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A Remark on Multilinear Singular Integrals with Rough Kernels[1]

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In this paper, the authors establish the $(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$-boundedness of some multilinear singular integrals with rough kernels in the Hardy spaces on unit spheres, where $1 < q \le p < \infty$. © 2001 Academic Press

Key Words: multilinear singular integral; Hardy space; atom.

1. INTRODUCTION

Let $m \in \{0\} \cup \mathbb{N}$ and $A(x)$ be a suitable function on $\mathbb{R}^n$. For $m \in \mathbb{N}$, let

$$R_m(A; x, y) = A(x) - \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \{0\} \cup \mathbb{N}$ for $i \in \{1, \dots, n\}$, $D^\alpha = (\partial/\partial y_1)^{\alpha_1} \cdots (\partial/\partial y_n)^{\alpha_n}$, and $R_0(A; x, y) = A(x)$. We consider the non-standard singular integral operator

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_m(A; x, y) f(y)\, dy$$

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and the corresponding maximal operator

$$T_* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|.$$

The following theorem on the $(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$-boundedness of the operator T_*, which was proved by "the method of rotations," can be found in [1].

THEOREM A. *Let $\Omega(x)$ be homogeneous of degree zero and integrable on S^{n-1}, the unit sphere on $\mathbb{R}^n$. Let $m \in \{0\} \cup \mathbb{N}$ and $A(x)$ have derivatives of order m in $L^r(\mathbb{R}^n)$ for some $1 < r \leq \infty$. If $1 < p, q < \infty$ and $1/q = 1/r + 1/p$, then the following propositions hold*:

(i) *if $\Omega(-x) = (-1)^{m+1}\Omega(x)$, then*

$$\|T_* f\|_{L^q(\mathbb{R}^n)} \leq C\|\Omega\|_{L^1(S^{n-1})} \sum_{|\alpha|=m} \|D^\alpha A\|_{L^r(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}; \tag{1.1}$$

(ii) *if $\Omega(-x) = (-1)^m \Omega(x)$, $\Omega \in L \log^+ L(S^{n-1})$ and*

$$\int_{S^{n-1}} \Omega(x) x^\alpha \, d\sigma(x) = 0 \qquad \text{for all } |\alpha| = m, \tag{1.2}$$

where and in what follows, $d\sigma(x)$ represents the Lebesgue measure on S^{n-1}, then (1.1) *holds with $\|\Omega\|_{L^1(S^{n-1})}$ replaced by $\|\Omega\|_{L \log^+ L(S^{n-1})}$.*

In the proof of Theorem A, Bajsanski and Coifman also established the following estimate:

(1.3) Let $N(y)$ be homogeneous of degree 0 and let $\phi(t)$ be such that

$$\int_0^\infty t^{m+n-1} \phi(t) \, dt < \infty$$

and that $t^s \phi(t)$ is decreasing for some $s \leq 0$. Let $A(x)$ have derivatives of order m in $L^r(\mathbb{R}^n)$ for some $1 < r \leq \infty$, and let $1 < p \leq \infty$, $1 < q \leq \infty$, $1/q = 1/r + 1/p$,

$$U_\varepsilon f(x) = \varepsilon^{-m-n} \int_{\mathbb{R}^n} N(x-y) \phi\left(\frac{|x-y|}{\varepsilon}\right) R_m(A; x, y) f(y) \, dy,$$

and

$$U_* f(x) = \sup_{\varepsilon > 0} \left\{ \varepsilon^{-m-n} \int_{\mathbb{R}^n} \left| N(x-y) \phi\left(\frac{|x-y|}{\varepsilon}\right) R_m(A; x, y) f(y) \right| dy \right\}.$$

Then

$$\|U_* f\|_{L^q(\mathbb{R}^n)} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{L^r(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \int_{S^{n-1}} |N(x)| \, d\sigma(x).$$

Moreover, $U_\varepsilon f(x)$ converges in both $L^q(\mathbb{R}^n)$ and almost everywhere, and its limit is equal to

$$f(x) \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\{ \int_{\mathbb{R}^n} N(y) y^\alpha \phi(|y|) \, dy \right\} D^\alpha A(x).$$

See [1, (3.1)]. We will also need to use this estimate later in the proof of our main theorem.

We point that if $m = 0$, (1.1) is equivalent to the $L^p(\mathbb{R}^n)$-boundedness for the maximal operator of the classical Calderón–Zygmund convolution type singular integral, which was proved by Calderón and Zygmund in [2]. In [7, 10], it was proved that the classical Calderón–Zygmund convolution type singular integral is bounded on $L^p(\mathbb{R}^n)$ if $\Omega \in H^1(S^{n-1})$. Later, in [8], Fan and Pan improved the Calderón–Zygmund result in [2] and showed that the $L^p(\mathbb{R}^n)$-boundedness for the corresponding maximal operator still holds if $\Omega \in L \log^+ L(S^{n-1})$ is replaced by $\Omega \in H^1(S^{n-1})$; in fact, Fan and Pan's result is more general. But Fan and Pan's proof strongly depends on Fourier transform. Obviously, their method cannot be used for non-convolution type operators. So it is still an unsolved problem whether (1.1) holds under the weaker assumption that $\Omega \in H^1(S^{n-1})$ when $m > 0$. Fortunately, in [9], Grafakos and Stefanov rebuild the $L^p(\mathbb{R}^n)$-boundedness by "the method of rotation." This provides us with a new idea to improve Bajsanski and Coifman's theorem. We can prove the following theorem.

THEOREM 1. *Let* $m \in \{0\} \cup \mathbb{N}$ *and* $\Omega(-x) = (-1)^m \Omega(x)$, $\Omega \in H^1(S^{n-1})$ *and satisfy the moment condition* (1.2). *Let* $A(x)$ *be the same as in Theorem* A. *Then* (1.1) *holds with* $\|\Omega\|_{L^1(S^{n-1})}$ *replaced by* $\|\Omega\|_{H^1(S^{n-1})}$.

We remark that Theorem 1 with $m = 0$ is obtained in [8].

2. PROOF OF THEOREM 1

Before proving Theorem 1, we first establish some properties of the Hardy spaces on the unit sphere which play an essential role in the proof of Theorem 1. Let us begin with the definition of the space $H^1(S^{n-1})$. For $f \in L^1(S^{n-1})$ and $x \in S^{n-1}$, we define

$$P^+ f(x) = \sup_{0<t<1} \left| \int_{S^{n-1}} P_{tx}(y) f(y) \, d\sigma(y) \right|,$$

where

$$P_{tx}(y) = \frac{1 - t^2}{|y - tx|^n}$$

for $y \in S^{n-1}$.

DEFINITION 1. An integrable function f on S^{n-1} is in the space $H^1(S^{n-1})$ if and only if

$$\|P^+ f\|_{L^1(S^{n-1})} = \int_{S^{n-1}} |P^+ f(x)|\, d\sigma(x) < \infty$$

and we define

$$\|f\|_{H^1(S^{n-1})} = \|P^+ f\|_{L^1(S^{n-1})}.$$

A very useful characterization of the space $H^1(S^{n-1})$ is its atomic decomposition. Let us first recall the definition of atoms.

DEFINITION 2. A function $a(\eta)$ on S^{n-1} is a regular atom if $a(\eta) \in L^\infty(S^{n-1})$, and there exist $\xi \in S^{n-1}$ and $\rho \in (0, 2]$ such that

(i) $\operatorname{supp} a \subset S^{n-1} \cap B(\xi, \rho)$, where $B(\xi, \rho) = \{y \in \mathbb{R}^n : |y - \xi| < \rho\}$;

(ii) $\|a\|_{L^\infty(S^{n-1})} \leq \rho^{-n+1}$;

(iii) $\int_{S^{n-1}} a(y)\, d\sigma(y) = 0$.

A function $a(\eta)$ on S^{n-1} is an exceptional atom if $a(\eta) \in L^\infty(S^{n-1})$ and

$$\|a\|_{L^\infty(S^{n-1})} \leq 1.$$

The following can be found in [3, 5].

LEMMA 1. *For any $f \in H^1(S^{n-1})$ there are complex numbers λ_j and atoms* (*regular or exceptional*) *a_j such that*

$$f = \sum_j \lambda_j a_j$$

and

$$\|f\|_{H^1(S^{n-1})} \sim \sum_j |\lambda_j|.$$

Here are some additional properties of the Hardy space $H^1(S^{n-1})$.

LEMMA 2. *Suppose that $\Omega \in H^1(S^{n-1})$, $\Omega(-x) = (-1)^m \Omega(x)$, and* (1.2) *holds. Then there exist $\{\Omega_j\}_{j=1}^\infty$ such that for every j, $\Omega_j \in C^\infty(S^{n-1})$,*

$\Omega_j(-x) = (-1)^m \Omega_j(x)$,

$$\int_{S^{n-1}} \Omega_j(x) x^\alpha \, d\sigma(x) = 0 \qquad \textit{for all } |\alpha| = m,$$

and $\lim_{j\to\infty} \|\Omega_j - \Omega\|_{H^1(S^{n-1})} = 0$.

Proof. In [6], it is proved that the Bochner–Riesz means of the spherical harmonic expansion of Ω converges to Ω in $H^1(S^{n-1})$-norm. While the Bochner–Riesz means of spherical harmonic expansion are $C^\infty(S^{n-1})$ functions, we deduce from this that $C^\infty(S^{n-1})$ is dense in $H^1(S^{n-1})$. Now, suppose that $\{f_j\}_{j=1}^\infty$ are $C^\infty(S^{n-1})$ functions and

$$\lim_{j\to\infty} \|f_j - \Omega\|_{H^1(S^{n-1})} = 0.$$

Take

$$g_j(x) = \frac{f_j(x) + (-1)^m f_j(-x)}{2}.$$

Then $g_j(-x) = (-1)^m g_j(x)$ and $\lim_{j\to\infty} \|g_j - \Omega\|_{H^1(S^{n-1})} = 0$. Let $\{\alpha^{(k)}\}_{k=1}^{N_0}$ be all the multi-index such that $|\alpha^{(k)}| = m$. Denote by A the matrix

$$\begin{pmatrix} \int_{S^{n-1}} x^{\alpha^{(1)}} x^{\alpha^{(1)}} \, d\sigma(x) & \cdots & \int_{S^{n-1}} x^{\alpha^{(1)}} x^{\alpha^{(N_0)}} \, d\sigma(x) \\ \int_{S^{n-1}} x^{\alpha^{(2)}} x^{\alpha^{(1)}} \, d\sigma(x) & \cdots & \int_{S^{n-1}} x^{\alpha^{(2)}} x^{\alpha^{(N_0)}} \, d\sigma(x) \\ \vdots & \ddots & \vdots \\ \int_{S^{n-1}} x^{\alpha^{(N_0)}} x^{\alpha^{(1)}} \, d\sigma(x) & \cdots & \int_{S^{n-1}} x^{\alpha^{(N_0)}} x^{\alpha^{(N_0)}} \, d\sigma(x) \end{pmatrix}.$$

Since $x^{\alpha^{(k)}}$, $k = 1, 2, \ldots, N_0$, are linearly independent, we have $\det(A) \neq 0$. Let

$$c_{j,\alpha^{(k)}} = \int_{S^{n-1}} g_j(x) x^{\alpha^{(k)}} \, d\sigma(x), \qquad k = 1, 2, \ldots, N_0,$$

and

$$(a_{j,\alpha^{(1)}}, a_{j,\alpha^{(2)}}, \ldots, a_{j,\alpha^{(N_0)}}) = (c_{j,\alpha^{(1)}}, c_{j,\alpha^{(2)}}, \ldots, c_{j,\alpha^{(N_0)}}) A^{-1}.$$

By using the moment condition (1.2), it is easy to verify that for every k,

$$|c_{j,\alpha^{(k)}}| \leq \|g_j - \Omega\|_{H^1(S^{n-1})} \to 0$$

as $j \to \infty$. Thus, $|a_{j,\alpha^{(k)}}| \to 0$ as $j \to \infty$. Take

$$\Omega_j(x) = g_j(x) - \sum_{k=1}^{N_0} a_{j,\alpha^{(k)}} x^{\alpha^{(k)}}.$$

Then it is not difficult to check that $\{\Omega_j\}_{j=1}^{\infty}$ is just the sequence we are looking for. This finishes the proof of Lemma 2.

The following lemma with $s = n$ is included in [11, Theorem 2]; see also [9]. However, our proof is simple and elementary, which is different from that in [11].

LEMMA 3. *If* $\Omega \in H^1(S^{n-1})$ *and*

$$\int_{S^{n-1}} \Omega(x)\, d\sigma(x) = 0,$$

then for any $s \in \mathbb{R}$, $(\Omega(x)/|x|^s)\chi_{\{1/2<|x|<2\}}(x) \in H^1(\mathbb{R}^n)$.

Proof. By Lemma 1 and $\int_{S^{n-1}} \Omega(x)\, d\sigma(x) = 0$, we can write

$$\Omega(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where $\{a_j\}_{j=1}^{\infty}$ is a sequence of regular atoms. Suppose a_j is supported in $B(\xi_j, \rho_j) \cap S^{n-1}$ with $|\xi_j| = 1$. If ρ_j is large, it is easy to check that $(a_j(x)/|x|^s)\chi_{\{1/2<|x|<2\}}$ is an $H^1(\mathbb{R}^n)$-atom times a constant C. Here, a function $a(x)$ on $\mathbb{R}^n$ is called an $H^1(\mathbb{R}^n)$-atom if $\operatorname{supp} a \subset B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$ and some $r > 0$, $\|a\|_{L^\infty(\mathbb{R}^n)} \le r^{-n}$ and $\int_{\mathbb{R}^n} a(x)\, dx = 0$. If ρ_j is small, we split $(a_j(x)/|x|^s)\chi_{\{1/2<|x|<2\}}$ into no more than $[3/4\rho_j]$ parts along the radial direction, where $[s]$ is the biggest integer no more than s. Then it is not difficult to check that every part is an $H^1(\mathbb{R}^n)$-atom times a constant no more than $C\rho_j$. Thus, we have decomposed $(\Omega(x)/|x|^s)\chi_{\{1/2<|x|<2\}}$ into the sum of a sequence of $H^1(\mathbb{R}^n)$-atoms with the sum of the coefficients no more than $C\sum_{j=1}^{\infty}|\lambda_j|$. Therefore, $(\Omega(x)/|x|^s)\chi_{\{1/2<|x|<2\}} \in H^1(\mathbb{R}^n)$ and

$$\left\| \frac{\Omega(x)}{|x|^s} \chi_{\{1/2<|x|<2\}} \right\|_{H^1(\mathbb{R}^n)} \le C\|\Omega\|_{H^1(S^{n-1})}.$$

This finishes the proof of Lemma 3.

LEMMA 4. *Let Ω be the same as in Theorem* 1. *Define*

$$\mathscr{N}_1(x) = \mathscr{R}\left(\frac{\Omega(\cdot)}{|\cdot|^{n+m}}\right)(x) = \text{p.v.}\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^{n+m}}\, dy.$$

Here, $\mathscr{R}$ *is the Riesz transform and* $\mathscr{N}_1(x)$ *is a vector-valued function. Then*

(4a) $\mathscr{N}_1(x)$ *is homogeneous of degree* $-n-m$;

(4b) $\mathscr{N}_1(-x) = (-1)^{m+1}\mathscr{N}_1(x)$;

(4c) $\int_{S^{n-1}} |\mathscr{N}_1(x)|\, d\sigma(x) \le C\|\Omega\|_{H^1(S^{n-1})}$.

Proof. The proofs of (4a) and (4b) are trivial. To see (4c), for $3/4 < |x| < 3/2$, we write

$$\begin{aligned}\mathscr{N}_1(x) &= \text{p.v.}\int_{|y|<1/2} \frac{x-y}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^{n+m}}\, dy + \text{p.v.}\int_{1/2\le|y|<2} \cdots + \int_{|y|\ge 2} \cdots \\ &= I_1(x) + I_2(x) + I_3(x).\end{aligned}$$

By the argument in [1, p. 13], we can see that $\Omega(-x) = (-1)^m\Omega(x)$ and the moment condition (1.2) imply that

$$\int_{S^{n-1}} \Omega(x) x^\alpha\, d\sigma(x) = 0 \qquad \text{for all } |\alpha| \le m.$$

Thus,

$$\begin{aligned}|I_1(x)| &= \left| \int_{|y|<1/2} \left[\frac{(x-y)}{|x-y|^{n+1}} - \sum_{|\alpha|\le m} D^\alpha\left(\frac{(\cdot)}{|\cdot|^{n+1}}\right)(x)(-y)^\alpha \right] \frac{\Omega(y)}{|y|^{n+m}}\, dy \right| \\ &\le C\int_{|y|<1/2} \frac{|\Omega(y)|}{|y|^{n-1}}\, dy = C\|\Omega\|_{L^1(S^{n-1})}.\end{aligned}$$

Note that the Riesz transforms map $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. By Lemma 3, we know that $I_2(x) \in L^1(\mathbb{R}^n)$. Finally, it is easy to see that

$$|I_3(x)| \le C\int_{|y|>2} \frac{|\Omega(y)|}{|y|^{2n+m}}\, dy \le C\|\Omega\|_{L^1(S^{n-1})}.$$

Now, we have shown that $\mathscr{N}_1(x)\chi_{\{3/4<|x|<3/2\}}(x)\in L^1(\mathbb{R}^n)$. This is equivalent to $\mathscr{N}_1(x)\in L^1(S^{n-1})$. Moreover,

$$\begin{aligned}\int_{S^{n-1}}|\mathscr{N}_1(x)|\,d\sigma(x) &= C\int_{3/4<|x|<3/2}|\mathscr{N}_1(x)|\,dx\\ &\le C\|\Omega\|_{L^1(S^{n-1})}+C\left\|\frac{\Omega(\cdot)}{|\cdot|^{n+m}}\chi_{\{1/2<|\cdot|<2\}}\right\|_{H^1(\mathbb{R}^n)}\\ &\le C\|\Omega\|_{H^1(S^{n-1})}.\end{aligned}$$

This finishes the proof of Lemma 4.

LEMMA 5. *Let $\phi(t)$ be a C^∞ function on $(0,\infty)$ which satisfies $\phi(t)=0$ for $t\le 1/4$, $\phi(t)=1$ for $t\ge 3/4$, and $0\le\phi(t)\le 1$ for all t. Define the vector-valued function*

$$\mathscr{N}_2(x)=\mathscr{R}\left(\frac{\Omega(\cdot)}{|\cdot|^{n+m}}\phi(|\cdot|)\right)(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^{n+1}}\frac{\Omega(y)}{|y|^{n+m}}\phi(|y|)\,dy.$$

Then

(5a) $|\mathscr{N}_2(x)-\mathscr{N}_1(x)|\le C|x|^{-(n+m+1)}$ *for* $|x|\ge 1$;

(5b) $|\mathscr{N}_2(x)|\le G(x)$ *for* $|x|\le 1$, *where $G(x)$ is homogeneous of degree zero and*

$$\|G\|_{L^1(S^{n-1})}\le C\|\Omega\|_{H^1(S^{n-1})};$$

(5c) *if $g(x)$ is a vector-valued $C_0^\infty(\mathbb{R}^n)$ function, then*

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n+m}}\phi\left(\frac{|x-y|}{\varepsilon}\right)(\mathscr{R}\cdot g)(y)\,dy\\ &\quad=\varepsilon^{-n-m}\int_{\mathbb{R}^n}\mathscr{N}_2\left(\frac{x-y}{\varepsilon}\right)\cdot g(y)\,dy.\end{aligned}$$

Proof. If $|x|\ge 1$, since $\phi(t)=1$ for $t\ge 3/4$, we have

$$\begin{aligned}|\mathscr{N}_2(x)-\mathscr{N}_1(x)|&\le\left|\int_{|y|\le 3/4}\left[\frac{(x-y)}{|x-y|^{n+1}}-\sum_{|\alpha|\le m}D^\alpha\left(\frac{(\cdot)}{|\cdot|^{n+1}}\right)(x)(-y)^\alpha\right]\right.\\ &\qquad\left.\times[\phi(y)-1]\frac{\Omega(y)}{|y|^{n+m}}\,dy\right|\\ &\le C|x|^{-(n+m+1)}\int_{|y|<3/4}\frac{|\Omega(y)|}{|y|^{n-1}}\,dy\\ &= C\|\Omega\|_{L^1(S^{n-1})}|x|^{-(n+m+1)}.\end{aligned}$$

For the case $|x| < 1$, notice first that if $|x| < 1/8$, $|\mathcal{N}_2(x)| \le C\|\Omega\|_{L^1(S^{n-1})}$ because the singularity of the integrand in the integral representation of $\mathcal{N}_2$ is away from x. If $1/8 \le |x| \le 1$, write

$$\begin{aligned}
&|\mathcal{N}_2(x) - \phi(|x|)\mathcal{N}_1(x)| \\
&\le \int_{|y|<1/16} \left| \frac{(x-y)}{|x-y|^{n+1}} - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha\left(\frac{(\cdot)}{|\cdot|^{n+1}}\right)(x)(-y)^\alpha \right| \\
&\quad \times \frac{|\Omega(y)|}{|y|^{n+m}} |\phi(|y|) - \phi(|x|)|\, dy \\
&\quad + \int_{1/16\le|y|\le 2} \frac{1}{|x-y|^n} \frac{|\Omega(y)|}{|y|^{n+m}} |\phi(|y|) - \phi(|x|)|\, dy \\
&\quad + \int_{|y|>2} \frac{1}{|x-y|^n} \frac{|\Omega(y)|}{|y|^{n+m}} |\phi(|y|) - \phi(|x|)|\, dy \\
&= J_1(x) + J_2(x) + J_3(x).
\end{aligned}$$

Trivial computation leads to

$$J_1(x) \le C\int_{|y|<1/16} \frac{|\Omega(y)|}{|y|^{n-1}}\, dy \le C\|\Omega\|_{L^1(S^{n-1})}$$

and

$$J_3(x) \le C\int_{|y|>2} \frac{|\Omega(y)|}{|y|^{2n+m}}\, dy \le C\|\Omega\|_{L^1(S^{n-1})}.$$

For the second term, we use that ϕ is a Lipschitz function to obtain

$$\begin{aligned}
J_2(x) &\le C\int_{1/16\le|y|\le 2} \frac{|\Omega(y)|}{|y|^{n-1/2}|x-y|^{n-1}}\, dy \\
&\le C|x|^{n-3/2} \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-1/2}|x-y|^{n-1}}\, dy.
\end{aligned}$$

Therefore, (5b) follows if we set

$$\begin{aligned}
G(x) = C\Bigg[&\|\Omega\|_{L^1(S^{n-1})} + |x|^{n+m}|\mathcal{N}_1(x)| \\
&+ |x|^{n-3/2} \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-1/2}|x-y|^{n-1}}\, dy \Bigg].
\end{aligned}$$

Finally, (5c) is easy to check; see [1, 2] for the ideas. This completes the proof of Lemma 5.

Proof of Theorem 1. We will follow a procedure similar to that in [1, Sect. 7]. By Lemma 2, we may assume that $\Omega \in C^\infty(S^{n-1})$ and $g(x) = \mathscr{R}f(x)$. Let $\mathscr{N}_1(x)$ be the same as in Lemma 4. Define

$$S_\varepsilon g(x) = \int_{|x-y|>\varepsilon} \mathscr{N}_1(x-y) R_m(A; x, y) \cdot g(y)\, dy$$

and S_* its corresponding maximal operator. By Lemma 4, Theorem A(i), and the $L^p(\mathbb{R}^n)$-boundedness of the Riesz transforms, we obtain

$$\|S_* g\|_{L^q(\mathbb{R}^n)} \le C\|\Omega\|_{H^1(S^{n-1})} \sum_{|\alpha|=m} \|D^\alpha A\|_{L^r(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Since $f(x) = C(\mathscr{R} \cdot g)(x)$ with C a constant, to finish the proof, it suffices to estimate the operator

$$\Delta g(x) = \sup_{\varepsilon>0} \left| T_\varepsilon(\mathscr{R} \cdot g)(x) - S_\varepsilon g(x) \right|.$$

Let ϕ and $\mathscr{N}_2$ be the same as in Lemma 5. We represent $|T_\varepsilon(\mathscr{R} \cdot g)(x) - S_\varepsilon g(x)|$ as the sum of the following three terms:

$$\begin{aligned} D_\varepsilon g(x) &= T_\varepsilon(\mathscr{R} \cdot g)(x) \\ &\quad - \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m}} \phi\left(\frac{|x-y|}{\varepsilon}\right) R_m(A; x, y)(\mathscr{R} \cdot g)(y)\, dy, \\ E_\varepsilon g(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m}} \phi\left(\frac{|x-y|}{\varepsilon}\right) \mathscr{R} \\ &\quad \cdot [R_m(A; x, \cdot) g(\cdot)](y)\, dy - S_\varepsilon g(x), \\ F_\varepsilon g(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m}} \phi\left(\frac{|x-y|}{\varepsilon}\right) \\ &\quad \times \{R_m(A; x, y)(\mathscr{R} \cdot g)(y) - \mathscr{R}[R_m(A; x, \cdot) g(\cdot)](y)\}\, dy. \end{aligned}$$

The corresponding maximal operators are denoted by D_*, E_*, and F_*, respectively. It is easy to see that

$$|D_\varepsilon g(x)| \le C\varepsilon^{-n-m} \int_{|x-y|<\varepsilon} |\Omega(x-y) R_m(A; x, y)(\mathscr{R} \cdot g)(y)|\, dy.$$

Applying (1.3) with the case $\phi = \chi_{[0,1]}$, we can obtain a desired estimate for $\|D_* g\|_{L^q(\mathbb{R}^n)}$. Using Lemma 5, by an estimate similar to that for $E_\varepsilon g$ in [1], we can show

$$|E_\varepsilon g(x)| \leq C\varepsilon^{-n-m}\int_{|x-y|>\varepsilon}\left(\frac{|x-y|}{\varepsilon}\right)^{-n-m-1}|R_m(A;x,y)g(y)|\,dy$$
$$+ C\varepsilon^{-n-m}\int_{|x-y|<\varepsilon}|G(x-y)R_m(A;x,y)g(y)|\,dy.$$

Applying (1.3) again to each part on the right-hand side of the above inequality, we can obtain a desired estimate for $\|E_* g\|_{L^q(\mathbb{R}^n)}$. Finally, as we pointed at the end of Section 1, Theorem 1 is true if $m = 0$. Thus, the term F_ε can be estimated by induction on m as in [1] on F_ε, which leads to a desired estimate for $\|F_* g\|_{L^q(\mathbb{R}^n)}$.

This finishes the proof of Theorem 1.

Finally, we remark that in [4], a conclusion similar to Theorem A was proved to be true for this kind of operator with several remainders. By the idea in [4], we can also extend Theorem 1 to a several remainders case. For brevity, we state the result without a proof.

THEOREM 2. *Let $\Omega(x)$ be homogeneous of degree zero and integrable on S^{n-1}. Let $k \in \mathbb{N}$, $\{m_j\}_{j=1}^k \subset \{0\} \cup \mathbb{N}$, and $M = \sum_{j=1}^k m_j$. Suppose that $A_j(x)$ has derivatives of order m_j in $L^{r_j}(\mathbb{R}^n)$ for some $1 < r_j \leq \infty$, $j = 1, \ldots, k$. If $1 < p, q < \infty$ and $1/q = \sum_{j=1}^k 1/r_j + 1/p$, then the operator*

$$\overline{T}_* f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{j=1}^k R_{m_j}(A_j;x,y)f(y)\,dy\right|$$

has the following properties:

(i) *if $\Omega(-1) = (-1)^{M+1}\Omega(x)$, then*

$$\|\overline{T}_* f\|_{L^q(\mathbb{R}^n)} \leq C\|\Omega\|_{L^1(S^{n-1})}\prod_{j=1}^k \sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{L^{r_j}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}; \quad (2.1)$$

(ii) *if $\Omega(-x) = (-1)^M\Omega(x)$, $\Omega \in H^1(S^{n-1})$ and*

$$\int_{S^{n-1}}\Omega(x)x^\alpha\,d\sigma(x) = 0 \qquad \textit{for all } |\alpha| = M,$$

then (2.1) *holds with $\|\Omega\|_{L^1(S^{n-1})}$ replaced by $\|\Omega\|_{H^1(S^{n-1})}$.*

Part (i) of the preceding theorem was proved in [4], while (ii) is new.

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