Counter example $\left\{ \begin{array}{l} E[XY] = E[X]E[Y] \not\rightarrow X \,\&\, Y \text{ independant} \\ X \,\&\, Y \text{ indepenet} \longrightarrow E[XY] = E[X]E[Y] \end{array} \right\}$

$$P_{xy}(XY) = \begin{cases} \frac{1}{4} & \text{if } (x,y) = (1,1) \text{ or } (-1,1) \\ \frac{1}{2} & \text{if } (x,y) = (0,0) \end{cases}$$

$$E[XY] = \frac{1}{4}(\overset{X}{1}) \times (\overset{Y}{1}) + \frac{1}{4}(-1) \times (1) + \underbrace{\frac{1}{2}(0) \times (0)}_{0} = \frac{1}{4} - \frac{1}{4} = 0$$

$$E[X] = \left[\frac{1}{4}(-1) + \frac{1}{2}(0) + \frac{1}{4}(1)\right] = 0 \in P(X=x) = \begin{cases} P(X=0) = \frac{1}{2} \\ P(X=1) = \frac{1}{4} \\ P(X=-1) = \frac{1}{4} \end{cases}$$

$$E[Y] = \left[\frac{1}{2}(1) + \frac{1}{2}(0)\right] = \frac{1}{2} \in P(Y=y) = \begin{cases} P(Y=0) = \frac{1}{2} \\ P(Y=1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{cases}$$

$$\Rightarrow E[XY] = \underbrace{E[X]}_{0} E[Y] = 0$$

☆ But not independent since: $\begin{cases} P(1,1) = \frac{1}{4} \\ P(X=1)\,P(Y=1) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \end{cases}$

$$P(X=1, Y=1) \neq P(X=1)\,P(Y=1) \Rightarrow \boxed{X \,\&\, Y \text{ are } \underline{\text{not}} \text{ independent}}$$ ☆

$$\Rightarrow Var(X+Y) = Var(X) + Var(Y) + 2\underbrace{\{E[XY] - E[X]E[Y]\}}_{0}$$

$$Var(X+Y) = Var(X) + Var(Y)$$ but $X \,\&\, Y$ are dependent