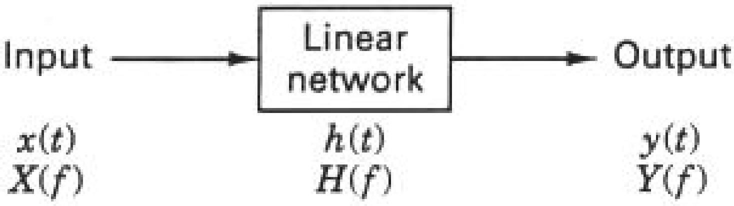
UNIT 3:
Time-domain representations for LTI systems – 2

3.1 Properties of impulse response representation:

Impulse Response

Def. Linear system: system that satisfies superposition theorem.



For any system, we can define its impulse response as:

$$h(t) = y(t) \qquad \text{when} x(t) = \delta(t)$$

For linear time invariant system, the output can be modeled as the convolution of the impulse response of the system with the input.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

For casual system, it can be modeled as convolution integral.

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$

3.2 Differential equation representation:

General form of differential equation is

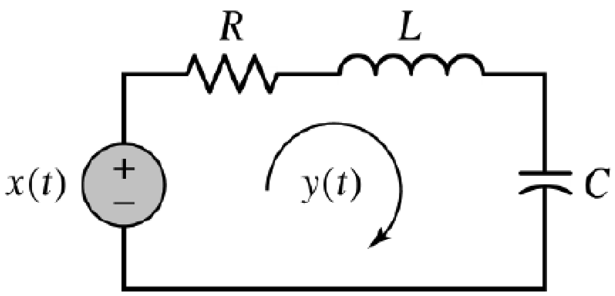
$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

where *ak* and *bk* are coefficients, $x(.)$ is input and $y(.)$ is output and order of differential or difference equation is (M,N).

Example of Differential equation

• Consider the RLC circuit as shown in figure below. Let $x(t)$ be the input voltage source and $y(t)$ be the output current. Then summing up the voltage drops around the loop gives

$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\,d\tau = x(t)$$



3.3 Solving differential equation:

A wide variety of continuous time systems are described the linear differential equations:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

- Just as before, in order to solve the equation for $y(t)$, we need the ICs. In this case, the ICs are given by specifying the value of y and its derivatives 1 through $N-1$ at $t = 0-$
- Note: the ICs are given at $t = 0-$ to allow for impulses and other discontinuities at $t = 0$.
- Systems described in this way are
- linear time-invariant (LTI): easy to verify by inspection
- Causal**:** the value of the output at time t depends only on the output and the input at times $0 \le t \le t$
- As in the case of discrete-time system, the solution $y(t)$ can be decomposed into $y(t) = y_h(t)+y_p(t)$, where homogeneous solution or zero-input response (ZIR), $yh(t)$ satisfies the equation

- The zero-state response (ZSR) or particular solution $y_p(t)$ satisfies the equation

$$y_h^N(t) + \sum_{i=0}^{N-1} a_i y_h^{(i)}(t) = \sum_{i=0}^{m} b_i x^{(M-i)}(t), \qquad t \ge 0$$

with ICs $y_p(0^-) = y_p^{(1)}(0^-) = \ldots = y_p^{(N-1)}(0^-) = 0.$

Homogeneous solution (ZIR) for CT

- A standard method for obtaining the homogeneous solution or (ZIR) is by setting all terms involving the input to zero.

$$\sum_{i=0}^{N} a_i y_h^{(i)}(t) = 0, \qquad t \ge 0$$

and homogeneous solution is of the form

$$y_h(t) = \sum_{i=1}^{N} C_i e^{r_i t}$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^k = 0$$

and $C_1, \ldots, C_N$ are solved using ICs.

Homogeneous solution (ZIR) for DT

- The solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

is

$$y_h[n] = \sum_{i=1}^{N} c_i r_i^n$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^{N-k} = 0$$

and $C_1, \ldots, C_N$ are solved using ICs.

Example 1 (ZIR)

- Solution of

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$$

is

$$y_h(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

- Solution of $y[n] - 9/16 y[n-2] = x[n-1]$ is $y_h[n] = c_1(3/4)^n + c_2(-3/4)$

Example 2 (ZIR)

- Consider the first order recursive system described by the difference equation $y[n] - \rho y[n-1] = x[n]$, find the homogeneous solution.
- The homogeneous equation (by setting input to zero) is $y[n] - \rho y[n-1] = 0$.
- The homogeneous solution for $N = 1$ is $y_h[n] = c_1 r_1^n$.
- r_1 is obtained from the characteristics equation $r_1 - \rho = 0$, hence $r_1 = \rho$
- The homogeneous solution is $y_h[n] = c_1 \rho^n$

Example 3 (ZIR)

- Consider the RC circuit described by $y(t) + RC\frac{d}{dt}y(t) = x(t)$
- The homogeneous equation is $y(t) + RC\frac{d}{dt}y(t) = 0$
- Then the homogeneous solution is

$$y_h(t) = c_1 e^{r_1 t}$$

where r_1 is the root of characteristic equation $1 + RCr_1 = 0$

- This gives $r_1 = -\frac{1}{RC}$
- The homogeneous solution is

$$y_h(t) = c_1 e^{\frac{-t}{RC}}$$

Particular solution (ZSR)

- Particular solution or ZSR represents solution of the differential or difference equation for the given input.
- To obtain the particular solution or ZSR, one would have to use the method of integrating factors.
- y_p is not unique.
- Usually it is obtained by assuming an output of the same general form as the input.
- If $x[n] = \alpha_n$ then assume $y_p[n] = c\alpha^n$ and find the constant c so that $y_p[n]$ is the solution of given equation

1.1.3 Examples

Example 1 (ZSR)

- Consider the first order recursive system described by the difference equation $y[n] - \rho y[n-1] = x[n]$, find the particular solution when $x[n] = (1/2)^n$.
- Assume a particular solution of the form $y_p[n] = c_p(1/2)^n$.
- Put the values of $y_p[n]$ and $x[n]$ in the equation then we get $c_p(\frac{1}{2})^n - \rho c_p(\frac{1}{2})^{n-1} = (\frac{1}{2})^n$
- Multiply both the sides of the equation by $(1/2)^n$ we get $c_p = 1/(1-2\rho)$.
- Then the particular solution is

$$y_p[n] = \frac{1}{1-2\rho}(\frac{1}{2})^n$$

- For $\rho = (1/2)$ particular solution has the same form as the homogeneous solution
- However no coefficient c_p satisfies this condition and we must assume a particular solution of the form $y_p[n] = c_p n(1/2)^n$.
- Substituting this in the difference equation gives $c_p n(1-2\rho) + 2\rho c_p = 1$
- Using $\rho = (1/2)$ we find that $c_p = 1$.

Example 2 (ZSR)

- Consider the RC circuit described by $y(t) + RC\frac{d}{dt}y(t) = x(t)$
- Assume a particular solution of the form $y_p(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$.
- Replacing $y(t)$ by $y_p(t)$ and $x(t)$ by $\cos(\omega_0 t)$ gives

$$c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) - RC\omega_0 c_1\sin(\omega_0 t) + RC\omega_0 c_2\cos(\omega_0 t) = \cos(\omega_0 t)$$

- The coefficients c_1 and c_2 are obtained by separately equating the coefficients of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, gives

$$c_1 = \frac{1}{1+(RC\omega_0)^2} \qquad \textit{and} \qquad c_2 = \frac{RC\omega_0}{1+(RC\omega_0)^2}$$

- Then the particular solution is

$$y_p(t) = \frac{1}{1+(RC\omega_0)^2}\cos(\omega_0 t) + \frac{RC\omega_0}{1+(RC\omega_0)^2}\sin(\omega_0 t)$$

Complete solution

- Find the form of the homogeneous solution y_h from the roots of the characteristic equation
- Find a particular solution y_p by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution
- Determine the coefficients in the homogeneous solution so that the complete solution $y = y_h + y_p$ satisfies the initial conditions

3.4 Difference equation representation:

- A wide variety of discrete-time systems are described by linear difference equations:

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \qquad n = 0, 1, 2, \ldots$$

  where the coefficients $a_1, \ldots, a_N$ and $b_0, \ldots, b_M$ do not depend on n. In order to be able to compute the system output, we also need to specify the initial conditions (ICs) $y[-1], y[-2] \ldots y[-N]$

- Systems of this kind are
  - linear time-invariant (LTI): easy to verify by inspection
  - causal: the output at time n depends only on past outputs $y[n-1], \ldots, y[n-N]$ and on current and past inputs $x[n], x[n-1], \ldots, x[n-M]$

- Systems of this kind are also called Auto Regressive Moving-Average (ARMA) filters. The name comes from considering two special cases.
- auto regressive (AR) filter of order N, $AR(N)$: $b_0 = \ldots = b_M = 0$

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = 0 \qquad n = 0, 1, 2, \ldots$$

  In the AR case, the system output at time n is a linear combination of N past outputs; need to specify the ICs $y[-1], \ldots, y[-N]$.
- moving-average (MA) filter of order $N, AR(N)$: $a_0 = \ldots = a_N = 0$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad n = 0, 1, 2, \ldots$$

  In the MA case, the system output at time n is a linear combination of the current input and M past inputs; no need to specify ICs.
- An ARMA(N, M) filter is a combination of both.

- Let us first rearrange the system equation:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \qquad n = 0, 1, 2, \ldots$$

- at $n=0$

$$y[0] = \underbrace{-\sum_{k=1}^{N} a_k y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[-k]}_{depends\ on\ input\ x[0] \to x[-M]}$$

- at $n=1$

$$y[1] = \sum_{k=1}^{N} a_k y[1-k] + \sum_{k=0}^{M} b_k x[1-k]$$

After rearranging

$$y[1] = -a_1 y[0] \underbrace{- \sum_{k=1}^{N-1} a_{k+1} y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{depends\ on\ input x[1] \ldots x[1-M]}$$

- at $n=2$

$$y[2] = \sum_{k=1}^{N} a_k y[2-k] + \sum_{k=0}^{M} b_k x[2-k]$$

After rearranging

$$y[2] = -a_1 y[1] - a_2 y[0] \underbrace{- \sum_{k=1}^{N-1} a_{k+1} y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{depends\ on\ input\ x[2] \ldots x[2-M]}$$

Example of Difference equation

- An example of II order difference equation is

$$y[n] + y[n-1] + \frac{1}{4} y[n-2] = x[n] + 2x[n-1]$$

- Memory in discrete system is analogous to energy storage in continuous system
- Number of initial conditions required to determine output is equal to maximum memory of the system

Initial Conditions

Initial Conditions summarise all the information about the systems past that is needed to determine the future outputs.

- In discrete case, for an N^{th} order system the N initial value are

$$y[-N], y[-N+1], \ldots, y[-1]$$

- The initial conditions for N^{th}-order differential equation are the values of the first N derivatives of the output

$$y(t)|_{t=0}, \frac{d}{dt}y(t)|_{t=0}, \frac{d^2}{dt^2}y(t)|_{t=0}, \ldots \frac{d^{N-1}}{dt^{N-1}}y(t)|_{t=0}$$

Solving difference equation

- Consider an example of difference equation $y[n] + ay[n-1] = x[n], \quad n = 0, 1, 2\ldots$ with $y[-1] = 0$ Then

$$\begin{aligned}
y[0] &= -ay[-1] + x[0] \\
y[1] &= -ay[0] + x[1] \\
&= -a(-ay[-1] + x[0]) + x[1] \\
&= a^2y[-1] - ax[0]) + x[1] \\
y[2] &= -ay[1] + x[2] \\
&= -a(-a^2y[-1] - ax[0] + x[1]) + x[2] \\
&= a^3y[-1] + a^2x[0] - ax[1] + x[2]
\end{aligned}$$

and so on

- We get $y[n]$ as a sum of two terms:
  $y[n] = (-a)^{n+1}y[-1] + \sum_{i=0}^{n}(-a)^{n-i}x[i], \qquad n = 0, 1, 2, \ldots$
- First term $(-a)^{n+1}y[-1]$ depends on IC's but not on input

- Second term $\sum_{i=0}^{n}(-a)^{n-i}x[i]$ depends only on the input, but not on the IC's
- This is true for any ARMA (auto regressive moving average) system: the system output at time n is a sum of the AR-only and the MA-only outputs at time n.
- Consider an ARMA (N,M) system $y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i], \qquad n = 0,1,2,\ldots$ with the initial conditions $y[-1],\ldots,y[-N]$.
- Output at time n is:

$$y[n] = y_h[n] + y_p[n]$$

  where $y_h[n]$ and $y_p[n]$ are homogeneous and particular solutions
- First term depends on IC's but not on input
- Second term depends only on the input, but not on the IC's
- Note that $y_h[n]$ is the output of the system determined by the ICs only (setting the input to zero), while $y_p[n]$ is the output of the system determined by the input only (setting the ICs to zero).
- $y_h[n]$ is often called the zero-input response (ZIR) usually referred as homogeneous solution of the filter (referring to the fact that it is determined by the ICs only)
- $y_p[n]$ is called the zero-state response (ZSR) usually referred as particular solution of the filter (referring to the fact that it is determined by the input only, with the ICs set to zero).

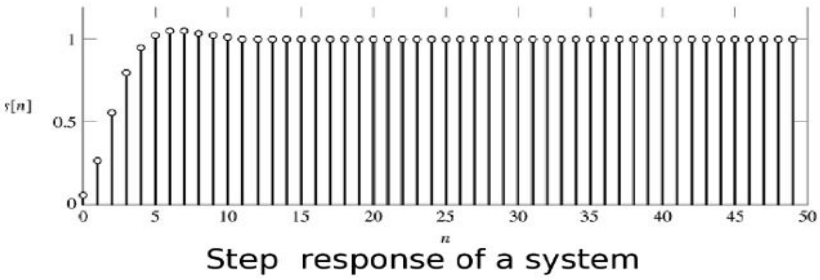


Figure 1.2: Step response

- Consider the output decomposition $y[n] = y_h[n] + y_p[n]$ of an ARMA (N, M) filter

$$y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i], \qquad n = 0, 1, 2, \ldots$$

  with the ICs $y[-1], \ldots, y[-N]$.

- The output of an ARMA filter at time n is the sum of the ZIR and the ZSR at time n.

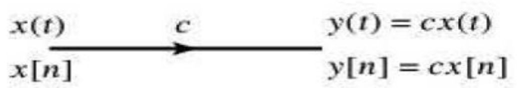
Example of difference equation

- example: A system is described by $y[n] - 1.143y[n-1] + 0.4128y[n-2] = 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$

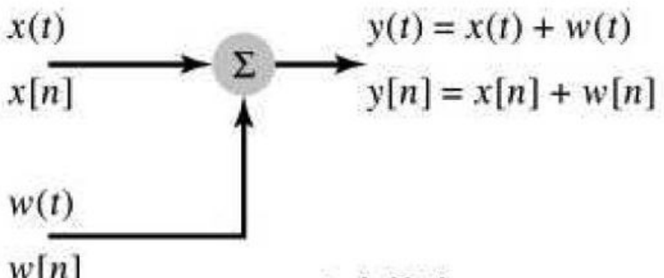
- Rewrite the equation as $y[n] = 1.143y[n-1] - 0.4128y[n-2] + 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$

3.5 Block Diagram representation:

- A block diagram is an interconnection of elementary operations that act on the input signal
- This method is more detailed representation of the system than impulse response or differential/difference equation representations
- The impulse response and differential/difference equation descriptions represent only the input-output behavior of a system, block diagram representation describes how the operations are ordered
- Each block diagram representation describes a different set of internal computations used to determine the system output
- Block diagram consists of three elementary operations on the signals:
  - Scalar multiplication: $y(t) = cx(t)$ or $y[n] = x[n]$, where c is a scalar
  - Addition: $y(t) = x(t) + w(t)$ or $y[n] = x[n] + w[n]$.
- Block diagram consists of three elementary operations on the signals:
  - Integration for continuous time LTI system: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
    Time shift for discrete time LTI system: $y[n] = x[n-1]$
- Scalar multiplication: $y(t) = cx(t)$ or $y[n] = x[n]$, where c is a scalar

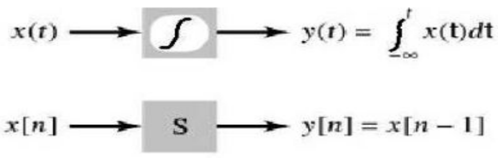


Scalar Multiplication



Addition

- Addition: $y(t) = x(t) + w(t)$ or $y[n] = x[n] + w[n]$
- Integration for continuous time LTI system: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
  Time shift for discrete time LTI system: $y[n] = x[n-1]$



Integration and timeshifting

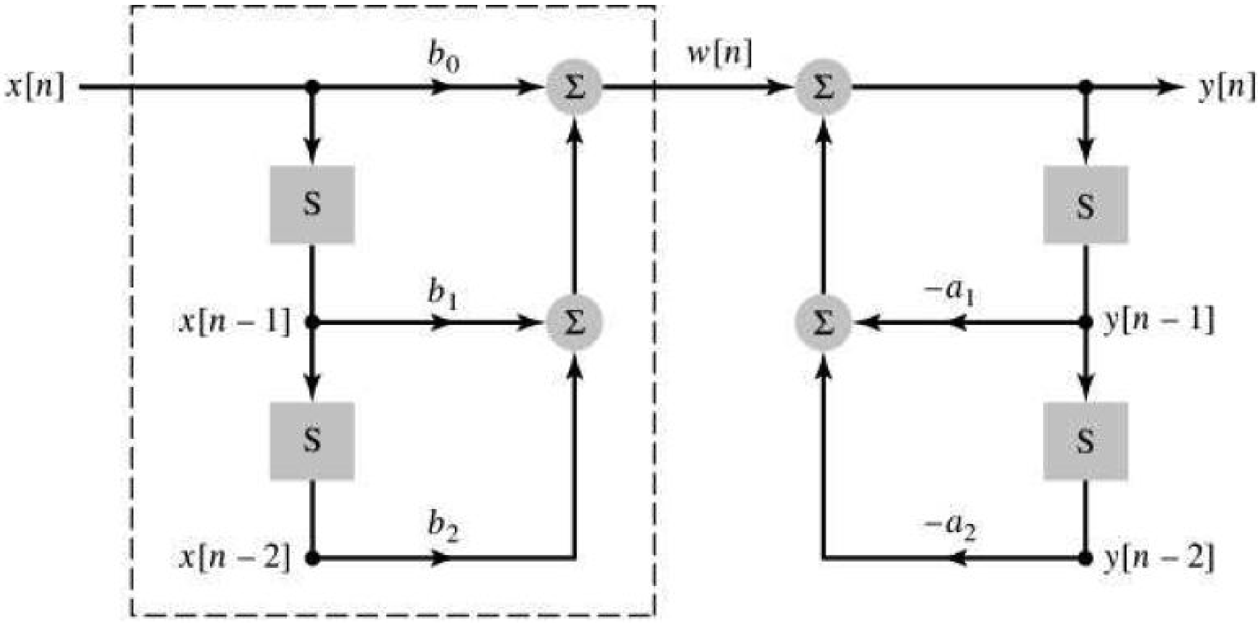


Figure 1.10: Example 1: Direct form I

Example 1

- Consider the system described by the block diagram as in Figure 1.10
- Consider the part within the dashed box
- The input $x[n]$ is time shifted by 1 to get $x[n-1]$ and again time shifted by one to get $x[n-2]$. The scalar multiplications are carried out and

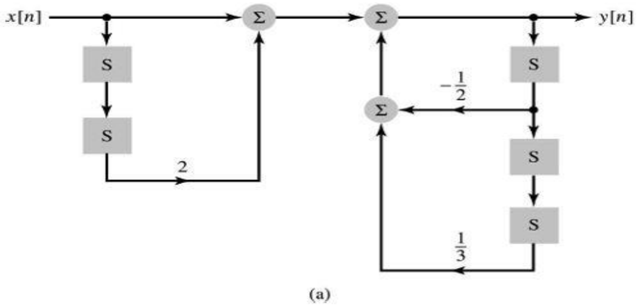


Figure 1.11: Example 2: Direct form I

they are added to get $w[n]$ and is given by

$$w[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2].$$

- Write $y[n]$ in terms of $w[n]$ as input $y[n] = w[n] - a_1y[n-1] - a_2y[n-2]$
- Put the value of $w[n]$ and we get $y[n] = -a_1y[n-1] - a_2y[n-2] + b_0x[n] + b_1x[n-1] + b_2x[n-2]$
  and $y[n] + a_1y[n-1] + a_2y[n-2] = b_0x[n] + b_1x[n-1] + b_2x[n-2]$
- The block diagram represents an LTI system

Example 2

- Consider the system described by the block diagram and its difference equation is $y[n] + (1/2)y[n-1] - (1/3)y[n-3] = x[n] + 2x[n-2]$

Example 3

- Consider the system described by the block diagram and its difference equation is $y[n] + (1/2)y[n-1] + (1/4)y[n-2] = x[n-1]$

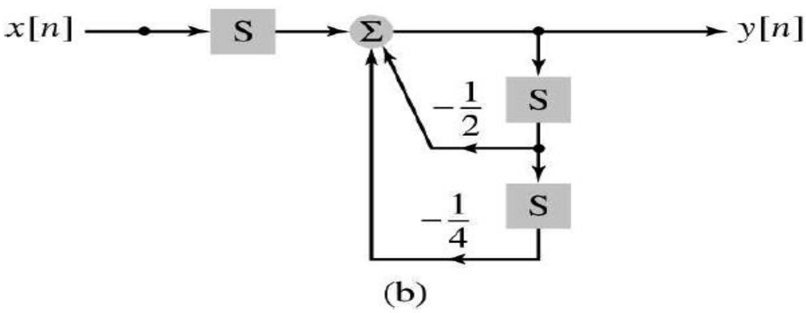


Figure 1.12: Example 3: Direct form I

- Block diagram representation is not unique, direct form II structure of Example 1
- We can change the order without changing the input output behavior Let the output of a new system be $f[n]$ and given input $x[n]$ are related by

$$f[n] = -a_1 f[n-1] - a_2 f[n-2] + x[n]$$

- The signal $f[n]$ acts as an input to the second system and output of second system is

$$y[n] = b_0 f[n] + b_1 f[n-1] + b_2 f[n-2].$$

- The block diagram representation of an LTI system is not unique

Continuous time

- Rewrite the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

as an integral equation. Let $v^{(0)}(t) = v(t)$ be an arbitrary signal, and set

$$v^{(n)}(t) = \int_{-\infty}^{t} v^{(n-1)}(\tau) d\tau, \qquad n = 1, 2, 3, \ldots$$

where $v^{(n)}(t)$ is the n-fold integral of $v(t)$ with respect to time

- Rewrite in terms of an initial condition on the integrator as

$$v^{(n)}(t) = \int_{0}^{t} v^{(n-1)}(\tau) d\tau + v^{(n)}(0), \quad n = 1, 2, 3, \ldots$$

- If we assume zero ICs, then differentiation and integration are inverse operations, ie.

$$\frac{d}{dt} v^{(n)}(t) = v^{(n-1)}(t), \quad t > 0 \text{ and } n = 1, 2, 3, \ldots$$

- Thus, if $N \geq M$ and integrate N times, we get the integral description of the system

$$\sum k = 0^N a_k y^{(N-k)}(t) = \sum k = 0^M b_k x^{(N-k)}(t)$$

- For second order system with $a_0 = 1$, the differential equation can be

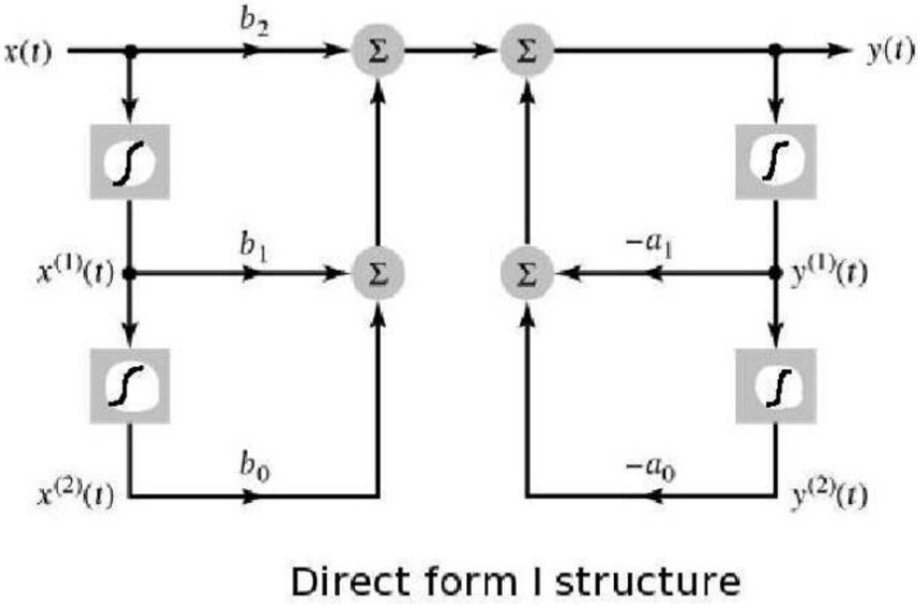
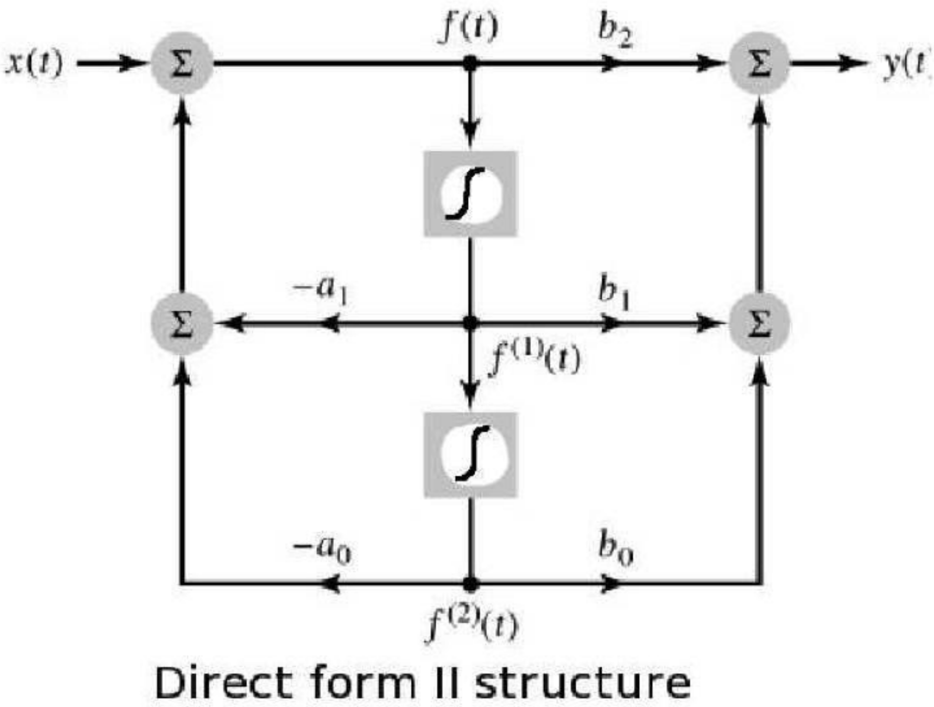


Figure 1.25: Direct form I

written as

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + a_1 x^{(1)}(t) + b_0 x^{(2)}(t)$$



Recommended Questions

1. Show that
   (a) $x(t) * \delta(t) = x(t)$
   (b) $x(t) * \delta(t - t_0) = x(t - t_0)$
   (c) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$
   (d) $x(t) * u(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$

   (a) By definition (2.6) and Eq. (1.22) we have

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, d\tau = x(\tau)|_{\tau=t} = x(t)$$

   (b) By Eqs. (2.7) and (1.22) we have

$$x(t) * \delta(t-t_0) = \delta(t-t_0) * x(t) = \int_{-\infty}^{\infty} \delta(\tau - t_0) x(t-\tau)\, d\tau$$
$$= x(t-\tau)|_{\tau=t_0} = x(t-t_0)$$

   (c) By Eqs. (2.6) and (1.19) we have

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau$$

since $u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases}$.

2. Evaluate y (t) = x (t) * h(t), where x (t) and h (t) are shown in Fig. 2-6 (a) by analytical technique, and (b) by a graphical method.

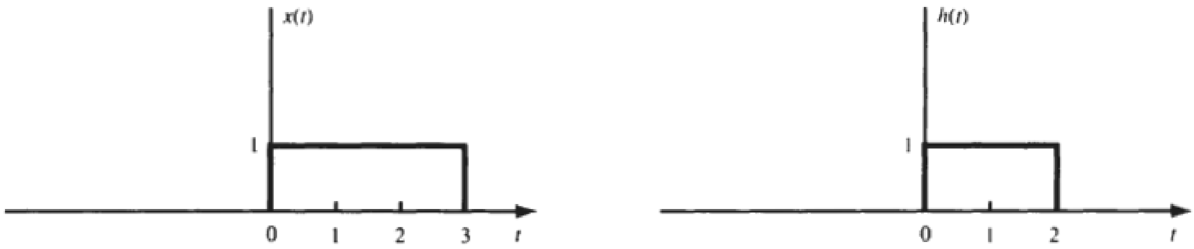


Fig. 2-6

3. Consider a continuous-time LTI system described by

4. $$y(t) = \mathbf{T}\{x(t)\} = \frac{1}{T}\int_{t-T/2}^{t+T/2} x(\tau)\, d\tau$$
   a. Find and sketch the impulse response h(t) of the system.
   b. Is this system causal?

5. Let y (t) be the output of a continuous-time LTI system with input x(t) . Find the output of the system if the input is $x^{1}(t)$, where $x^{1}(t)$ is the first derivative of x(t) .

6. Verify the BIBO stability condition for continuous-time LTI systems.

7. Consider a stable continuous-time LTI system with impulse response $h(t)$ that is real and even. Show that Cos ***wt*** and sin ***wt*** are Eigen functions of this system with the same real Eigen value.

8. The continuous-time system shown in Fig. 2-19 consists of two integrators and two scalar multipliers. Write a differential equation that relates the output y(t) and the input ***x(*** t).

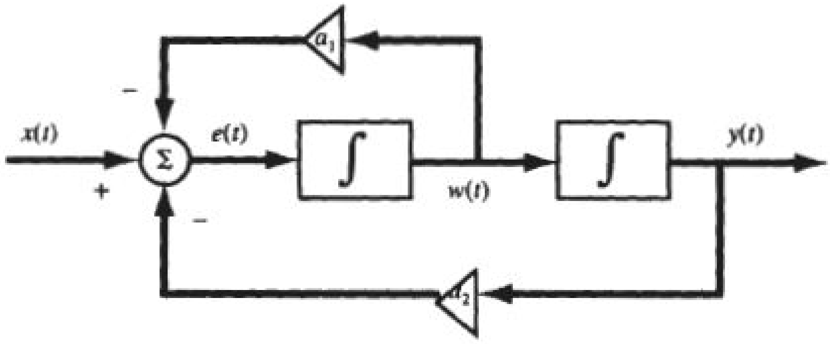


Fig. 2-19