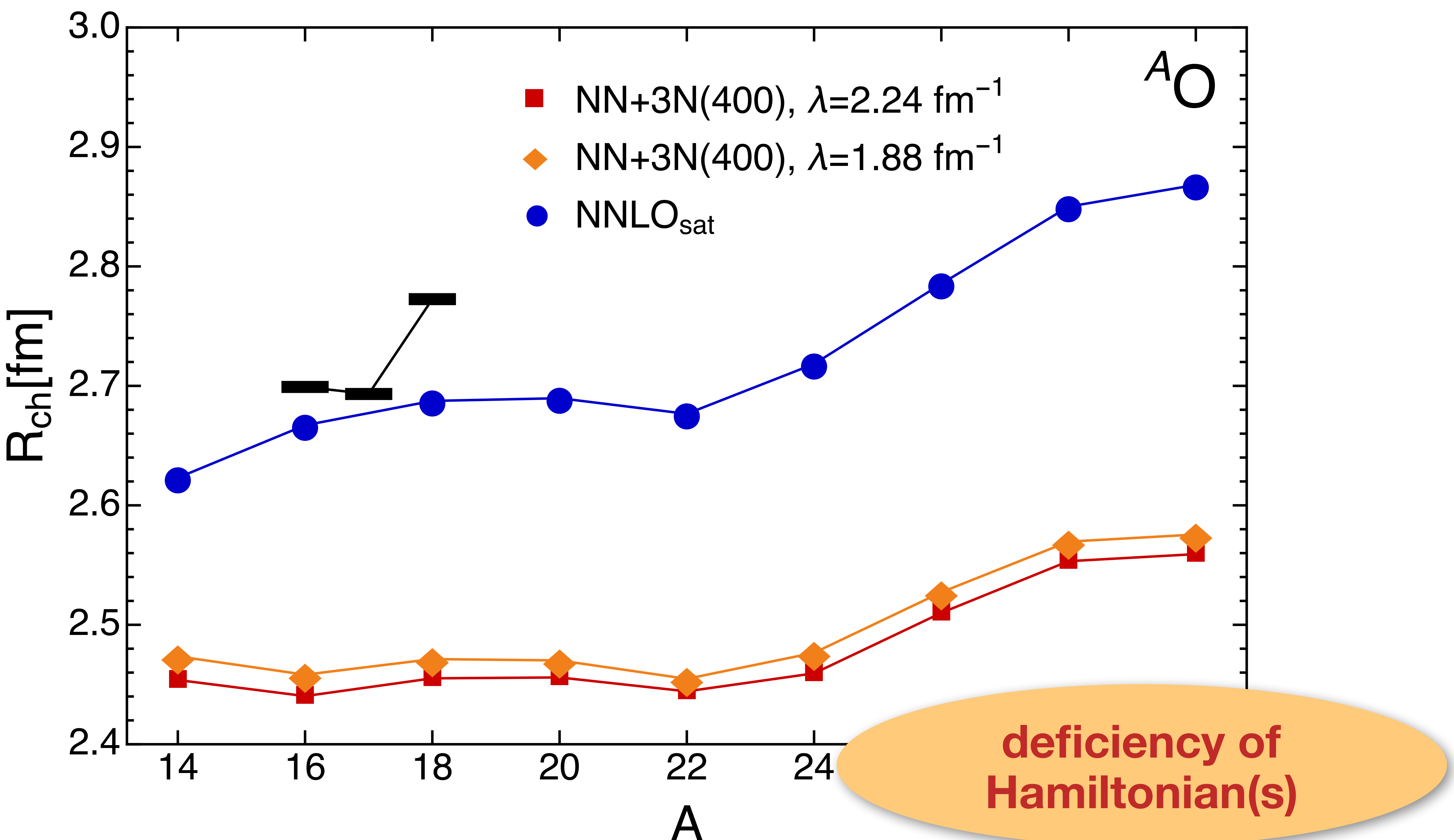


- Multi-Reference IM-SRG with number-projected Hartree-Fock-Bogoliubov as reference state
- consistent results from different many-body methods

correct drip line

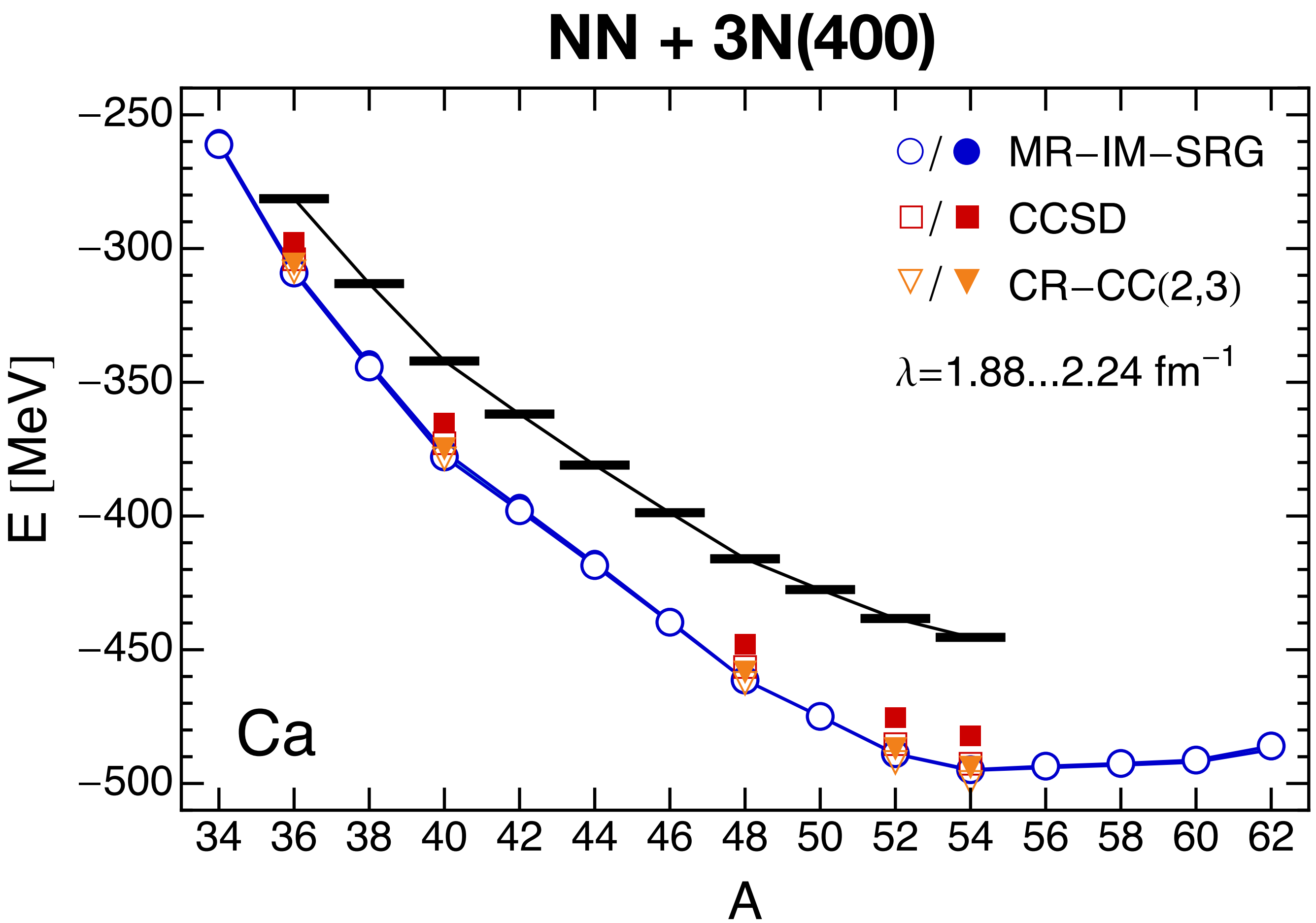
Oxygen Radii

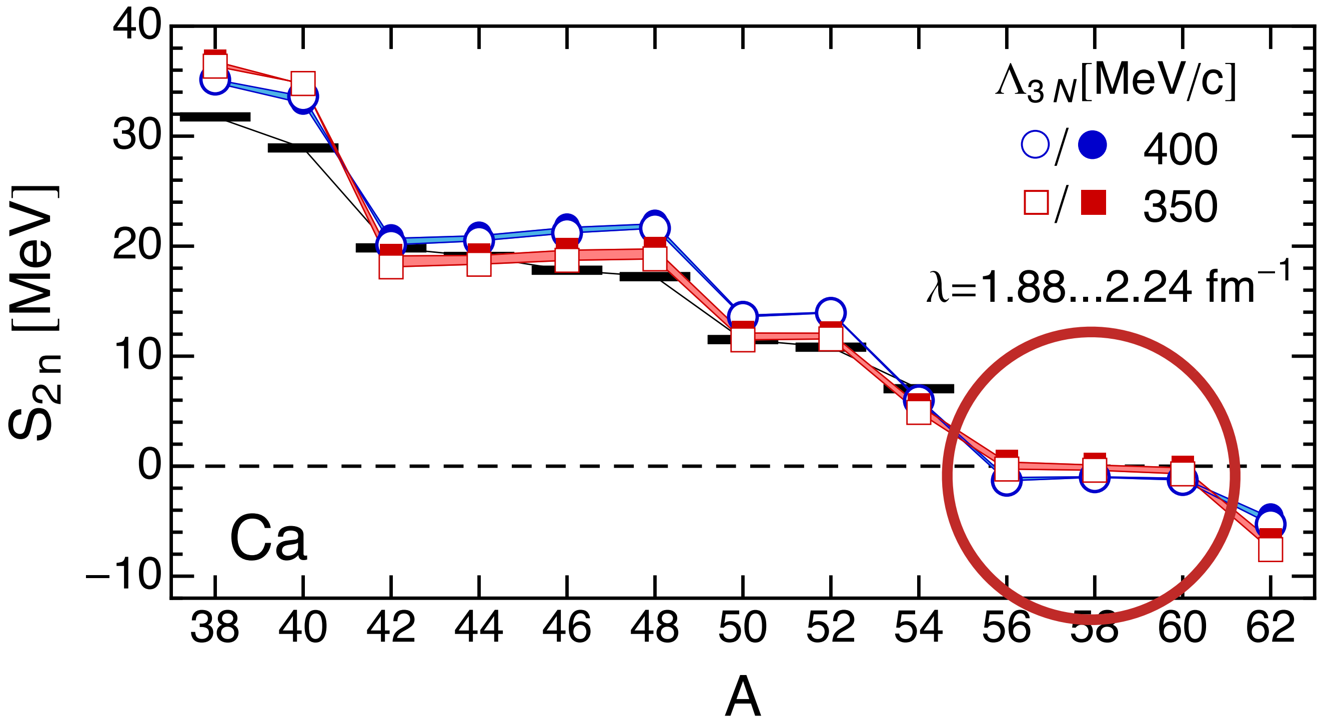
V. Lapoux, V. Somà, C. Barbieri, H. H., J. D. Holt, and S. R. Stroberg, under peer review



Calcium Isotopes

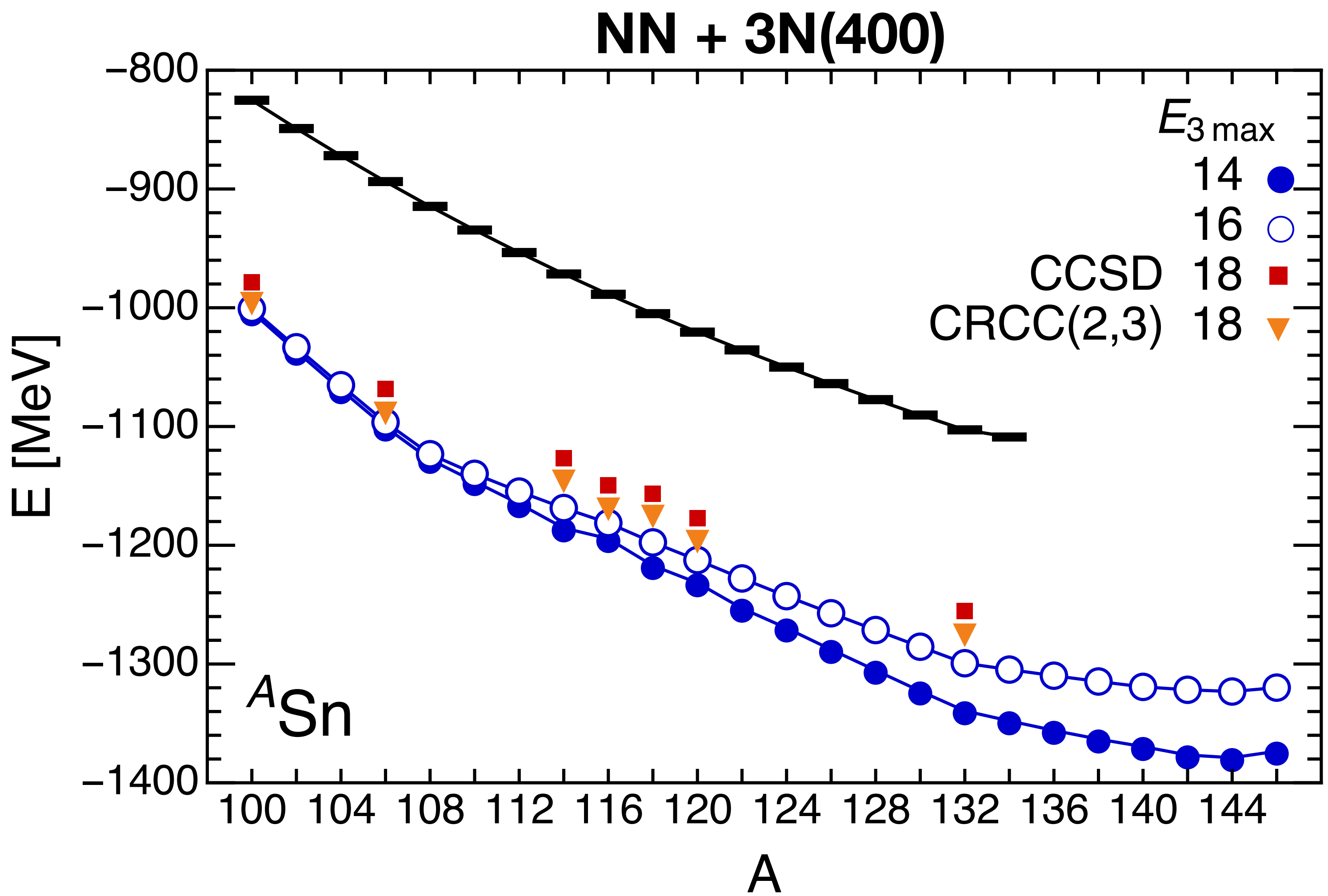
*HH et al., PRC **90**, 041302(R) (2014)*





- differential observables (S_{2n}, spectra,...) filter out interaction components that cause overbinding
- predict flat trends for g.s. energies/S_{2n} beyond ^{54}Ca - await experimental data
- ^{52}Ca, ^{54}Ca magic for these NN+3N interactions
- no continuum coupling yet, other S_{2n} uncertain-ties < 1MeV

The Mass Frontier: Tin



E_{3max}	memory (float) [GB]
14	5
16	~20
18	100+

- systematics of overbinding similar to Ca/Ni
- not converged with respect to 3N matrix element truncation:

$$e_1 + e_2 + e_3 \leq E_{3max}$$

($e_{1,2,3}$: SHO energy quantum numbers)

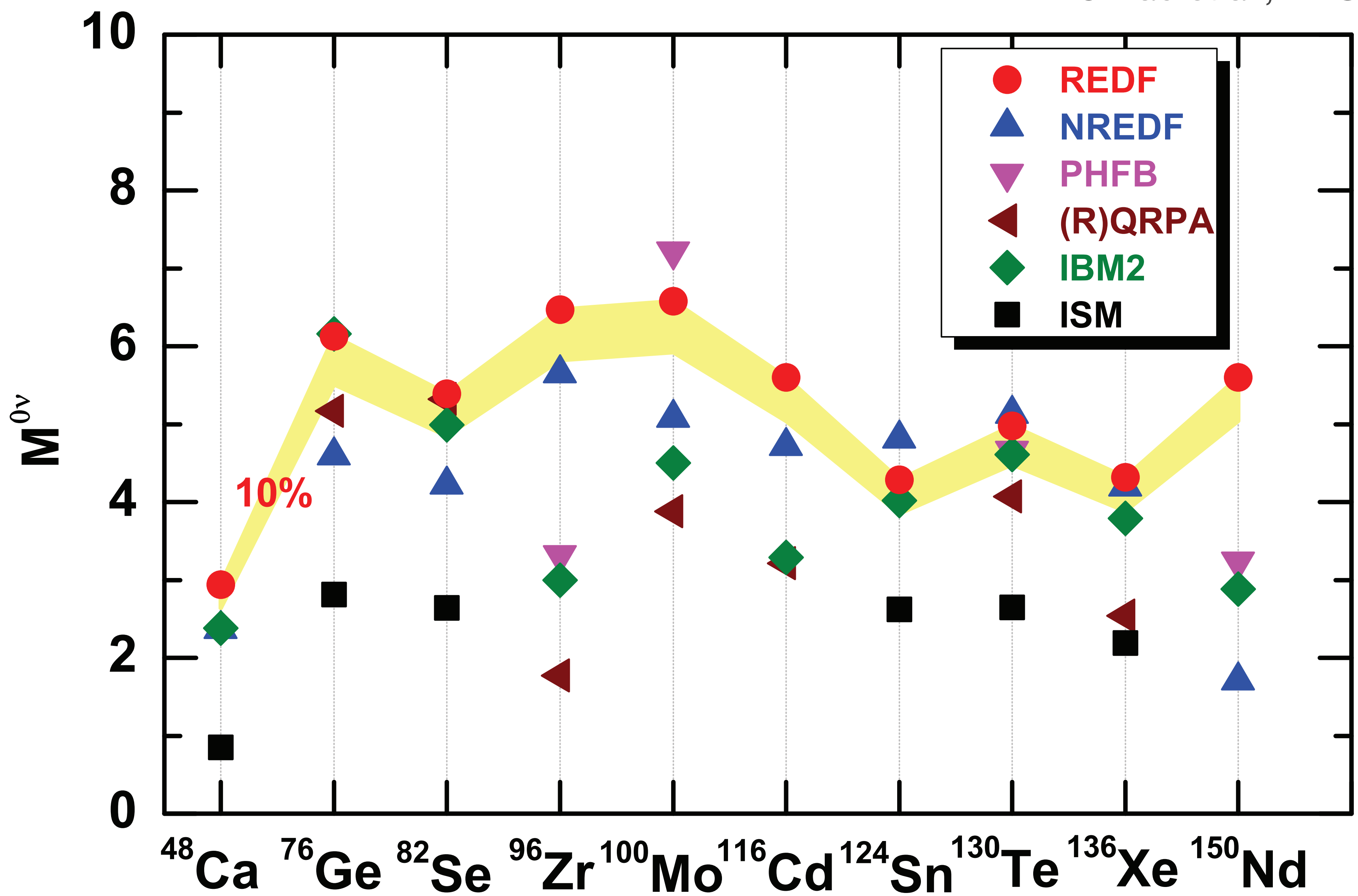
- need technical improvements to go further

Neutrinoless Double Beta Decay:
Ground-State to Ground-State Decay

with **J. Yao**, C. Jiao, J. Engel

Nuclear Matrix Elements

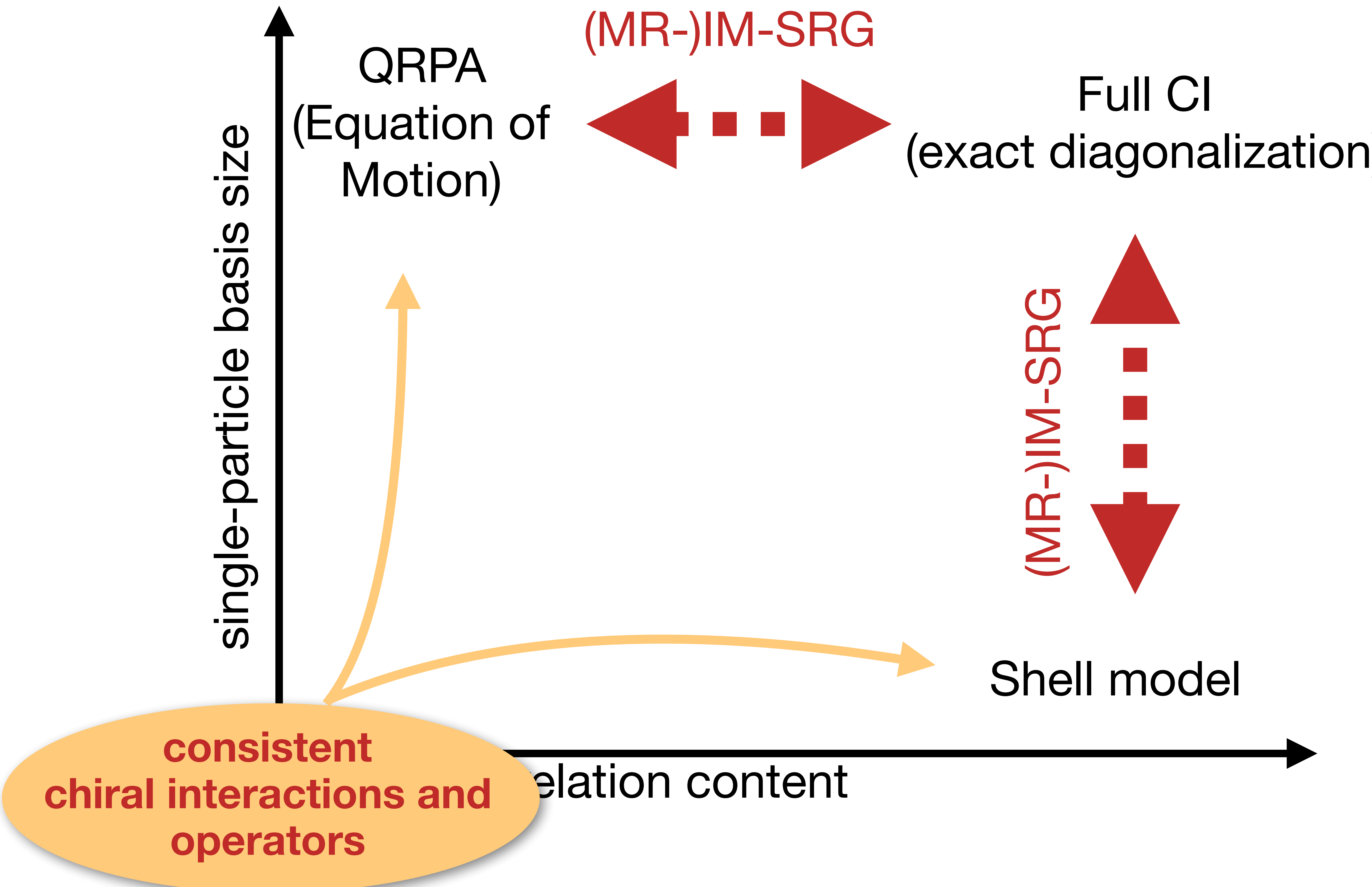
J. Yao et al., PRC **91**, 024316 (2015)



- inputs tailored to specific methods: phenome... EDFs, Shell Model interactions, …
- quenched g_A, "renormalization" of operato...

comparing apples and oranges

Many-Body Approaches



MR-IM-SRG References States

available

- **number-projected Hartree-Fock Bogoliubov** vacua:

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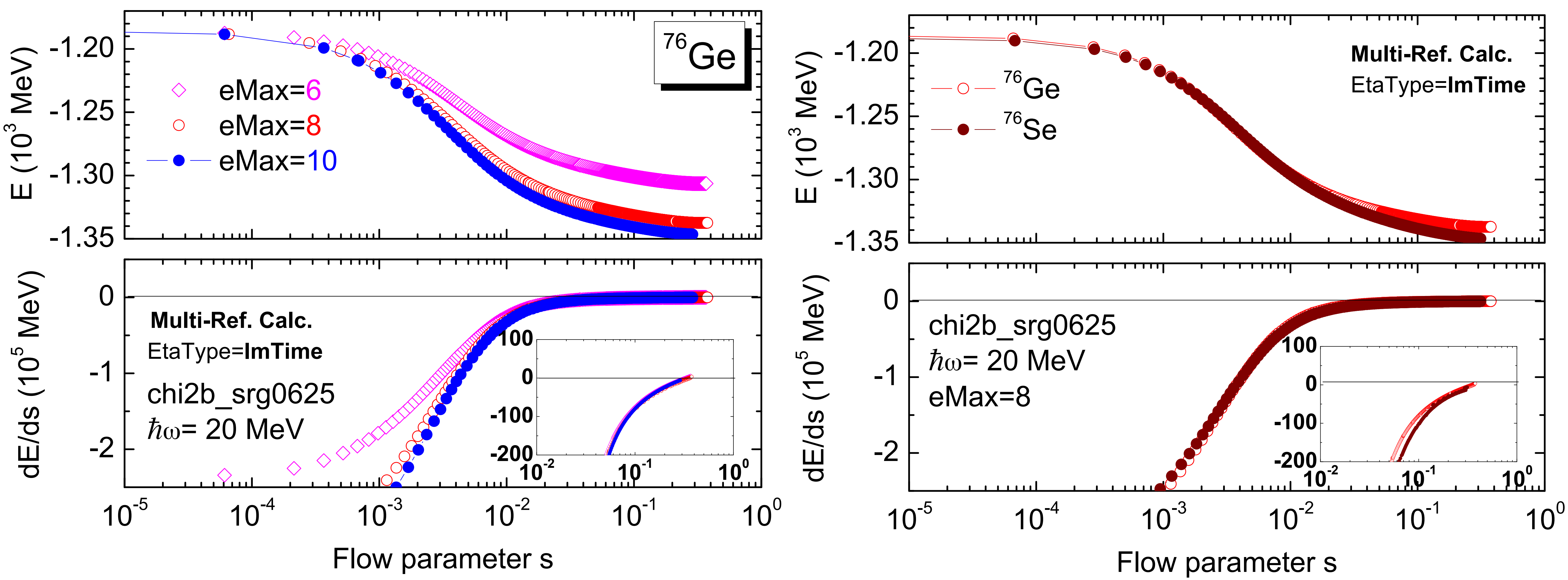
future

- **Generator Coordinate Method** (w/projections):

$$\left|\Phi\right\rangle = \int dq\, f(q)\, P_{J=0M=0} P_Z P_N \left|q\right\rangle$$

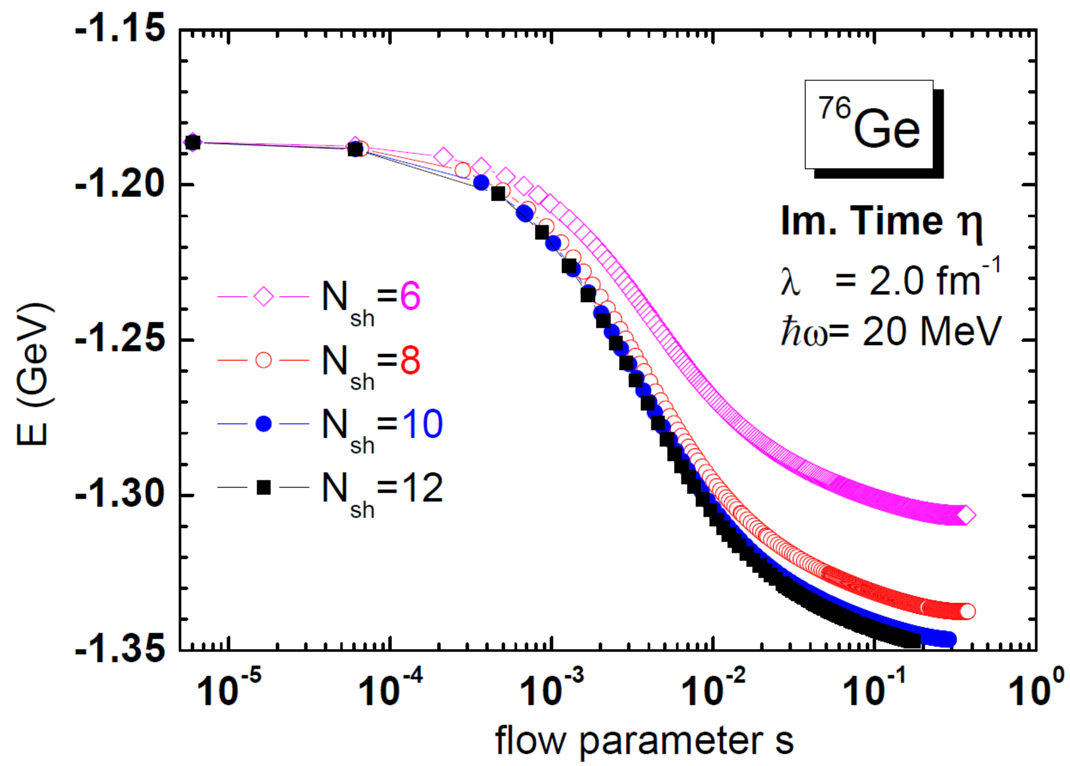
- Density Matrix Renormalization Group, Tensor Network States, …

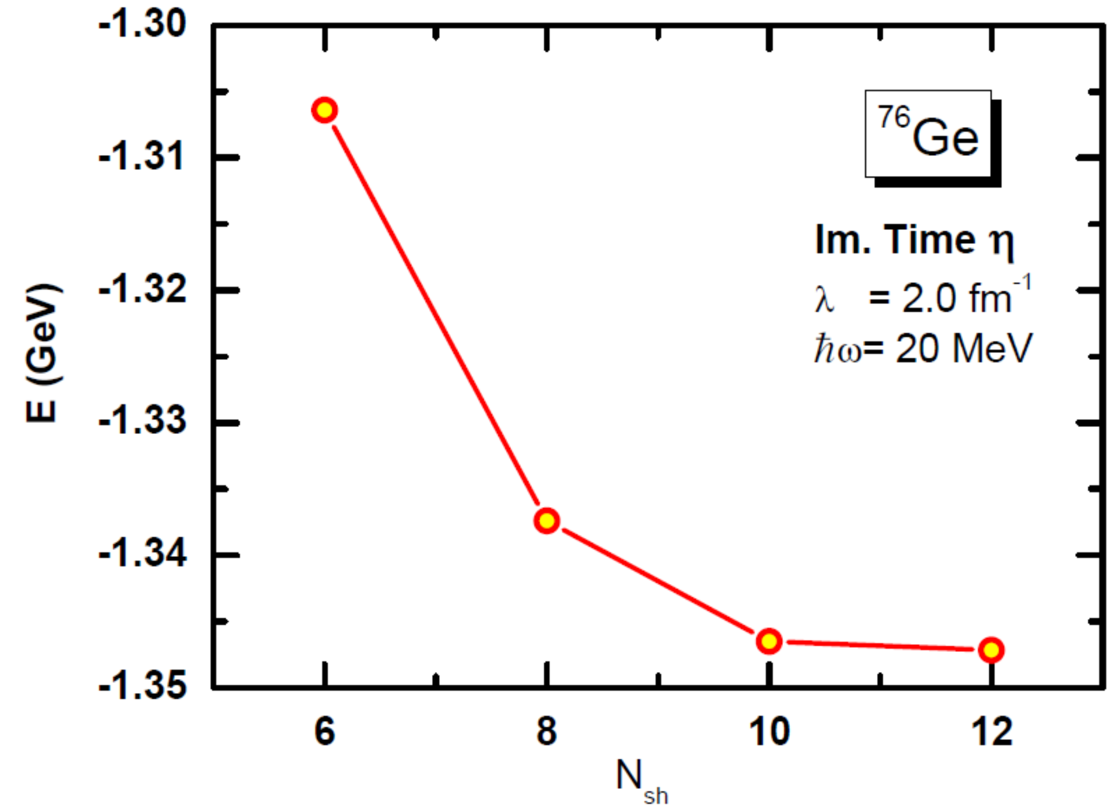
$^{76}Ge \longrightarrow {}^{76}Se$

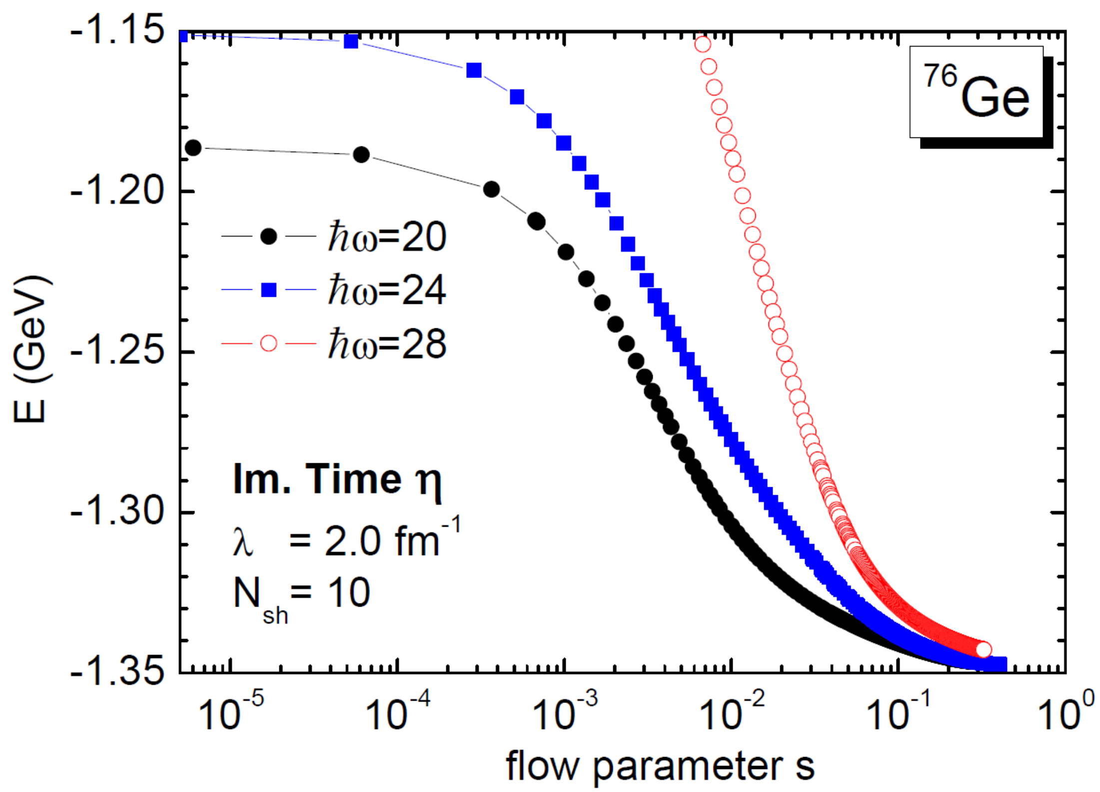


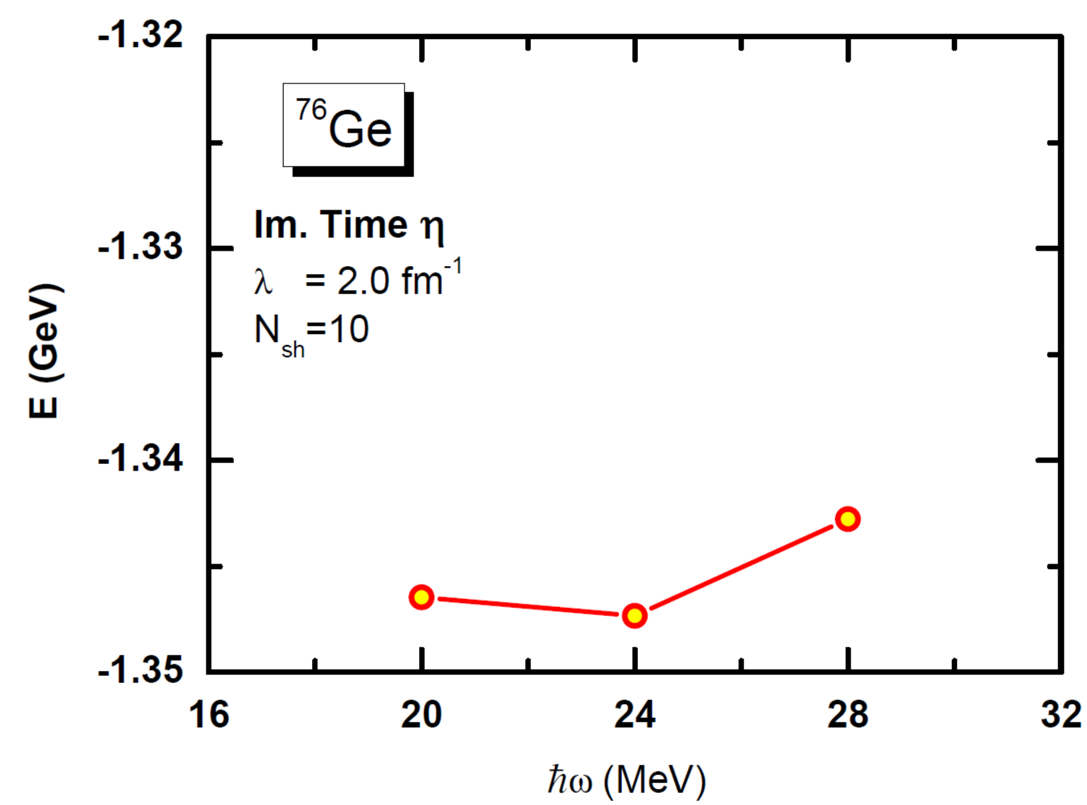
proof of principle: MR-IM-SRG based on **(intrinsically deformed) GCM state converges** ^{76}Ge,^{76}Se ground-state energies

$^{76}Ge \longrightarrow {}^{76}Se$









Direct DBD Calculation

- **direct** MR-IM-SRG (Magnus) calculation of **initial and final states**:

$$\left|\Psi_{I,F}\right\rangle = e^{\overline{\Omega}_{I,F}}\left|\Phi_{I,F}\right\rangle$$

- evaluate NME for transition operator in **closure approximation**:

$$M_{0\nu\beta\beta} = \left\langle \Phi_F \right| e^{-\overline{\Omega}_F} O_{0\nu\beta\beta} e^{\overline{\Omega}_I} \left|\Phi_I\right\rangle$$

- explore possible expansions and check consistency, e.g.,

$$e^{-\overline{\Omega}_F} = e^{-(\overline{\Omega}_I + \delta\overline{\Omega})} = e^{-\delta\overline{\Omega}} e^{-\overline{\Omega}_I} + \ldots$$

in progress

Neutrinoless Double Beta Decay:

Explicit Treatment of Excited States

N. M. Parzuchowski, S. K. Bogner, in preparation

S. R. Stroberg, H. H., J. D. Holt, S. K. Bogner, A. Schwenk, PRC93, 051301(R) (2016)

S. K. Bogner, H. H., J. D. Holt, A. Schwenk, S. Binder, A. Calci, J. Langhammer, R. Roth, Phys. Rev. Lett. 113, 142501 (2014)

K. Tsukiyama, S. K. Bogner, and A. Schwenk, Phys. Rev. C **85**, 061304(R) (2012)

Equation-of-Motion Method

- describe "excited states" based on reference state:

$$\left|\Phi_k\right\rangle = Q_k^\dagger \left|\Phi_0\right\rangle$$

- **(MR-)IM-SRG effective Hamiltonian** in EOM approach:

$$\left[H(s), Q_k^\dagger(s)\right] = \omega_k(s) Q_k^\dagger(s), \quad \omega_k(s) = E_k(s) - E_0(s)$$

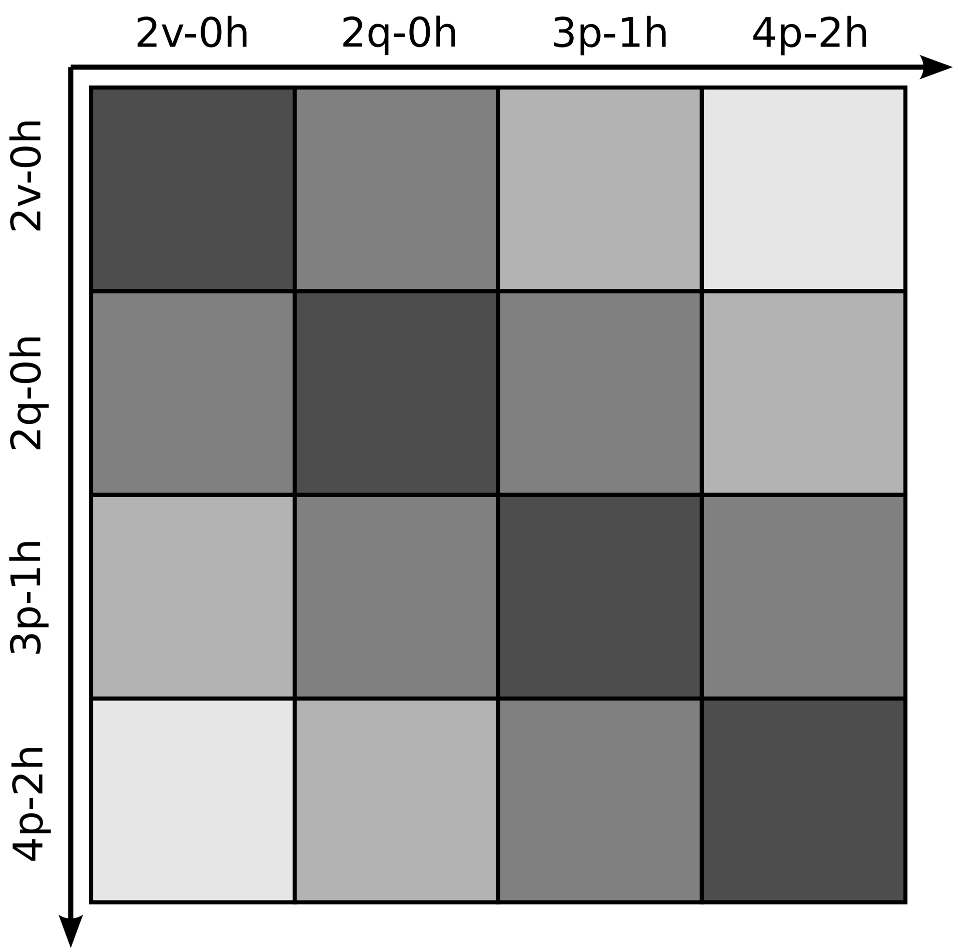
- ansatz for excitation operator (**g.s. correlations built into Hamiltonian**):

$$Q_k^\dagger(s) = \sum_{ph} q_h^p(s) :A_h^p: + \frac{1}{4} \sum_{pp'hh'} q_{hh'}^{pp'}(s) :A_{hh'}^{pp'}:$$

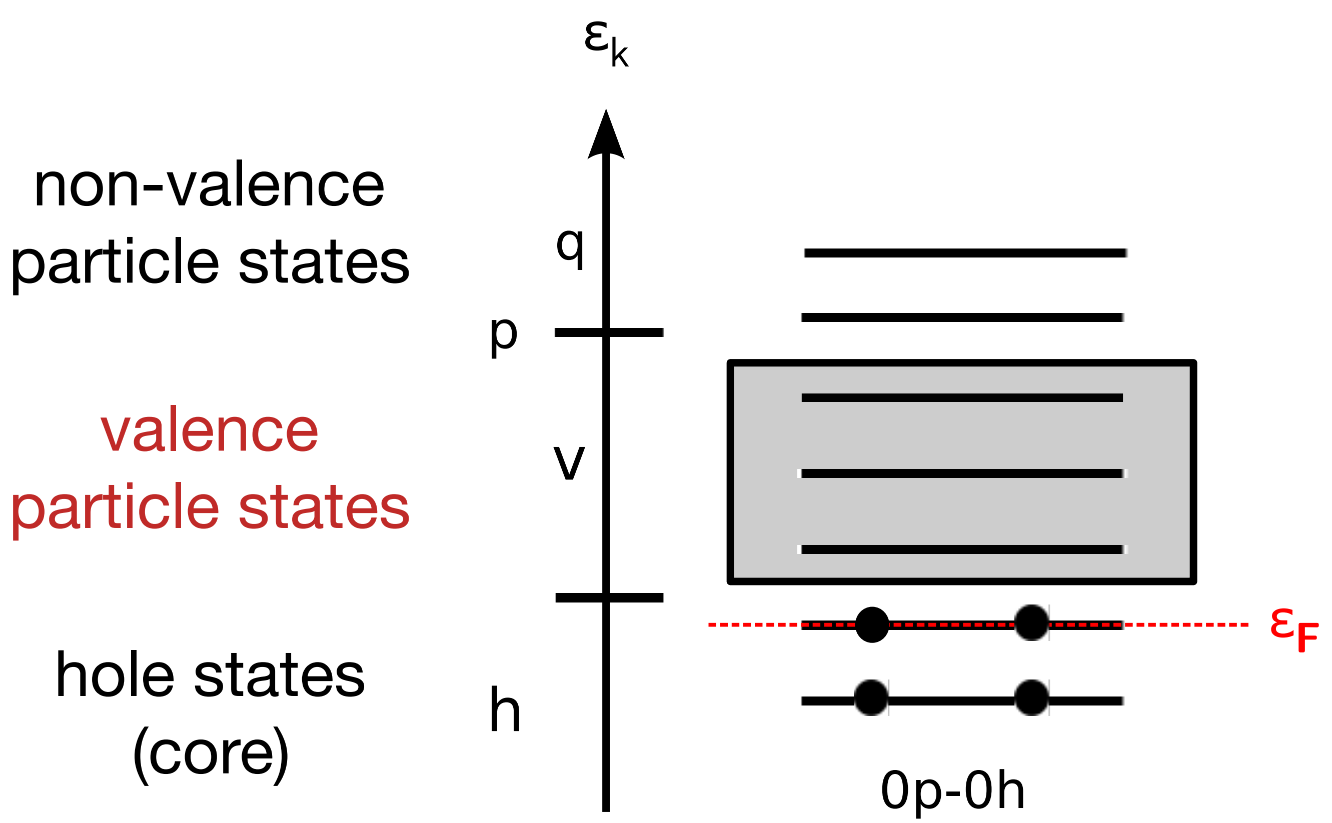
- **polynomial** effort vs. factorial scaling of Shell Model

- can exploit multi-reference capabilities (commutator formulation identical to flow equations)

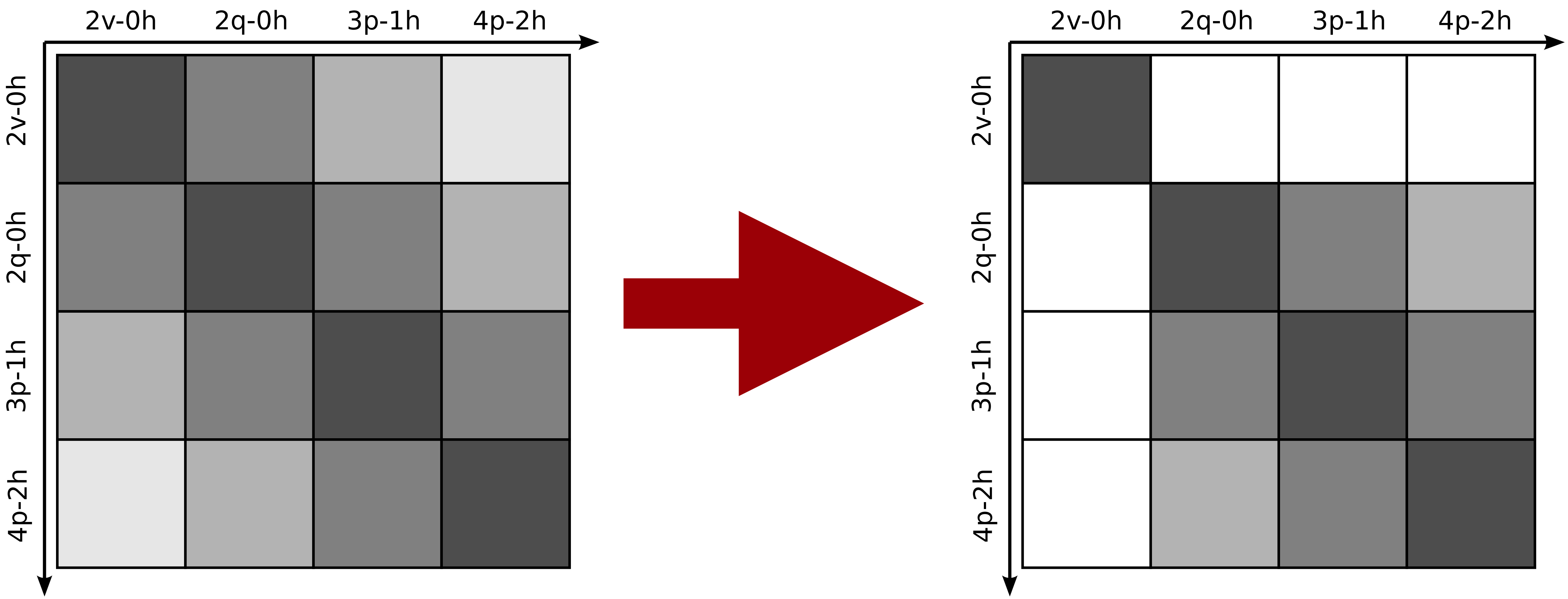
Valence Space Decoupling



$\langle i | H | j \rangle$



Valence Space Decoupling



$\langle i | H | j \rangle$ $\qquad$ $\langle i | H(\infty) | j \rangle$

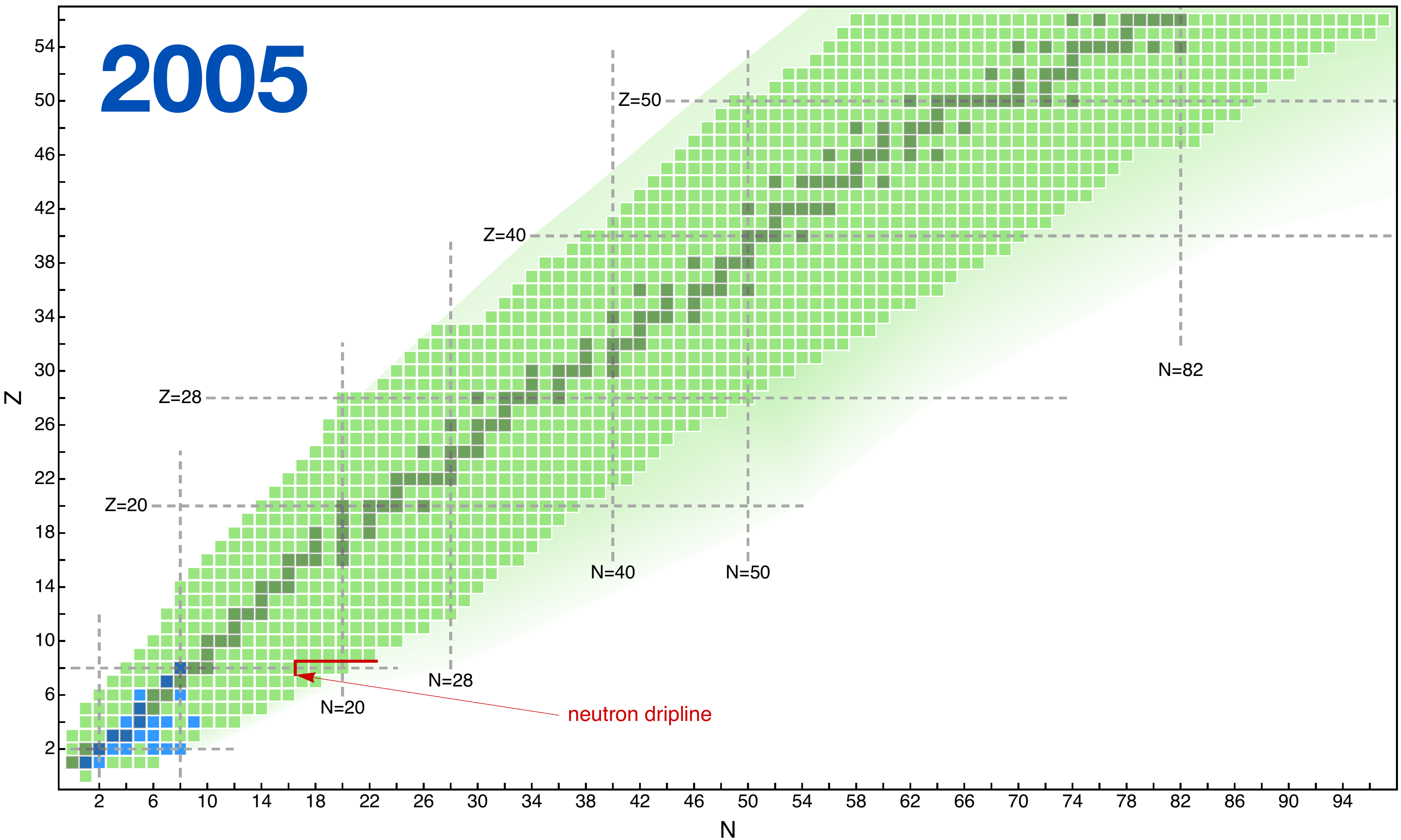
consistent interaction and DBD operator for Shell Model
(talk by R. Stroberg)

off-diagonal Hamiltonian:

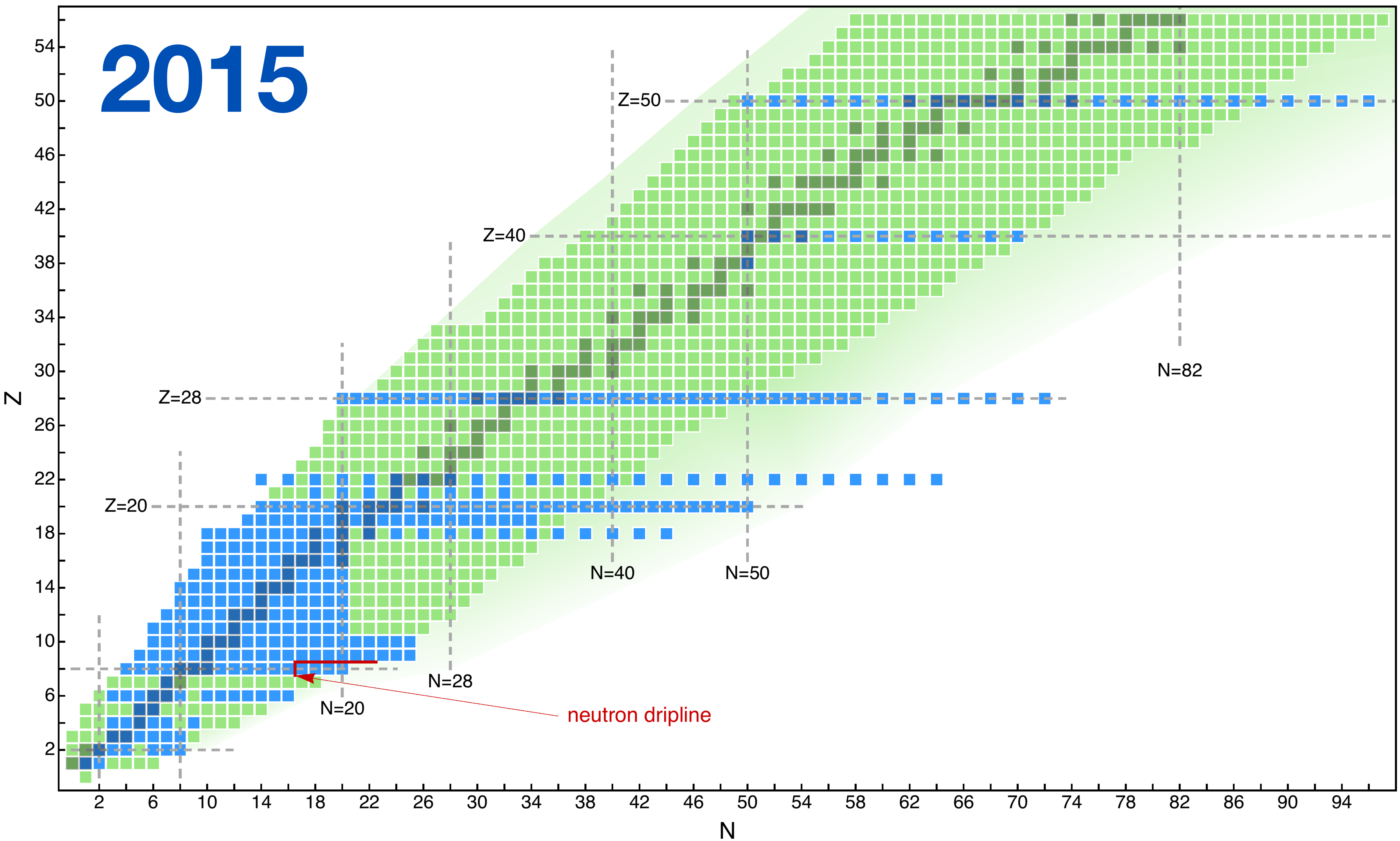
$$\left\{H^{od}\right\} = \{f_{h'}^{h}, f_{p'}^{p}, f_{h}^{p}, f_{v}^{q}, \Gamma_{hh'}^{pp'}, \Gamma_{hv}^{pp'}, \Gamma_{vv'} \}$$

Epilogue

Progress in *Ab Initio* Calculations



Progress in *Ab Initio* Calculations



Summary

- towards ***ab initio*** **NMEs:** interaction, operators, many-body method with **systematic uncertainties** & convergence to exact result
- rapidly **growing capabilities**: g.s. energies, spectra, radii, transitions, …

➡ **ingredients for NME calculation, plus validation through other observables**

- test new generation of chiral Hamiltonians, **greatly improved optimization** - also **more accurate (?)**
    - $\mathrm{NNLO}_{\mathrm{sat}}$, $\mathrm{NNLO}_{\mathrm{sim}}$, EKM / LENPIC interactions, local NNLO, etc.

Acknowledgments

S. K. Bogner, M. Hjorth-Jensen, T. D. Morris, N. M. Parzuchowski, F. Yuan
NSCL, Michigan State University

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ICER

Supplements

Magnus Formulation of the In-Medium SRG

T. D. Morris, N. M. Parzuchowski, S. K. Bogner, in preparation

T. D. Morris, N. M. Parzuchowski, S. K. Bogner, PRC **92**, 034331 (2015)

H. H., S. K. Bogner, **T. D. Morris**, A. Schwenk, and K. Tuskiyama, Phys. Rept. **621**, 165 (2016)

W. Magnus, Comm. Pure and Appl. Math **VII**, 649-673 (1954)

Magnus Series Formulation

- explicit exponential ansatz for unitary transformation:

$$U(s) = \mathcal{S} \exp \int_0^s ds' \eta(s') \equiv \exp \Omega(s)$$

- flow equation for **Magnus** operator :

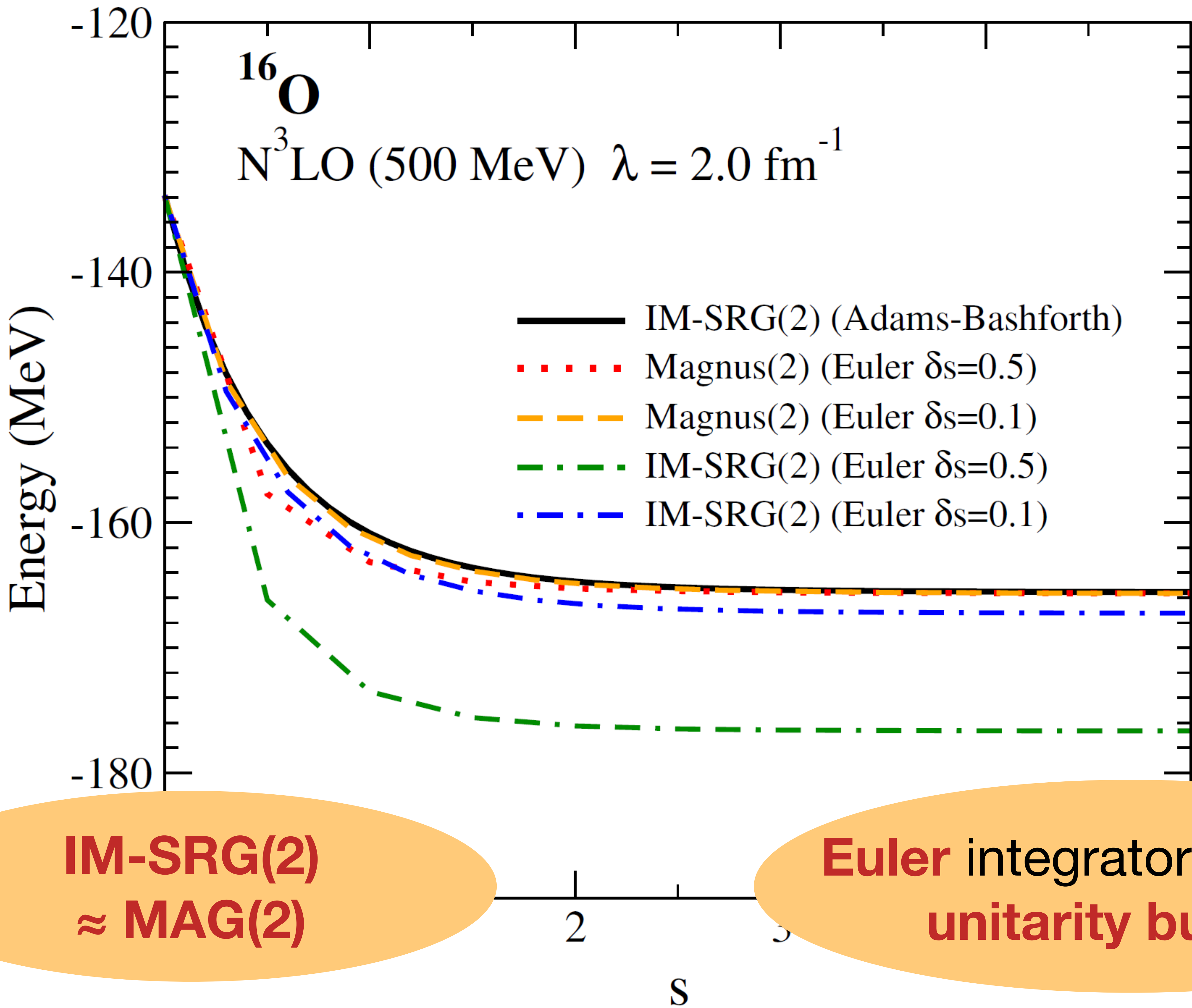
$$\frac{d}{ds}\Omega = \sum_{k=0}^{\infty} \frac{B_k}{k!} \mathrm{ad}_{\Omega}^{k}(\eta) \ , \quad \mathrm{ad}_{\Omega}(O) = [\Omega, O]$$

(B_k: Bernoulli numbers)

- construct $O(s) = U(s) O_0 U^{\dagger}(s)$ using Baker-Campbell-Hausdorff expansion (**Hamiltonian + effective operators**)

- MAG(2): **two-body truncation** (as in NO2B, IM-SRG(2))

Magnus vs. Direct Integration



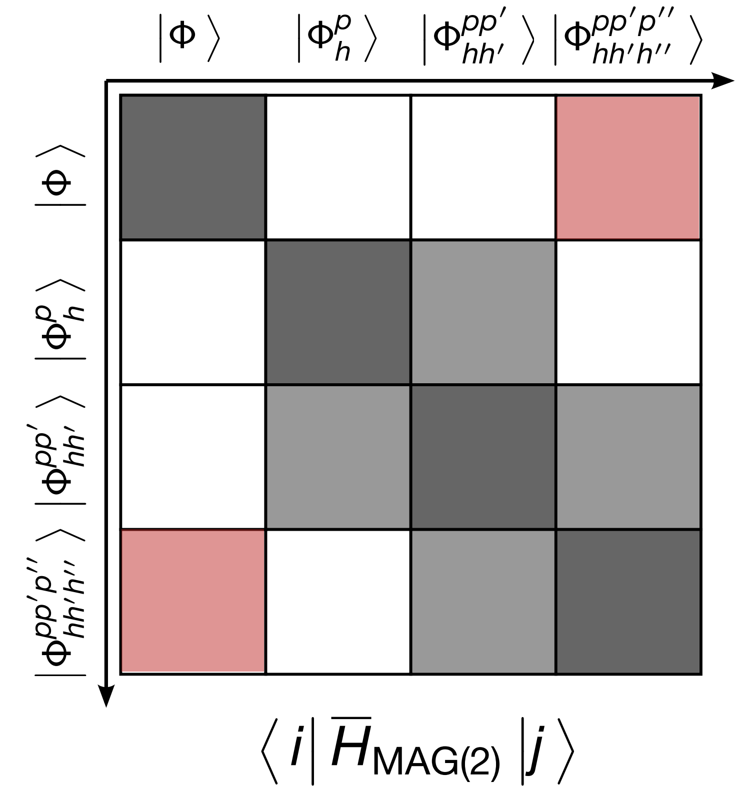
IM-SRG(2) ≈ MAG(2)

Euler integrator sufficient, **unitarity built in!**

Approximating IM-SRG(3)

- final Hamiltonian (IM-SRG(2), MAG(2)):

$$\overline{H}_{\mathrm{MAG(2)}} = \left(e^{\Omega_{1,2}} H e^{-\Omega_{1,2}}\right) = \overline{H}_{0,1,2} + \overline{H}_3 + \ldots$$

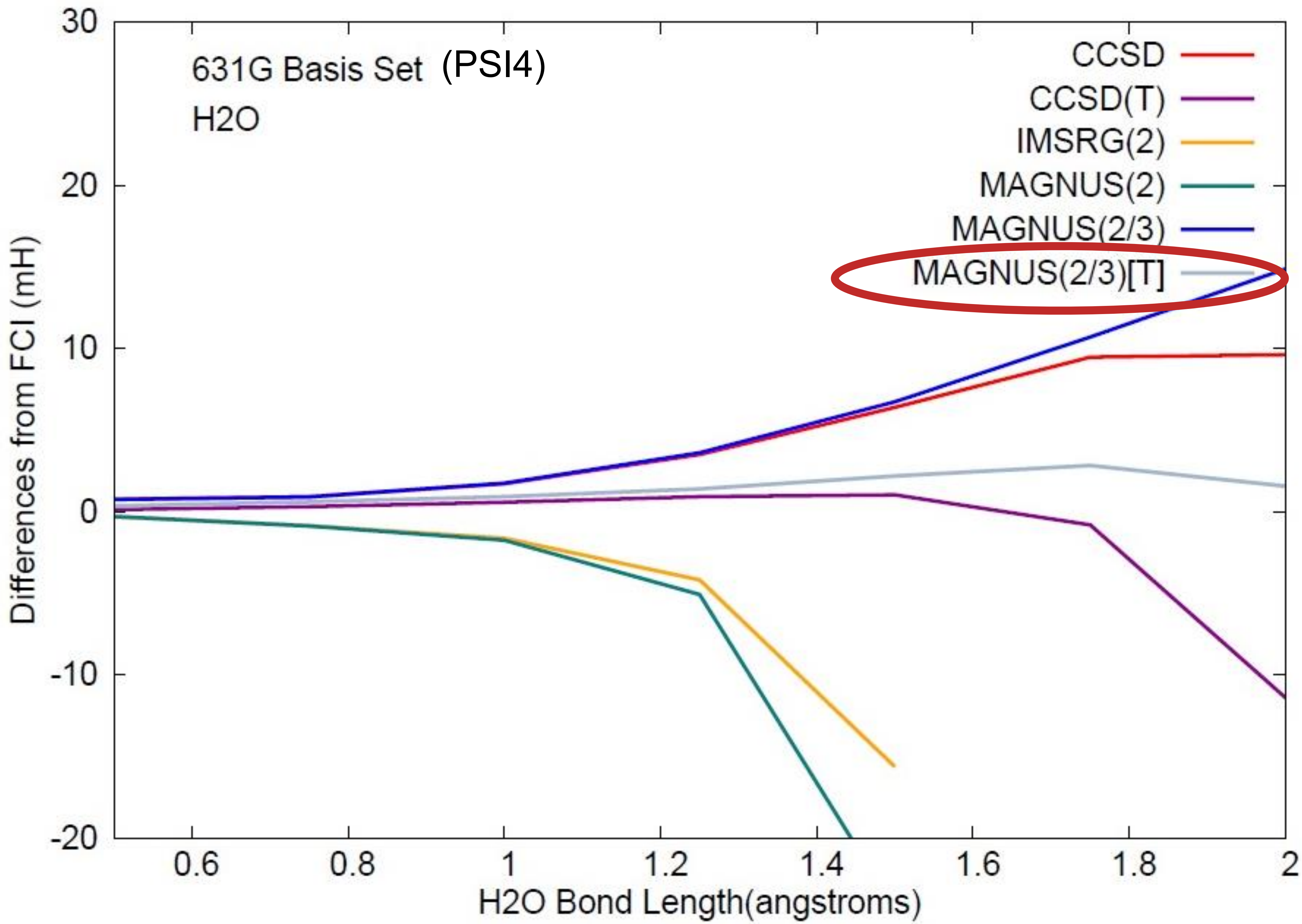


- energy contribution of $\overline{H}_3$ (cf. 0B flow):

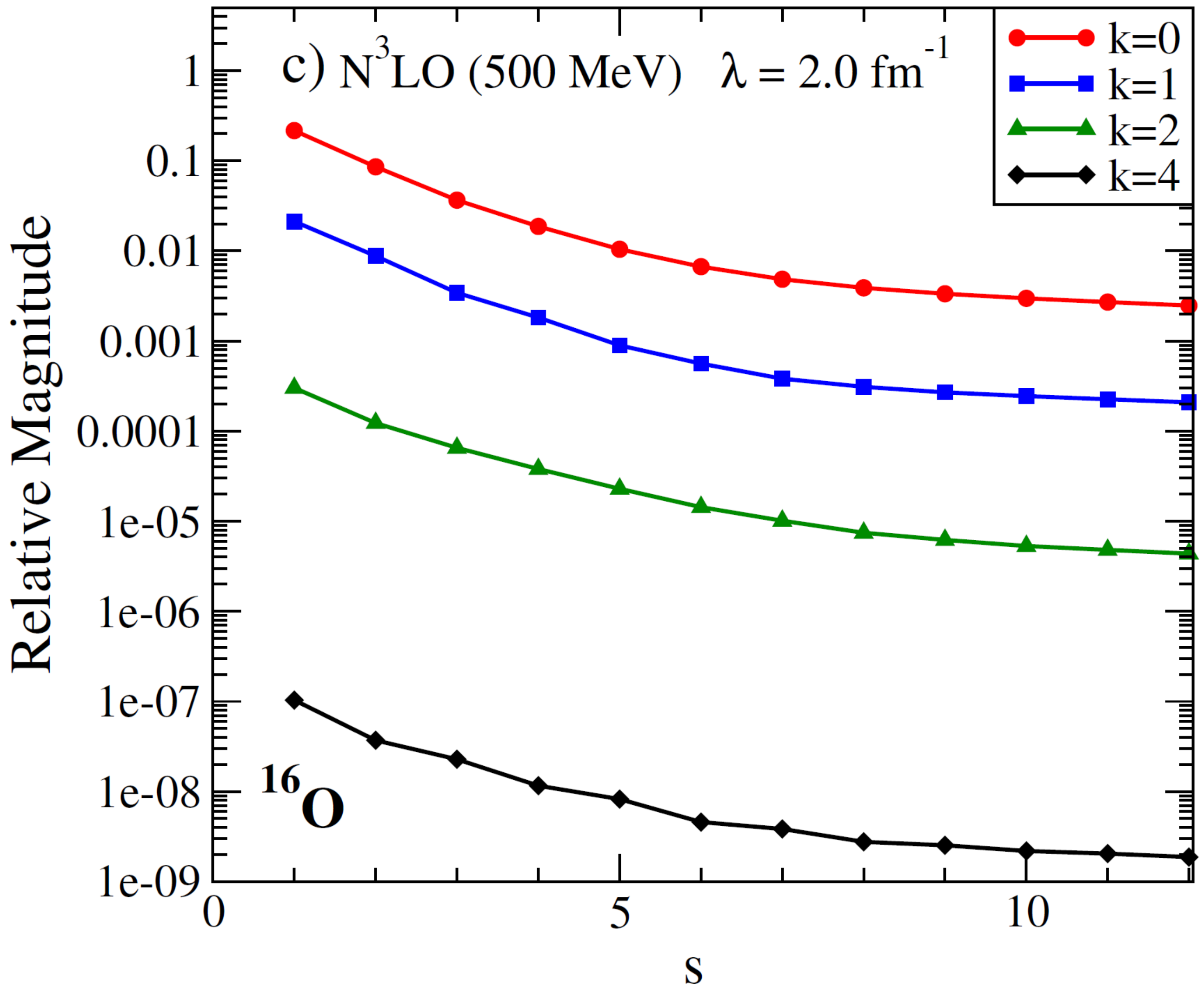
$$\Delta E_3 = \frac{1}{36} \sum_{pp'p''hh'h''} \frac{\left|(\overline{H}_3)_{pp'p''hh'h''}\right|^2}{\overline{\Delta}_{pp'p''hh'h''}}$$

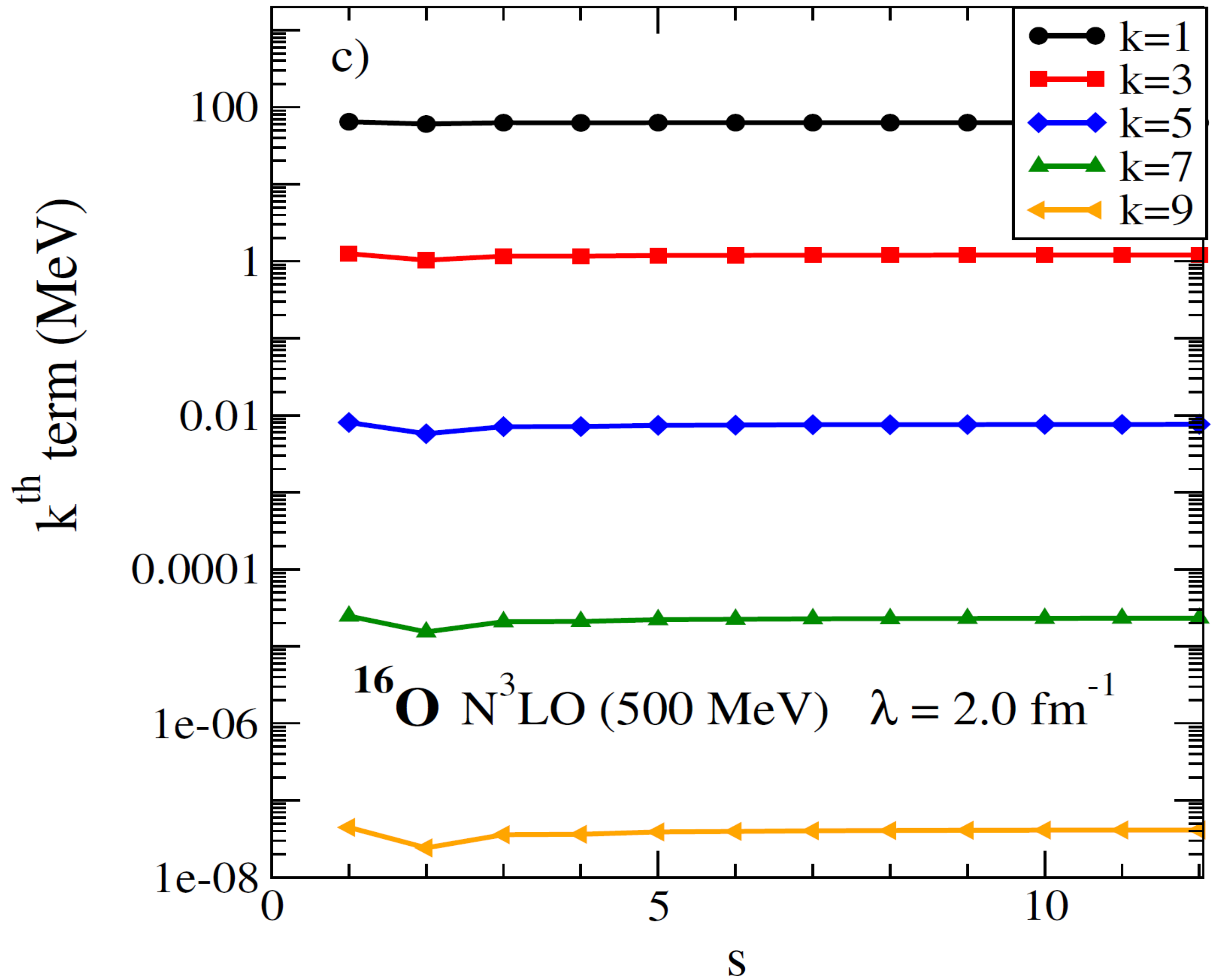
- **family of non-iterative methods**: level of approximation for $\overline{H}_3$ and energy denominator $\overline{\Delta}$
- generalizes to **arbitrary observables, excited states**
- multi-reference variant in development

Example: Bond Breaking in Water



Convergence



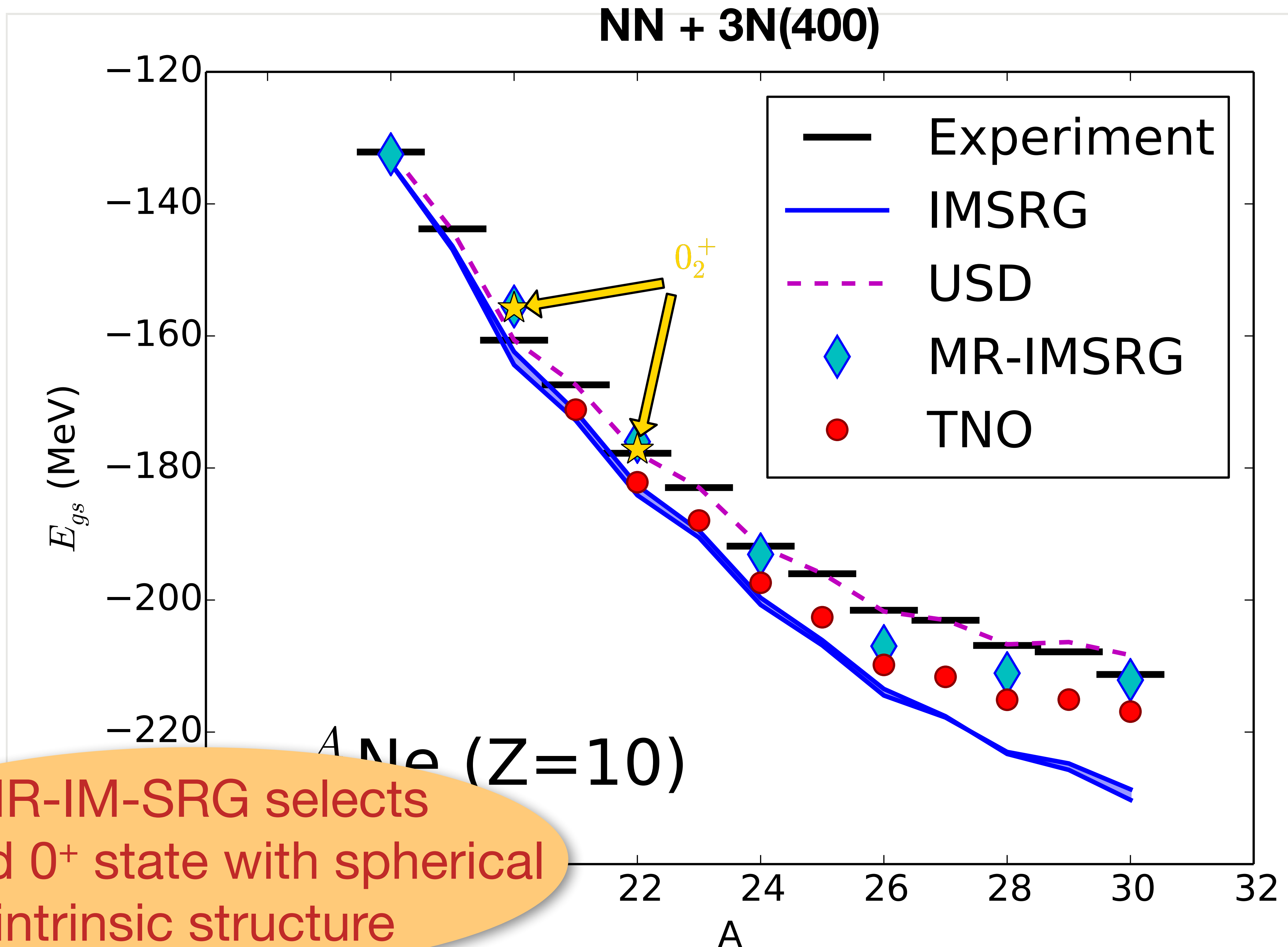


$$\frac{d}{ds}\Omega = \sum_k \frac{B_k}{k!} \mathrm{ad}_\Omega^k(\eta)$$

$$H(s) = \sum_k \frac{1}{k!} \mathrm{ad}_\Omega^k(H_0)$$

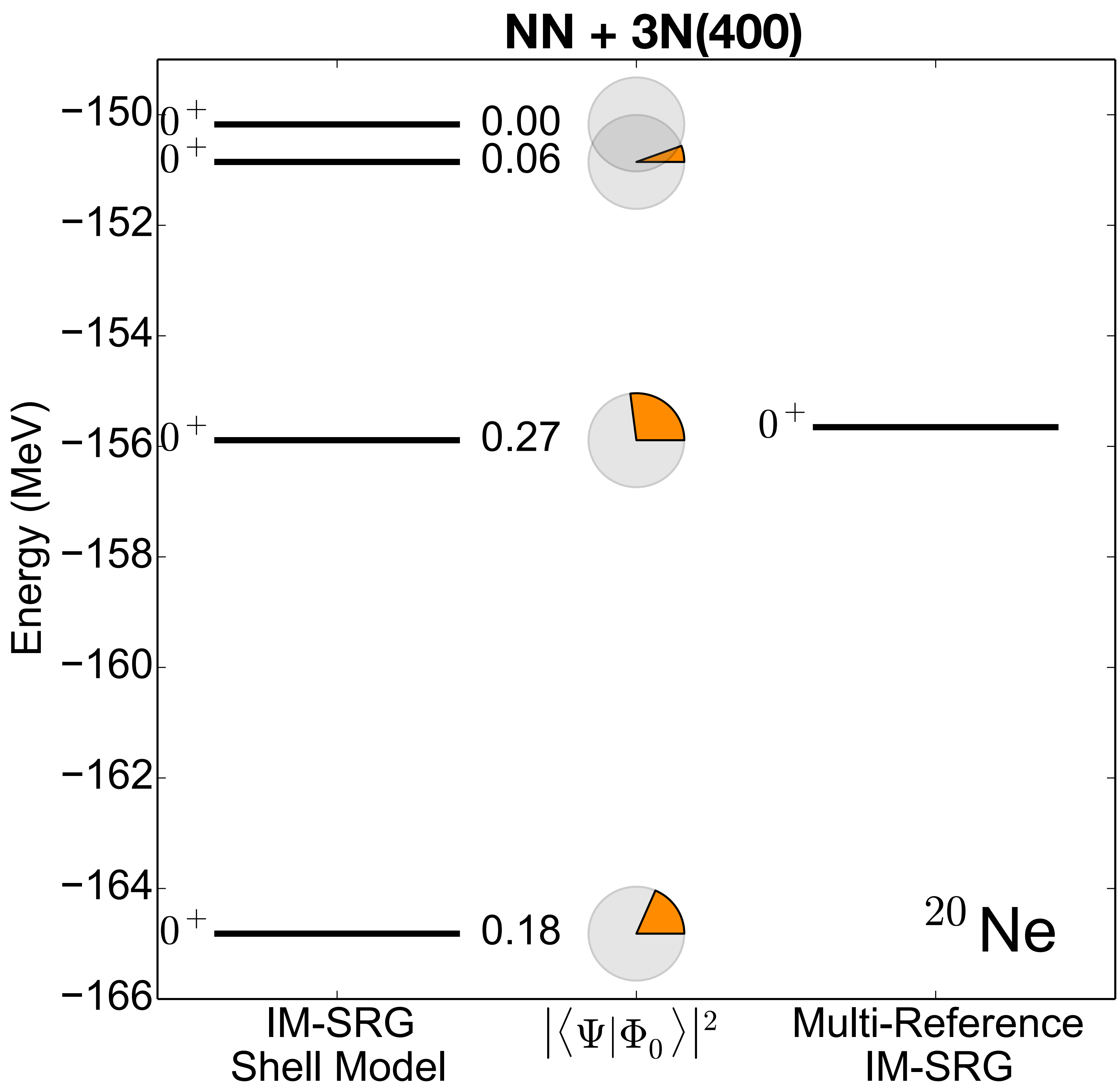
➡ **ODE and BCH expansions converge rapidly and monotonically**

Neon Isotopes

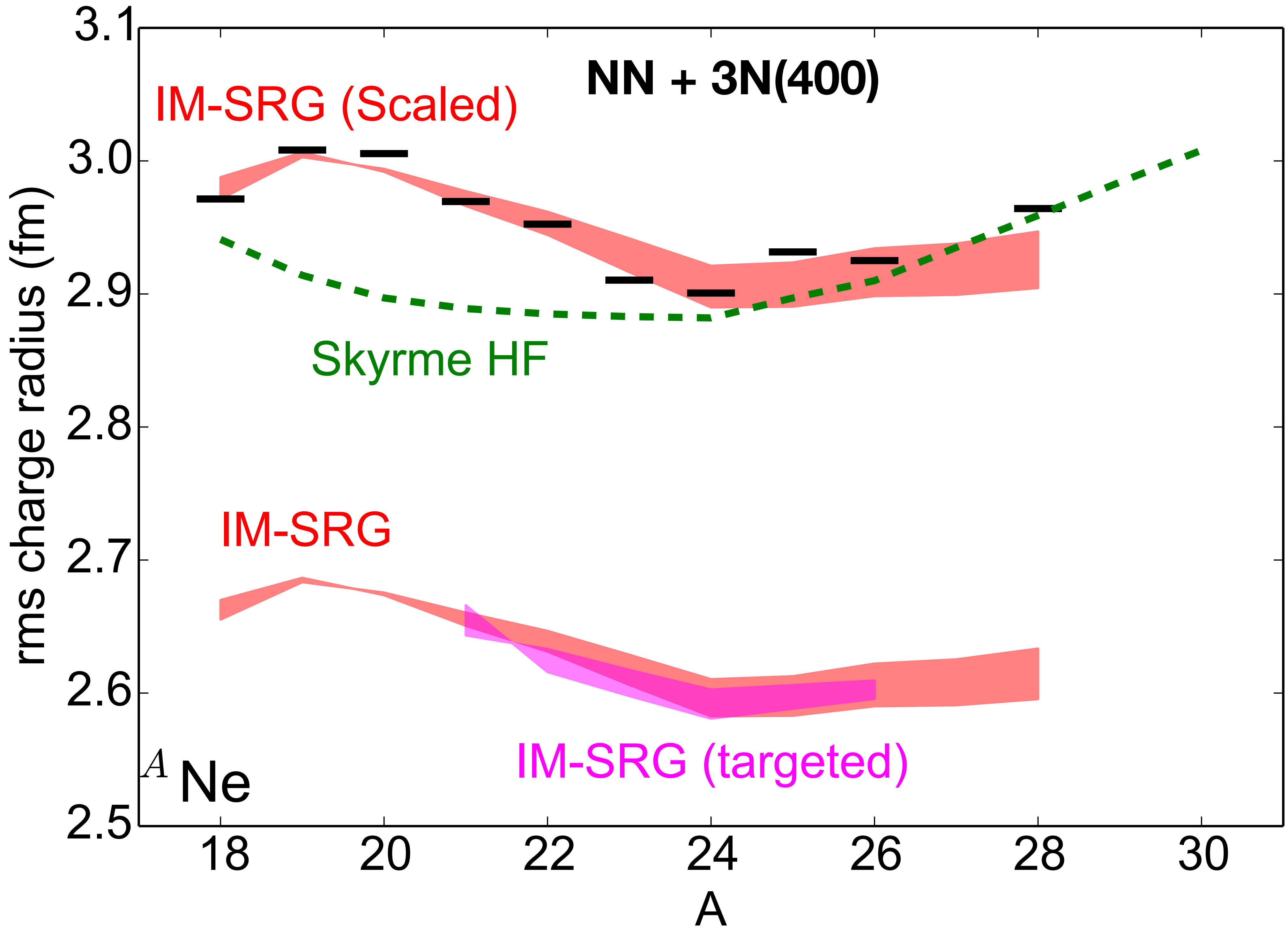


MR-IM-SRG selects excited 0⁺ state with spherical intrinsic structure

Deformation: ^{20}Ne



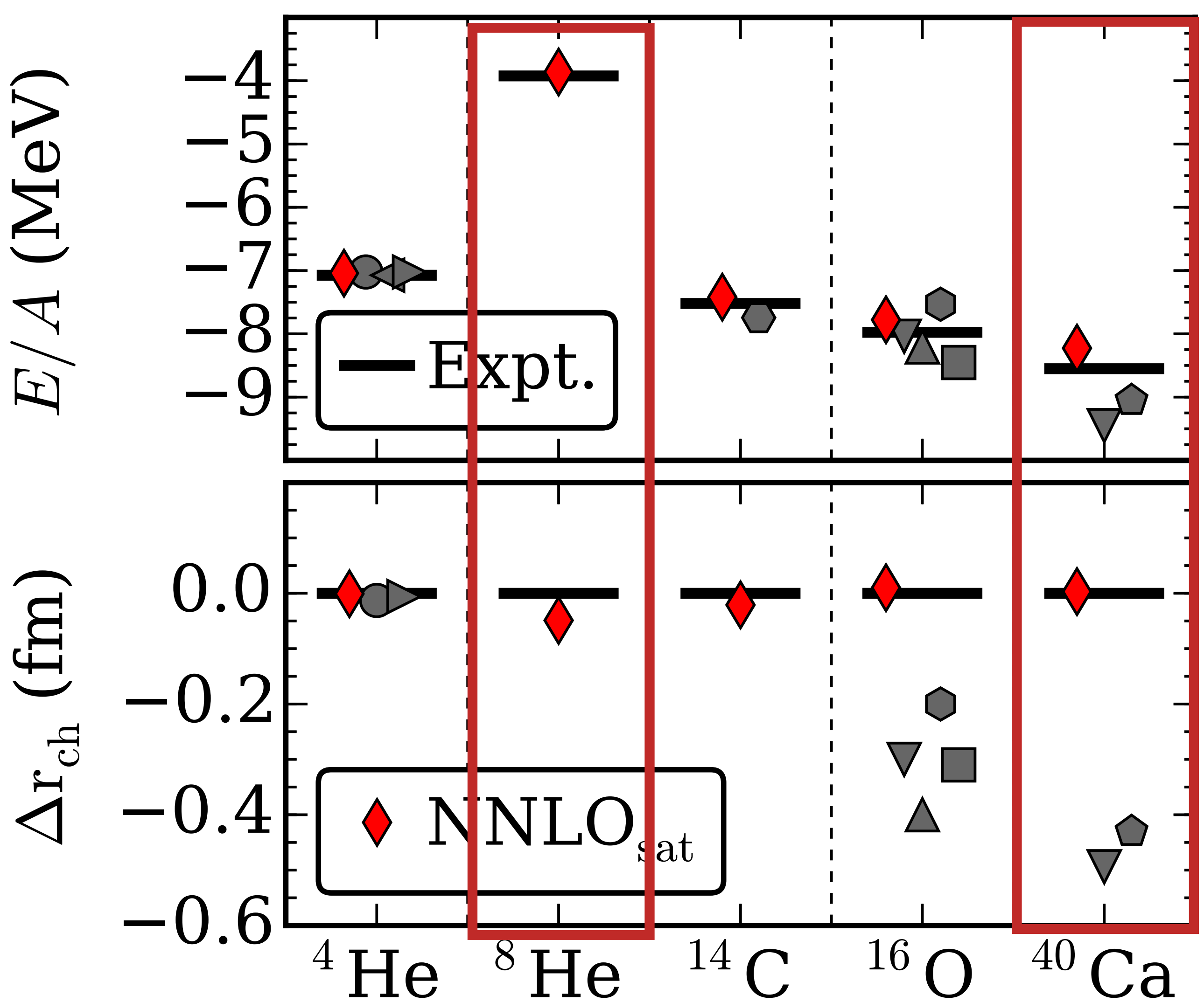
Neon Radii



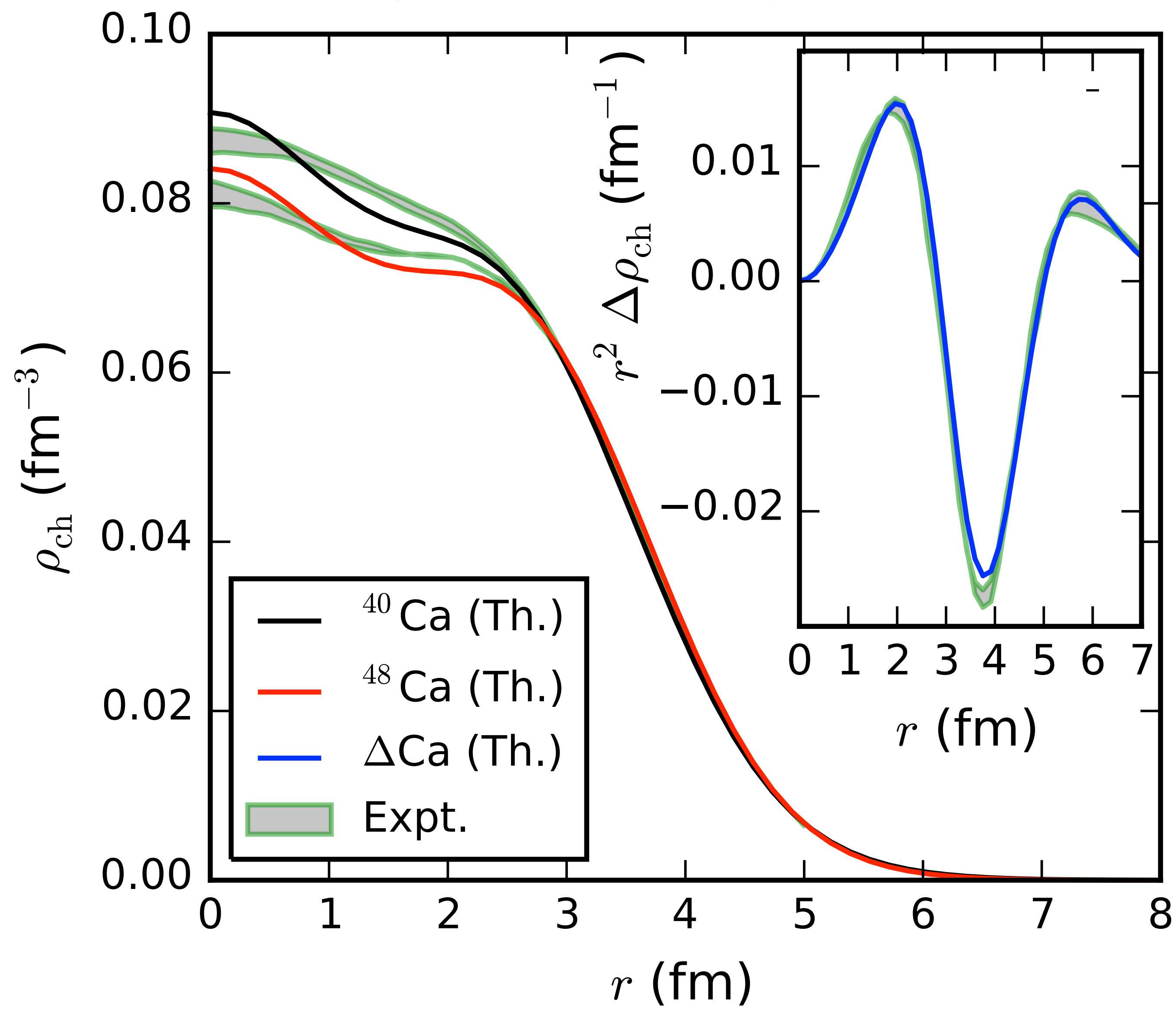
NN + 3N(400)
IM-SRG (Scaled)
Skyrme HF
IM-SRG
IM-SRG (targeted)
^{A}Ne
rms charge radius (fm)
A
3.1
3.0
2.9
2.8
2.7
2.6
2.5
18
20
22
24
26
28
30

$\mathrm{NNLO}_{\mathrm{sat}}$

A. Ekström et al., PRC ***91****, 051301(R) (2015)*



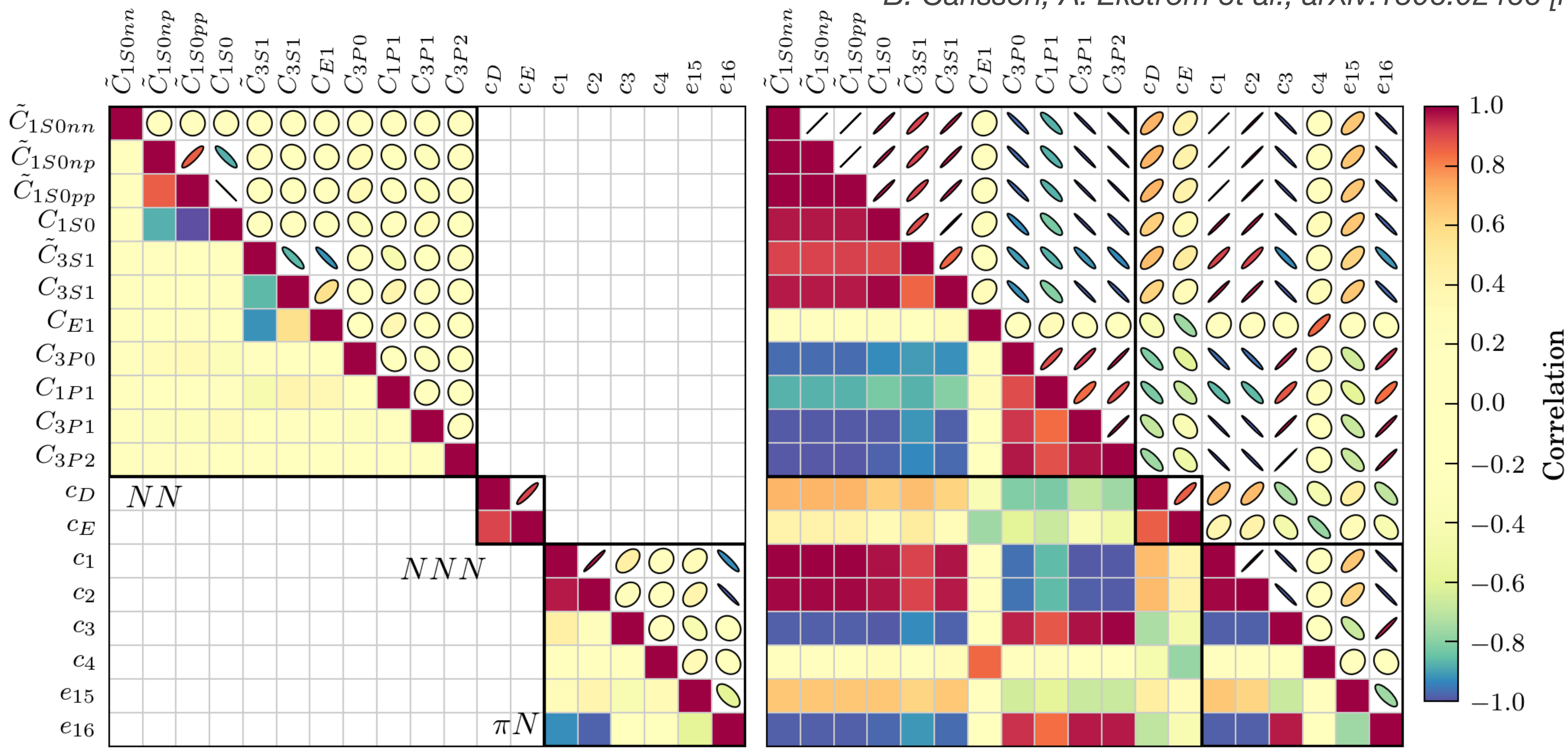
G. Hagen et al., Nature Physics ***12****, 186 (2015)*



- accurate description of ^{8}He, ^{40,48}Ca g.s. energies & radii, ^{40,48}Ca charge distributions
- predictions for electric dipole polarizability, neutron skin, weak form factor of ^{48}Ca

Optimization of Correlated LECs

B. Carlsson, A. Ekström et al., arXiv:1506.02466 [nucl-th]



- chiral LECs in NN, 3N, πN sectors are correlated
- sequential vs. simultaneous optimization, NNLO, NN+3N:

$$E(^4\text{He}) = 28^{+8}_{-18}\ \text{MeV} \quad \text{vs.} \quad E(^4\text{He}) = 28.26^{+4}_{-5}\ \text{MeV}$$

Induced Interactions

- SRG is a unitary transformation in A-body space
- up to A-body interactions are induced during the flow:

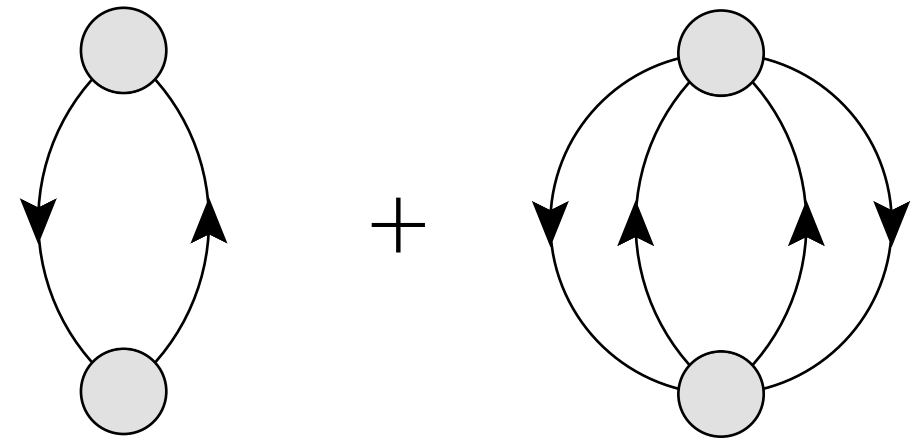
$$\frac{dH}{d\lambda} = [[\sum a^\dagger a, \sum \underbrace{a^\dagger a^\dagger aa}_{\text{2-body}}], \sum \underbrace{a^\dagger a^\dagger aa}_{\text{2-body}}] = \ldots + \sum \underbrace{a^\dagger a^\dagger a^\dagger aaa}_{\text{3-body}} + \ldots$$

- state-of-the-art: evolve in three-body space, truncate induced four- and higher many-body forces (Jurgenson, Furnstahl, Navratil, PRL 103, 082501; Hebeler, PRC 85, 021002; Wendt, PRC 87, 061001)
- λ-dependence of eigenvalues is a diagnostic for size of omitted induced interactions

IM-SRG(2) Flow Equations

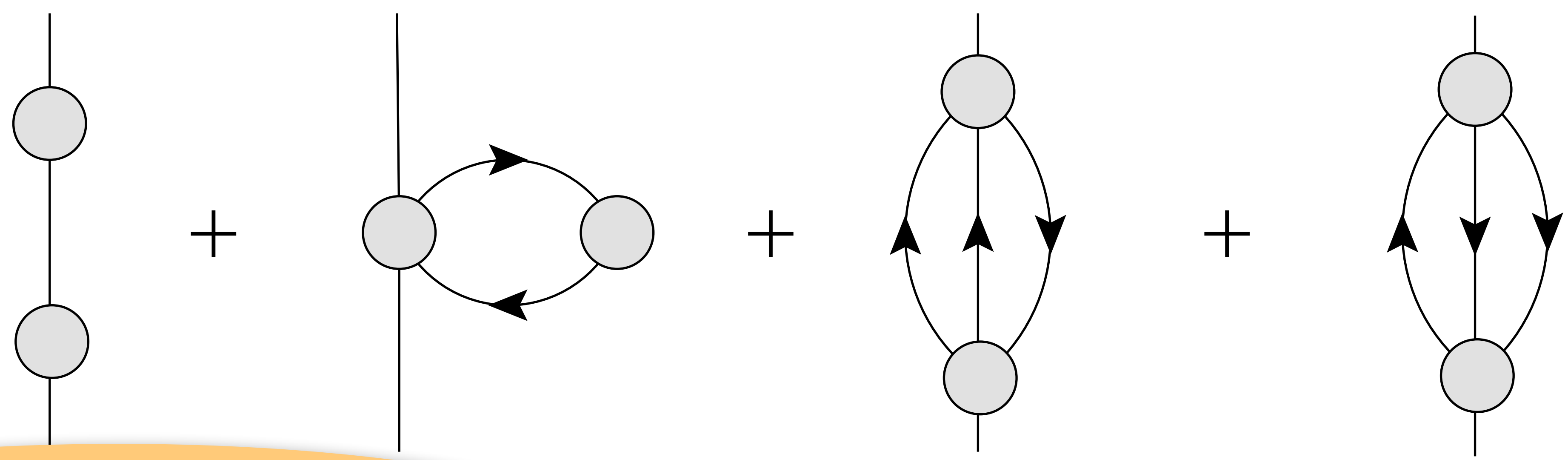
0-body Flow

$$\frac{dE}{ds} =$$



1-body Flow

$$\frac{df}{ds} =$$



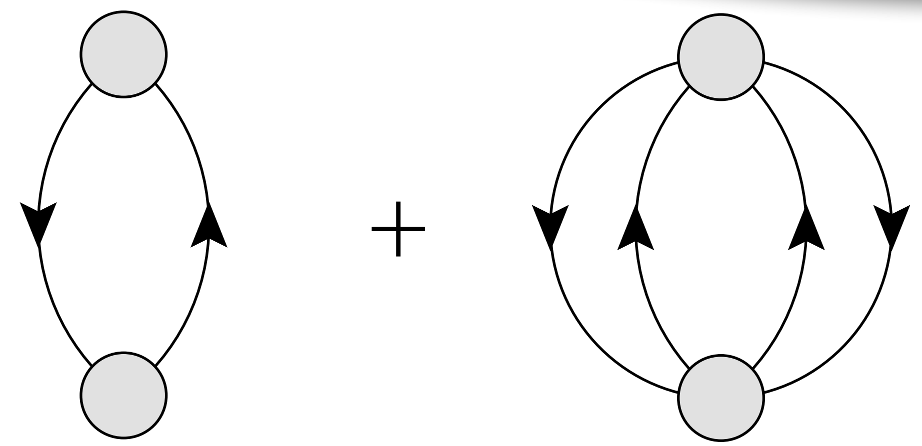
IM-SRG(2): truncate ops. at two-body level

IM-SRG(2) Flow Equations

0-body Flow

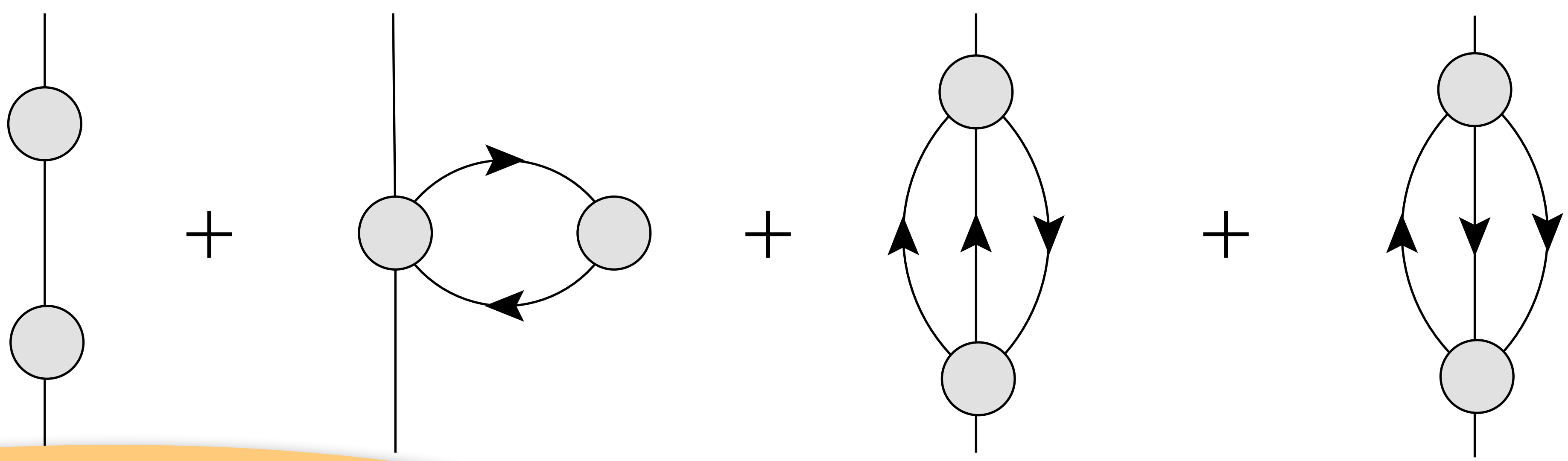
~ 2nd order MBPT for *H(s)*

$$\frac{dE}{ds} =$$



1-body Flow

$$\frac{df}{ds} =$$



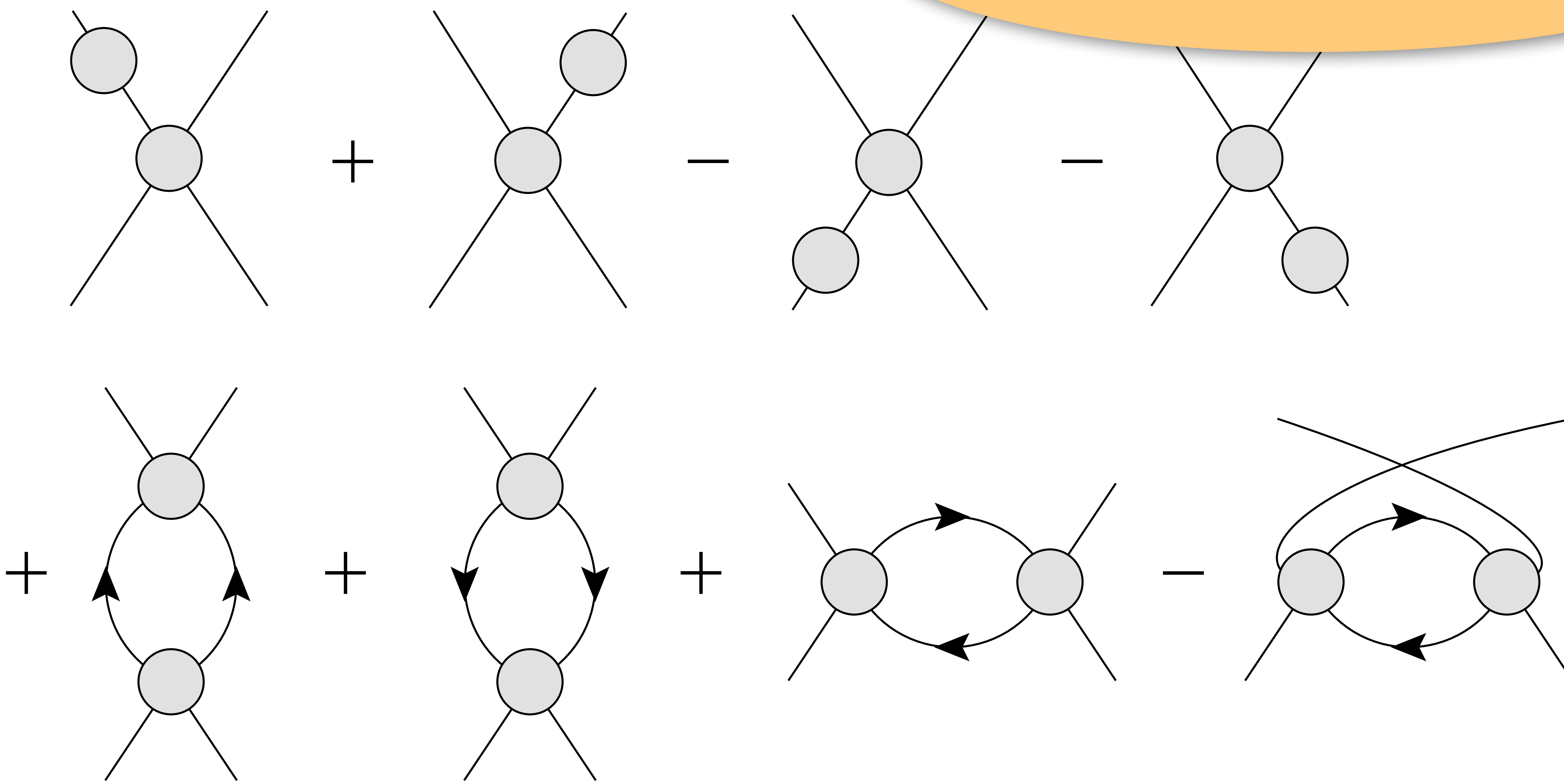
IM-SRG(2): truncate ops. at two-body level

IM-SRG(2) Flow Equations

O(N^6) scaling
(before particle/hole distinction)

2-body Flow

$$\frac{d\Gamma}{ds} =$$



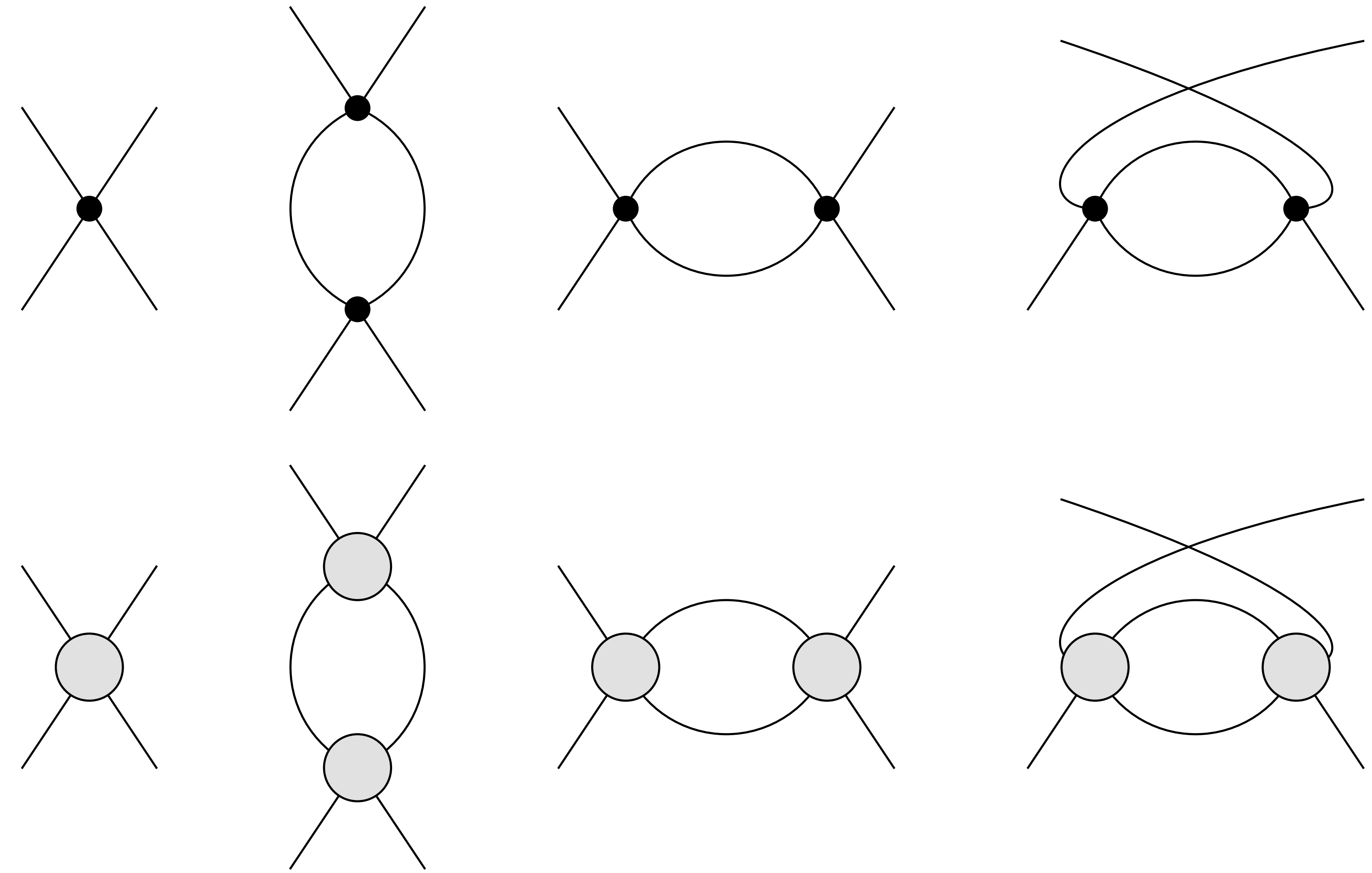
s channel — ladders

t channel, *u* channel — rings

In-Medium SRG Flow: Diagrams

$\Gamma(\delta s) \sim$

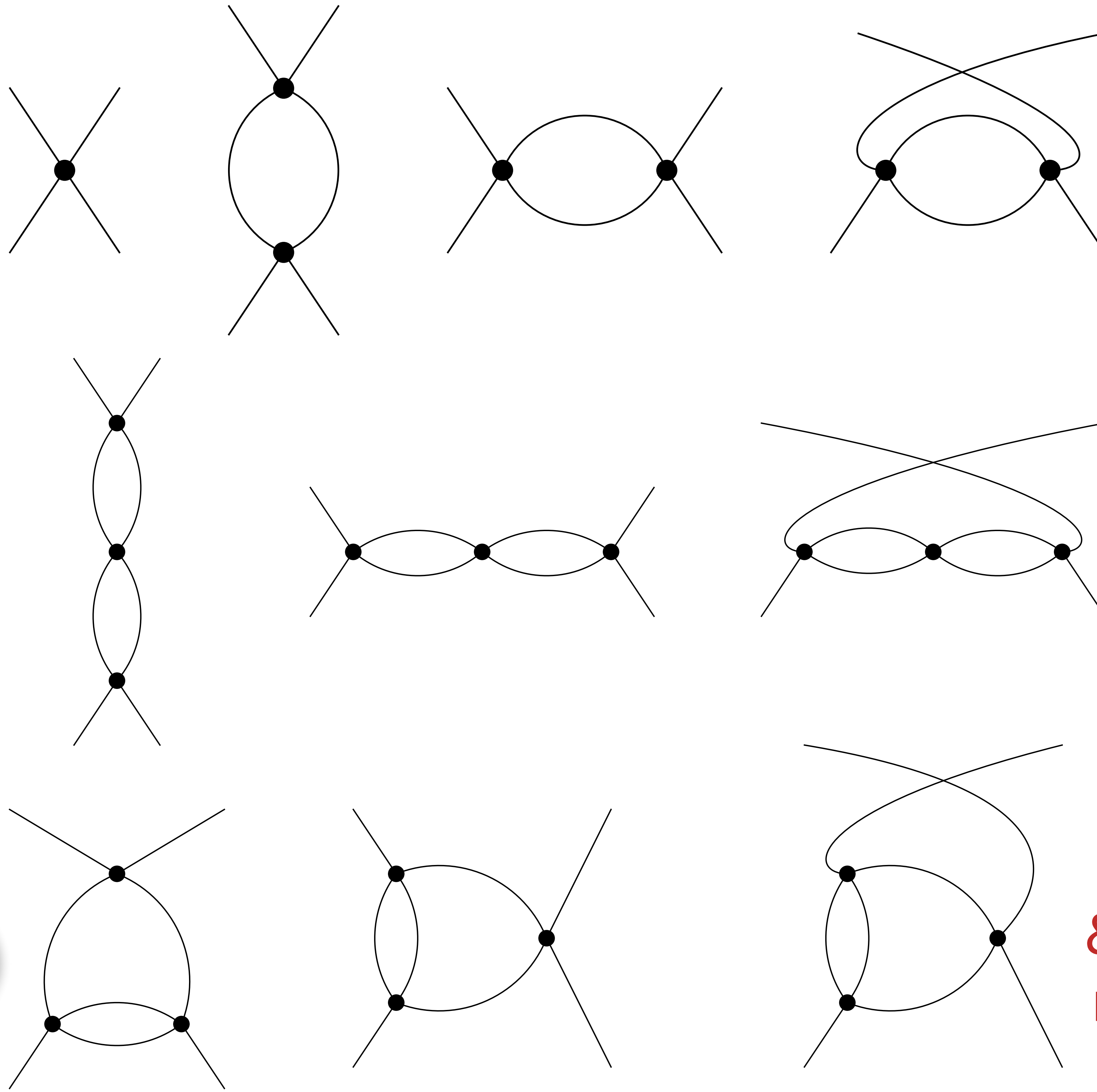
$\Gamma(2\delta s) \sim$



In-Medium SRG Flow: Diagrams

$\Gamma(\delta s) \sim$

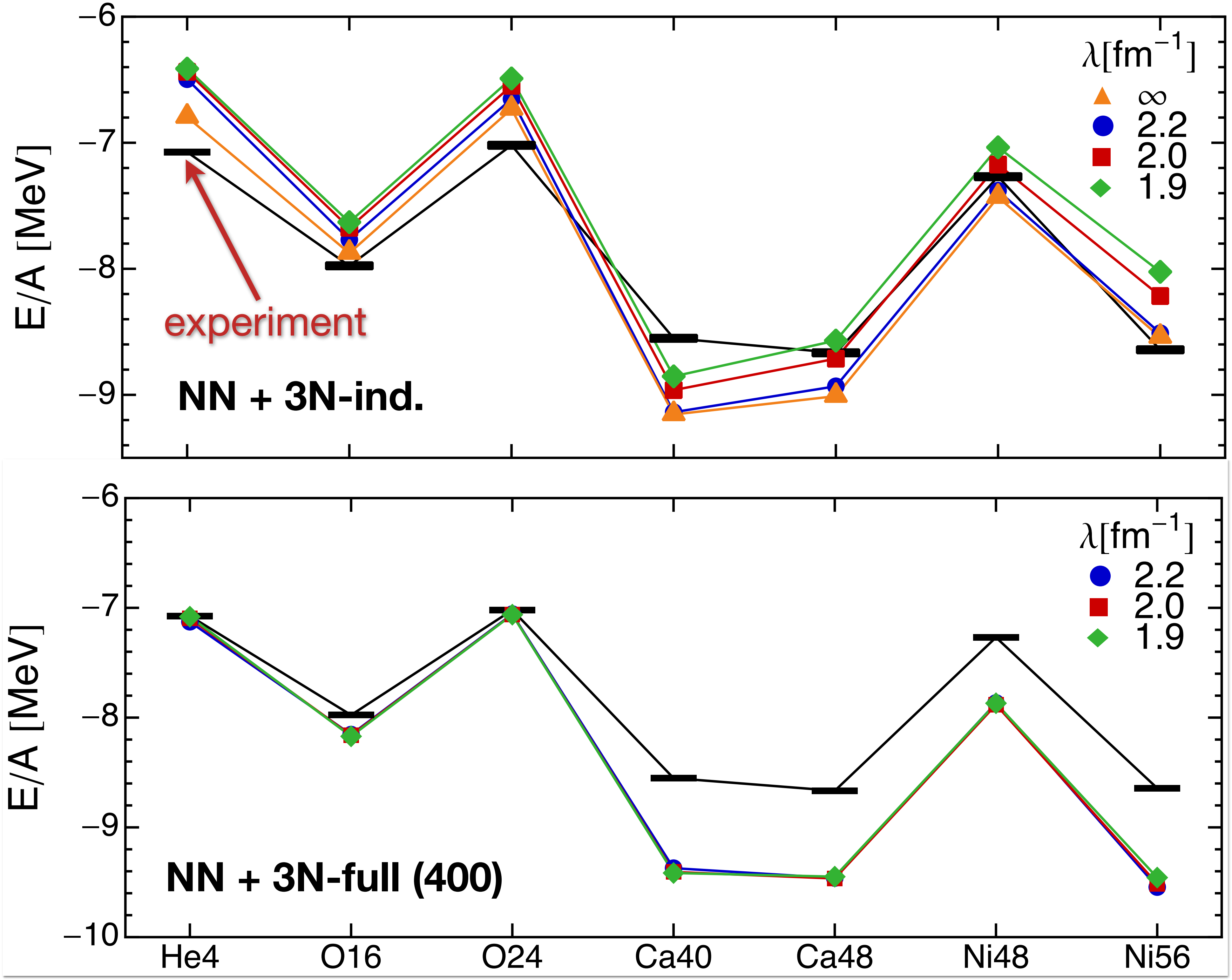
$\Gamma(2\delta s) \sim$



non-perturbative resummation

& many more...

Results: Closed-Shell Nuclei



Phys. Rev. C **87**, 034307 (2013), arXiv: 1212.1190 [nucl-th]

Choice of Generator

- Wegner: $$\eta^{I} = \left[H_d, H_{od}\right]$$
- White: (J. Chem. Phys. 117, 7472)

$$\eta^{II} = \sum_{ph} \frac{f_h^p}{\Delta_h^p} :A_h^p: + \frac{1}{4} \sum_{pp'hh'} \frac{\Gamma_{hh'}^{pp'}}{\Delta_{hh'}^{pp'}} :A_{hh'}^{pp'}: - \text{H.c.}$$

$\Delta_h^p, \Delta_{hh'}^{pp'}$: approx. 1p1h, 2p2h excitation energies

- "imaginary time": (Morris, Bogner)

$$\eta^{III} = \sum_{ph} \text{sgn}\left(\Delta_h^p\right) f_h^p :A_h^p: + \frac{1}{4} \sum_{pp'hh'} \text{sgn}\left(\Delta_{hh'}^{pp'}\right) \Gamma_{hh'}^{pp'} :A_{hh'}^{pp'}: - \text{H.c.}$$

- off-diagonal matrix elements are suppressed like $e^{-\Delta^2 s}$ (Wegner), e^{-s} (White), and $e^{-|\Delta| s}$ (imaginary time)
- g.s. energies ($s \to \infty$) differ by $\ll 1\%$

MR-IM-SRG Flow Equations

0-body flow:

$$\frac{dE}{ds} = \sum_{ab}(n_a - n_b)\eta^a_b f^b_a + \frac{1}{2}\sum_{abcd}\eta^{ab}_{cd}\Gamma^{cd}_{ab}n_a n_b \bar{n}_c \bar{n}_d$$

O(N⁴)

$$+ \frac{1}{4}\sum_{abcd}\left(\frac{d}{ds}\Gamma^{ab}_{cd}\right)\lambda^{ab}_{cd} + \frac{1}{4}\sum_{abcdklm}\left(\eta^{ab}_{cd}\Gamma^{kl}_{am} - \Gamma^{ab}_{cd}\eta^{kl}_{am}\right)\lambda^{bkl}_{cdm}$$

$O(N^4)$ **$O(N^7)$**

- storage of full 3B density matrix too expensive in general

➡ exploit structure of specific reference states:

- Projected HFB: $O(N^3)$ storage, scaling reduced to $O(N^4)$

$$\lambda^{abc}_{def} = \bar{\lambda}_{abc}\,\delta^a_d\delta^b_e\delta^c_f + \widetilde{\lambda}_{a|be}\,\delta^a_d\delta^{b\bar{c}}\delta_{e\bar{f}} + \text{perm.}$$

- NCSM / active-space CI: small non-zero block only

Particle-Number Projected HFB

- HFB ground state is a superposition of states with different particle number:

$$\left|\Psi\right\rangle = \sum_{A=N,N\pm2,\ldots} c_A \left|\Psi_A\right\rangle, \quad \left|\Psi_N\right\rangle \equiv P_N\left|\Psi\right\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{i\phi(\hat{N}-N)}\left|\Psi\right\rangle$$

- calculate irreducible densities (project only once), e.g.,

$$\lambda^k_l = \frac{\left\langle\Psi\right| A^k_l P_N \left|\Psi\right\rangle}{\left\langle\Psi\middle|\Psi\right\rangle}, \quad \lambda^{kl}_{mn} = \frac{\left\langle\Psi\right| A^{kl}_{mn} P_N \left|\Psi\right\rangle}{\left\langle\Psi\middle|\Psi\right\rangle} - \lambda^k_m\lambda^l_m + \lambda^k_n\lambda^l_m$$

- work in natural orbitals (= HFB canonical basis):

$$\lambda^k_l = n_k\delta^k_l \left(= v_k^2\delta^k_l\right), \quad 0 \leq n_k \leq 1$$

- in NO basis, $\lambda^{kl}_{mn}, \lambda^{klm}_{nop}$ require only $N^2/2$, $N^3/4$ storage