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Borel Type Subalgebras of the q-Schur^m Algebra*

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In [G1], J. A. Green investigated certain subalgebras, called Borel subalgebras, of the Schur algebra associated with the Borel subgroups of the general linear group. Besides their combinatorial definition, these algebras are quasi-hereditary and give rise to a triangular decomposition of the Schur algebra with which Weyl and co-Weyl modules can be described as induced modules by using tensor and hom functors (see [Sa]). In [PW], part of Green's work has been generalized to the q-Schur algebra. Recently, a new class of quasi-hereditary algebras, called the q-Schur^2 algebras, associated with the Hecke algebra of the Weyl group of type B has been introduced by Du and Scott [DS] (see [DJM1] for a Morita equivalent version). Associated with Ariki–Koike Hecke algebras, a more general class of quasi-hereditary algebras, called cyclotomic q-Schur algebras, has been introduced by Dipper, James, and Mathas in [DJM2]. Since these algebras do not occur naturally in the context of Lie theory or quantum groups (cf. [DS1]), it would be interesting to find possible

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connections between the representation theories of these algebras and relevant quantum groups. As an attempt to this problem, we will investigate in this paper some Lie-theoretic structure of these algebras, and especially seek the existence of their Borel type subalgebras.

We shall aim at the q-Schur^2 algebra first, since the construction for the q-Schur^2 algebra in [DS] is almost parallel to that for the q-Schur algebra. Especially, the existence of both a natural basis (i.e., the counterpart for the centralizer algebra of a permutation module) and a Green–Murphy type (or cellular) basis for a q-Schur^2 algebra guarantees that we can mimic Green's construction in this case. Surprisingly, the work for Borel type subalgebras in the $m = 2$ case can be easily generalized to the cyclotomic q-Schur algebras, though there is no natural basis available in the work [DJM2] for $m > 2$. In this generalization, we first aim at a subclass of cyclotomic q-Schur algebras, called q-Schur^m algebras indexed by bidegree (n, r) as for the q-Schur algebras and will prove that Borel type subalgebras exist in a q-Schur^m algebra. It is also interesting to note that a Borel type subalgebra of the q-Schur^m algebra of degree (n, r) is isomorphic to a Borel subalgebra of the q-Schur algebra of degree (N, r) for some $N = N(m, n, r)$. We will also explain how we can easily get the Borel type subalgebras for an arbitrary cyclotomic q-Schur algebra.

It is worth pointing out that the notion of Borel subalgebras for an arbitrary quasi-hereditary algebra has been introduced by Scott [Sc]. It would be interesting to know if the Borel type subalgebras of q-Schur^m algebras fit the definition given in [Sc]. If it was the case, it would imply that the higher derived functors vanished in the case discussed in (4.10) and (5.16(f)). Thus, we would have an analogue of the Borel–Bott–Weil theorem for q-Schur^m algebras.

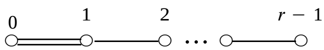
We organize the paper as follows. Section 1 collects results on q-Schur^2 algebras and related combinatorics. Candidates $\mathscr{S}_R^{2,\succcurlyeq}$ and $\mathscr{S}_R^{2,\preccurlyeq}$ for Borel type subalgebras are introduced as subspaces. In Section 2, we prove that these subspaces are subalgebras, where we discover an important connection with the subalgebra structure on the Borel subalgebras $\mathscr{S}_R^{1,\succcurlyeq}$ and $\mathscr{S}_R^{1,\preccurlyeq}$ of the q-Schur algebra $\mathscr{S}_R^1$. This important observation indicates that a somewhat easy generalization exists. In Section 3, we will prove that a q-Schur^2 algebra is a product of the Borel type subalgebras, and hence, we obtain a triangular decomposition of the q-Schur^2 algebra. The representation theory is investigated in Section 4, where the quasi-heredity of $\mathscr{S}_R^{2,\succcurlyeq}$ and $\mathscr{S}_R^{2,\preccurlyeq}$ is obtained by using the criterion established in [DR] and is used to determine the PIMs and some induced standard and costandard modules. In Section 5, we shall define the Borel type subalgebras $\mathbf{S}_R^{m,\succcurlyeq}$ and $\mathbf{S}_R^{m,\preccurlyeq}$ for the q-Schur^m algebra $\mathbf{S}_R^m$ and show how all results in Sections 2–4 for the $m = 2$ case are generalized to $\mathbf{S}_R^m$ for arbitrary m. Finally, we determine the tilting modules and the Ringel duals of these

Borel type algebras. We will see that $\mathbf{S}_R^{m,\succcurlyeq}$ and $\mathbf{S}_R^{m,\preccurlyeq}$ are Ringel dual to each other.

Throughout, unless specified, R denotes a commutative ring with 1.

1. THE q-SCHUR^2 ALGEBRA

Let W be the Weyl group of type B_r. As a Coxeter group, we denote the set of Coxeter generators of W by $S = \{s_0, s_1, \dots, s_{r-1}\}$ with relations described in the Coxeter diagram



In Section 5, W will be identified with the wreath product $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_r$ for $m = 2$, where $\mathfrak{S}_r = \mathfrak{S}_{\{1,2,\dots,r\}}$ is the symmetric group on r letters.

Let $\mathscr{Z} = \mathbb{Z}[q_0, q_0^{-1}, q, q^{-1}]$ be the Laurent polynomial ring in the indeterminates q_0, q and let $q_{s_0} = q_0$ and $q_{s_i} = q$ for $1 \le i \le r-1$. The generic Hecke algebra $\mathscr{H}$ associated to W is an associative algebra over $\mathscr{Z}$ with a $\mathscr{Z}$-basis $\{T_w \mid w \in W\}$ and multiplication defined by

$$(T_s - q_s)(T_s + 1) = 0, \qquad \text{if } s \in S,$$
$$T_x T_y = T_{xy}, \qquad \text{if } l(xy) = l(x) + l(y).$$

Here l is the length function on W. For a commutative ring R which is a $\mathscr{Z}$-algebra, let $\mathscr{H}_R = \mathscr{H} \otimes_{\mathscr{Z}} R$ be the Hecke algebra over R. For simplicity, we shall continue to use T_w for $T_w \otimes 1$ and q_s for $q_s \otimes 1$.

We need the notion of multi-compositions. Let $\mathbb{Z}^+$ be the set of nonnegative integers. Fix $n, r \in \mathbb{Z}^+$ with $n > 0$. A *composition* λ of r with n parts is a sequence $(\lambda_1, \dots, \lambda_n)$ such that $\lambda_i \in \mathbb{Z}^+$ and $|\lambda| = \sum_i \lambda_i = r$, and λ is called a *partition* if the sequence is weakly decreasing. For any positive integer m, an *m-composition* λ of r is defined to be a sequence of compositions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ such that $r = \sum_{i=1}^m |\lambda^{(i)}|$ and λ is called an *m-partition* if each $\lambda^{(i)}$ is a partition. Here the number of parts in each $\lambda^{(i)}$ may be different. Denote by $\Lambda_m(r)^+$ the set of all m-partitions of r. Putting $a_0 = 0$ and $a_i = a_{i-1} + |\lambda^{(i)}|$ for all $i \ge 1$, the sequence $\mathbf{a} = (a_0, a_1, \dots, a_m)$ is called the *cumulative norm sequence* (or simply, c.n.s.) of λ. Let $\trianglelefteq$ be the *dominance order* on m-compositions. Thus, $\lambda \trianglelefteq \mu$ means that, for every i, $1 \le i \le m$,

$$\sum_{j=0}^{i-1} a_j + \sum_{t=1}^{k} \lambda_t^{(i)} \le \sum_{j=0}^{i-1} b_j + \sum_{t=1}^{k} \mu_t^{(i)}, \qquad \forall k,$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_m)$ resp. $\mathbf{b} = (b_0, b_1, \ldots, b_m)$ is the c.n.s. of λ resp. μ.

Let $\Lambda(n, r)$ be the set of all compositions of r with n parts, and let, for $m > 0$,

$$\begin{cases} \Lambda_m(n,r) = \bigcup_{\lambda \in \Lambda(m,r)} \Lambda(\max_{n,r}, \lambda_1) \\ \qquad\qquad \times \cdots \times \Lambda(\max_{n,r}, \lambda_{m-1}) \times \Lambda(n, \lambda_m) \\ \Lambda_m(n,r)^+ = \Lambda_m(n,r) \cap \Lambda_m(r)^+, \end{cases} \tag{1.1}$$

where $\max_{n,r}$ is the maximum of n, r. Note that $\Lambda(n, r) = \Lambda_1(n, r)$ and that $\Lambda_2(n, r)$ is denoted $\Pi(n, r)$ in [DS]. Note also that $\Lambda_m(n, r)^+$ is a coideal of the poset $\Lambda_m(r)^+ = \Lambda_m(r, r)^+$. An m-composition $\lambda \in \Lambda_m(n, r)$ will sometimes be viewed as a single composition by concatenating $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}$. To indicate the difference, the latter will be denoted by $\overline{\lambda} \in \Lambda(N, r)$, where N is defined by

$$N = N(m, n, r) = (m-1)\max_{n,r} + n. \tag{1.2}$$

For example, $\overline{\lambda} = (30 \cdots 0201) \in \Lambda(9, 6)$ if $\lambda = ((30 \cdots 0), (201)) \in \Lambda_2(3, 6)$. Clearly, the map $\lambda \mapsto \overline{\lambda}$ defines a bijection from $\Lambda_m(n, r)$ to $\Lambda(N, r)$. Note that, for $\lambda, \mu \in \Lambda_m(n, r)$, we have $\lambda \lhd \mu$ if and only if $\overline{\lambda} \lhd \overline{\mu}$.

The subgroup of W generated by $\{s_1, \ldots, s_{r-1}\}$ will be identified with $\mathfrak{S}_r$. For a (1-)composition λ of r, let

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1, \ldots, \lambda_1\}} \times \cdots \times \mathfrak{S}_{\{\lambda_1 + \cdots + \lambda_{n-1} + 1, \ldots, r\}} \tag{1.3}$$

be the Young subgroup of $\mathfrak{S}_r$ corresponding to λ, and $\mathscr{D}_\lambda$ the set of distinguished representatives of right $\mathfrak{S}_\lambda$-cosets. If μ is another composition of r, then $\mathscr{D}_{\lambda\mu} = \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$ is the set of distinguished representatives of double $\mathfrak{S}_\lambda - \mathfrak{S}_\mu$ cosets, and, for $d \in \mathscr{D}_{\lambda\mu}$, $d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$ is a Young (or parabolic) subgroup. For convenience, we will use in the sequel the notation

$$\mathfrak{S}_{\lambda d \cap \mu} = d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu, \qquad \mathfrak{S}_{\lambda \cap d\mu} = \mathfrak{S}_\lambda \cap d\mathfrak{S}_\mu d^{-1}. \tag{1.4}$$

To any 2-composition $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2(n, r)$, we associate a so-called quasi-parabolic subgroup W_λ of W (see [DS]). By definition, we have $W_\lambda = C_\lambda \mathfrak{S}_{\tilde{\lambda}}$, where C_λ is the subgroup of W generated by $t_i = s_{i-1} \cdots s_1 s_0 s_1 \cdots s_{i-1}$ for $1 \le i \le |\lambda^{(1)}|$. Let $\hat{\lambda} = (|\lambda^{(1)}|, \lambda^{(2)})$. Then $W_{\hat{\lambda}}$ is the minimal parabolic subgroup of W containing W_λ. For quasi-parabolic subgroups, the distinguished coset representatives are introduced in [DS, Sect. 2]. Let $\mathscr{D}_{\lambda^{(1)}}$ (resp. $\mathscr{D}_{\hat{\lambda}}$) be the set of distinguished representatives in the right coset $\mathfrak{S}_{\lambda^{(1)}} \setminus \mathfrak{S}_a$ with $a = |\lambda^{(1)}|$ (resp. $W_{\hat{\lambda}} \setminus W$). Then $\mathscr{D}_\lambda = \mathscr{D}_{\lambda^{(1)}} \mathscr{D}_{\hat{\lambda}}$

(resp. $\mathscr{D}_{\lambda\mu} = \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$) is the set of distinguished representatives in the right W_λ-cosets (resp. double W_λ-W_μ-cosets) in the sense of [DS, (2.2.5)]. The reader should not confuse the notation $\mathscr{D}_{\lambda\mu}$ for type A with that for type B.

For $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2(n, r)$, let $x_\lambda = x_{\bar\lambda}\pi_\lambda$ where $\pi_\lambda = \prod_{i=1}^{|\lambda^{(1)}|}(q^{i-1} + T_{t_i})$, and $x_{\bar\lambda} = \sum_{w \in \mathfrak{S}_{\bar\lambda}} T_w$. The element x_λ serves as the generator of the trivial representation for W_λ. Following [DS, Sect. 3], we introduce the endomorphism algebras

$$\begin{aligned} \mathscr{S}^2 = \mathscr{S}_q^2(n, r) &= \operatorname{End}_{\mathscr{H}}\left(\bigoplus_{\lambda \in \Lambda_2(n,r)} x_\lambda \mathscr{H} \right) \\ \mathscr{S}_R^2 = \mathscr{S}_q^2(n, r; R) &= \operatorname{End}_{\mathscr{H}_R}\left(\bigoplus_{\lambda \in \Lambda_2(n,r)} x_\lambda \mathscr{H}_R \right). \end{aligned} \tag{1.5}$$

These endomorphism algebras are called the *q-Schur*^{2} *algebras* of degree (n, r) (see also [DJM1] for a Morita equivalent version).

For any $d \in \mathscr{D}_{\lambda\mu}$, the conjugate intersection $W_\lambda^d \cap W_\mu = d^{-1}W_\lambda d \cap W_\mu$ is a subgroup of W, which will be denoted by W_ν with $\nu = \lambda d \cap \mu$ (cf. [DS, (2.2.8)]). Similarly, write $W_{\hat\lambda}^{\hat d} \cap W_{\hat\mu} = W_{\hat\nu}$. Then $W_{\hat\nu}$ is a parabolic subgroup of W. Let $\pi_{\mu\setminus\hat\nu} \in \mathscr{H}_R$ be the element obtained by deleting the product $\pi_{\hat\nu}$ from π_μ. So $\pi_\mu = \pi_{\hat\nu}\pi_{\mu\setminus\hat\nu}$. For any $\lambda, \mu \in \Lambda_2(n, r)$ and $d \in \mathscr{D}_{\lambda\mu}$, there is a unique element $\varphi_{\lambda\mu}^d$ in $\mathscr{S}_R^2$ such that

$$\varphi_{\lambda\mu}^d(x_\nu h) = \delta_{\mu,\nu} x_\lambda T_{u\hat d}\pi_{\mu\setminus\hat\nu} T_v T_{\mathscr{D}_{\bar\lambda d \cap \bar\mu} \cap W_{\bar\mu}} h, \tag{1.6}$$

where $d = u\hat d v$ is the *right distinguished decomposition* of d (see [DS, Sect. 2.3]), i.e., $\hat d \in \mathscr{D}_{\hat\lambda\hat\mu}$, $u \in \mathscr{D}_{\lambda^{(1)}}$, and $\hat d v^{-1} \in \mathscr{D}_{\hat\lambda}$, and $T_X = \sum_{w \in X} T_w$ for any subset $X \subset W$. Moreover, the set

$$\left\{ \varphi_{\lambda\mu}^d \mid \lambda, \mu \in \Lambda_2(n, r), d \in \mathscr{D}_{\lambda\mu} \right\} \tag{1.7}$$

forms a basis for $\mathscr{S}_R^2$. We shall call it the *natural basis* for $\mathscr{S}_R^2$.

Write $X_{W_\lambda d W_\mu} = \varphi_{\lambda\mu}^d(x_\mu)$. Let ι be the anti-automorphism of $\mathscr{H}_R$ sending T_w to $T_{w^{-1}}$. By [DS, (4.2.2.2)]

$$X_{W_\lambda d W_\mu}^\iota = X_{W_\mu d^{-1} W_\lambda}. \tag{1.8}$$

So ι induces an anti-involution

$$\iota\colon \mathscr{S}_R^2 \to \mathscr{S}_R^2 \qquad \text{such that } \left(\varphi_{\lambda\mu}^d\right)^\iota = \varphi_{\mu\lambda}^{d^{-1}}. \tag{1.9}$$

Now we mimic Green's construction in [G1] to introduce certain subspaces of $\mathscr{S}_R^2$ via its natural basis given in (1.7). First, we need some combinatorics.

Let $I(n,r) = \{\mathbf{i} = (i_1, i_2, \dots, i_r) \mid 1 \le i_j \le n \text{ for } 1 \le j \le r\}$, where n, r are positive integers. Then, the symmetric group $\mathfrak{S}_r$ acts on $I(n,r)$ by place permutation: $\mathbf{i}w = (i_{w(1)}, \dots, i_{w(r)})$ for any $\mathbf{i} \in I(n,r)$ and $w \in \mathfrak{S}_r$. Following [G1, Sect. 3], $I(n,r)$ is a poset with the partial order $\preccurlyeq$ defined by setting $\mathbf{i} \preccurlyeq \mathbf{j}$ if $i_k \le j_k$ for all k with $1 \le k \le r$. The *weight* $\mathrm{wt}(\mathbf{i})$ of $\mathbf{i}$ is a composition $(\lambda_1, \dots, \lambda_n)$ of r, where $\lambda_j = \#\{i_k \in \mathbf{i} \mid i_k = j\}$. Obviously, we have $\mathrm{wt}(\mathbf{i}) \trianglerighteq \mathrm{wt}(\mathbf{j})$ if $\mathbf{i} \preccurlyeq \mathbf{j}$. For any $\lambda \in \Lambda(n,r)$, let

$$\mathbf{i}_\lambda = \Big(\underbrace{1, \dots, 1}_{\lambda_1}, \dots, \underbrace{n, \dots, n}_{\lambda_n}\Big).$$

If $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in \Lambda_m(n,r)$, then we define $\mathbf{i}_\lambda = \mathbf{i}_{\bar\lambda} \in I(N,r)$, where N is defined in (1.2).

(1.10) LEMMA. *Let $\lambda, \mu \in \Lambda_m(n,r)$ be of c.n.s.* $\mathbf{a}$, $\mathbf{b}$, *respectively. Then $\mathbf{i}_\lambda w \succcurlyeq \mathbf{i}_\mu$ if and only if, for every i with $1 \le i \le m$ and every k, $w(j) \le b_{i-1} + \sum_{t=1}^k \mu_t^{(i)}$ for all j with $a_{i-1} < j \le a_{i-1} + \sum_{t=1}^k \lambda_t^{(i)}$.*

Proof. Since $\mathbf{i}_\lambda = \mathbf{i}_{\bar\lambda}$ for any $\lambda \in \Lambda_m(n,r)$, we have $\mathbf{i}_\lambda w \succcurlyeq \mathbf{i}_\mu$ if and only if $\mathbf{i}_{\bar\lambda} w \succcurlyeq \mathbf{i}_{\bar\mu}$. So we may assume $\lambda = \bar\lambda = (\lambda_1, \dots, \lambda_N)$ and $\mu = \bar\mu = (\mu_1, \dots, \mu_N)$. Thus, $\mathbf{i}_\lambda w \succcurlyeq \mathbf{i}_\mu$ if and only if

$$\begin{aligned}
&i_{w(1)}, \dots, i_{w(\mu_1)} \ge 1,\\
&i_{w(\mu_1+1)}, \dots, i_{w(\mu_1+\mu_2)} \ge 2,\\
&\cdots,\\
&i_{w(\mu_1+\cdots+\mu_{N-1}+1)}, \dots, i_{w(r)} \ge N,
\end{aligned}$$

which are equivalent to

$$\begin{aligned}
&w(1), \dots, w(\lambda_1) \le \mu_1,\\
&w(\lambda_1+1), \dots, w(\lambda_1+\lambda_2) \le \mu_1 + \mu_2,\\
&\cdots,\\
&w(\lambda_1+\cdots+\lambda_{N-1}+1), \dots, w(r) \le r,
\end{aligned}$$

as required. ∎

(1.11) DEFINITION. For any $\mu \in \Lambda_m(n,r)$, let

$$\Omega_m^{\succcurlyeq}(\mu) = \left\{(\lambda, d) \mid \lambda \in \Lambda_m(n,r), d \in \mathscr{D}_{\bar\lambda\bar\mu} \text{ and } \mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\mu\right\},$$

$$\Omega_m^{\preccurlyeq}(\mu) = \left\{(\lambda, d) \mid \lambda \in \Lambda_m(n,r), d \in \mathscr{D}_{\bar\mu\bar\lambda} \text{ and } \mathbf{i}_\mu d \preccurlyeq \mathbf{i}_\lambda\right\},$$

and

$$\mathscr{S}_R^{2,\succcurlyeq} = \mathscr{S}_R^{2,\succcurlyeq}(n,r) = R\text{-span}\left\{\varphi_{\lambda\mu}^{d} \mid (\lambda, d) \in \Omega_2^{\succcurlyeq}(\mu), \mu \in \Lambda_2(n,r)\right\},$$

$$\mathscr{S}_R^{2,\preccurlyeq} = \mathscr{S}_R^{2,\preccurlyeq}(n,r) = R\text{-span}\left\{\varphi_{\mu\lambda}^{d} \mid (\lambda, d) \in \Omega_2^{\preccurlyeq}(\mu), \mu \in \Lambda_2(n,r)\right\}.$$

Clearly, $(\lambda, d) \in \Omega_m^{\succcurlyeq}(\mu)$ if and only if $(\lambda, d^{-1}) \in \Omega_m^{\preccurlyeq}(\mu)$, and therefore, we have, by (1.9), $\mathscr{S}_R^{2,\preccurlyeq} = (\mathscr{S}_{\mathrm{R}}^{2,\succcurlyeq})^{\iota}$.

2. THE SUBALGEBRA STRUCTURE ON $\mathscr{S}_R^{2,\succcurlyeq}$ AND $\mathscr{S}_R^{2,\preccurlyeq}$

In this section, we will prove that the vector spaces $\mathscr{S}_R^{2,\succcurlyeq}$ and $\mathscr{S}_R^{2,\preccurlyeq}$ are actually subalgebras of $\mathscr{S}_R^2$. We shall see that the subalgebra structures of $\mathscr{S}_R^{2,\succcurlyeq}$ and $\mathscr{S}_R^{2,\preccurlyeq}$ are closely related to the subalgebra structure of the Borel subalgebras $\mathscr{S}_R^{1,\succcurlyeq}$ and $\mathscr{S}_R^{1,\preccurlyeq}$ for the corresponding q-Schur algebra. Let us first look at the q-Schur algebra case.

For a commutative ring R, let $R[M_n(q)]$ be the associative algebra over R generated by X_{ij} with $1 \le i, j \le n$ such that

$$\begin{cases} (1) \quad X_{ij}X_{ik} = qX_{ik}X_{ij}, & \text{if } j > k, \\ (2) \quad X_{ji}X_{ki} = X_{ki}X_{ji}, & \text{if } j > k, \\ (3) \quad X_{ij}X_{rs} = q^{-1}X_{rs}X_{ij}, & \text{if } i > r, j < s, \\ (4) \quad X_{ij}X_{rs} - X_{rs}X_{ij} = (q^{-1} - 1)X_{is}X_{rj}, & \text{if } i < r, j < s. \end{cases} \tag{2.1}$$

As an R-module, $R[M_n(q)]$ has a basis $\{\prod_{ij} X_{ij}^{t_{ij}} \mid t_{ij} \in \mathbb{Z}^+\}$, where the products are formed with respect to any fixed order of the X_{ij}'s (see, for example, [DPW, (1.1)] with $\alpha = q, \beta = 1$ there). Let $A_q(n,r)$ be the rth homogeneous component of $R[M_q(n)]$. Then $A_q(n,r)$ has a basis

$$\left\{X_{\lambda\mu}^{d} = X_{\mathbf{i}_\lambda d, \mathbf{i}_\mu} \mid \lambda, \mu \in \Lambda(n,r), d \in \mathscr{D}_{\lambda\mu}\right\},$$

where $\mathscr{D}_{\lambda\mu}$ denotes the set of distinguished representatives for the double cosets $\mathfrak{S}_\lambda \setminus \mathfrak{S}_r / \mathfrak{S}_\mu$, and $X_{\mathbf{ij}} = X_{i_1, j_1} X_{i_2, j_2} \cdots X_{i_r, j_r}$ if $\mathbf{i} = (i_1, \ldots, i_r)$ and $\mathbf{j} = (j_1, \ldots, j_r)$. Denote by $A_q(n,r)^*$ the linear dual of $A_q(n,r)$. Then, by [DPW, (5.5)],

$$A_q(n,r)^* \cong \mathrm{End}_{\mathscr{H}_R}\left(\oplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathscr{H}_R\right),$$

where $\mathscr{H}_R$ is the Hecke algebra associated to the symmetric group $\mathfrak{S}_r$ over the commutative ring R.

(2.2) DEFINITIONS AND RESULTS. The algebra

$$\mathscr{S}_R^1 = \mathscr{S}_q(n, r; R) = \mathrm{End}_{\mathscr{H}_R}\big(\oplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathscr{H}_R\big)$$

is called the *q-Schur algebra* of degree (n, r) (cf. [DJ2]). The natural basis for a q-Schur algebra is given as follows: For $\lambda, \mu \in \Lambda(n, r), d \in \mathscr{D}_{\lambda\mu}$, let $\psi_{\lambda\mu}^d \in \mathscr{S}_R^1$ be defined by

$$\psi_{\lambda\mu}^d(x_\nu h) = \delta_{\mu\nu} \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w h = \delta_{\mu\nu} T_{\mathfrak{S}_\lambda d \mathfrak{S}_\mu} h.$$

Then $\mathscr{S}_R^1$ has an R-basis $\{\psi_{\lambda\mu}^d \mid \lambda, \mu \in \Lambda(n, r), d \in \mathscr{D}_{\lambda\mu}\}$. Moreover, if we identify $\psi_{\lambda\mu}^d$ with its image under the isomorphism above, the basis $\{\psi_{\lambda\mu}^d \mid \lambda, \mu \in \Lambda(n, r), d \in \mathscr{D}_{\lambda\mu}\}$ is the dual of the basis $\{X_{\lambda,\mu}^d \mid \lambda, \mu \in \Lambda(n, r), r \in \mathscr{D}_{\lambda\mu}\}$ for $A_q(n, r)$. Thus, we have $\psi_{\lambda\mu}^d(X_{\rho\nu}^{d_1}) = \delta_{\lambda\rho}\delta_{\mu\nu}\delta_{d,d_1}$ (see [DPW, (5.7)]).

The following theorem was first obtained by Parshall and Wang (see [PW, Sect. 11.2]). The proof below is different.

(2.3) THEOREM. *Suppose $\lambda, \mu, \nu \in \Lambda(n, r)$ and $d_1 \in \mathscr{D}_{\lambda\mu}, d_2 \in \mathscr{D}_{\mu\nu}$.*

(a) *If $\mathbf{i}_\lambda d_1 \succcurlyeq \mathbf{i}_\mu$ and $\mathbf{i}_\mu d_2 \succcurlyeq \mathbf{i}_\nu$, then $\psi_{\lambda\mu}^{d_1}\psi_{\mu\nu}^{d_2} = \sum_{\substack{d \in \mathscr{D}_{\lambda\nu} \\ \mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\nu}} a_d \psi_{\lambda\nu}^d$ for some $a_d \in R$.*

(b) *If $\mathbf{i}_\lambda d_1 \preccurlyeq \mathbf{i}_\mu$ and $\mathbf{i}_\mu d_2 \preccurlyeq \mathbf{i}_\nu$, then $\psi_{\lambda\mu}^{d_1}\psi_{\mu\nu}^{d_2} = \sum_{\substack{d \in \mathscr{D}_{\lambda\nu} \\ \mathbf{i}_\lambda d \preccurlyeq \mathbf{i}_\nu}} a_d \psi_{\lambda\nu}^d$ for some $a_d \in R$.*

(c) *Let $\mathscr{S}_R^{1,\succcurlyeq} = \mathscr{S}_R^{1,\succcurlyeq}(n, r)$ (resp. $\mathscr{S}_R^{1,\preccurlyeq} = \mathscr{S}_R^{1,\preccurlyeq}(n, r)$) be the free R-submodule of $\mathscr{S}_R^1$ spanned by $\{\psi_{\lambda\mu}^d \mid (\lambda, d) \in \Omega_1^{\succcurlyeq}(\mu), \mu \in \Lambda_1(n, r)\}$ (resp. $\{\psi_{\mu\lambda}^d \mid (\lambda, d) \in \Omega_1^{\preccurlyeq}(\mu), \mu \in \Lambda_1(n, r)\}$). Then the subspaces $\mathscr{S}_R^{1,\succcurlyeq}$ and $\mathscr{S}_R^{1,\preccurlyeq}$ are subalgebras for the q-Schur algebra $\mathscr{S}_R^1$.*

Proof. By embedding $\mathscr{S}_R^1$ into $\mathscr{S}_R^2$ via $\psi_{\lambda\mu}^d \mapsto \varphi_{(\lambda,0)(\mu,0)}^d$ (or a direct construction), we obtain an anti-automorphism ι on $\mathscr{S}_R^1$ of order 2 which turns (b) into (a). The statement (c) follows easily from (a) and (b). So it remains to prove (b).

Suppose $\lambda, \mu \in \Lambda(n, r), d \in \mathscr{D}_{\lambda\mu}$ with $\mathbf{i}_\lambda d \preccurlyeq \mathbf{i}_\mu$. We claim that $\psi_{\lambda\mu}^d(X_{\mathbf{ik}}) \neq 0$ implies $\mathbf{i} \preccurlyeq \mathbf{k}$. Indeed, we have, by the hypothesis and (2.2), $\mathbf{k} = \mathbf{i}_\mu w$ for some w. If $l(w) = 0$, i.e., $w = e$, then $\mathbf{i}_\mu = \mathbf{k}$. So $\mathbf{i} = \mathbf{i}_\lambda d \preccurlyeq \mathbf{i}_\mu = \mathbf{k}$. Assume now $l(w) > 0$. Write $w = w't$ with $t = (a, a+1)$ and $l(w) = l(w') + 1$. Then $k_a > k_{a+1}$. If $i_a \le i_{a+1}$, then, by (2.1(3)), $X_{\mathbf{ik}} = qX_{\mathbf{i}t,\mathbf{k}t} = qX_{\mathbf{i}t,\mathbf{i}_\mu w'}$. By induction, $\mathbf{i}t \preccurlyeq \mathbf{i}_\mu w' = \mathbf{k}t$, which implies $\mathbf{i} \preccurlyeq \mathbf{k}$. If $i_a > i_{a+1}$, then by (2.1(4)(3)),

$$\begin{aligned} X_{i_a k_a} X_{i_{a+1} k_{a+1}} &= X_{i_{a+1} k_{a+1}} X_{i_a k_a} - (q^{-1} - 1) X_{i_{a+1} k_a} X_{i_a k_{a+1}} \\ &= X_{i_{a+1} k_{a+1}} X_{i_a k_a} - (1 - q) X_{i_a k_{a+1}} X_{i_{a+1} k_a}. \end{aligned}$$

Thus, $X_{\mathbf{ik}} = X_{\mathbf{i}t,\mathbf{k}t} - (1-q)X_{\mathbf{i},\mathbf{k}t}$. Since $\psi^d_{\lambda\mu}(X_{\mathbf{ik}}) \neq 0$, we have either $\psi^d_{\lambda\mu}(X_{\mathbf{i}t,\mathbf{k}t}) \neq 0$ or $\psi^d_{\lambda\mu}(X_{\mathbf{i},\mathbf{k}t}) \neq 0$. By induction, either $\mathbf{i}t \preccurlyeq \mathbf{k}t$ or $\mathbf{i} \preccurlyeq \mathbf{k}t$. If $\mathbf{i}t \preccurlyeq \mathbf{k}t$, then $\mathbf{i} \preccurlyeq \mathbf{k}$. Assume $\mathbf{i} \preccurlyeq \mathbf{k}t$. By definition, $i_{a+1} \leq k_a$ and $i_a \leq k_{a+1}$, which implies $i_{a+1} < i_a \leq k_{a+1} < k_a$. Hence, $\mathbf{i} \preccurlyeq \mathbf{k}$. This completes the proof of our claim.

Let $\psi^{d_1}_{\mu\nu}\psi^{d_2}_{\mu\nu} = \sum_{d \in \mathscr{D}_{\lambda\nu}} a_d \psi^d_{\lambda\nu}$. We hope to prove that if $\mathbf{i}_\lambda d_1 \preccurlyeq \mathbf{i}_\mu$ and $\mathbf{i}_\mu d_2 \preccurlyeq \mathbf{i}_\nu$, then $a_d \neq 0$ implies $\mathbf{i}_\lambda d \preccurlyeq \mathbf{i}_\nu$. Suppose now $a_d \neq 0$. Then $\psi^{d_1}_{\lambda\mu}\psi^{d_2}_{\mu\nu}(X_{\mathbf{ij}}) \neq 0$, where $(\mathbf{i},\mathbf{j}) = (\mathbf{i}_\lambda d, \mathbf{i}_\nu)$. If we denote by Δ the comultiplication on $A_q(n,r)$, then,

$$\psi^{d_1}_{\lambda\mu}\psi^{d_2}_{\mu\nu}(X_{\mathbf{ij}}) = \mathrm{mult} \circ \left(\psi^{d_1}_{\lambda\mu} \otimes \psi^{d_2}_{\mu\nu}\right) \circ \Delta(X_{\mathbf{ij}})$$

$$= \sum_{\mathbf{k} \in I(n,r)} \psi^{d_1}_{\lambda\mu}(X_{\mathbf{ik}})\psi^{d_2}_{\mu\nu}(X_{\mathbf{kj}}) \neq 0.$$

So there is a $\mathbf{k} \in I(n,r)$ such that $\psi^{d_1}_{\lambda\mu}(X_{\mathbf{ik}}) \neq 0$ and $\psi^{d_2}_{\mu\nu}(X_{\mathbf{kj}}) \neq 0$. Thus, $\mathbf{i} \preccurlyeq \mathbf{k}$ and $\mathbf{k} \preccurlyeq \mathbf{j}$ by the above claim, and hence, $\mathbf{i} \preccurlyeq \mathbf{j}$, or $\mathbf{i}_\lambda d \preccurlyeq \mathbf{i}_\nu$. ∎

In the rest of this section, we shall try to find the relation between $\varphi^d_{\lambda\mu}$ and $\psi^d_{\bar\lambda\bar\mu}$ for $\lambda, \mu \in \Lambda_2(n,r)$ and $d \in \mathscr{D}_{\bar\lambda\bar\mu}$ and then we prove that $\mathscr{S}^{2,\succcurlyeq}_R$ and $\mathscr{S}^{2,\preccurlyeq}_R$ are subalgebras via (2.3). First, we recall a result on the Weyl group of type B_r.

For $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2(n,r)$, let C_λ be the subgroup of W generated by t_i with $1 \leq i \leq |\lambda^{(1)}|$ (as given in Section 1). Recall from (1.6) that $\hat\nu \in \Lambda_2(n,r)$ is defined by $W^{\hat d}_{\hat\lambda} \cap W_{\hat\mu} = W_{\hat\nu}$ where $\lambda, \mu \in \Lambda_2(n,r)$ and $\hat d$ is distinguished, uniquely determined by d. In particular, we have (cf. [DS, Sect. 4.2])

$$C_{\hat\nu} = C^{\hat d}_{\hat\lambda} \cap C_{\hat\mu} = C^{\hat d}_{\lambda} \cap C^{\hat d}_{\mu} = Z_{C_\lambda \cap C_\mu}(\hat d), \tag{2.4}$$

where $Z_{C_\lambda \cap C_\mu}(\hat d)$ is the centralizer of $\hat d$ in $C_\lambda \cap C_\mu$.

(2.5) THEOREM. *The subspaces $\mathscr{S}^{2,\succcurlyeq}_R$ and $\mathscr{S}^{2,\preccurlyeq}_R$ are subalgebras of $\mathscr{S}^2_R$.*

Proof. We claim first that, for any $\lambda, \mu \in \Lambda_2(n,r)$, $d \in \mathscr{D}_{\bar\lambda\bar\mu}$ with $\mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\mu$,

$$\varphi^d_{\lambda\mu}(x_\mu) = x_\lambda T_{\mu\hat d}\pi_{\mu\setminus\hat\nu}T_v h_{\bar\lambda d \cap \bar\mu} = T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}}\pi_\mu = \psi^d_{\bar\lambda\bar\mu}(x_{\bar\mu})\pi_\mu, \tag{2.6}$$

where $h_{\bar\lambda d \cap \bar\mu} = T_{\mathscr{D}_{\bar\lambda d \cap \bar\mu} \cap \mathfrak{S}_{\bar\mu}}$ and $d = u\hat d v$ is the right distinguished decomposition of d (see (1.6)). Indeed, since $\mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\mu$ implies $d(j) \leq |\mu^{(1)}|$ for all $j \leq |\lambda^{(1)}|$ (see (1.10)), we have $d^{-1}C_\lambda d \subseteq C_\mu$. Let $d = u\hat d v$ be the right distinguished decomposition of d with $u \in \mathscr{D}_{\lambda^{(1)}}$, $v^{-1} \in \mathscr{D}_{\mu^{(1)}}$ and $\hat d \in \mathscr{D}_{\hat\lambda\hat\mu}$.

Then, $\hat{d}^{-1}C_\lambda\hat{d} \subseteq C_\mu$, since C_λ is a normal subgroup of $W(B_a)$ with $a = |\lambda^{(1)}|$. For any $t_j \in C_\lambda$, write $t_j\hat{d} = \hat{d}(\hat{d}^{-1}t_j\hat{d})$. We have $l(t_j) + l(\hat{d}) = l(\hat{d}) + l(\hat{d}^{-1}t_j\hat{d})$ as $\hat{d}^{-1}t_j\hat{d} \in C_\mu$. Consequently, $l(\hat{d}^{-1}t_j\hat{d}) = l(t_j)$. Comparing the numbers of s_0 occurring in $\hat{d}^{-1}t_j\hat{d}$ and t_j, we have $\hat{d}^{-1}t_j\hat{d} = t_j$, and hence, $C_\lambda = \hat{d}^{-1}C_\lambda\hat{d}$. Thus, C_λ is a subgroup of C_μ, and hence, $C_{\hat{\nu}} = C_\lambda$ and $\pi_{\lambda\setminus\hat{\nu}} = 1$. Let $d^{-1} = u_1\hat{d}_1v_1$ be the right distinguished decomposition for d^{-1}. Then, by (1.6),

$$X_{W_\mu d^{-1}W_\lambda} = \varphi_{\mu\lambda}^{d^{-1}}(x_\lambda) = x_\mu T_{u_1\hat{d}_1}\pi_{\lambda\setminus\hat{\nu}}T_{v_1}h_{\bar{\lambda}d_1\cap\bar{\mu}} = \pi_\mu T_{\mathfrak{S}_{\bar{\mu}}d^{-1}\mathfrak{S}_{\bar{\lambda}}}.$$

Thus, $\varphi_{\lambda\mu}^{d}(x_\mu) = X_{W_\lambda dW_\mu} = (X_{W_\mu d^{-1}W_\lambda})^\iota = T_{\mathfrak{S}_{\bar{\lambda}}d\mathfrak{S}_{\bar{\mu}}}\pi_\mu$. The last equality in (2.6) follows from the definition of $\psi_{\bar{\lambda}\bar{\mu}}^{d}$.

If $\varphi_{\lambda\mu}^{d_1}, \varphi_{\nu\rho}^{d_2} \in \mathscr{S}_R^{2,\succcurlyeq}$, then $\varphi_{\lambda\mu}^{d_1}\varphi_{\nu\rho}^{d_2} = 0$ unless $\mu = \nu$. Let $d_2 = u\hat{d}_2v$ be the right distinguished decomposition of d_2. By repeatedly applying (2.6), we have

$$\begin{aligned}
\varphi_{\lambda\mu}^{d_1}\varphi_{\mu\rho}^{d_2}(x_\rho) &= \varphi_{\lambda\mu}^{d_1}\Big(x_\mu T_{u\hat{d}_2}\pi_{\rho\setminus\hat{\mu}}T_v h_{\bar{\lambda}d_2\cap\bar{\mu}}\Big)\\
&= T_{\mathfrak{S}_{\bar{\lambda}}d_1\mathfrak{S}_{\bar{\mu}}}\pi_\mu T_{u\hat{d}_2}\pi_{\rho\setminus\hat{\mu}}T_v h_{\bar{\lambda}d_2\cap\bar{\mu}}\\
&= T_{\mathfrak{S}_{\bar{\lambda}}\cap\mathscr{D}_{\bar{\lambda}\cap d_1\bar{\mu}}}T_{d_1}T_{\mathfrak{S}_{\bar{\mu}}d_2\mathfrak{S}_{\bar{\rho}}}\pi_\rho\\
&= \psi_{\bar{\lambda}\bar{\mu}}^{d_1}\psi_{\bar{\mu}\bar{\rho}}^{d_2}(x_{\bar{\rho}})\pi_\rho\\
&= \sum_{d\in\mathscr{D}_{\bar{\lambda}\bar{\rho}},\,\mathbf{i}_{\bar{\lambda}}d\succcurlyeq\mathbf{i}_{\bar{\rho}}} a_d\psi_{\bar{\lambda}\bar{\rho}}^{d}(x_{\bar{\rho}})\pi_\rho,
\end{aligned} \tag{2.7}$$

where the notation $\bar{\lambda}\cap d_1\bar{\mu}$ is given in (1.4), and the last equality follows from (2.3(a)). Noting $\mathbf{i}_\lambda = \mathbf{i}_{\bar{\lambda}}$, we obtain

$$\varphi_{\lambda\mu}^{d_1}\varphi_{\mu\rho}^{d_2} = \sum_{d\in\mathscr{D}_{\bar{\lambda}\bar{\rho}}\mathbf{i}_\lambda d\succcurlyeq\mathbf{i}_\rho} a_d\varphi_{\lambda\rho}^{d} \in \mathscr{S}_R^{2,\succcurlyeq} \tag{2.8}$$

for some $a_d \in R$. Hence the subspace $\mathscr{S}_R^{2,\succcurlyeq}$ is a subalgebra of $\mathscr{S}_R^2$. Applying (1.9), we see that $\mathscr{S}_R^{2,\preccurlyeq}$ is also a subalgebra of $\mathscr{S}_R^2$. ∎

(2.9) COROLLARY. *We have the following isomorphisms of R-algebras*

$$\mathscr{S}_R^{2,\succcurlyeq}(n,r) \cong \mathscr{S}_R^{1,\succcurlyeq}(\max_{n,r}+n,r),$$
$$\mathscr{S}_R^{2,\preccurlyeq}(n,r) \cong \mathscr{S}_R^{1,\preccurlyeq}(\max_{n,r}+n,r).$$

Proof. This immediately follows from (2.7) and (2.8). ∎

3. THE TRIANGULAR DECOMPOSITION OF $\mathcal{S}_R^2$

In [G1], Green proved that a Schur algebra is a product of its two (opposite) Borel subalgebras. In this section, we are going to generalize this result to q-Schur^2 algebras. Let $\mathcal{S}_R^{2,\succcurlyeq}\mathcal{S}_R^{2,\preccurlyeq}$ denote the submodule of $\mathcal{S}_R^2$ generated by all products ab with $a \in \mathcal{S}_R^{2,\succcurlyeq}$ and $b \in \mathcal{S}_R^{2,\preccurlyeq}$. We shall prove that $\mathcal{S}_R^2 = \mathcal{S}_R^{2,\succcurlyeq}\mathcal{S}_R^{2,\preccurlyeq}$ and call this the *triangular decomposition* of $\mathcal{S}_R^2$. Using the embedding of a q-Schur algebra into a q-Schur^2 algebra, we will reobtain the triangular decomposition of a q-Schur algebra, which was first established by Parshall and Wang [PW] in the context of quantum linear groups. Our proof below is in fact a simple application of the Green–Murphy type basis for the q-Schur^2 algebra given in [DS]. These basis elements have double indices of semi-standard tableaux. Let us briefly recall some combinatorics first.

As usual, we identify each $\mu \in \Lambda(n, r)$ with its Young diagram, which consists of boxes arranged in a manner as illustrated by the example $\mu = (4021) \in \Lambda(4, 7)$ for which we have

$$\mu = \begin{array}{cccc} \square & \square & \square & \square \\ - & - & - & \\ \square & \square & & \\ \square & & & \end{array}$$

A μ-tableau $\mathbf{t}$ is obtained by replacing each box by one of the numbers $1, 2, \ldots, r$, and $\mathbf{t}$ is called *regular* if the set of entries in $\mathbf{t}$ is equal to $\{1, \ldots, r\}$. A regular μ-tableau $\mathbf{t}$ is row-standard if its entries are increasing along each row. Let $\mathbf{t}^{\mu}$ be the regular μ-tableau in which the numbers $1, 2, \ldots, r$ appear in order along successive rows. For example, for $\mu = (4021)$

$$\mathbf{t}^{\mu} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ - & - & - & \\ 5 & 6 & & \\ 7 & & & \end{array}$$

The symmetric group $\mathfrak{S}_r$ acts on the regular tableaux by permuting its entries. The set of distinguished representatives $\mathcal{D}_{\mu}$ can be characterised as

$$\mathcal{D}_{\mu} = \{d \in \mathfrak{S}_r \mid \mathbf{t}^{\mu}d \text{ is row standard}\}.$$

A μ-tableau $\mathbf{t}$ is said to be of *type* λ if the number of entries i in $\mathbf{t}$ is equal to λ_i. For any $\mu \in \Lambda(n, r)^+$, a μ-tableau is said to be *semi-standard* if its entries are weakly increasing along each row and strictly increasing down each column. Let $\mathfrak{T}(\mu, \lambda)$ (resp. $\mathfrak{T}^{ss}(\mu, \lambda)$) be the set of all μ-tableaux (resp. semi-standard μ-tableaux) of type λ.

For $w \in \mathfrak{S}_r$ and $\mathfrak{s} \in \mathfrak{T}^{ss}(\mu, \lambda)$, let $\delta(w, \mathfrak{s}) \in \mathscr{D}_\lambda$ be defined by the row-standard λ-tableau $\mathbf{t}^\lambda \delta(w, \mathfrak{s})$ for which i belongs to row a if the place occupied by i in $\mathbf{t}^\mu w$ is occupied by a in $\mathfrak{s}$. Thus, we obtain a map $\delta(*, *)\colon \mathfrak{S}_r \times \mathfrak{T}(\mu, \lambda) \to \mathscr{D}_\lambda$. In particular, the map $\delta(1, *)$ gives a bijection between $\mathfrak{T}(\mu, \lambda)$ and $\mathscr{D}_\lambda$, and $\mathfrak{s}$ has non-decreasing rows (i.e., $\mathfrak{s}$ is *weakly* row-standard) if and only if $\delta(1, \mathfrak{s}) \in \mathscr{D}_{\lambda\mu}$ [DJ1, (1.7)].

We now introduce appropriate terminology for multi-compositions. For $\mu = (\mu^{(1)}, \dots, \mu^{(m)}) \in \Lambda_m(n, r)$, a multi-tableau $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_m)$ is called a μ-tableau if

$$\overline{\mathfrak{s}} = \begin{array}{c} \mathfrak{s}_1 \\ \vdots \\ \mathfrak{s}_m \end{array}$$

is a $\overline{\mu}$-tableau. Clearly, each $\mathfrak{s}_i$ is a $\mu^{(i)}$-tableau. The multi-tableau $\mathfrak{s}$ is said to be *regular* (resp. row-standard) if $\overline{\mathfrak{s}}$ is regular (resp. row standard). For $\mu, \lambda \in \Lambda_m(n, r)$, a μ-tableau $\mathfrak{s}$ is said to be of type λ if the $\overline{\mu}$-tableau $\overline{\mathfrak{s}}$ is of type $\overline{\lambda}$. Let $\mathbf{t}^\mu$ be the regular tableau such that $\overline{\mathbf{t}}^\mu = \mathbf{t}^{\overline{\mu}}$. For example,

$$\mathbf{t}^\mu = \left(\mathbf{t}^{\mu^{(1)}}, \dots, \mathbf{t}^{\mu^{(m)}}\right) = \boxed{\begin{array}{ccccccc} 1 & 2 & 3 & \quad & 5 & 6 & \qquad 9 \\ 4 & & & & 7 & 8 & \end{array}}$$

if $\mu = ((310 \cdots 0), (220 \cdots 0), (10)) \in \Lambda_3(2, 9)$. Here the rows in each single tableau corresponding to 0 parts at the end of each $\lambda^{(i)}$ are omitted for visual clarity.

The symmetric group $\mathfrak{S}_r$ acts on regular μ-tableaux diagonally, i.e., $\mathfrak{s}w = (\mathfrak{s}_1 w, \dots, \mathfrak{s}_m w)$. Note that $\overline{\mathfrak{s}}w = \overline{\mathfrak{s}w}$.

The notion of semi-standard multi-tableaux has been introduced by Du and Scott in [DS] for $m = 2$ (see also a version given in [DJM1]), and by Dipper, James, and Mathas for arbitrary m in [DJM2]. The following definition is a generalized version of [DS, (1.2.2)].

(3.1) DEFINITION. For $\mu \in \Lambda_m(n, r)^+$ and $\lambda \in \Lambda_m(n, r)$, a μ-tableau $\mathfrak{s}$ of type λ is said to be *semi-standard* if

(a) $\mathfrak{s}_1, \dots, \mathfrak{s}_m$ have non-decreasing rows and strictly increasing columns,

(b) all entries in $\mathfrak{s}_j$ are strictly bigger than $(j-1)\max_{n,r}$ for $2 \le j \le m$.

Note that if we write every entry in $\mathfrak{s}$ as $i + (j-1)\max_{n,r}$ with $1 \le i < \max_{n,r}$ and replace it by the symbol i_j, the definition above is turned into the definition given in [DJM2, (4.4)].

Let $\mathfrak{T}^{ss}_m(\mu, \lambda)$ be the set of all semi-standard μ-tableaux of type λ. For any $\mathfrak{s} \in \mathfrak{T}^{ss}_m(\mu, \lambda)$, let

$$\delta(\mathfrak{s}) = \delta(1, \bar{\mathfrak{s}}). \tag{3.2}$$

Since $\mathfrak{s}$ is a semi-standard μ-tableau of type λ, $\bar{\mathfrak{s}}$ is a weakly row-standard $\bar{\mu}$-tableau of type $\bar{\lambda}$, and hence, $\delta(\mathfrak{s}) \in \mathscr{D}_{\bar{\lambda}\bar{\mu}}$.

(3.3) LEMMA. *For any* $\mathfrak{s} \in \mathfrak{T}^{ss}_m(\mu, \lambda)$, *we have* $\mathbf{i}_\lambda \delta(\mathfrak{s}) \succcurlyeq \mathbf{i}_\mu$.

Proof. Let $\mathbf{a}$ resp. $\mathbf{b}$ be the c.n.s. of λ resp. μ. For a tableau $\mathbf{t}$, write $\mathrm{row}_{\mathbf{t}}(a) = i$, if a is an entry in $\mathbf{t}$ whose row index is i. By (1.10), it suffices to prove that for every i with $1 \le i \le m$ and every k, $\delta(\mathfrak{s})(j) \le b_{i-1} + \sum_{t=1}^{k} \mu_t^{(i)}$ for all j with $a_{i-1} < j \le a_{i-1} + \sum_{t=1}^{k} \lambda_t^{(i)}$. Suppose there are i_0, j_0, k_0 with $a_{i_0-1} < j_0 \le a_{i_0-1} + \sum_{t=1}^{k_0} \lambda_t^{(i)}$ such that $\delta(\mathfrak{s})(j_0) > b_{i_0-1} + \sum_{t=1}^{k_0} \mu_t^{(i)}$. Then $\mathrm{row}_{\bar{\mathbf{t}}^\mu}(\delta(\mathfrak{s})(j_0)) \ge (i_0 - 1)\max_{n,r} + (k_0 + 1)$. Since $\mathfrak{s}$ is a semi-standard μ-tableau of type λ, we have by (3.1(a)) that $a \ge (i_0 - 1)\max_{n,r} + (k_0 + 1)$, where a is the entry of $\bar{\mathfrak{s}}$ in the place which is occupied by $\delta(\mathfrak{s})(j_0)$ in $\bar{\mathbf{t}}^\mu$. By definition, $a = \mathrm{row}_{\bar{\mathbf{t}}^\lambda \delta(\mathfrak{s})}(\delta(\mathfrak{s})(j_0)) \ge (i_0 - 1)\max_{n,r} + (k_0 + 1)$. However, $a = \mathrm{row}_{\bar{\mathbf{t}}^\lambda \delta(\mathfrak{s})}(\delta(\mathfrak{s})(j_0)) = \mathrm{row}_{\bar{\mathbf{t}}^\lambda}(j_0)$, since the symmetric group $\mathfrak{S}_r$ acts on $\bar{\mathbf{t}}^\lambda$ by entry permutation. Therefore, $a \le (i_0 - 1)\max_{n,r} + k_0$, a contradiction. ∎

The first part of the following theorem guarantees the existence of the expected triangular decomposition of $\mathscr{S}^2_R$.

(3.4) THEOREM [DS, Sect. 6]. (a) *The set*

$$\Big\{\varphi_{\lambda\mu}^{\delta(\mathfrak{s})}\varphi_{\mu\nu}^{\delta(\mathfrak{t})^{-1}} \mid \mu \in \Lambda_2(n, r)^+, \lambda, \nu \in \Lambda_2(n, r), \\ \mathfrak{s} \in \mathfrak{T}^{ss}_2(\mu, \lambda), \mathfrak{t} \in \mathfrak{T}^{ss}_2(\mu, \nu)\Big\}$$

is an R-basis of $\mathscr{S}^2_R$.

(b) *For any commutative Noetherian ring R, the algebra* $\mathscr{S}^2_R$ *is quasi-hereditary.*

We shall generalize this result to the q-Schur^m algebra $\mathbf{S}^m_R$ in order to get its triangular decomposition in Section 5.

(3.5) THEOREM. *Let* $\mathscr{S}^{2,\succcurlyeq}_R$ *and* $\mathscr{S}^{2,\preccurlyeq}_R$ *be the subalgebras of* $\mathscr{S}^2_R$ *as defined in* (1.11). *Then* $\mathscr{S}^2_R$ *has the following triangular decomposition*

$$\mathscr{S}^2_R = \mathscr{S}^{2,\succcurlyeq}_R \mathscr{S}^{2,\preccurlyeq}_R = \sum_{\mu \in \Lambda_2(n,r)^+} \mathscr{S}^{2,\succcurlyeq}_R \varphi^1_{\mu\mu} \mathscr{S}^{2,\preccurlyeq}_R .$$

Proof. Applying (3.3) for $m = 2$, we have $\varphi_{\lambda\mu}^{\delta(\mathfrak{s})} \in \mathscr{S}^{2,\succcurlyeq}_R$ for any $\mathfrak{s} \in \mathfrak{T}^{ss}_2(\mu, \lambda)$. Let ι be the anti-involution on $\mathscr{H}_R$ as in (1.8). By (1.9), $\varphi_{\mu\nu}^{\delta(\mathbf{t})^{-1}} = (\varphi_{\nu\mu}^{\delta(\mathbf{t})})^\iota \in (\mathscr{S}^{2,\succcurlyeq}_R)^\iota = \mathscr{S}^{2,\preccurlyeq}_R$ for every $\mathbf{t} \in \mathfrak{T}^{ss}_2(\mu, \nu)$. So the basis given in (3.4) is contained in $\mathscr{S}_R^{2,\succcurlyeq}\mathscr{S}_S^{2,\preccurlyeq}$, and therefore, $\mathscr{S}_R^2=\mathscr{S}_R^{2,\succcurlyeq}\mathscr{S}_R^{2,\preccurlyeq}$. Finally, for any $\mathfrak{s}\in\mathfrak{T}_2^{ss}(\mu,\lambda)$ and $\mathbf{t}\in\mathfrak{T}_2^{ss}(\mu,\nu)$,

$$\varphi_{\lambda\mu}^{\delta(\mathfrak{s})}\varphi_{\mu\nu}^{\delta(\mathbf{t})^{-1}} = \varphi_{\lambda\mu}^{\delta(\mathfrak{s})}\varphi_{\mu\mu}^{1}\varphi_{\mu\nu}^{\delta(\mathbf{t})^{-1}} \in \mathscr{S}_R^{2,\succcurlyeq}\varphi_{\mu\mu}^{1}\mathscr{S}_R^{2,\preccurlyeq}.$$

So the last equality follows. ∎

4. THE QUASI-HEREDITY OF $\mathscr{S}_R^{2,\preccurlyeq}$ AND $\mathscr{S}_R^{2,\succcurlyeq}$

In this section, we will discuss the representation theory of the Borel type subalgebras $\mathscr{S}_R^{2,\preccurlyeq}$ and $\mathscr{S}_R^{2,\succcurlyeq}$ of a q-Schur^2 algebra $\mathscr{S}_R^2$. We will show that both $\mathscr{S}_R^{2,\preccurlyeq}$ and $\mathscr{S}_R^{2,\succcurlyeq}$ are quasi-hereditary. The proof is based on the notion of a standardly based algebra introduced by the authors in [DR]. Then, using the triangular decomposition, we will also prove that the standard modules and costandard modules for $\mathscr{S}_R^2$ can be induced from a left (resp. right) standard modules for $\mathscr{S}_R^{2,\preccurlyeq}$ (resp. $\mathscr{S}_R^{2,\succcurlyeq}$). We should point out that, by (2.9), the quasi-heredity (4.5) of $\mathscr{S}_R^{2,\preccurlyeq}$ and $\mathscr{S}_R^{2,\succcurlyeq}$ follows from that of the Borel subalgebras of q-Schur algebras (see [DR, (5.6.1)]). For completeness, we include a direct proof below. First, we recall the definition of standardly based algebras.

(4.1) DEFINITION. Assume that R is a commutative ring with 1. Let A be an R-algebra and $(\Lambda,\le)$ a poset. A is called a *standardly based algebra on* Λ (or *standardly based*) if the following conditions hold.

(a) For any $\lambda\in\Lambda$, there are index sets $I(\lambda)$, $J(\lambda)$ and subsets

$$\mathscr{A}^\lambda=\left\{a_{i,j}^\lambda \mid (i,j)\in I(\lambda)\times J(\lambda)\right\}$$

of A such that the union $\mathscr{A}=\bigcup_{\lambda\in\Lambda}\mathscr{A}^\lambda$ is disjoint and forms an R-basis for A.

(b) For any $a\in A$, $a_{i,j}^\lambda\in\mathscr{A}$, we have

$$a\cdot a_{i,j}^\lambda \equiv \sum_{i'\in I(\lambda)} f_{i',\lambda}(a,i)a_{i',j}^\lambda \qquad \mathrm{mod}\ A(>\lambda)$$

$$a_{i,j}^\lambda\cdot a \equiv \sum_{j'\in J(\lambda)} f_{\lambda,j'}(j,a)a_{i,j'}^\lambda, \qquad \mathrm{mod}\ A(>\lambda),$$

where $A(>\lambda)$ is the R-submodule of A spanned by $\mathscr{A}^\mu$ with $\mu>\lambda$, and $f_{i',\lambda}(a,i)$, $f_{\lambda,j'}(j,a)\in R$ are independent of j and i, respectively. Such a base $\mathscr{A}$ is called a *standard base* for the algebra A.

Note that a cellular algebra [GL] must be standardly based. However, the converse may not be true. We shall prove that $\mathscr{S}_R^{2,\succcurlyeq}$ and $\mathscr{S}_R^{2,\preccurlyeq}$ are standardly based algebras whose standard bases are not cellular bases.

(4.2) DEFINITION. For any $\lambda \in \Lambda$, let f_λ: $J(\lambda) \times I(\lambda) \to R$ be a function, whose value $f_\lambda(j, i')$ at $(j, i') \in J(\lambda) \times I(\lambda)$ is defined by

$$a_{ij}^\lambda a_{i'j'}^\lambda \equiv f_\lambda(j, i') a_{ij'}^\lambda \qquad \mathrm{mod}\ A(> \lambda).$$

The function f_λ induces a bilinear form β_λ: $A^\lambda \times A^\lambda \to R$ such that $\beta_\lambda(a_{ij}^\lambda, a_{i'j'}^\lambda) = f_\lambda(j', i)$, where A^λ is the free R-submodule of A spanned by a_{ij}^λ for all $(i, j) \in I(\lambda) \times J(\lambda)$. We say A is a standardly *full*-based algebra if $\mathrm{im}(\beta_\lambda) = R$ for all $\lambda \in \Lambda$ (compare [DR, (1..3.1)]).

The following result has been proved by the authors in [DR, (3.2.1), (4.2.7)].

(4.3) THEOREM. (a) *Let R be a commutative Noetherian ring. If A is a standardly full-based algebra, then A is a quasi-hereditary algebra over R in the sense of* [CPS2].

(b) *If R is a commutative local Noetherian ring, then A is split quasi-hereditary if and only if A is a standardly full-based algebra.*

If A is quasi-hereditary with poset Λ, then we use $\Delta(A, \lambda)$ and $\nabla(A, \lambda)(\lambda \in \Lambda)$ to denote the standard and costandard (left) A-modules. Since A^{op} is also quasi-hereditary, the left A^{op}-modules $\Delta(A^{\mathrm{op}}, \lambda)$ and $\nabla(A^{\mathrm{op}}, \lambda)$ are naturally right A-modules by shifting the left action of A^{op} to the right action of A. We shall denote these right modules by $\Delta^{\mathrm{op}}(A, \lambda)$ and $\nabla^{\mathrm{op}}(A, \lambda)$.

Let A be a standardly based algebra on the poset Λ with a standard base as in (4.1). For each $\lambda \in \Lambda$ and $(i, j) \in I(\lambda) \times J(\lambda)$, let $\Delta(\lambda, j)$ (resp. $\Delta(i, \lambda)$) be the left (resp. right) A-submodule generated by the elements $a_{ij}^\lambda + A(> \lambda)$ with j fixed (resp. with i fixed). The following result (see [DR, (3.2.2)]) identifies these modules with the standard and costandard modules under the quasi-heredity structure of A.

(4.4) THEOREM. *If R is a commutative local Noetherian ring and A is standardly full-based, then the module* $\Delta(\lambda, j)$ *(resp.* $\Delta(i, \lambda)$*) is isomorphic to the left standard module* $\Delta(A, \lambda)$ *(resp. right standard modules* $\Delta^{\mathrm{op}}(A, \lambda)$*) of A under its quasi-heredity structure as considered in* (4.3).

We are now ready to prove the main results of this section.

(4.5) THEOREM. (a) *The subalgebras* $\mathscr{S}_R^{2,\succcurlyeq}$ *and* $\mathscr{S}_R^{2,\preccurlyeq}$ *are standardly full-based, and hence are quasi-hereditary over a commutative Noetherian ring R.*

(b) *If R is a field, then all the costandard modules $\nabla(\mathscr{S}_R^{2,\succcurlyeq}, \lambda)$ (resp. standard modules $\Delta(\mathscr{S}_R^{2,\preccurlyeq}, \lambda)$) for $\mathscr{S}_R^{2,\succcurlyeq}$ (resp. $\mathscr{S}_R^{2,\preccurlyeq}$) are simple and of dimension one.*

Proof. For any $\mu \in \Lambda_2(n,r)$, let $\leq$ be the dominance order $\trianglelefteq$ on $\Lambda = \Lambda_2(n,r)$. Let $I(\mu) = \Omega_2^{\succcurlyeq}(\mu)$ and $J(\mu) = \{\mathbf{i}_\mu\}$ (see (1.11) for the definition of $\Omega_2^{\succcurlyeq}(\mu)$). For any $\varphi_{\lambda,\mu}^d \in \mathscr{S}_R^{2,\succcurlyeq}$, write $\varphi_{\lambda,\mu}^d = \varphi_{\mathbf{i}_\lambda d, \mathbf{i}_\mu}$. It is easy to see that (4.1)(a) holds. Now, (2.8) implies (4.1)(b). So $\mathscr{S}_R^{2,\succcurlyeq}$ is standardly based. Moreover, it is full in the sense of (4.2) since $\varphi_{\mu,\mu}^1 \varphi_{\mu,\mu}^1 = \varphi_{\mu,\mu}^1$ for every $\mu \in \Lambda_2(n,r)$. So $\mathscr{S}_R^{2,\succcurlyeq}$ is quasi-hereditary over R by (4.3). Similarly, we have $\mathscr{S}_R^{2,\preccurlyeq}$ is quasi-hereditary. Thus (a) follows. The statement (b) follows immediately from (a) and (4.4), easily. ∎

(4.6) PROPOSITION. *Suppose that the base ring R is a field.*

(a) *Let $V_\lambda = \mathscr{S}_R^{2,\succcurlyeq}\varphi_{\lambda\lambda}^1$ for each $\lambda \in \Lambda_2(n,r)$. Then $\{V_\lambda \mid \lambda \in \Lambda_2(n,r)\}$ is a complete set of non-isomorphic principal indecomposable $\mathscr{S}_R^{2,\succcurlyeq}$-modules. Moreover,* $\dim_R V_\lambda = \prod_{i=1}^{\max_{n,r}+n}\binom{\lambda_i + i - 1}{i-1}$, *where λ_i is the ith component of $\bar{\lambda}$, i.e., $\bar{\lambda} = (\lambda_1, \ldots, \lambda_{\max_{n,r}+n})$.*

(b) *Let* $\operatorname{rad} V_\lambda$ *be the radical of V_λ. Then* $\operatorname{rad} V_\lambda$ *is spanned by $\varphi_{\mu\lambda}^d \in \mathscr{S}_R^{2,\succcurlyeq}$ with $\varphi_{\mu\lambda}^d \neq \varphi_{\lambda\lambda}^1$. Thus, the dimension of any irreducible $\mathscr{S}_R^{2,\succcurlyeq}$-module is one.*

Proof. (a) Since any element in $\varphi_{\lambda\lambda}^1\mathscr{S}_R^{2,\succcurlyeq}\varphi_{\lambda\lambda}^1$ is an R-linear combination of $\varphi_{\lambda\lambda}^d$ with $\mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\lambda$ and $d \in \mathscr{D}_{\lambda\lambda}$, we must have $d = 1$. Thus, $\varphi_{\lambda\lambda}^1$ is a primitive idempotent element, since $\varphi_{\lambda\lambda}^1\mathscr{S}^{2,\succcurlyeq}\varphi_{\lambda\lambda}^1 = R\varphi_{\lambda\lambda}^1$. On the other hand, the identity element has the decomposition $1 = \sum_{\lambda\in\Lambda_2(n,r)}\varphi_{\lambda\lambda}^1$. So $\{V_\lambda \mid \lambda \in \Lambda_2(n,r)\}$ is a complete set of non-isomorphic principal indecomposable $\mathscr{S}_R^{2,\succcurlyeq}$-modules. The last assertion follows from [Sa, (6.1)] by counting the elements in $\Omega_2^{\preccurlyeq}(\lambda)$.

(b) The first assertion follows from a straightforward computation. The last assertion follows from the fact that $\{V_\lambda/\operatorname{rad} V(\lambda) \mid \lambda \in \Lambda_2(n,r)\}$ is the complete set of non-isomorphic simple $\mathscr{S}_R^{2,\succcurlyeq}$-modules. ∎

We are now going to look at the relationship between the standard modules and costandard modules for q-Schur^2 algebras (see (3.4)) and their Borel type subalgebras.

Let R be a commutative ring. For any $\lambda \in \Lambda_2(n,r)$, let $\chi_\lambda\colon \mathscr{S}_R^{2,\succcurlyeq} \to R$ be an R-linear map such that

$$\chi_\lambda\left(\varphi_{\mu\nu}^d\right) = \begin{cases} 1, & \text{if } \mu = \nu = \lambda \text{ and } d = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

Clearly, χ_λ is an algebra homomorphism, and via χ_λ, R can be made into a rank-one (left or right) $\mathscr{S}_R^{2,\succcurlyeq}$-module R_λ. One may define similarly algebra homomorphism $\chi_\lambda\colon \mathscr{S}_R^{2,\preccurlyeq} \to R$ and (left or right) $\mathscr{S}_R^{2,\preccurlyeq}$-module $\hat{R}_\lambda$. The following result can be proved easily.

(4.8) PROPOSITION. *If R is a commutative local Noetherian ring, then $\hat{R}_\lambda$ (resp. R_λ) is isomorphic to the left standard module $\Delta(\mathscr{S}_R^{2,\preccurlyeq}, \lambda)$ (resp. right standard module $\Delta^{\mathrm{op}}(\mathscr{S}_R^{2,\preccurlyeq}, \lambda)$).*

Proof. Obviously, $\Delta(\mathscr{S}_R^{2,\preccurlyeq}, \lambda)$ is a free R-module spanned by $\overline{\varphi}^1_{\lambda\lambda} = \varphi^1_{\lambda\lambda} + \mathscr{S}_R^{2,\preccurlyeq}(> \lambda)$, where $\mathscr{S}_R^{2,\preccurlyeq}(> \lambda)$ is the submodule of $\mathscr{S}_R^{2,\preccurlyeq}$ spanned by $\varphi^d_{\mu\nu} \in \mathscr{S}_R^{2,\preccurlyeq}$ with $\mu > \lambda$. Obviously, $\varphi\overline{\varphi}^1_{\lambda\lambda} = \chi_\lambda(\varphi)\overline{\varphi}^1_{\lambda\lambda}$ for $\varphi \in \mathscr{S}_R^{2,\preccurlyeq}$. Thus $\Delta(\mathscr{S}_R^{2,\preccurlyeq}, \lambda) \cong \hat{R}_\lambda$. One can prove $\Delta^{\mathrm{op}}(\mathscr{S}_R^{2,\succcurlyeq}, \lambda) \cong R_\lambda$ similarly. ∎

(4.9) THEOREM. *Let R be a commutative local Noetherian ring and $\lambda \in \Lambda_2(n, r)$. Then we have*

$$\text{(a)}\quad \mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \Delta\left(\mathscr{S}_R^{2,\preccurlyeq}, \lambda\right) \cong \begin{cases} \Delta\left(\mathscr{S}_R^2, \lambda\right), & \text{if } \lambda \in \Lambda_2(n,r)^+ \\ 0, & \text{otherwise, and} \end{cases}$$

$$\text{(b)}\quad \Delta^{\mathrm{op}}\left(\mathscr{S}_R^{2,\succcurlyeq}, \lambda\right) \otimes_{\mathscr{S}_R^{2,\succcurlyeq}} \mathscr{S}_R^2 \cong \begin{cases} \Delta^{\mathrm{op}}\left(\mathscr{S}_R^2, \lambda\right), & \text{if } \lambda \in \Lambda_2(n,r)^+ \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We prove (a). The proof of (b) is similar. By (4.8), it suffices to prove $\mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \hat{R}_\lambda = 0$ for $\lambda \notin \Lambda_2(n,r)^+$. By (3.4), $\mathscr{S}_R^2$ has an R-basis $\varphi^{d(\mathfrak{s})}_{\rho\mu}\varphi^{d(\mathfrak{t})^{-1}}_{\mu\nu}$, where $\mathfrak{s} \in \mathfrak{T}_2^{ss}(\mu, \rho)$ and $\mathfrak{t} \in \mathfrak{T}_2^{ss}(\mu, \nu)$. By (3.3), $\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu} \in \mathscr{S}_R^{2,\preccurlyeq}$. If $\lambda \in \Lambda_2(n,r) \setminus \Lambda_2(n,r)^+$, then $\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu} \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} 1 = 1 \otimes \chi_\lambda(\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu})1 = 0$. Thus, $\mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \hat{R}_\lambda = 0$.

Suppose now $\lambda \in \Lambda_2(n,r)^+$. By (4.8), it is equivalent to prove $\mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \hat{R}_\lambda \cong \Delta(\mathscr{S}_R^2, \lambda)$. The proof is similar to that given by J. A. Green in [G2]. Recall from [DS, Sect. 6], that the Green–Murphy basis element $\varphi^{\delta(\mathfrak{s})}_{\lambda\mu}\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu}$ is denoted by $\Phi^\mu_{\mathfrak{s}\mathfrak{t}}$, where $\mathfrak{s} \in \mathfrak{T}_2^{ss}(\mu, \lambda)$, $\mathfrak{t} \in \mathfrak{T}_2^{ss}(\mu, \nu)$, and $\mu \in \Lambda_2(n,r)^+$, $\lambda, \nu \in \Lambda_2(n,r)$. Let $\mathscr{S}_R^2(> \mu)$ be the submodule of $\mathscr{S}_R^2$ spanned by $\Phi^\rho_{\mathfrak{s}'\mathfrak{t}'}$ with $\rho > \mu$, where $\leq$ is the dominance order $\trianglelefteq$ on $\Lambda_2(n,r)$. Then the standard module $\Delta(\mathscr{S}_R^2, \lambda)$, $\lambda \in \Lambda_2(n,r)^+$, for the q-Schur[2] algebra $\mathscr{S}_R^2$ satisfies $\Delta(\mathscr{S}_R^2, \lambda) \cong \mathscr{S}_R^2\overline{\varphi}^1_{\lambda\lambda}$, where $\overline{\varphi}^1_{\lambda\lambda} = \varphi^1_{\lambda\lambda} + \mathscr{S}_R^2(> \lambda)$. Write $\mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \hat{R}_\lambda = \mathscr{S}_R^2(1 \otimes 1)$. It is enough to find two $\mathscr{S}_R^2$-homomorphisms $f\colon \mathscr{S}_R^2\overline{\varphi}^1_{\lambda\lambda} \to \mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \hat{R}_\lambda$ and $g\colon \mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \hat{R}_\lambda \to \mathscr{S}_R^2\overline{\varphi}^1_{\lambda\lambda}$ such that $f(\overline{\varphi}^1_{\lambda\lambda}) = 1 \otimes 1$ and $g(1 \otimes 1) = \overline{\varphi}^1_{\lambda\lambda}$.

We first define f. Consider the $\mathscr{S}_R^2$-homomorphism $\tilde{f}\colon \mathscr{S}_R^2\varphi^1_{\lambda\lambda} \to \mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \hat{R}_\lambda$ satisfying $f(\varphi^1_{\lambda\lambda}) = 1 \otimes 1$. For $\varphi^{d(\mathfrak{s})}_{\rho\nu}\varphi^{d(\mathfrak{t})^{-1}}_{\nu\mu} \in \mathscr{S}_R^2(> \lambda)$, $\varphi^{d(\mathfrak{s})}_{\rho\nu}\varphi^{d(\mathfrak{t})^{-1}}_{\nu\mu} \otimes 1 = 0$, since $\nu > \lambda$. It follows that $\tilde{f}(\mathscr{S}_R^2(> \lambda)) = 0$. So $\tilde{f}$ induces an $\mathscr{S}_R^2$-homomorphism f from $\mathscr{S}_R^2\overline{\varphi}^1_{\lambda\lambda}$ to $\mathscr{S}_R^2 \otimes_{\mathscr{S}_R^{2,\preccurlyeq}} \hat{R}_\lambda$ such that $f(\overline{\varphi}^1_{\lambda\lambda}) = 1 \otimes 1$.

To define g, we note that, for any $\varphi^d_{\rho,\mu} \in \mathscr{S}_R^{2,\preccurlyeq}$, if $\rho \neq \lambda$, then $\varphi^d_{\rho,\mu} \in \mathscr{S}_R^2(> \lambda)$ by [DS, (6.1.4)]. Consequently, we have $\varphi\overline{\varphi}^1_{\lambda\lambda} = \chi_\lambda(\varphi)\overline{\varphi}^1_{\lambda\lambda}$ for all $\varphi \in \mathscr{S}_R^{2,\preccurlyeq}$. Now, using the universal property of tensor product, g is defined. ∎

For $\lambda \in \Lambda_2(n,r)$, let $H^p(\lambda) = \operatorname{Ext}^p_{\mathscr{S}_R^{2,\succcurlyeq}}({}_{\mathscr{S}_R^{2,\succcurlyeq}}\mathscr{S}_R^2, R_\lambda)$. Then, $H^0(\lambda)$ may be regarded as a module induced from a Borel subalgebra. The following result generalized the classical Borel–Weil theorem (compare the remarks after [G1, (5.9)]).

(4.10) COROLLARY. *Let $\nabla(\mathscr{S}_R^2, \mu)$ be the (left) costandard module for $\mathscr{S}_R^2$ associated to $\mu \in \Lambda_2(n,r)^+$, and $\lambda \in \lambda_2(n,r)$. Then*

$$H^0(\lambda) \cong \begin{cases} \nabla\left(\mathscr{S}_R^2, \lambda\right), & \text{if } \lambda \in \Lambda_2(n,r)^+ \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By the adjoint isomorphism (see, e.g., [CR, 2.19]), we see that

$$H^0(\lambda) \cong \operatorname{Hom}_R\left(\Delta^{\mathrm{op}}\left(\mathscr{S}_R^{2,\succcurlyeq}, \lambda\right) \otimes_{\mathscr{S}_R^{2,\succcurlyeq}} \mathscr{S}_R^2, R\right).$$

Now, the result follows from (4.9). ∎

(4.11) *Remark.* As mentioned in the Introduction, it would be interesting to know if $H^p(\lambda) = 0$ for all $\lambda \in \Lambda_2(n,r)$ and $p > 0$.

5. THE GENERAL CASE

In [AK], Ariki and Koike introduced certain algebras associated to the complex reflection groups $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_r$, which are now known as Ariki–Koike Hecke algebras. When $m = 1$ and 2, these algebras are isomorphic to the Hecke algebra of types A_{r-1} and B_r, respectively. In [DJM2], Dipper, James, and Mathas introduced some associated endomorphism algebras, called cyclotomic q-Schur algebras. In this section, we generalize the results in Sections 2–4 to cyclotomic q-Schur algebras. For simplicity, we will mainly work on a subclass of cyclotomic q-Schur algebras, called q-Schur^m algebras. We will see that results for other cyclotomic q-Schur algebras can be obtained easily from that for q-Schur^m algebras.

Though the natural basis like (1.7) is not available for a q-Schur^m algebra $\mathbf{S}_R^m$ with $m > 2$, having a close look at the work in previous sections, we notice that only part of the basis elements in (1.7), which are involved in the definition of Green–Murphy type basis (3.4), have been used. Note also that a Murphy type basis for $\mathbf{S}_R^m$ is introduced in [DJM2].

So, if we can prove this basis is a Green type basis (cf. [G2]), i.e., is of the form (3.4), then our generalization to arbitrary m will be almost straightforward. So we aim to construct those elements $\varphi^d_{\lambda\mu}$ involved in (1.11). To do this, it suffices to generalize the relation (2.6). The next combinatorial lemma will lead a generalization of (2.6). Recall the c.n.s. of a multi-composition in Section 1.

(5.1) LEMMA. *If* $\lambda, \mu \in \Lambda_m(n,r)$ *and* $d \in \mathscr{D}_{\bar{\lambda}\bar{\mu}}$ *with* $\mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\mu$, *then* $d = w_m \cdots w_1$ *with* $l(d) = \sum_{i=1}^m l(w_i)$, *where* $w_j \in \mathfrak{S}_{\{a_{j-1}+1,\dots,b_j\}}$ *for all* $1 \le j \le m$, *where* $\mathbf{a} = (a_0, a_1, \dots, a_m)$ *resp.* $\mathbf{b} = (b_0, b_1, \dots, b_m)$ *is the c.n.s. of* λ *resp.* μ.

Proof. We first construct inductively a sequence of elements $x_m, \dots, x_1$ with $x_j \in \mathfrak{S}_{\{a_{j-1},\dots,b_j\}}$ such that $d^{-1} = x_1 \cdots x_m$. Applying (1.10) to d, we have, for every i with $1 \le i \le m$, $d(j) \le b_{i-1}$ for all $j \le a_{i-1}$. Thus, for $i = m$, we obtain $\{d(1), d(2), \dots, d(a_{m-1})\} \subseteq \{1, 2, \dots, b_{m-1}\}$. Taking complements, we have

$$\{b_{m-1}+1, \dots, r\} \subseteq \{d(j) \mid a_{m-1}+1 \le j \le r\}.$$

So we may choose $x_m \in \mathfrak{S}_{\{a_{m-1}+1,\dots,b_m\}}$ such that $x_m d(j) = j$ for all $b_{m-1}+1 \le j \le b_m = r$ and $x_m d(j) = d(j)$ for all $j \le a_{m-1}$. Now, since $d(j) \le b_{m-2}$ for all $j \le a_{m-2}$, it follows that

$$\{d(1), d(2), \dots, d(a_{m-2})\} \subseteq \{1, 2, \dots, b_{m-2}\},$$

and hence,

$$\begin{aligned}\{b_{m-2}+1, \dots, b_{m-1}, b_{m-1}+1, \dots, b_m\} &\subseteq \{d(j) \mid a_{m-2}+1 \le j \le r\}\\ &= \{x_m d(j) \mid a_{m-2}+1 \le j \le r\}.\end{aligned}$$

Therefore,

$$\{b_{m-2}+1, \dots, b_{m-1}\} \subseteq \{x_m d(j) \mid a_{m-2}+1 \le j \le b_{m-1}\}.$$

Thus, there is an $x_{m-1} \in \mathfrak{S}_{\{a_{m-2}+1,\dots,b_{m-1}\}}$ such that $x_{m-1}x_m d(j) = j$ for all j with $b_{m-2}+1 \le j \le r$, and $x_{m-1}x_m d(j) = d(j)$ for $j \le a_{m-2}$. Continue this process. After finitely many steps, we will find $x_2, \dots, x_m$ with $x_i \in \mathfrak{S}_{\{a_{i-1}+1,\dots,b_i\}}$, such that $x_2 \cdots x_m d(j) = j$ for all $j \ge b_1 + 1$ and $x_2 \cdots x_m d(j) = d(j)$ for all $j \le a_1$. Finally, choose $x_1 \in \mathfrak{S}_{\{1,\dots,b_1\}}$ such that $x_1 \cdots x_m d(j) = j$ for all $j \le b_1$. Since $x_1(j) = j$ for all $j \ge b_1 + 1$, we have $x_1 \cdots x_m d(j) = j$ for all $j \ge b_1 + 1$. Thus, $x_1 \cdots x_m d = 1$, and $d^{-1} = x_1 \cdots x_m$.

Next, applying induction on m and noting the exchange condition for Coxeter groups, one sees easily that there are $y_j \in \mathfrak{S}_{\{a_{j-1}+1,\dots,b_j\}}$ such that $d^{-1} = y_1 y_2 \cdots y_m$ and $l(d^{-1}) = \sum_{i=1}^m l(y_i)$. Putting $w_i = y_i^{-1}$, the result follows. ∎

(5.2) *Remark*. This lemma can be used to give a weak version of the middle equality in (2.6). That is, we want to prove that, for $\lambda, \mu \in \Lambda_2(n,r)$ and $d \in \mathscr{D}_{\bar\lambda\bar\mu}$ with $\mathbf{i}_\lambda \succcurlyeq \mathbf{i}_\mu$, $T_{\mathfrak{S}_{\bar\lambda} \mathrm{d} \mathfrak{S}_{\bar\mu}} \pi_\mu \in x_\lambda \mathscr{H}_R$.

Write $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, $\mu = (\mu^{(1)}, \mu^{(2)})$ and put $a = |\lambda^{(1)}|$ and $b = |\mu^{(1)}|$. Then the c.n.s. of λ and μ are $(0, a, r)$ and $(0, b, r)$, respectively, and $a \le b$. Since $\mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\mu$, by (5.1), we can write $d = w_2 w_1$ with $w_1 \in \mathfrak{S}_{\{1,\dots,b\}}$, and $w_2 \in \mathfrak{S}_{\{a+1,\dots,r\}}$. Writing $T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}} = x_{\bar\lambda} T_d h_1$ for some $h_1 \in \mathfrak{S}_{\bar\mu}$ and noting that π_μ is in the center of $\mathscr{H}_R(W_{\hat\mu})$, we have

$$\begin{aligned} T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}} \pi_\mu &= x_{\bar\lambda} T_d h_1 \pi_\mu = x_{\bar\lambda} T_{w_2} T_{w_1} \pi_\mu h_1 \\ &= x_{\bar\lambda} T_{w_2} \pi_\mu T_{w_1} h_1 = x_{\bar\lambda} T_{w_2} \pi_\lambda \pi_{\mu\setminus\lambda} T_{w_1} h_1 \\ &= x_{\bar\lambda} \pi_\lambda T_{w_2} \pi_{\mu\setminus\lambda} T_{w_1} h_1 \in x_\lambda \mathscr{H}_R. \end{aligned}$$

Note that, with (5.2), $\varphi^d_{\lambda\mu}$ can be defined *easily* for those $\lambda, \mu \in \Lambda_2(n,r)$ and $d \in \mathscr{D}_{\bar\lambda\bar\mu}$ with $\mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\mu$ (compare the general definition in [DS, (4.2.6)]). We are now ready to generalize (5.2) to the Ariki–Koike Hecke algebras.

In the rest of the paper, let $\mathbf{H}_R$ be the Ariki–Koike Hecke algebra associated to the complex reflection group $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_r$. Then $\mathbf{H}_R$ is an associative algebra over a commutative ring R containing $q, q^{-1}, u_1, \dots, u_r$ with generators T_i, $0 \le i \le r-1$, and relations

$$\begin{gathered} T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \text{for } 1 \le i \le r-2 \\ T_i T_j = T_j T_i, \qquad \text{if } |i-j| \ge 2 \\ (T_i - q)(T_i + 1) = 0, \qquad \text{if } i \ne 0 \\ (T_0 - u_1) \cdots (T_0 - u_m) = 0. \end{gathered}$$

Note that the subalgebra generated by T_i with $1 \le i \le r-1$ is the Hecke algebra $\mathbf{H}_R(\mathfrak{S}_r)$ of type A_{r-1}, and, when $m = 2$ and $u_1 = q_0$ and $u_2 = -1$, $\mathbf{H}_R$ is the Hecke algebra of type B_r as defined at the beginning of Section 1. We will use the notation $\mathbf{H}_R(\mathfrak{S}_\lambda)$ for the subalgebra of $\mathbf{H}_R$ defined by a parabolic subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_r$. Ariki and Koike proved that the set

$$\left\{ L_1^{c_1} \cdots L_r^{c_r} T_w \mid w \in \mathfrak{S}_r, \text{ and } 0 \le c_i \le m-1, i = 1, 2, \dots, r \right\}, \quad (5.3)$$

is a R-basis of $\mathbf{H}_R$, where $L_i = q^{1-i}T_{i-1} \cdots T_1T_0T_1 \cdots T_{i-1}$. Moreover, we have the results (see [AK, (2.1)–(2.2)]),

$$\begin{cases} (1) \quad L_i \text{ and } L_j \text{ commute for all } 1 \le i, j \le r \\ (2) \quad \text{If } a \in R \text{ and } i \ne k, \text{ then } T_i \text{ commutes with } \prod_{i=1}^{k}(L_i - a). \end{cases} \tag{5.4}$$

Suppose that $\lambda \in \Lambda_m(n,r)$ is of c.n.s. $\mathbf{a} = (a_0, \dots, a_m)$. Following [DJM2], we denote

$$\pi_\lambda = \prod_{k=2}^{m} \prod_{i=1}^{a_{k-1}} (L_i - u_k), \qquad x_\lambda = \pi_\lambda x_{\bar\lambda}. \tag{5.5}$$

(5.6) LEMMA. *Let $\lambda, \mu \in \Lambda_m(n,r)$ be of c.n.s.* $\mathbf{a}$ *and* $\mathbf{b}$, *respectively. If $d \in \mathscr{D}_{\bar\lambda\bar\mu}$ with $\mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\mu$, then $T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}}\pi_\mu \in x_\lambda \mathbf{H}_R$.*

Proof. We first note that if $T_i \in \mathbf{H}_R(\mathfrak{S}_{\bar\mu})$, then $i \ne b_j$ for all j. By (5.4)(2) and (5.5), all the elements in $\mathbf{H}_R(\mathfrak{S}_{\bar\mu})$, commute with π_μ. Since $T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}} = x_{\bar\lambda} T_d h$ for some $h \in \mathbf{H}_R(\mathfrak{S}_{\bar\mu})$, we need only prove $T_d\pi_\mu \in \pi_\lambda \mathbf{H}_R$. For simplicity, we put $\pi_{b_k} = \prod_{i=1}^{b_k}(L_i - u_{k+1})$. Then, $\pi_\mu = \pi_{b_1} \cdots \pi_{b_{m-1}}$. Since $a_k \le b_k$, the product π_{a_k} is part of π_{b_k}. Let $\pi_{b_k \setminus a_k}$ denote the product obtained by deleting π_{a_k} from π_{b_k}.

By (5.1), $T_d = T_{w_m} \cdots T_{w_2}T_{w_1}$ for some $w_j \in \mathfrak{S}_{\{a_{j-1}+1, \dots, b_j\}}$, $1 \le j \le m$. Thus, for every $1 \le j \le m$, (5.4)(2) implies that T_{w_j} commutes with the product $\pi_{a_{j-1}} \prod_{k=j}^{m-1} \pi_{b_k}$, and $T_{w_m} \cdots T_{w_{j+1}}$ commutes with π_{a_l} for all $l < j$. Taking $j = 1$, we obtain $T_{w_1}\pi_\mu = \pi_\mu T_{w_1}$, and, for $j = 2$, we have by (5.4)(1)

$$\begin{aligned} T_{w_2}T_{w_1}\pi_\mu &= T_{w_2}\pi_\mu T_{w_1} \\ &= T_{w_2}\pi_{a_1}(\pi_{b_2} \cdots \pi_{b_{m-1}})\pi_{b_1\setminus a_1}T_{w_1} \\ &= \pi_{a_1}(\pi_{b_2} \cdots \pi_{b_{m-1}})T_{w_2}\pi_{b_1\setminus a_1}T_{w_1}. \end{aligned}$$

Now, if we rewrite the product in the parentheses as a product of $\pi_{a_2}(\pi_{b_3} \cdots \pi_{b_{m-1}})$ and $\pi_{b_2\setminus a_2}$, then the former commutes with T_{w_3}. Continue this process. We finally obtain

$$T_d\pi_\mu = \pi_\lambda T_{w_m}\pi_{b_{m-1}\setminus a_{m-1}}T_{w_{m-1}} \cdots \pi_{b_1\setminus a_1}T_{w_1} \in \pi_\lambda \mathbf{H}_R,$$

as desired. ∎

Note that, for $d = \delta(\mathfrak{z})$, (5.6) holds by (3.3). This case was proved in [DJM2, (4.10)].

Cyclotomic q-Schur algebras $\mathbf{S}_R(\Lambda)$ associated to $\mathbf{H}_R$ are introduced in [DJM2] for a finite set Λ of m-compositions of r such that $\Lambda \cap \Lambda_m(r)^+$ is a coideal of $\Lambda_m(r)^+$. We are interested in those cyclotomic q-Schur algebras defined by $\Lambda_m(n,r)$.

(5.7) DEFINITION. The *q-Schur^m algebra* $\mathbf{S}_R^m$ of degree (n, r) is the cyclotomic q-Schur algebra associated to the poset $\Lambda_m(n, r)$. In other words, we define

$$\mathbf{S}_R^m = \mathbf{S}_R^m(n, r) = \operatorname{End}_{\mathbf{H}_R}\big(\oplus_{\lambda \in \Lambda_m(n,r)} x_\lambda \mathbf{H}_R\big).$$

The q-Schur^m algebra is a natural generalization of the q-Schur algebra and the q-Schur^2 algebra. When $m = 2$ and $u_1 = q_0$, $u_2 = -1$, $\mathbf{S}_R^2$ is isomorphic to $\mathscr{S}_R^2$. Many nice properties of q-Schur^2 algebras such as the quasi-heredity (over fields), a cellular basis, etc., have also been established in [DJM2] for a cyclotomic q-Schur algebra. In particular, as in the $m = 2$ case, we have, for any m-compositions λ, μ, the isomorphism [DJM2, (5.17(ii))] $\operatorname{Hom}_{\mathbf{H}_R}(x_\mu \mathbf{H}_R, x_\lambda \mathbf{H}_R) \cong x_\lambda \mathbf{H}_R \cap \mathbf{H}_R x_\mu$, and a Murphy type basis for $x_\lambda \mathbf{H}_R \cap \mathbf{H}_R x_\mu$ [DJM2, (6.3)]. This leads to a nice basis, i.e., a cellular basis, for any cyclotomic q-Schur algebra (see [DJM2, (6.6)]).

Let us give a little more details about this construction. For any standard μ-tableau $\mathbf{s}$, let $\mathfrak{f}_\lambda(\mathbf{s})$ be the μ-tableau of type λ obtained from $\mathbf{s}$ by replacing each entry a in $\mathbf{s}$ by i_k if a is in the row i of the kth component of $\mathbf{t}^\lambda$. Take a semi-standard μ-tableau $\mathfrak{s}$ of type λ and consider the inverse image $\mathbf{T}_{\mathfrak{s},\lambda} = \mathfrak{f}_\lambda^{-1}(\mathfrak{s})$. Thus, $\mathbf{T}_{\mathfrak{s},\lambda}$ is the set of all standard μ-tableaux whose image under the map $\mathfrak{f}_\lambda$ is $\mathfrak{s}$. Let $\trianglelefteq$ be the partial order on the set of standard μ-tableaux (see [DJM2, (3.1.1)]). Thus, for standard μ-tableaux $\mathbf{s}$, $\mathbf{t}$, $\mathbf{s} \trianglelefteq \mathbf{t}$ means $\mathbf{s}\downarrow k \trianglelefteq \mathbf{t}\downarrow k$ for $1 \le k \le r$, where $\mathbf{s}\downarrow k$ is a multi-composition determined by the entries $1, 2, \ldots, k$ in $\mathbf{s}$. For example, if

$$\mathbf{s} = \boxed{\begin{array}{ccccc} 1 & 2 & 8 & \quad 6 & 7 \\ 3 & 5 & & & 9 \\ 4 & & & & \end{array}}$$

then $\mathbf{s}\downarrow 6 = ((221)(10))$ and $\mathbf{s}\downarrow 8 = ((321)(20))$. Among the elements in the set $\mathbf{T}_{\mathfrak{s},\lambda}$, there is a unique standard μ-tableau, which is maximal with respect to $\trianglelefteq$. Such a tableau was denoted by first$(\mathfrak{s})$ in [DJM2]. Let $d(\mathfrak{s})$ be a distinguished coset representative in $\mathscr{D}_\mu$ defined by $\mathbf{t}^\mu d(\mathfrak{s}) = \text{first}(\mathfrak{s})$. Then, if we define, for $\mathfrak{s} \in \mathfrak{T}_m^{ss}(\mu, \lambda)$, $\mathfrak{t} \in \mathfrak{T}_m^{ss}(\mu, \nu)$, $\Phi_{\mathfrak{s}\mathfrak{t}} \in \mathbf{S}_R^m$ such that

$$\Phi_{\mathfrak{s}\mathfrak{t}}^\mu(x_\rho) = \delta_{\nu\rho} h_1 T_{d(\mathfrak{s})^{-1}} x_\mu T_{d(\mathfrak{t})} h_2, \tag{5.8}$$

where $h_1 = \sum_{x \in \mathfrak{S}_{\bar\lambda} \cap \mathscr{D}_{\bar\mu d(\mathfrak{s}) \cap \bar\lambda}} T_{x^{-1}}$ and $h_2 = \sum_{x \in \mathfrak{S}_{\bar\nu} \cap \mathscr{D}_{\bar\mu d(\mathfrak{t}) \cap \bar\nu}} T_x$, then, by [DJM2, (6.6)], the set

$$\{\Phi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s} \in \mathfrak{T}^{ss}(\nu, \lambda), \mathfrak{t} \in \mathfrak{T}^{ss}(\nu, \mu), \lambda, \mu \in \Lambda_m(n, r), \nu \in \Lambda_m(n, r)^+\} \tag{5.9}$$

forms a basis for the q-Schur^m algebra $\mathbf{S}_R^m(n, r)$. Note that the element $\Phi_{\mathfrak{s}\mathfrak{t}}^{\mu}(x_\nu)$ is in fact a sum of Murphy's basis elements, which is denoted m_{ST} in [DJM2, (6.2)] (see the proof of (4.10) there). Note also that we adopt the Φ-notation used in [DS] instead of the φ-notation in [DJM2] as we have already used φ for natural basis elements.

As a consequence of (5.9), we obtain the quasi-heredity of the q-Schur^m algebra. For a field R, the following result is given in [DJM2, (6.18)]. The general case follows from [DR, (3.2.1)] (see (4.3a)).

(5.10) THEOREM. *Let R be a commutative Noetherian ring. Then the algebra $\mathbf{S}_R^m$ is quasi-hereditary.*

We are now going to define the Borel type subalgebras of a q-Schur^m algebra $\mathbf{S}_R^m(n, r)$. For $\lambda, \mu \in \Lambda_m(n, r)$ and $d \in \mathscr{D}_{\bar\lambda\bar\mu}$ with $\mathbf{i}_\lambda d \succcurlyeq \mathbf{i}_\mu$, let $\varphi_{\lambda\mu}^d \in \mathbf{S}_R^m$ be defined by

$$\varphi_{\lambda\mu}^d(x_\nu h) = \delta_{\mu\nu} T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}} \pi_\mu h, \qquad \nu \in \Lambda_m(n, r), h \in \mathbf{H}_R. \tag{5.11}$$

This is well-defined by (5.6). Let ι be the R-linear anti-automorphism on $\mathbf{H}_R$ sending T_i to T_i for $0 \le i \le r-1$ (see [GL, (5.5)]). Then $(\pi_\lambda)^\iota = \pi_\lambda$ for any $\lambda \in \Lambda_m(n, r)$ (see (5.5) for the definition of π_λ). It implies that $(\pi_\lambda T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}})^\iota = T_{\mathfrak{S}_{\bar\mu} d^{-1} \mathfrak{S}_{\bar\lambda}} \pi_\lambda$ for any $\lambda, \mu \in \Lambda_m(n, r)$. Thus we can define $\varphi_{\lambda\mu}^d = (\varphi_{\mu\lambda}^{d^{-1}})^\iota$ if $\mathbf{i}_\lambda d \preccurlyeq \mathbf{i}_\mu$.

(5.12) LEMMA. *For any $\mu \in \Lambda_m(n, r)$, let $\Omega_m^{\succcurlyeq}(\mu)$ and $\Omega_m^{\preccurlyeq}(\mu)$ be defined in (1.11). Then*

(a) *the set $\{\varphi_{\lambda\mu}^d \mid (\lambda, d) \in \Omega_m^{\succcurlyeq}(\mu), \mu \in \Lambda_m(n, r)\}$ is R-linearly independent.*

(b) *the set $\{\varphi_{\mu\lambda}^d \mid (\lambda, d) \in \Omega_m^{\preccurlyeq}(\mu), \mu \in \Lambda_m(n, r)\}$ is R-linearly independent.*

Proof. Suppose $\sum_{\substack{\mu \in \Lambda_m(n,r) \\ (\lambda, d) \in \Omega_m^{\succcurlyeq}(\mu)}} a_{\lambda,\mu}^d \varphi_{\lambda\mu}^d = 0$ with $a_{\lambda\mu}^d \in R$. Applying the left hand side to x_μ, we have $\sum_{(\lambda, d) \in \Omega_m^{\succcurlyeq}(\mu)} a_{\lambda\mu}^d T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}} \pi_\mu = 0$. Since π_μ is a monic polynomial in L_i, by (5.3), we have $\sum_{(\lambda, d) \in \Omega_m^{\succcurlyeq}(\mu)} a_{\lambda\mu}^d T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}} L_1^{a_1} \cdots L_r^{a_r} = 0$, where $L_1^{a_1} \cdots L_r^{a_r}$ is the highest term of π_μ. This implies $a_{\lambda\mu}^d = 0$, for all $(\lambda, d) \in \Omega_m^{\succcurlyeq}(\mu)$, by (5.3) again. Thus, (a) follows. The statement (b) follows from (a). ∎

(5.13) THEOREM. *Maintain the notation above. Let* $\mathbf{S}_R^{m,\succcurlyeq} = \mathbf{S}_R^{m,\succcurlyeq}(n,r)$ *(resp.* $\mathbf{S}_R^{m,\preccurlyeq} = \mathbf{S}_R^{m,\preccurlyeq}(n,r)$*) be the free R-submodule of* $\mathbf{S}_R^m$ *spanned by* $\varphi_{\lambda\mu}^d$ *with* $(\lambda, d) \in \Omega_m^{\succcurlyeq}(\mu)$ *(resp.* $\varphi_{\mu\lambda}^d$ *with* $(\lambda, d) \in \Omega_m^{\preccurlyeq}(\mu)$*) for all* $\mu \in \Lambda_m(n,r)$*. Then* $\mathbf{S}_R^{m,\succcurlyeq}$ *and* $\mathbf{S}_R^{m,\preccurlyeq}$ *are subalgebras of the q-Schur*^{m} *algebra* $\mathbf{S}_R^m$*, and we have, for* $N = N(m,n,r)$ *as in* (1.2)*, the following isomorphisms of R-algebras*:

$$\mathbf{S}_R^{m,\succcurlyeq}(n,r) \cong \mathbf{S}_R^{1,\succcurlyeq}(N,r), \qquad \mathbf{S}_R^{m,\preccurlyeq}(n,r) \cong \mathbf{S}_R^{1,\preccurlyeq}(N,r).$$

Proof. By Definition (5.11), we have $\varphi_{\lambda\mu}^d(x_\mu) = T_{\mathfrak{S}_{\bar\lambda} d \mathfrak{S}_{\bar\mu}} \pi_\mu = \psi_{\bar\lambda\bar\mu}^d(x_{\bar\mu})\pi_\mu$ for $(\lambda, d) \in \Omega_m^{\succcurlyeq}(\mu)$ and $\mu \in \Lambda_m(n,r)$. Now, the multiplicative closure condition and the isomorphisms can be proved similarly as in (2.7) and (2.8). ∎

These two subalgebras are called the *Borel type* subalgebras of $\mathbf{S}_R^m$.

Before generalizing (3.4) to $\mathbf{S}_R^m$ via (5.9), we observe from (3.3) that $\varphi_{\lambda\mu}^{\delta(\mathfrak{s})} \in \mathbf{S}_R^{m,\succcurlyeq}$ if $\mathfrak{s} \in \mathfrak{T}_m^{ss}(\mu, \lambda)$. So we must find some relation between this element $\delta(\mathfrak{s})$ and the element $d(\mathfrak{s})$ used in the definition (5.8).

We first recall from (3.2) the definition of $\delta(\mathfrak{s})$. Let $\mathfrak{s} \in \mathfrak{T}_m^{ss}(\mu, \lambda)$. Then $\delta(\mathfrak{s})$ is the distinguished coset representative defined by row standard tableau $\mathbf{t}^{\bar\lambda}\delta(\mathfrak{s})$ for which i belongs to row a if the place occupied by i in $\mathbf{t}^{\bar\mu}$ is occupied by a in $\bar{\mathfrak{s}}$. For example, if $\mu = ((3210\cdots0), (210\cdots0), (111)) \in \Lambda_3(3,12)^+$ and $\lambda = ((2110\cdots0), (220\cdots0), (112)) \in \Lambda_3(3,12)$, we take

$\mathfrak{s} =$

1	1	14	13	13	25
2	27		14		26
3					27

Then

$\mathbf{t}^\mu =$

1	2	3	7	8	10
4	5		9		11
6					12

and

$\mathbf{t}^\lambda\delta(\mathfrak{s}) =$

1	2	7	8	10	
4		3	9	11	
6				5	12

.

Here, again rows corresponding to those 0-parts are omitted. Note that the standard μ-tableau $\mathbf{t}^\mu\delta(\mathfrak{s})^{-1}$ is obtained by replacing all the numbers i in $\mathfrak{s}$ by the sequence obtained by reading the ith row in $\mathbf{t}^{\bar\lambda}$, the replacements in $\mathfrak{s}$ are made from left to right, down successive rows (compare [DS, (1.2.2)]). Thus, in the previous example, we have

$$\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1} = \boxed{\begin{array}{llllll} 1 & 2 & 7 & 5 & 6 & 9 \\ 3 & 11 & & 8 & & 10 \\ 4 & & & & & 12 \end{array}}\,.$$

Clearly, the standard μ-tableau $\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}$ is the maximal element first$(\mathfrak{s})$ in $\mathbf{T}_{\mathfrak{s},\lambda}$ under the partial order $\trianglelefteq$ on standard tableaux. Therefore, we have the result

$$\text{for any } \mathfrak{s} \in \mathfrak{T}^{ss}_{m}(\mu, \lambda), \qquad \mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1} = \text{first}(\mathfrak{s});$$
$$\text{therefore, } d(\mathfrak{s}) = \delta(\mathfrak{s})^{-1}. \tag{5.14}$$

This result implies immediately the following.

(5.15) THEOREM. *Maintain the notation in* (5.8) *and* (5.11). *For any* $\mu \in \Lambda_m(n,r)^+$, $\lambda, \nu \in \Lambda_m(n,r)$, *and* $\mathfrak{s} \in \mathfrak{T}^{ss}_m(\mu, \lambda)$, $\mathfrak{t} \in \mathfrak{T}^{ss}_m(\mu, \nu)$, *we have* $\varphi^{\delta(\mathfrak{s})}_{\lambda,\mu} \in \mathbf{S}^{m,\succcurlyeq}_R$, $\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu,\nu} \in \mathbf{S}^{m,\preccurlyeq}_R$, *and* $\Phi_{\mathfrak{s}\mathfrak{t}} = \varphi^{\delta(\mathfrak{s})}_{\lambda\mu}\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu}$. *Thus, the set*

$$\Big\{\varphi^{\delta(\mathfrak{s})}_{\lambda\mu}\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu} \mid \lambda, \nu \in \Lambda_m(n,r), \mu \in \Lambda_m(n,r)^{+},$$
$$\mathfrak{s} \in \mathfrak{T}^{ss}_m(\mu, \lambda), \mathfrak{t} \in \mathfrak{T}^{ss}_m(\mu, \nu)\Big\}$$

forms a basis for $\mathbf{S}^m_R$.

Proof. We have seen that part of the first assertion follows from (3.3). Let $\mathfrak{t}^{\mu}$ be the unique element in $\mathfrak{T}^{ss}_m(\mu, \mu)$. By (5.14) and (5.8), one checks easily that $\varphi^{\delta(\mathfrak{s})}_{\lambda\mu} = \Phi_{\mathfrak{s}\mathfrak{t}^{\mu}}$, $\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu} = \Phi_{\mathfrak{t}^{\mu}\mathfrak{t}}$ and $\Phi_{\mathfrak{s}\mathfrak{t}^{\mu}}\Phi_{\mathfrak{t}^{\mu}\mathfrak{t}} = \Phi_{\mathfrak{s}\mathfrak{t}}$. Hence we have $\Phi_{\mathfrak{s}\mathfrak{t}} = \varphi^{\delta(\mathfrak{s})}_{\lambda\mu}\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu}$. The last assertion follows from (5.9). ∎

As in (4.7), we have an algebra homomorphism χ_λ from $\mathbf{S}^{m,\succcurlyeq}_R$ to R. Let R_λ be the $\mathbf{S}^{m,\succcurlyeq}_R$-module induced by χ_λ.

(5.16) THEOREM. *Let* $\mathbf{S}^{m,\succcurlyeq}_R$ *(resp.* $\mathbf{S}^{m,\preccurlyeq}_R$*) be the Borel type subalgebras of* $\mathbf{S}^m_R$.

(a) *The q-Schur*^{m} *algebra* $\mathbf{S}^m_R$ *has a triangular decomposition*

$$\mathbf{S}^m_R = \mathbf{S}^{m,\succcurlyeq}_R\mathbf{S}^{m,\preccurlyeq}_R = \sum_{\lambda \in \Lambda_m(n,r)^+} \mathbf{S}^{m,\succcurlyeq}_R\varphi^1_{\lambda\lambda}\mathbf{S}^{m,\preccurlyeq}_R\,.$$

(b) *The Borel type subalgebras* $\mathbf{S}^{m,\succcurlyeq}_R$ *and* $\mathbf{S}^{m,\preccurlyeq}_R$ *are quasi-hereditary with simple costandard modules and simple standard modules, respectively, if* R *is a commutation local Noetherian ring.*

(c) *The set* $\{\varphi^1_{\lambda\lambda} \mid \lambda \in \Lambda_m(n,r)\}$ *is the complete set of primitive idempotents in* $\mathbf{S}_R^{m,\succcurlyeq}$ *and* $\mathbf{S}_R^{m,\preccurlyeq}$ *if R is a field.*

(d) *Suppose R is a field, and let* $V_\lambda = \mathbf{S}_R^{m,\succcurlyeq}\varphi^1_{\lambda\lambda}$ *for* $\lambda \in \Lambda_m(n,r)$. *Then* $V_\lambda = \Delta(\mathbf{S}_R^{m,\succcurlyeq}, \lambda)$ *and* $\{V_\lambda \mid \lambda \in \Lambda_m(n,r)\}$ *is a complete set of all principal indecomposable* $\mathbf{S}_R^{m,\succcurlyeq}$*-modules. Moreover, if we write* $\overline{\lambda} = (\lambda_1, \ldots, \lambda_N)$, *where N is defined in* (1.2), *then*

$$\dim_R V_\lambda = \prod_{i=1}^{N} \binom{\lambda_i + i - 1}{i - 1}.$$

(e) *Suppose R is a field. Then* $\{V_\lambda/\operatorname{rad} V_\lambda \mid \lambda \in \Lambda_m(n,r)\}$ *is the complete set of non-isomorphic simple* $\mathbf{S}_R^{m,\succcurlyeq}$*-modules and any simple module is of dimension one.*

(f) *Suppose that R is a commutative local Noetherian ring. Let* $\Delta(\mathbf{S}_R^m, \lambda)$ *(resp.* $\nabla(\mathbf{S}_R^m, \lambda)$*) be the left standard module (resp. costandard module) of* $\mathbf{S}_R^m$ *with respect to* $\lambda \in \Lambda_m(n,r)^+$. *Let* $\Delta(\mathbf{S}_R^{m,\preccurlyeq}, \lambda)$ *(resp.* $\Delta^{\mathrm{op}}(\mathbf{S}_R^{m,\succcurlyeq}, \lambda)$*) be the left (resp. right) standard module for* $\mathbf{S}_R^{m,\preccurlyeq}$ *(resp. for* $\mathbf{S}_R^{m,\succcurlyeq}$*). Then*

$$\mathbf{S}_R^m \otimes_{\mathbf{S}_R^{m,\preccurlyeq}} \Delta\big(\mathbf{S}_R^{m,\preccurlyeq}, \lambda\big) \cong \begin{cases} \Delta(\mathbf{S}_R^m, \lambda), & \text{if } \lambda \in \Lambda_m(n,r)^+, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\operatorname{Hom}_{\mathbf{S}_R^{m,\succcurlyeq}}\big({}_{\mathbf{S}_R^{m,\succcurlyeq}}\mathbf{S}_R^m, R_\lambda\big) \cong \begin{cases} \nabla(\mathbf{S}_R^m, \lambda), & \text{if } \lambda \in \Lambda_m(n,r)^+, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. All statements can be proved formally by using arguments similar to those given in Section 4.

(5.17) *Remark.* (1) We remark that an arbitrary cyclotomic q-Schur algebra $\mathbf{S}_R(\Lambda)$ defined in [DJM2] is Morita equivalent to a centralizer subalgebra of a q-Schur^m algebra defined by a coideal of $\Lambda_m(n,r)^+$ for some n. This is because, by [DJM2, (3.9i)], $\mathscr{S}_R(\Lambda)$ is Morita equivalent to $\mathbf{S}_R(\Lambda^+)$, where $\Lambda^+ = \Lambda \cap \Lambda_m(r)^+$ and hence, is Morita equivalent to the subalgebra $e\mathbf{S}_R^m(n,r)e$, where $e = \sum_{\lambda \in \Lambda'} \varphi^1_{\lambda\lambda}$ for any subset Λ' of $\Lambda_m(n,r)$ whose intersection with $\Lambda_m(r)^+$ is Λ^+. Moreover, the Borel type subalgebras of $e\mathbf{S}_R^m(n,r)e$ are $\mathbf{S}_R^{m,\succcurlyeq} \cap e\mathbf{S}_R^m e$ and $\mathbf{S}_R^{m,\preccurlyeq} \cap e\mathbf{S}_R^m e$.

(2) It would be nice to prove that the Borel type subalgebra $\mathbf{S}_R^{m,\preccurlyeq}$ is a *Borel subalgebra* of the quasi-hereditary algebra $\mathbf{S}_R^m$ in the sense of [Sc], and to establish the Borel–Bott–Weil theorem in this generality as described in (4.11).

6. TILTING MODULES AND RINGEL DUALS

In this section, we discuss the tilting modules and Ringel duals for Borel type subalgebras $\mathbf{S}_R^{m,\succcurlyeq}$ and $\mathbf{S}_R^{m,\preccurlyeq}$ over a *field* R.

Let A be a quasi-hereditary algebra over a field R. Then the category of left A-modules ${}_A\mathscr{C}$ is a *highest weight category* on a poset $(\Lambda, \leq)$ in the following sense (see [CPS1]).

(a) For each $\lambda \in \Lambda$, there is a simple A-module $L(\lambda)$ such that $\{L(\lambda) \mid \lambda \in \Lambda\}$ is the set of the non-isomorphic left A-modules.

(b) For each $\lambda \in \Lambda$, the standard module $\Delta(\lambda)$ has simple head $L(\lambda)$ and all other composition factors $L(\mu)$ satisfy $\mu < \lambda$.

(c) The projective cover $P(\lambda)$ of $L(\lambda)$ has a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_t = P(\lambda)$ such that any section $F_i'/F_{i-1}' \cong \Delta(\mu)$ with $\mu > \lambda$ if $i \neq t$ and $\mu = \lambda$ if $i = t$. The section F_t/F_{t-1} is called the top section.

The conditions (b) and (c) can be replaced by (b′) and (c′) as follows.

(b′) The costandard module $\nabla(\lambda)$ has socle $L(\lambda)$ and all other composition factors $L(\mu)$ satisfy $\mu < \lambda$;

(c′) The injective envelope $I(\lambda)$ has a filtration $0 = F_0' \subset F_1' \subset \cdots \subset F_{t'}' = I(\lambda)$ such that any section $F_i'/F_{i-1}' \cong \Delta(\mu)$ with $\mu > \lambda$ if $i \neq 1$ and $\mu = \lambda$ if $i = 1$. The section F_1/F_0 is called the bottom section.

We shall say a left A-module M has a Δ-filtration (resp. ∇-filtration) if M has a filtration whose sections are of forms $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) for some $\lambda \in \Lambda$. A module M is called a *tilting module* if it has Δ-filtration and ∇-filtration. Let ${}_A\mathscr{C}(tilt)$ be the subcategory of tilting modules. In [R], Ringel proved that, for each $\lambda \in \Lambda$, there is a unique indecomposable $X(\lambda) \in {}_A\mathscr{C}(tilt)$, called partial tilting module with respect to λ, such that $X(\lambda)$ has a Δ-filtration with bottom section $\Delta(\lambda)$ and a ∇-filtration with top section $\nabla(\lambda)$. A module $X \in {}_A\mathscr{C}(tilt)$ is called a full tilting module if X has a decomposition $X = \bigoplus_{\lambda \in \Lambda} X(\lambda)^{\oplus m_\lambda(X)}$ such that $m_\lambda(X) > 0$ for all $\lambda \in \Lambda$. For full tilting modules X and Y, Ringel proved that the endomorphism algebras $\mathrm{End}_A(X)$ and $\mathrm{End}_A(Y)$ are Morita equivalent. Such an endomorphism algebra is called a *Ringel dual* of the quasi-hereditary algebra A. It is known that the Ringel dual is a quasi-hereditary algebra.

(6.1) THEOREM. *Let R be a field. The Borel type subalgebras $\mathbf{S}_R^{m,\succcurlyeq}$ and $\mathbf{S}_R^{m,\preccurlyeq}$, viewed as regular modules, are full tilting modules, and therefore, they are Ringel dual to each other.*

Proof. By (5.16)(d), $V_\lambda = \mathbf{S}_R^{m,\succcurlyeq}\varphi_{\lambda\lambda}^1$ is a principal indecomposable $\mathbf{S}_R^{m,\succcurlyeq}$-module, and is isomorphic to the standard module $\Delta(\lambda) = \Delta(\mathbf{S}_R^{m,\succcurlyeq}, \lambda)$. Hence $0 \subset V_\lambda$ is a Δ-filtration of V_λ with bottom section $\Delta(\lambda)$. On the other hand, V_λ has a basis $\varphi^d_{\mu,\lambda}$ with $(\mu, d) \in \Omega^{\succcurlyeq}_m(\lambda)$. If we order these basis elements $\varphi^d_{\mu,\lambda}$ as $\varphi_1, \varphi_2, \dots,$ where $\varphi_i = \varphi_{\mu_{(i)},\lambda}$, such that $\mu_{(i)} \trianglelefteq \mu_{(j)}$ implies $i < j$, and define M_i as the R-space spanned by φ_j with $j \le i$, then M_i is an $\mathbf{S}^{m,\succcurlyeq}_R$-module, and $M_t = V_\lambda$ where $t = \#\Omega^{\succcurlyeq}_m(\lambda)$. Consider the filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = V_\lambda$ of V_λ. Obviously, its section M_i/M_{i-1} is of dimensional one for all $1 \le i \le t$. Suppose that M_i/M_{i-1} is spanned by $\overline{\varphi}^d_{\mu\lambda}$. Then, for any $\varphi^{d'}_{\rho\nu} \in \mathbf{S}^{m,\succcurlyeq}_R$, we have $\varphi^{d'}_{\rho\nu}\overline{\varphi}^d_{\mu\lambda} = 0$ if $\rho \ne \mu$. If $\rho = \mu$, then $\nu = \mu$, $d' = 1$, and $\varphi^{d'}_{\rho\nu}\overline{\varphi}^d_{\mu\lambda} = \overline{\varphi}^d_{\mu\lambda}$. Thus, $\varphi^{d'}_{\rho\nu}\overline{\varphi}^d_{\mu\lambda} = \chi_\mu(\varphi^{d'}_{\rho\nu})\overline{\varphi}^d_{\mu\lambda}$. So M_i/M_{i-1} is isomorphic to R_μ defined by χ_μ. On the other hand, by definition, $\nabla(\mu) \cong \mathrm{Hom}_R(\Delta^{\mathrm{op}}(\mu), R)$, where $\Delta^{\mathrm{op}}(\mu)$ is the right standard $\mathbf{S}^{m,\succcurlyeq}_R$-module defined by χ_μ, too (see (4.8)). Therefore, $M_i/M_{i-1} \cong \nabla(\mu)$. In particular, $M_t/M_{t-1} \cong \nabla(\lambda)$. So V_λ is the partial tilting module corresponding to λ. Since $1 = \sum_{\lambda \in \Lambda_m(n,r)} \varphi^1_{\lambda\lambda}$, we have that $\mathbf{S}^{m,\succcurlyeq}_R = \bigoplus_{\lambda \in \Lambda_m(n,r)} V_\lambda$, and so, $\mathbf{S}^{m,\succcurlyeq}_R$ is a full tilting $\mathbf{S}^{m,\succcurlyeq}_R$-module. Consequently, the Ringel dual $\mathrm{End}_{\mathbf{S}^{m,\succcurlyeq}_R}(\mathbf{S}^{m,\succcurlyeq}_R)$ $(\cong (\mathbf{S}^{m,\succcurlyeq}_R)^{\mathrm{op}})$ of $\mathbf{S}^{m,\succcurlyeq}_R$ is isomorphic to $\mathbf{S}^{m,\preccurlyeq}_R$. ∎

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