Nonarchimedean Bornologies, Cyclic Homology and Rigid Cohomology

Guillermo Cortiñas[1], Joachim Cuntz[2], Ralf Meyer, Georg Tamme[3]

Abstract. Let V be a complete discrete valuation ring with residue field k and with fraction field K of characteristic 0. We clarify the analysis behind the Monsky–Washnitzer completion of a commutative V-algebra using spectral radius estimates for bounded subsets in complete bornological V-algebras. This leads us to a functorial chain complex for commutative k-algebras that computes Berthelot's rigid cohomology. This chain complex is related to the periodic cyclic homology of certain complete bornological V-algebras.

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1 Introduction

The problem of defining a p-adic cohomology theory with good properties (and a Frobenius) for an algebraic variety over a field k of non-zero characteristic p has a long history. In the breakthrough paper [27] by Monsky and Washnitzer, such a theory for smooth affine varieties was constructed as follows. Take a complete discrete valuation ring V of mixed characteristic with uniformizer π and residue field $k = V/\pi V$ (for example, V the ring of Witt vectors $W(k)$ if k is perfect). Let K be the fraction field of V. Choose a V-algebra R which is a lift

mod π of the coordinate ring of the variety and which is smooth over V (such a lift exists by [13]). Monsky–Washnitzer then introduce the 'weak' or dagger-completion $R^\dagger$ of R and define their cohomology as the de Rham cohomology of $R^\dagger \otimes_V K$. The construction of a weak completion has become a basis for the definition of cohomology theories in this context ever since. The Monsky–Washnitzer theory has been generalized by Berthelot [3] to "rigid cohomology," which is a satisfactory cohomology theory for general varieties over k. Its definition uses certain de Rham complexes on rigid analytic spaces.

The definition of the Monsky–Washnitzer cohomology depends on the choice of certain smooth algebras over V. Also Berthelot's definition of rigid cohomology depends on choices. So the chain complexes that compute them are not yet functorial for algebra homomorphisms. Only their homology is functorial. Functorial complexes that compute rigid cohomology have been constructed by Besser [5]. However, the construction is based on some abstract existence statements, and is not at all explicit.

One aim of our article is the construction of a natural and explicit chain complex that computes rigid cohomology. A second aim is linking rigid cohomology to cyclic homology. For cyclic homology in characteristic 0 it was recognized long ago that analytic versions of the theory can be treated in an elegant way using bornological structures on the underlying algebras, see [10, 25]. On the other hand, the relevance of a bornological point of view in the context of dagger completions has been highlighted recently by Bambozzi [2]. For our purposes here, we again find that it is natural to work in a framework based on bornological structures. This allows to generalize the weak completions of Monsky–Washnitzer to bornological versions of J-adic completions for an ideal J in a commutative V-algebra. The development of the corresponding techniques is our third project.

To prepare the ground for our theory, in Section 2 we first recall some basics on bornological V-modules and K-vector spaces. We are particularly interested in completions and completed tensor products, which play a crucial role in our theory. We also relate bornological V-modules to inductive systems of V-modules, carrying over well known results for bornological vector spaces over $\mathbb{R}$ and $\mathbb{C}$. Section 3 contains our bornological interpretation of the weak completions used for Monsky–Washnitzer cohomology and rigid cohomology. The main point here is the spectral radius $\varrho(S)$ of a bounded subset S in a complete bornological algebra, which is concerned with the growth rate of the powers S^n, $n \in \mathbb{N}$. This is a non-negative *real* number or ∞. The Monsky–Washnitzer completion of a finitely generated, commutative V-algebra R is the smallest completion of R in which all finitely generated V-submodules S of R have $\varrho(S) \le 1$. This completion makes sense also for an infinitely generated noncommutative algebra R and is denoted by $\overline{R_{\mathrm{lg}}}$ in this generality. Similarly, $\overline{\underline{R}_{J,\alpha}}$ for an ideal $J \lhd R$ with $\pi \in J$ and $\alpha \in [0,1]$ is the smallest completion of $\underline{R} := R \otimes K$ in which all finitely generated V-submodules S of J have $\varrho(S) \le \epsilon^\alpha$ where $\epsilon = |\pi|$ is the absolute value of the uniformizer of V. Section 3 first

describes these completions more explicitly and proves some basic properties. Then it relates them to Monsky–Washnitzer completions of finitely generated subalgebras of R and certain generalized tube algebras for $J \lhd R$.

The first step then, in our natural construction of a complex computing rigid cohomology, is to present a commutative k-algebra A by the free commutative V-algebra $R \coloneqq V[A]$ generated by the set A. This comes with a canonical surjective V-algebra homomorphism $p \colon R \to A$. Let $J \lhd R$ be its kernel. Since the algebra R is in general neither finitely generated nor Noetherian, many results of Monsky and Washnitzer do not apply to it. But our bornological approach also works fine for such infinitely generated algebras. We define a family of weak completions $\overline{\underline{R}_{J,\alpha}}$ of $\underline{R} \coloneqq R \otimes K$ that depend on the ideal J and $\alpha \in [0,1]$. Set $\mathbb{N} \coloneqq \mathbb{Z}_{\geq 0}$. The de Rham complexes of these weak completions for $\alpha = 1/m$, $m \in \mathbb{N}_{\geq 1}$, form a projective system. Its homotopy projective limit is a chain complex that is naturally associated to A. We show that it computes the rigid cohomology if A is of finite type over k. The first step to see this is a homotopy invariance result: the completions $\overline{\underline{R}_{J,\alpha}}$ and $\overline{\underline{R}'_{J',\alpha}}$ for another free commutative V-algebra R' with a surjection $p' \colon R' \to A$ and $J' \coloneqq \ker(p')$ are homotopy equivalent with 'dagger continuous' homotopies. This kind of homotopy is defined using a weak completion of the polynomial algebra $K[t]$. As a consequence of the homotopy, the de Rham chain complexes for $\overline{\underline{R}_{J,\alpha}}$ and $\overline{\underline{R}'_{J',\alpha}}$ are homotopy equivalent. When R' is finitely generated, we then show in Section 6 that the homotopy limit of the de Rham complexes for the system $\left(\overline{\underline{R}'_{J',1/m}} \right)$ computes rigid cohomology using results of Große-Klönne in [18].

To establish the link to cyclic homology, we compute in Section 4 the Hochschild and periodic cyclic homology of the completions $\overline{R_{\mathrm{lg}}}$ and $\overline{\underline{R}_{J,\alpha}}$. This is based on flatness results for Monsky–Washnitzer completions of torsion-free, finitely generated, commutative V-algebras, which allow to compute their Hochschild homology. We obtain an analogue of Connes' computation, in [9], of the periodic cyclic homology for algebras of smooth functions on manifolds in this setting, showing that the periodic cyclic homology of $\overline{\underline{R}_{J,\alpha}}$ is naturally isomorphic to the cohomology of the de Rham complex for $\overline{\underline{R}_{J,\alpha}}$ made periodic. This result is formally very similar to the theorem of Feigin–Tsygan in [14, Theorem 5], [15, Theorem 6.1.1], which relates cyclic homology in characteristic 0 to Grothendieck's infinitesimal cohomology (for a different proof, covering also the non-affine case, see [8, Theorem 6.7]). Our proof uses the fact that different flat resolutions give quasi-isomorphic complexes and in fact, we find that this method also gives a very short new proof of the Feigin–Tsygan Theorem. We further prove in this section that periodic cyclic homology for complete bornological K-algebras is invariant under dagger-continuous homotopies.

On the basis of the results in Section 4 we obtain in Section 5 a second chain complex that models rigid cohomology made periodic. Namely, we take the cyclic bicomplexes of $\overline{\underline{R}_{J,\alpha}}$ for $\alpha = 1/m$, which compute the periodic cyclic homology of these algebras. In Section 4 we had seen that the periodic cyclic homology of $\overline{\underline{R}_{J,\alpha}}$ is naturally isomorphic to the cohomology of the de Rham

complex for $\overline{R_{J,\alpha}}$ made periodic. Thus we see that periodic rigid cohomology is isomorphic to the periodic cyclic homology (as defined in [7]) of the pro-algebra given by the projective system of algebras $\overline{R_{J,1/m}}$; here pro-algebras are needed, and they produce the homotopy projective limit above.

Our article develops a general conceptual framework for the study of bornological completions that generalize the weak completions of Monsky–Washnitzer, and contains much more material than what is needed for the proof of our result in Theorem 6.5. In the last section we describe a quick route to the construction of the natural complexes that compute rigid cohomology. This approach uses only elementary properties of bornological completions and avoids the finer analysis of spectral radius or linear growth completions. It also brings to light more clearly the analogy with Grothendieck's original construction of infinitesimal cohomology in characteristic 0.

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1.1 Notation

Let V be a complete discrete valuation ring and let π be a generator for the maximal ideal in V. Let K be the fraction field of V, that is, $K = V[\pi^{-1}]$. Let $k = V/\pi V$ be the residue field. In the sections on periodic cyclic homology and rigid cohomology, we need K to have characteristic 0. In the earlier sections, this assumption is not needed and not made.

Every element of K is written uniquely as $x = u\pi^{\nu(x)}$, where $u \in V \smallsetminus \pi V$ and $\nu(x) \in \mathbb{Z} \cup \{-\infty\}$ is the *valuation* of x. We fix $0 < \epsilon < 1$, and define the *absolute value* $|\ |\colon K \to \mathbb{R}_{\geq 0}$ by $|x| = \epsilon^{\nu(x)}$ for $x \neq 0$ and $|0| = 0$.

If M is a V-module, let $\underline{M}$ be the associated K-vector space $M \otimes K$. A V-module M is *flat* if and only if the canonical map $M \to \underline{M}$ is injective, if and only if it is *torsion-free*, that is, $\pi x = 0$ implies $x = 0$.

2 Bornological modules over discrete valuation rings

This section defines (convex) bornological V-modules, convergence of sequences in them, and related notions of separatedness and completeness, completion, and completed tensor products. We also establish some basic results about these notions. We show, for example, that the category of complete bornological modules is equivalent to the category of inductive systems of complete V-modules with injective transition maps (Proposition 2.10). We use the notation in Section 1.1 without further comment.

Definition 2.1 (compare [21]). Let M be a V-module. A (convex) *bornology* on M is a family $\mathcal{B}$ of subsets of M, called *bounded* subsets, satisfying

- every finite subset is in $\mathcal{B}$;

- subsets and finite unions of sets in $\mathcal{B}$ are in $\mathcal{B}$;

- if $S \in \mathcal{B}$, then the V-submodule generated by S also belongs to $\mathcal{B}$.

A *bornological V-module* is a V-module equipped with a bornology.

The condition on unions holds if and only if $\mathcal{B}$ with inclusion as partial order is directed. The third condition is a convexity condition and is related in [2] to standard notions of convexity for bornological $\mathbb{R}$-vector spaces. One may also consider bornologies that do not satisfy the convexity condition above, but we shall not do so in this article.

DEFINITION 2.2. A *bornological K-vector space* is a bornological V-module such that multiplication by π is an invertible map with bounded inverse.

EXAMPLE 2.3. Let M be a V-module. The collection $\mathcal{F}$ of all subsets of finitely generated submodules of M is a bornology, called the *fine* bornology.
The fine bornology is the smallest possible bornology: a finitely generated submodule of M is the set of V-linear combinations of the elements of a finite subset and hence bounded in any bornology. If M itself is finitely generated, then all bounded subsets are bounded in the fine bornology, so this is the only bornology on M in this case.

EXAMPLE 2.4. Let M be a V-module. A (nonarchimedean) *seminorm* on M is a map $\|\ \|: M \to \mathbb{R}_{\geq 0}$ such that

$$\|ax\| = |a|\,\|x\|, \qquad \text{for all } a \in V,\ x \in M,$$
$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad \text{for all } x, y \in M.$$

A *norm* is a seminorm for which $\|x\| = 0$ implies $x = 0$. If M admits a norm, M must be torsion-free. Let $(I, \leq)$ be a directed set and let $\|\ \|_i$ for $i \in I$ be seminorms on M with $\|\ \|_i \leq \|\ \|_j$ for $i, j \in I$ with $i \leq j$. Call a subset of M bounded if it is bounded with respect to all these seminorms. This is a bornology on M. It is analogous to the standard (von Neumann) bornology on a locally convex topological vector space over $\mathbb{R}$, see [25, Example 1.11].
A seminorm on a K-vector space X is determined up to equivalence by its unit ball M; it is equivalent to the *gauge seminorm*

$$\|x\|_M = \inf\{|a|^{-1} : ax \in M\}. \tag{1}$$

Indeed, by [29, Lemma 2.2.ii], we have

$$\epsilon \|x\|_M \leq \|x\| \leq \|x\|_M \qquad (x \in X). \tag{2}$$

Thus the bornology associated to a seminorm determines the seminorm up to equivalence.

An *inductive system* in a category $\mathcal{C}$ is a functor $X: I \to \mathcal{C}$ from a directed set I. We identify such a functor with the collection $X = \{X_i, \phi_{ij}: X_i \to X_j\}$ of the objects $X_i = X(i)$ and the transition maps $\phi_{ij} = X(i < j)$. The inductive

systems in $\mathcal{C}$ form a category ind-$\mathcal{C}$, where the homomorphisms from X to $Y\colon J \to \mathcal{C}$ are given by

$$\hom_{\text{ind-}\mathcal{C}}(X,Y) = \varprojlim_{i} \varinjlim_{j} \hom_{\mathcal{C}}(X_i, Y_j).$$

PROPOSITION 2.5. *The category of bornological V-modules and bounded V-module maps is equivalent to the full subcategory of the category of inductive systems of V-modules consisting of those inductive systems $(X_i, \varphi_{ij}\colon X_i \to X_j)$ that have injective maps φ_{ij}.*

Proof. Let X be a bornological V-module. Let $\mathcal{B}$ be the directed set of bounded subsets, and let $\mathcal{B}' \subseteq \mathcal{B}$ be the subset of all bounded V-submodules. Convexity means that $\mathcal{B}'$ is cofinal in $\mathcal{B}$, so it is again directed. The V-submodules $S \in \mathcal{B}'$ with the inclusion maps $S \hookrightarrow T$ for $S \subseteq T$ form an inductive system of V-modules with injective transition maps. A bounded map $(X_1, \mathcal{B}_1) \to (X_2, \mathcal{B}_2)$ maps any bounded V-submodule of X_1 into some bounded V-submodule of X_2 and thus gives a morphism of inductive systems. Hence we have defined a functor A from bornological V-modules to inductive systems.

Conversely, an inductive system of V-modules $(X_i, \varphi_{ij}\colon X_i \to X_j)$ has an inductive limit V-module X. We equip X with the "inductive limit bornology": a subset of X is bounded if and only if it is contained in the image of X_i for some i. A morphism of inductive systems induces a bounded V-module map on the inductive limits, so we get a functor L from inductive systems of V-modules to bornological V-modules. The inductive limit comes with natural bounded V-module maps $X_i \to X$. These are injective if all the transition maps $\varphi_{ij}\colon X_i \to X_j$ are injective. In this case, the inductive system $A \circ L(X_i, \varphi_{ij})$ is isomorphic to (X_i, φ_{ij}) because the bounded V-submodules X_i form a cofinal subset in $\mathcal{B}'$. Given a bornological V-module $(X, \mathcal{B})$, the inclusion maps $S \hookrightarrow X$ for $S \in \mathcal{B}'$ induce a natural isomorphism $L \circ A(X) \cong X$. Hence the functors A and L provide the desired equivalence of categories. $\qquad\square$

Let S be a V-module. The π-*adic completion* of S is the V-module

$$\widehat{S} = \varprojlim_{m \to \infty} S/\pi^m S,$$

equipped with the map $S \to \widehat{S}$, $x \mapsto [x \bmod \pi^m S]_{m \in \mathbb{N}}$. The module S is π-*adically complete* if the map $S \to \widehat{S}$ is an isomorphism, and π-*adically separated* if the map $S \to \widehat{S}$ is injective, that is, $\bigcap_{m \in \mathbb{N}} \pi^m S = \{0\}$. We give each $S/\pi^m S$ the discrete topology, and then $\widehat{S}$ the projective limit topology. The topology on S induced by $\widehat{S}$ is called the π-*adic topology* on S. Thus a sequence $(x_n)_{n \in \mathbb{N}}$ in S converges to $x \in S$ if and only if for each $m \in \mathbb{N}$ there are only finitely many $n \in \mathbb{N}$ with $x_n - x \notin \pi^m S$. The π-adic topology on S is generated by the pseudometric $d_S(x,y) = \epsilon^{\sup\{m : x-y \in \pi^m S\}}$, which is a metric if and only if S is π-adically separated.

Definition 2.6. Let X be a bornological V-module. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and let $x \in X$. If $S \subseteq X$ is bounded, then $(x_n)_{n \in \mathbb{N}}$ *S-converges* to x if there is a sequence $(\delta_n)_{n \in \mathbb{N}}$ in V with $\lim \delta_n = 0$ in the π-adic topology and $x_n - x \in \delta_n \cdot S$ for all $n \in \mathbb{N}$. A sequence in X *converges* if it S-converges for some bounded subset S of X. If S is bounded, then $(x_n)_{n \in \mathbb{N}}$ is *S-Cauchy* if there is a sequence $(\delta_n)_{n \in \mathbb{N}}$ in V with $\lim \delta_n = 0$ and $x_n - x_m \in \delta_l \cdot S$ for all $n, m, l \in \mathbb{N}$ with $n, m \geq l$. A sequence in X is *Cauchy* if it is S-Cauchy for some bounded $S \subseteq X$. The bornological V-module X is *separated* if limits of convergent sequences are unique. It is *complete* if it is separated and for every bounded $S \subseteq X$ there is a bounded $S' \subseteq X$ so that all S-Cauchy sequences are S'-convergent.

Example 2.7. Any V-module M is complete in the fine bornology of Example 2.3 because all finitely generated V-modules are π-adically complete. A finite-dimensional bornological K-vector space carries a unique *separated* bornology, namely, the fine bornology.

Proposition 2.8. *Let X be a bornological V-module and $\mathcal{B}$ its family of bounded subsets. A sequence in X converges if and only if it is contained in a bounded V-submodule $Y \subseteq X$ and converges in Y in the π-adic topology. The bornological V-module X is separated if and only if all bounded V-submodules are π-adically separated, if and only if bounded, π-adically separated V-submodules are cofinal in $\mathcal{B}$ for inclusion. It is complete if and only if bounded, π-adically complete V-submodules are cofinal in $\mathcal{B}$.*

Proof. We may choose S in Definition 2.6 as a bounded V-submodule and assume that $x_0 \in S$. Then $x = (x - x_0) + x_0 \in \delta_0 \cdot S + S \subseteq S$ and $x_n = (x_n - x) + x \in \delta_n \cdot S + S \subseteq S$ for all $n \in \mathbb{N}$. A sequence (x_n) in S S-converges to $x \in S$ if and only if it converges in the π-adic topology on S. If a sequence has two limit points $x \neq y$, then we may find a single bounded V-module S so that it S-converges towards both x and y. Hence S is not π-adically separated. Conversely, if there is such an S, then there is a π-adically convergent sequence in S with two limit points. Hence the bornological V-module X is not separated. Since submodules of π-adically separated V-modules remain separated, all bounded V-submodules are π-adically separated once this happens for a cofinal set of bounded V-submodules.

If a sequence (x_n) is Cauchy, then there is a bounded V-submodule S such that $x_n \in S$ for all $n \in \mathbb{N}$ and the sequence is S-Cauchy in the standard quasi-metric defining the π-adic topology on S. In particular, if S is π-adically complete, then any S-Cauchy sequence is S-convergent. Thus X is complete if bounded, π-adically complete V-submodules are cofinal in $\mathcal{B}$. Conversely, assume that X is complete and let $S \in \mathcal{B}$. Then S is contained in a bounded V-submodule S' by convexity. Choose a bounded V-submodule $S'' \subseteq X$ so that S'-Cauchy sequences are S''-convergent. Any point in the π-adic completion $\widehat{S'}$ of S' is the limit of a Cauchy sequence in S'. Such sequences are S'-Cauchy and hence S''-convergent. Writing a point in $\widehat{S'}$ as a limit of a Cauchy sequence

and taking its limit in S'' defines a bounded V-module map $\widehat{S}' \to S''$. The kernel of this map is π-adically closed because X is separated. Hence its image is a quotient of a π-adically complete V-module by a closed submodule; such quotients remain π-adically complete. Hence the image of $\widehat{S}'$ is a bounded, π-adically complete V-submodule of X containing S. This proves that such submodules are cofinal in $\mathcal{B}$. $\qquad\square$

Remark 2.9. Let X be torsion-free. Then $y = \lambda x$ for $x \in X$ and $\lambda \in V$ with $\lambda \ne 0$ determines x. Thus $\lambda^{-1} y = x$ is well-defined. Then X is not separated if and only if X contains a bounded K-vector subspace. If $0 \ne x \in \bigcap \pi^l S$ for some bounded V-submodule S, then the V-submodule generated by $\pi^{-n} x$ for $n \in \mathbb{N}$ is a bounded K-vector subspace in X.

Proposition 2.10. *The category* $\overline{\mathfrak{Bor}}(V)$ *of complete bornological V-modules and bounded V-module maps is equivalent to the full subcategory of the category of inductive systems of π-adically complete V-modules consisting of all inductive systems* $(X_i, \varphi_{ij} : X_i \to X_j)$ *with injective maps* φ_{ij}. *A similar statement holds for separated bornological V-modules and inductive systems of π-adically separated V-modules.*

Proof. Let $\mathcal{B}$ be the family of bounded subsets of X. Proposition 2.8 shows that the π-adically complete or separated bounded V-modules of X are cofinal in $\mathcal{B}$ if and only if X is complete or separated, respectively. Now copy the proof of Proposition 2.5 with these cofinal subsets in the place of $\mathcal{B}'$. $\qquad\square$

Example 2.11. Let M be a V-module with a *seminorm* as in Example 2.4. If this is a norm, M becomes an ultra-metric space; when it is complete, we call M a *Banach V-module*. A *Banach K-vector space* is a Banach V-module where multiplication by π is bijective. Since $\|ax\| = |a| \|x\|$, every torsion element $x \in M$ has $\|x\| = 0$; thus normed modules are torsion-free or, equivalently, flat. If M is any torsion-free V-module, then the valuation $\nu : M \to \mathbb{N} \cup \{\infty\}$, $x \mapsto \sup\{n : x \in \pi^n M\}$, defines a seminorm $\|x\|_c = \epsilon^{\nu(x)}$, the *canonical* seminorm. Its associated topology is the π-adic topology. In particular, $\| \ \|_c$ is a norm if and only if M is flat and 0 is the only divisible submodule of M. If M is flat, then $(M, \| \ \|_c)$ is Banach if and only if M is π-adically complete. Thus Proposition 2.10 implies that any flat complete bornological V-module is a filtered union of Banach submodules with norm-decreasing inclusions. The norm $\| \ \|_c$ is always bounded above by 1. If M is π-adically separated, $\| \ \|$ is any other seminorm, and $x = \pi^n y$ with $y \notin \pi M$, then $\|x\| = \|x\|_c \|y\|$. Hence if also $\| \ \|$ is bounded, then the identity is bounded as a map $(M, \| \ \|_c) \to (M, \| \ \|)$. The identity is bounded as a map $(M, \| \ \|) \to (M, \| \ \|_c)$ if and only if $\| \ \|$ is bounded below on $M \smallsetminus \pi M$ if and only if πM is open in M.

We call a Banach module $(M, \| \ \|)$ *bornological* if $\| \ \|$ is equivalent to the canonical norm on each closed ball $B_\rho := \{x \in M : \|x\| \le \rho\}$ for $\rho > 0$. Equivalently, there is $\delta > 0$ so that $B_\delta \subset \pi \cdot M$ because then automatically $x \in \pi \cdot B_\rho$ if $\|x\| \le \min\{\delta, \epsilon \cdot \rho\}$. Thus any Banach K-vector space is bornological. The

V-module $M = \prod_{n \geq 0} \pi^n V$ with the supremum norm is a Banach V-module that is not bornological.

DEFINITION 2.12. A subset S in a bornological V-module is *bornologically closed* if all limits of convergent sequences in S again belong to S. The *bornological closure* of S is the smallest bornologically closed subset containing S. The *separated quotient* of a bornological V-module is the quotient by the bornological closure of $\{0\}$, equipped with the quotient bornology.

The bornological closure of S may not be easy to compute explicitly. We certainly have to add all limits of convergent sequences in S. There are usually more convergent sequences in this larger set. So we may have to repeat this step to arrive at a bornologically closed subset, compare [20, p. 14]. The notion of a closed subset defines a topology. The translations $x \mapsto x + y$ for a fixed element y and the maps $x \mapsto \lambda x$ for $\lambda \in V$ are then continuous and closed. This implies that the bornological closure of a V-submodule is again a V-submodule, compare [20, Corollaire 1 on p. 15]. A bornological V-module X is separated if and only if $\{0\}$ is bornologically closed. Thus the separated quotient $X/\overline{\{0\}}$ is indeed separated, and it is the largest quotient with this property: any bounded V-module map $X \to Y$ into a separated bornological V-module factors uniquely through $X/\overline{\{0\}}$.

LEMMA 2.13. *Let X be a complete bornological V-module. Any bornologically closed V-submodule $Y \subseteq X$, equipped with the subspace bornology, is complete.*

Proof. Let $S \subseteq Y$ be bounded. Since X is complete, Proposition 2.8 gives a π-adically complete, bounded V-submodule $T \subseteq X$ that contains S. Since Y carries the subspace bornology, $Y \cap T$ is bounded in Y. We claim that any S-Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ is $Y \cap T$-convergent. This claim implies the statement.

Since (x_n) is S-Cauchy, there is a null sequence $(\delta_k)_{k \in \mathbb{N}}$ in V with $x_n - x_m \in \delta_k \cdot S$ for all $n, m \in \mathbb{N}$ and $k \leq n, m$. We may arrange that the sequence $|\delta_k|$ is decreasing. Since $S \subseteq T$ and T is π-adically complete, the sequence (x_n) converges π-adically in T. Its limit x belongs to $Y \cap T$ because Y is bornologically closed in X. The Cauchy condition gives $y_n \in S$ with $x_n - x_{n+1} = \delta_n \cdot y_n$ for all $n \in \mathbb{N}$. Since $|\delta_n|$ is decreasing and converges to 0, the series

$$w_n := -\sum_{k=0}^{\infty} \frac{\delta_{n+k}}{\delta_n} y_{n+k}$$

is well defined and converges in T. The limit w_n belongs to $Y \cap T$ because Y is bornologically closed. We have

$$\delta_n w_n = -\lim_{N \to \infty} \sum_{k=0}^{N} \delta_{n+k} y_{n+k} = \lim_{N \to \infty} (x_n - x_{n+N+1}) = x_n - x.$$

This witnesses that (x_n) is $(Y \cap T)$-convergent to x. $\qquad\square$

DEFINITION 2.14. The *completion* of a bornological V-module X is a complete bornological V-module $\overline{X}$ with a bounded map $X \to \overline{X}$ that is universal in the sense that any map from X to a complete bornological V-module factors uniquely through it.

In order to construct completions, we first discuss inductive limits in the categories of separated and complete bornological V-modules. The category of bornological V-modules is complete and cocomplete, that is, any diagram has both a limit and a colimit. They are constructed by equipping the limit or colimit in the category of V-modules with a canonical bornology. Any limit may be described as a submodule of a product; hence limits of separated bornological V-modules remain separated. Thus the category of separated bornological V-modules has all limits, and these are just the same as in the larger category of bornological V-modules. In contrast, a colimit may be described as a quotient of a direct sum, and such a quotient need not remain separated. To make it so, we take the separated quotient. This has exactly the right universal property for a colimit in the category of separated bornological V-modules.

For a diagram of complete bornological V-modules, the limit in the category of bornological V-modules is easily seen to be complete again because it is the kernel of a certain map between products. Products inherit completeness from their factors, and the kernel of a bounded linear map between separated bornological V-modules is bornologically closed. Hence limits also inherit completeness. Quotients of complete bornological V-modules by bornologically closed V-submodules remain complete. Hence the separated quotient of the usual colimit is a complete bornological V-module and so has the right universal property for the colimit in the category of complete bornological V-modules.

PROPOSITION 2.15. *Completions of bornological V-modules always exist and may be constructed as follows. Write $X = \varinjlim (X_i)_{i \in I}$ as the inductive limit of the directed set of its bounded V-submodules. Then $\overline{X}$ is the separated quotient of the bornological inductive limit $\varinjlim (\widehat{X_i})_{i \in I}$. The completion functor commutes with colimits, that is, the completion of a colimit of a diagram of bornological V-modules is the* separated quotient *of the colimit of the diagram of completions.*

Proof. Let Y be a complete bornological V-module. Then $Y \cong \varinjlim (Y_j)_{j \in J}$ with π-adically complete V-modules Y_j by Proposition 2.10. Hence

$$\operatorname{Hom}(X, Y) \cong \varprojlim_i \varinjlim_j \operatorname{Hom}(X_i, Y_j) \cong \varprojlim_i \varinjlim_j \operatorname{Hom}(\widehat{X_i}, Y_j)$$

$$\cong \varprojlim_i \operatorname{Hom}(\widehat{X_i}, Y) \cong \operatorname{Hom}\left(\varinjlim_i \widehat{X_i}, Y\right) \cong \operatorname{Hom}\left(\varinjlim_i \widehat{X_i} \,\big/\, \overline{\{0\}}, Y\right).$$

Thus $\varinjlim_i \widehat{X_i} \big/ \overline{\{0\}}$, the separated quotient of the inductive limit of $(\widehat{X_i})_{i \in I}$, has the universal property that defines the completion. The completion functor is

defined as a left adjoint to the inclusion functor and therefore commutes with colimits, where the colimit in the category of complete bornological V-modules is the separated quotient of the usual colimit. $\qquad\square$

The separated quotient in Proposition 2.15 is hard to control. It may be needed because the maps $\widehat{X_i} \to \widehat{X_j}$ for $i \le j$ need not be injective, although the maps $X_i \to X_j$ are injective by construction:

EXAMPLE 2.16. Let $V = \mathbb{Z}_p$, $K = \mathbb{Q}_p$ for some prime p. Let $X_0 = \mathbb{Z}_p[t]$ and let X_1 be the $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p[t]$ generated by X_0 and by the polynomials $f_m = p^{-m}(1 + pt + (pt)^2 + \cdots + (pt)^{m-1})$ for $m \in \mathbb{N}$. Let $\widehat{X_0}$ and $\widehat{X_1}$ be the π-adic completions of the V-modules X_0 and X_1, respectively. The series $\sum_{i=0}^{\infty}(pt)^i$ converges towards a non-zero point in $\widehat{X_0}$ that is mapped to 0 in $\widehat{X_1}$ because

$$\sum_{i=0}^{n}(pt)^i = p^m f_m + \sum_{i=m+1}^{n}(pt)^i \in p^m X_1$$

for all $n \ge m$ and all $m \in \mathbb{N}$. Thus the map $\widehat{X_0} \to \widehat{X_1}$ induced by the inclusion $X_0 \to X_1$ is not injective.

Hence the transition maps in the inductive system $(\widehat{X_i})_{i \in I}$ in Proposition 2.15 need not be injective. This is why separated quotients may really be needed in Proposition 2.15. There are examples of separated bornological vector spaces over $\mathbb{R}$ for which the completion is zero. In particular, the canonical map $X \to \overline{X}$ may fail to be injective. We expect such examples also over V, but we have not tried hard to find one.

There is a canonical map $\overline{X} \to \widehat{X}$ by Proposition 2.15. This map may fail to be injective and surjective: with the fine bornology, any V-module X is bornologically complete so that $\overline{X} = X$ need not be π-adically separated or π-adically complete.

Next we consider tensor products, following the recipe for $\mathbb{R}$-vector spaces in [19]. Undecorated tensor products are taken over V. Let S and T be V-modules. Then $S \otimes T$ is the target of the universal V-bilinear map on $S \times T$: any V-bilinear map $S \times T \to U$ for a V-module U extends uniquely to a V-bilinear map $S \otimes T \to U$. The complete bornological tensor product of two V-modules is defined by an analogous universal property with respect to bounded bilinear maps. A bilinear map $b\colon S \times T \to U$ between bornological modules is *bounded* if $b(M \times N)$ is bounded whenever M and N are bounded in S and T.

DEFINITION 2.17. First let X and Y be bornological V-modules. Their *bornological tensor product* is a bornological V-module $X \otimes Y$ with a bounded V-bilinear map $b\colon X \times Y \to X \otimes Y$ that is universal in the sense that any bounded V-bilinear map $X \times Y \to W$ to a bornological V-module W factors uniquely through b. Secondly, let X and Y be complete bornological V-modules. Their *complete bornological tensor product* is a complete bornological V-module $X \overline{\otimes} Y$ with a bounded V-bilinear map $b\colon X \times Y \to X \overline{\otimes} Y$ that is universal in the sense

that any bounded V-bilinear map $X \times Y \to W$ to a complete bornological V-module W factors uniquely through b.

Such tensor products exist and are described more concretely in the following lemma; it also justifies using the same notation $\otimes$ both for the V-module tensor product and the bornological V-module tensor product.

LEMMA 2.18. *Let X and Y be bornological V-modules. Then their bornological tensor product $X \otimes Y$ is the usual V-module tensor product equipped with the bornology that is generated by the images of $S \otimes T$ in $X \otimes Y$, where S and T run through the bounded V-submodules in X and Y, respectively.*
Let X and Y be complete bornological V-modules. Then their complete bornological tensor product $X \overline{\otimes} Y$ is the bornological completion of the bornological tensor product $X \otimes Y$. Write $X \cong \varinjlim(X_i, \varphi_{ij})$ and $Y \cong \varinjlim(Y_l, \psi_{lm})$ for inductive systems of complete V-modules with injective maps φ_{ij} and ψ_{lm} as in Proposition 2.10. *Then $X \otimes Y$ is the colimit of the inductive system of bounded V-modules $(X_i \otimes Y_l, \varphi_{ij} \otimes \psi_{lm})$. The complete bornological tensor product is the separated quotient of the colimit of the inductive system of bounded V-modules $(X_i \widehat{\otimes} Y_l, \varphi_{ij} \widehat{\otimes} \psi_{lm})$, where $X_i \widehat{\otimes} Y_l$ for two V-modules denotes the π-adic completion of their tensor product.*

Proof. Let Z be a bornological V-module. A bilinear map $b \colon X \times Y \to Z$ induces a linear map $b_* \colon X \otimes Y \to Z$. The map b is bounded if and only if $b(S, T) \subseteq Z$ is bounded for all bounded V-submodules $S \subseteq X$, $T \subseteq Y$. This is equivalent to $b_*(\iota(S \otimes T))$ being bounded for all such S, T, where $\iota(S \otimes T)$ denotes the image of $S \otimes T$ in $X \otimes Y$. Hence there is a natural bijection between bounded bilinear maps $X \times Y \to Z$ and bounded linear maps $X \otimes Y \to Z$, where we equip $X \otimes Y$ with the bornology generated by the V-submodules $\iota(S \otimes T)$ for all bounded V-submodules $S \subseteq X$, $T \subseteq Y$. Since the sets of bounded V-submodules $S \subseteq X$, $T \subseteq Y$ are directed, so is the resulting set of V-submodules $\iota(S \otimes T) \subseteq X \otimes Y$. Hence a subset of $X \otimes Y$ is bounded if and only if it is contained in $\iota(S \otimes T)$ for some bounded V-submodules $S \subseteq X$, $T \subseteq Y$.
The bornological completion $\overline{X \otimes Y}$ is a complete bornological V-module with the property that bounded linear maps $\overline{X \otimes Y} \to Z$ for complete bornological V-modules Z correspond bijectively to bounded linear maps $X \otimes Y \to Z$. Since the latter correspond bijectively to bounded bilinear maps $X \times Y \to Z$, the completion $\overline{X \otimes Y}$ has the correct universal property for a complete bornological tensor product.
If $X = \varinjlim X_i$, $Y = \varinjlim Y_l$, then a bounded V-bilinear map $b \colon X \times Y \to Z$ is equivalent to a family of bounded V-bilinear maps $b_{il} \colon X_i \times Y_l \to Z$ that are compatible in the usual sense, $(b_{il}) \in \varprojlim_{il} \mathrm{Hom}^{(2)}(X_i \times Y_l, Z)$, where $\mathrm{Hom}^{(2)}$ denotes the set of bounded bilinear maps. If Z is complete, then such a family is equivalent to a family of bounded V-linear maps $(b_{il})_* \colon X_i \widehat{\otimes} Y_l \to Z$ with the same compatibility, $(b_{il})_* \in \varprojlim_{il} \mathrm{Hom}(X_i \widehat{\otimes} Y_l, Z)$. Thus $X \overline{\otimes} Y$ has the same universal property as the colimit of the inductive system $X_i \widehat{\otimes} Y_l$ in the

category of complete bornological V-modules. This is the separated quotient of the colimit. $\square$

Example 2.19. The fraction field $K = V[\pi^{-1}]$ is $\varinjlim_{\mathbb{N}} V$, where the transfer maps are multiplication by π. Hence if M is a flat, complete bornological module and $\{M_\lambda : \lambda \in \Lambda\}$ is the direct system of its complete bounded submodules, then

$$M \otimes K = \varinjlim_{(\lambda, j) \in \Lambda \times \mathbb{N}} M_\lambda \otimes (\pi^{-j} V) = M \overline{\otimes} K.$$

More generally, if M is a flat, complete bornological V-module and N is any V-module with the fine bornology, then $M \overline{\otimes} N = M \otimes N$. This holds because $M_\lambda \otimes N_\nu$ is π-adically complete if $N_\nu \subseteq N$ is finitely generated and M_λ is π-adically complete.

If X and Y are bornological K-vector spaces, then $X \otimes Y = X \otimes_K Y$, and the complete bornological tensor product $X \overline{\otimes} Y$ is a bornological K-vector space as well; thus we may define $X \overline{\otimes}_K Y$ to be $X \overline{\otimes} Y$.

Routine arguments show that the category of complete bornological V-modules with the tensor product $\overline{\otimes}$ and with V as tensor unit is a symmetric monoidal category. Even more, this is a closed symmetric monoidal category; the internal hom functor is obtained by equipping the V-module $\mathrm{Hom}(X, Y)$ of bounded V-linear maps $X \to Y$ with the *equibounded bornology*, where a set S of maps $X \to Y$ is *equibounded* if for each bounded subset $T \subseteq X$ the set of $s(t)$ with $s \in S$, $t \in T$ is bounded in Y. The V-module $\mathrm{Hom}(X, Y)$ with this bornology is complete if Y is complete. A bounded linear map $X \to \mathrm{Hom}(Y, Z)$ is equivalent to a bounded bilinear map $X \times Y \to Z$, which is then equivalent to a bounded linear map $X \overline{\otimes} Y \to Z$ by the universal property of the tensor product. Thus the internal Hom functor has the correct universal property for a closed monoidal category. We shall only use one consequence, which can also be proved directly:

Proposition 2.20. *The tensor product functor on the category of bornological V-modules commutes with colimits in each variable. The completed tensor product functor on the category of complete bornological V-modules commutes with (separated) colimits in each variable.*

The adjective "separated" clarifies that a colimit in the category of complete bornological V-modules is the separated quotient of the usual colimit.

Proof. Any functor with a right adjoint commutes with colimits, so this follows from the existence of an internal Hom functor. $\square$

Definition 2.21. A bounded V-linear map $\varphi \colon X \to Y$ is called a *bornological quotient map* if any bounded subset $S \subseteq Y$ of Y is $\varphi(R)$ for some bounded subset R of X. Equivalently, the map $X/\ker(\varphi) \to Y$ induced by φ is a bornological isomorphism. Any bornological quotient map is surjective. An *extension* of bornological V-modules is a diagram of bornological V-modules

$$0 \longrightarrow X' \overset{f}{\longrightarrow} X \overset{g}{\longrightarrow} X'' \longrightarrow 0 \tag{3}$$

that is exact in the algebraic sense and such that $X' \subseteq X$ has the subspace bornology and g is a bornological quotient map. Equivalently, g is a cokernel for f and f is a kernel for g. A *split extension* is an extension with a bounded V-linear section.

LEMMA 2.22. *Given a complete bornological V-module Y and an extension* (3) *of complete bornological V-modules, we have $g \overline{\otimes} \mathrm{id}_Y = \operatorname{coker}(f \overline{\otimes} \mathrm{id}_Y)$; moreover, $g \overline{\otimes} \mathrm{id}_Y$ is a bornological quotient map and $\ker(g \overline{\otimes} \mathrm{id}_Y)$ is the bornological closure of the image of $f \overline{\otimes} \mathrm{id}_Y$ in $X \overline{\otimes} Y$. The functor $- \overline{\otimes} Y$ maps split extensions of complete bornological V-modules again to split extensions.*

Proof. Since $- \overline{\otimes} Y$ commutes with colimits by Proposition 2.20, it preserves cokernels and direct sums. The cokernel of $f \overline{\otimes} \mathrm{id}_Y$ in the category of complete bornological V-modules is the quotient of $X \overline{\otimes} Y$ by the bornological closure of the image of $f \overline{\otimes} \mathrm{id}_Y$. This gives the second statement. The last statement follows from the fact that $- \overline{\otimes} Y$ preserves direct sums: if the sequence (3) admits a bounded V-linear section, then $X \cong X' \oplus X''$. $\square$

LEMMA 2.23. *Taking completions preserves bornological quotient maps.*

Proof. Proposition 2.15 describes bornological completions through the π-adic completions of the bounded V-submodules. This reduces the assertion to the fact that π-adic completions preserve surjections. $\square$

3   BORNOLOGICAL ALGEBRAS

A *bornological algebra R* over V is an associative V-algebra and a bornological V-module such that the multiplication map is bounded. That is, if $S, T \subseteq R$ are bounded subsets, then so is $S \cdot T$. We shall consider only bornological algebras with unit. The unit map $V \to R$ is automatically bounded.

We first define the spectral radius of a bounded V-submodule in a bornological K-algebra. Based on this, we define the linear growth bornology on a bornological V-algebra and a 1-parameter family of bornologies on $\underline{R} \coloneqq R \otimes K$ for a V-algebra R and an ideal $J \lhd R$ with $\pi \in J$. These bornologies yield bornological completions of R and $\underline{R}$. If R is finitely generated and commutative, then we identify the completion of R for the linear growth bornology with the Monsky–Washnitzer completion of R. The completions of $\underline{R}$ for the bornologies associated to J interpolate between the Monsky–Washnitzer completions of the tube algebras (see Definition 3.1.19 and Proposition 3.1.21) of R with respect to the powers J^m of the ideal J.

3.1   SPECTRAL RADIUS ESTIMATES

The spectral radius of a bounded subset in a bornological algebra over $\mathbb{R}$ or $\mathbb{C}$ is defined in [25, Definition 3.4]. We need some notation to carry this definition over to algebras over the fraction field K of V. To some extent, the definition also works for algebras over V.

Let R be a K-algebra and let $M \subseteq R$ be a V-submodule. Fix $r \in \mathbb{R}_{>0}$. Recall that we write $\epsilon = |\pi|$. There is a smallest $j \in \mathbb{Z}$ with $\epsilon^j \leq r$, namely, $\lceil \log_\epsilon(r) \rceil$. We abbreviate

$$r \star M := \pi^{\lceil \log_\epsilon(r) \rceil} M.$$

Let $s \in \mathbb{R}_{\geq 0}$, $j \in \mathbb{N}$ and $N \subseteq R$ a V-submodule. We shall use the following elementary properties of the operation $\star$:

$$\begin{cases} \pi \left((rs) \star M \right) \subseteq r \star (s \star M) \subseteq (rs) \star M \\ (r\epsilon^j) \star M = r \star (\pi^j \cdot M) \\ (r \star M)(s \star N) = r \star (s \star MN) = s \star (r \star MN) \\ r \leq s \Rightarrow r \star M \subseteq s \star M. \end{cases} \tag{4}$$

Let $\sum_{n \in \mathbb{N}} r^{-n} \star M^n$ be the set of all finite sums of elements in $\bigcup_{n \in \mathbb{N}} r^{-n} \star M^n$. This is a V-submodule of R.

DEFINITION 3.1.1. Let R be a bornological K-algebra and let $M \subseteq R$ be a bounded V-submodule. The *spectral radius* $\varrho(M)$ is defined as the infimum of the set of all numbers $r \in \mathbb{R}_{>0}$ for which $\sum_{n \in \mathbb{N}} r^{-n} \star M^n$ is bounded. It is ∞ if no such r exists.

Let R be a V-algebra. If $r \geq 1$, then we may define $\sum_{n \in \mathbb{N}} r^{-n} \star M^n$ exactly as above. This suffices to give meaning to the assertion $\varrho(M) \leq \varrho$ if $\varrho \geq 1$. If R is torsion-free, we may define $\varrho(M) \in \mathbb{R}_{\geq 0}$ for a bounded V-submodule $M \subseteq R$ by viewing M as a subset of $\underline{R} := R \otimes K$ with the tensor product bornology, see Example 2.19.

LEMMA 3.1.2. *Let R be a bornological V-algebra and $M \subseteq R$ a bounded V-submodule. Let $j, c \in \mathbb{Z}$, $c \geq 1$. Then $\varrho(\pi^j M^c) = \epsilon^j \varrho(M)^c$.*

Proof. We first treat the case $j = 0$. Let

$$S = V \cdot 1_R + r^{-1/c} \star M + r^{-2/c} \star M^2 + \cdots + r^{-(c-1)/c} \star M^{c-1},$$

$$T = \sum_{i=0}^\infty r^{-i} \star (M^c)^i, \qquad U = \sum_{n=0}^\infty r^{-n/c} \star M^n.$$

Then $T \subseteq S \cdot T \subseteq U$ by (4). So T is bounded if U is bounded. Hence $\varrho(M^c) \leq \varrho(M)^c$. Conversely, (4) also gives $\pi \cdot U \subseteq S \cdot T$. Since S is bounded anyway, πU is bounded if T is bounded. If $s > r$ is arbitrary, then $s^{-n/c} < \epsilon r^{-n/c}$ for all but finitely many n. Hence $\sum_{n=0}^\infty s^{-n/c} \star M^n \subseteq \sum_{n=0}^N s^{-n/c} \star M^n + \pi \cdot U$ for some $N \in \mathbb{N}$, and the first summand is bounded anyway. So $\sum_{n=0}^\infty s^{-n/c} \star M^n$ is bounded for all $s > r$ if T is bounded. Hence $\varrho(M^c) \geq \varrho(M)^c$ as well. Finally, the assertion extends to $j \neq 0$ because, by (4), $\sum r^{-n} \star (\pi^j M^c)^n = \sum (r\epsilon^{-j})^{-n} \star (M^c)^n$. $\quad\square$

PROPOSITION 3.1.3. *Let $(R, \mathcal{B})$ be a bornological V-algebra. The following are equivalent:*

(1) $\varrho(M) \leq 1$ *for all $M \in \mathcal{B}$;*

(2) $\sum_{j=0}^{\infty} \pi^j M^{cj+d}$ is bounded for all $M \in \mathcal{B}$, $c, d \in \mathbb{N}$;

(3) $\sum_{j=0}^{\infty} \pi^j M^{j+1}$ is bounded for all $M \in \mathcal{B}$;

(4) any bounded subset of R is contained in a bounded V-submodule M with $\pi \cdot M^2 \subseteq M$.

Proof. If $\sum r^{-n} \star M^n$ is bounded for some $r \in \mathbb{R}_{\geq 0}$, then it is also bounded for all $r' \geq r$. Hence $\varrho(M) \leq 1$ if and only $\sum r^{-n} \star M^n$ is bounded for all r of the form $r = \epsilon^{-1/c}$ with $c \in \mathbb{Z}_{\geq 1}$. The proof of Lemma 3.1.2 shows that $\sum_{n \in \mathbb{N}} \epsilon^{n/c} \star M^n$ is bounded if and only if $\sum_{j=1}^{\infty} \pi^j M^{cj}$ is. Then $\sum_{j=1}^{\infty} \pi^j M^{cj} \cdot M^d = \sum_{j=1}^{\infty} \pi^j M^{cj+d}$ is bounded as well. So (1) and (2) are equivalent.

The implication (2)$\Rightarrow$(3) is trivial. We prove (3)$\Rightarrow$(4). Let S be bounded. It is contained in a bounded V-submodule M. Then $U := \sum_{j \geq 0} \pi^j M^{j+1}$ is bounded by (3). We have $S \subseteq M \subseteq U$ by construction, and $\pi \cdot U \cdot U \subseteq U$ because

$$\pi \cdot \sum_{i=0}^{\infty} \pi^i M^{i+1} \cdot \sum_{j=0}^{\infty} \pi^j M^{j+1} = \sum_{i,j=0}^{\infty} \pi^{i+j+1} M^{i+j+2} = \sum_{j=1}^{\infty} \pi^j M^{j+1} \subseteq U. \qquad (5)$$

Finally, we prove (4)$\Rightarrow$(2). Let $M \subseteq R$ and $c, d \in \mathbb{N}$ be as in (2). Condition (4) gives a bounded V-submodule $U \subseteq R$ with $M^c \cup M^d \subseteq U$ and $\pi \cdot U \cdot U \subseteq U$. Hence $\pi M^c \cdot U \subseteq \pi U \cdot U \subseteq U$. By induction, $\pi^i M^{ci} \cdot U \subseteq U$ for all $i \in \mathbb{N}$. Thus $\pi^i M^{ci+d} = \pi^i M^{ci} \cdot M^d \subseteq \pi^i M^{ci} U \subseteq U$ for all $i \in \mathbb{N}$. So $\sum_{i \geq 0} \pi^i M^{ci+d}$ is bounded. $\qquad \square$

Let R be a flat V-algebra and let $M \subseteq R$ be a V-submodule. Let $R_M \subseteq R$ be the subset of all $x \in R$ for which there is some $l \in \mathbb{N}$ with $\pi^l x \in M$. In other words, $R_M = R \cap \underline{M}$; this is the largest V-submodule of R whose elements are absorbed by M. There is a unique seminorm on R_M with unit ball M taking values in $|K|$, namely, the gauge seminorm $\| \ \|_M$ of (1). The submodule M satisfies $\pi^m M^2 \subseteq M$ if and only if $R_M \cdot R_M \subseteq R_M$ and

$$\|x \cdot y\|_M \leq \epsilon^{-m} \cdot \|x\|_M \|y\|_M$$

for all $x, y \in R_M$. From this and (2) we obtain that M is the unit ball of a seminormed V-subalgebra of R if and only if it satisfies $\pi^m M^2 \subseteq M$ for some $m \in \mathbb{N}$.

A complete bornological V-algebra R is a *Banach algebra* if there is a norm $\| \ \|$ for which the multiplication map is bounded and $(R, \| \ \|)$ is a bornological Banach module (in the sense of Example 2.11) whose closed balls generate the bornology. A complete bornological algebra is a *local Banach algebra* if it is a filtered union of bornological Banach subalgebras such that the inclusion maps are norm-decreasing (see [11, Definition 2.11]).

For example, a complete bornological V-algebra with $\varrho(M) \leq 1$ for all bounded V-submodules M is a local Banach algebra.

Remark 3.1.4. An argument similar to that of Proposition 3.1.3 shows that if $(R, \mathcal{B})$ is a bornological K-algebra, then $\varrho(M) < \infty$ for all $M \in \mathcal{B}$ if and only if the following condition holds:

$$\forall M \in \mathcal{B} \quad (\exists M \subseteq U \in \mathcal{B}, \ l \in \mathbb{Z}) \quad \pi^l U^2 \subseteq U.$$

It follows from this and from the arguments above that a complete bornological K-algebra is a local Banach algebra if and only if $\varrho(M) < \infty$ for all bounded V-submodules M (compare [25, Theorem 3.10]).

LEMMA 3.1.5. *If R is a complete bornological V-algebra and $M \subseteq R$ a bounded V-submodule with $\varrho(M) < \epsilon^{-1}$, then $1 - \pi z$ is invertible in R for all $z \in M$. If R is a complete bornological K-algebra and $M \subseteq R$ a bounded V-submodule with $\varrho(M) < 1$, then $1 - z$ is invertible in R for all $z \in M$.*

Proof. If $\varrho(M) < \epsilon^{-1}$, then there is r with $\epsilon < r < \varrho(M)^{-1}$. Then $\sum r^n \star M^n$ is bounded, and $\lim_{n\to\infty}(\pi z)^n = 0$ in the π-adic topology on $\sum r^n \star M^n$. So $\sum_{n=0}^{\infty}(\pi z)^n$ is a bornological Cauchy series. It converges because R is complete. The limit is an inverse for $1 - \pi z$. If R is a K-algebra and $\varrho(M) < 1$, then the argument above for $\pi^{-1}M$ gives the second statement by Lemma 3.1.2. $\square$

Now we define certain weak completions of a bornological algebra through spectral radius estimates.

DEFINITION 3.1.6. Let $(R, \mathcal{B})$ be a bornological V-algebra. The *linear growth bornology* on R is the smallest algebra bornology $\mathcal{B}_{\mathrm{lg}}$ on R that contains $\mathcal{B}$, such that all V-submodules $M \in \mathcal{B}_{\mathrm{lg}}$ satisfy $\varrho(M; \mathcal{B}_{\mathrm{lg}}) \le 1$. A subset of R has *linear growth* with respect to $\mathcal{B}$ if it is in $\mathcal{B}_{\mathrm{lg}}$. Let $\overline{R_{\mathrm{lg}}}$ be the bornological completion of $(R, \mathcal{B}_{\mathrm{lg}})$.

DEFINITION 3.1.7. Let $(R, \mathcal{B})$ be a bornological V-algebra, J an ideal in R and $\alpha \in [0,1]$. Let $\underline{R} := R \otimes K$ and $\underline{\mathcal{B}}_{J,\alpha}$ the smallest K-vector space bornology on $\underline{R}$ such that $\varrho(M; \underline{\mathcal{B}}_{J,\alpha}) \le \epsilon^\alpha$ whenever $M \subseteq J$ is a V-submodule with $M \in \mathcal{B}$; here we abusively denote by M also its image in $\underline{R}$. Let $\overline{\underline{R}_{J,\alpha}}$ be the completion of $(\underline{R}, \underline{\mathcal{B}}_{J,\alpha})$.
We wrote $\varrho(M; \mathcal{B}_{\mathrm{lg}})$ and $\varrho(M; \underline{\mathcal{B}}_{J,\alpha})$ here to emphasize the bornologies for which these spectral radii are computed.

Remark 3.1.8. If $\pi^k \in J$ for some $k \in \mathbb{N}$, then any K-vector space bornology on $\underline{R}$ that contains the $M \in \mathcal{B}$ with $M \subseteq J$ must contain all of $\mathcal{B}$. Thus we have $\underline{\mathcal{B}}_{J,\alpha} \supseteq \mathcal{B}$ in Definition 3.1.7.

Remark 3.1.9. The condition $\varrho(M) \le 1$ in Definition 3.1.6 is the strongest spectral radius constraint that makes sense in bornological V-algebras. Even in $\underline{R}$, asking for $\varrho(M) < 1$ is unreasonable if $1_R \in M$ because then Lemma 3.1.5 implies that 0 is invertible in the completion, forcing the completion to be $\{0\}$. If $M \subseteq J$ for an ideal J, however, then it makes sense to require $\varrho(M) \le \epsilon^\alpha$ with $\alpha \ge 0$ as in Definition 3.1.7. If $\pi \in J$, $\varrho(M) \le \epsilon^\alpha$ with $\alpha > 1$ would once more

imply the invertibility of $0 = 1 - \pi^{-1}\pi$ in the completion, which is unreasonable. This is why we restrict to $\alpha \leq 1$ and why the ideal $J = R$ is not a good choice unless $\alpha = 0$, when we get the bornology on $\underline{R}$ induced by the linear growth bornology on R: $\underline{\mathcal{B}}_{R,0} = \underline{\mathcal{B}}_{\mathrm{lg}}$.

LEMMA 3.1.10. *Let $(R, \mathcal{B})$ be a bornological V-algebra and $T \subseteq R$. The following are equivalent:*

(1) *T has linear growth;*

(2) *T is contained in $\sum_{i=0}^{\infty} \pi^i S^{i+1}$ for some $S \in \mathcal{B}$;*

(3) *T is contained in $\sum_{i=0}^{\infty} \pi^i S^{ci+d}$ for some $S \in \mathcal{B}$, $c, d \in \mathbb{N}$.*

In particular, $\mathcal{B}_{\mathrm{lg}}$ is also the smallest algebra bornology containing $\mathcal{B}$ such that all V-submodules $M \in \mathcal{B}$ satisfy $\varrho(M; \mathcal{B}_{\mathrm{lg}}) \leq 1$.

Proof. The implication (2)$\Rightarrow$(3) is trivial, and (3)$\Rightarrow$(1) follows from (2)$\Rightarrow$(1) in Proposition 3.1.3. The subsets of R as in (2) form a bornology $\mathcal{B}'$ because $\sum \pi^i S_1^{i+1} + \sum \pi^i S_2^{i+1} \subseteq \sum \pi^i (S_1 + S_2)^{i+1}$ and $S_1 + S_2$ is bounded if S_1, S_2 are. We claim that $S_1 \cdot S_2$ is in $\mathcal{B}'$ as well, that is, $\mathcal{B}'$ makes R a bornological algebra. There are bounded V-submodules $U_1, U_2 \subseteq R$ with $S_j \subseteq \sum \pi^i U_j^{i+1}$ for $j = 1, 2$. A computation as in (5) gives $S_1 \cdot S_2 \subseteq \sum \pi^i (U_1 + U_2)^{i+2}$, so $S_1 \cdot S_2 \in \mathcal{B}'$. The bornology $\mathcal{B}'$ contains $\mathcal{B}$. Equation (5) shows that any subset of the form $U \coloneqq \sum_{i=0}^{\infty} \pi^i S^{i+1}$ satisfies $\pi U^2 \subseteq U$. Hence $\varrho(M; \mathcal{B}') \leq 1$ for all $M \in \mathcal{B}'$ by the equivalence of (1) and (4) in Proposition 3.1.3. Since the linear growth bornology is the smallest algebra bornology containing $\mathcal{B}$ with this property, (1)$\Rightarrow$(2). $\qquad\square$

LEMMA 3.1.11. *Let $(R, \mathcal{B})$ be a bornological algebra, $J \triangleleft R$ an ideal with $\pi^k \in J$ for some $k \in \mathbb{N}$, $\alpha \in [0, 1]$, and $S \subseteq \underline{R}$. We have $S \in \underline{\mathcal{B}}_{J,\alpha}$ if and only if there are $l \in \mathbb{N}$, $r > \epsilon^\alpha$, and a bounded V-submodule $M \subseteq J$ such that*

$$ S \subseteq \pi^{-l} \sum_{n=1}^{\infty} r^{-n} \star M^n. $$

Proof. Let $\mathcal{B}'$ be the family of subsets described in the statement of the lemma. This is a K-vector space bornology on $\underline{R}$. Notice that $\mathcal{B}' \supseteq \mathcal{B}$ by Remark 3.1.8. Let $\mathcal{B}''$ be some K-vector space bornology on $\underline{R}$ and let $M \subseteq J$ be a bounded V-submodule. Then $\varrho(M; \mathcal{B}'') \leq \epsilon^\alpha$ if and only if $\sum r^{-n} \star M^n \in \mathcal{B}''$ for all $r > \epsilon^\alpha$, if and only if $\pi^{-l} \sum r^{-n} \star M^n \in \mathcal{B}''$ for all $r > \epsilon^\alpha$, $l \in \mathbb{N}$. Thus $\mathcal{B}'$ is the smallest K-vector space bornology on $\underline{R}$ with $\varrho(M) \leq \epsilon^\alpha$ for all bounded V-submodules $M \subseteq J$. That is, $\mathcal{B}' = \underline{\mathcal{B}}_{J,\alpha}$. $\qquad\square$

LEMMA 3.1.12. *Let R be a bornological V-algebra. The multiplication on R is bounded as a map $R_{\mathrm{lg}} \times R_{\mathrm{lg}} \to R_{\mathrm{lg}}$ and extends uniquely to a bornological algebra structure on $\overline{R_{\mathrm{lg}}}$. We have $\varrho(M) \leq 1$ for all bounded V-submodules $M \subseteq \overline{R_{\mathrm{lg}}}$ and so $\overline{R_{\mathrm{lg}}}$ is a local Banach algebra.*

Proof. The multiplication $R_{\mathrm{lg}} \times R_{\mathrm{lg}} \to R_{\mathrm{lg}}$ is bounded by our definition of the linear growth bornology. Like any bounded bilinear map, it extends uniquely to the completions. It remains associative and unital on $\overline{R_{\mathrm{lg}}}$ by the universal property of completions. So $\overline{R_{\mathrm{lg}}}$ is a complete bornological V-algebra. Any bounded V-submodule M of $\overline{R_{\mathrm{lg}}}$ is contained in the image of the π-adic completion $\widehat{U}$ of U in $\overline{R_{\mathrm{lg}}}$ for some V-submodule U of R of linear growth. Since $\varrho(U; R_{\mathrm{lg}}) \leq 1$, there is a V-submodule $U' \subseteq R$ of linear growth with $U \subseteq U'$ and $\pi \widehat{U'} \cdot U' \subseteq U'$. The image of $\widehat{U'}$ in the completion satisfies $\pi \widehat{U'} \cdot \widehat{U'} \subseteq \widehat{U'}$. So $\varrho(M) \leq 1$ by Proposition 3.1.3. $\qquad\square$

PROPOSITION 3.1.13. *Let R be a bornological V-algebra, $J \lhd R$ with $\pi^k \in J$ for some $k \in \mathbb{N}$, and $\alpha \in [0,1]$. The multiplication on $\underline{R}$ is bounded for the bornology $\underline{\mathcal{B}}_{J,\alpha}$, and $\overline{\underline{R}_{J,\alpha}}$ is a local Banach K-algebra.*

Proof. Products of subsets of the form $\pi^{-l} \sum_{n=1}^{\infty} r^{-n} \star M^n$ as in Lemma 3.1.11 are again contained in a subset of this form by (4). Thus the multiplication on $\underline{R}$ is bounded for the bornology $\underline{\mathcal{B}}_{J,\alpha}$. Hence it extends to $\overline{\underline{R}_{J,\alpha}}$. Any bounded subset of $\overline{\underline{R}_{J,\alpha}}$ is contained in the image of the π-adic completion of $U := \pi^{-l} \sum_{n=1}^{\infty} r^{-n} \star M^n$ for some $l \in \mathbb{N}$ and some bounded V-submodule $M \subseteq J$. We have $U \cdot U \subseteq \pi^{-l} U$. Hence $\widehat{U} \cdot \widehat{U} \subseteq \pi^{-l} \widehat{U}$. So the gauge seminorm associated with $\pi^l \widehat{U}$ is submultiplicative. Thus $\overline{\underline{R}_{J,\alpha}}$ is a local Banach K-algebra, see Remark 3.1.4. $\qquad\square$

EXAMPLE 3.1.14. Let $J = \pi R$ and assume that any bounded $M \subseteq J$ is contained in πM_0 for a bounded V-submodule $M_0 \subseteq R$. Let $\alpha \in [0,1]$. The condition $\varrho(M) \leq \epsilon^\alpha$ for bounded $M \subseteq J$ is equivalent to $\varrho(M_0) \leq \epsilon^{\alpha-1}$ for bounded $M_0 \subseteq R$ by Lemma 3.1.2. This condition for all M_0 implies $\varrho(M_0) \leq \epsilon^{(\alpha-1)/c}$ for all $c \in \mathbb{N}_{\geq 1}$ because $\varrho(M_0)^c = \varrho(M_0^c) \leq \epsilon^{\alpha-1}$ as well. So $\varrho(M_0) \leq 1$. Hence $\underline{\mathcal{B}}_{\pi R, \alpha} = \underline{\mathcal{B}}_{\mathrm{lg}}$ and $\overline{\underline{R}_{\pi R, \alpha}} = \overline{R_{\mathrm{lg}}} = K \otimes \overline{R_{\mathrm{lg}}}$.

The following two universal properties are immediate from Lemma 3.1.10 and from the definition of the respective bornologies.

PROPOSITION 3.1.15. *Let R and S be bornological V-algebras. Assume that S is complete and that $\varrho(M) \leq 1$ for all bounded V-submodules M in S. Any bounded homomorphism from R to S extends uniquely to a bounded homomorphism $\overline{R_{\mathrm{lg}}} \to S$. If $\varrho(M) \leq 1$ for all bounded V-submodules M in R, then $R = R_{\mathrm{lg}}$.*

PROPOSITION 3.1.16. *Let R be a bornological V-algebra, $J \lhd R$ with $\pi^k \in J$ for some $k \in \mathbb{N}$, and $\alpha \in [0,1]$. Let S be a complete bornological K-algebra. A bounded V-algebra homomorphism $\varphi \colon R \to S$ extends to a bounded K-algebra homomorphism $\overline{\underline{R}_{J,\alpha}} \to S$ if and only if $\varrho(\varphi(M)) \leq \epsilon^\alpha$ for all bounded V-submodules $M \subseteq J$.*

PROPOSITION 3.1.17. *A bounded unital algebra homomorphism $\varphi \colon R \to S$ induces a bounded unital algebra homomorphism $\overline{\varphi_{\mathrm{lg}}} \colon \overline{R_{\mathrm{lg}}} \to \overline{S_{\mathrm{lg}}}$. If φ is a bornological quotient map, then so is $\overline{\varphi_{\mathrm{lg}}}$.*

Proof. The first assertion follows from Proposition 3.1.15. If φ is a bornological quotient map, then so is $\varphi_{\mathrm{lg}} \colon R_{\mathrm{lg}} \to S_{\mathrm{lg}}$ by the concrete description of the linear growth bornology in Lemma 3.1.10. Hence so is $\overline{\varphi_{\mathrm{lg}}}$ because taking completions preserves bornological quotient maps by Lemma 2.23.   □

PROPOSITION 3.1.18. *Let R and S be bornological V-algebras. Let $I \triangleleft R$ and $J \triangleleft S$ be ideals containing π^k for some $k \in \mathbb{N}$. Let $\alpha, \beta \in [0,1]$ satisfy $\alpha \leq \beta$. A bounded unital algebra homomorphism $\varphi \colon R \to S$ with $\varphi(I) \subseteq J$ extends uniquely to a bounded unital algebra homomorphism $\overline{R_{I,\alpha}} \to \overline{S_{J,\beta}}$. This extension is a bornological quotient map if $\varphi|_I \colon I \to J$ is a bornological quotient map and $\alpha = \beta$.*

Proof. If $M \subseteq I$ is bounded, then $\varphi(M)$ is bounded and contained in J. Thus $\varrho(\varphi(M)) \leq \epsilon^\beta \leq \epsilon^\alpha$ in $\overline{S_{J,\beta}}$. This verifies the criterion in Proposition 3.1.16 for φ to extend uniquely to $\overline{R_{I,\alpha}}$. If $\varphi|_I$ is a bornological quotient map, then for any bounded V-submodule $M \subseteq J$ there is a bounded V-submodule $N \subseteq I$ with $\varphi(N) = M$. Hence the map $\underline{R} \to \underline{S}$ induced by φ maps $\pi^{-l} \sum_{n=1}^\infty r^{-n} \star N^n$ onto $\pi^{-l} \sum_{n=1}^\infty r^{-n} \star M^n$. Thus it is a bornological quotient map from $(\underline{R}, \underline{\mathcal{B}}_{I,\alpha})$ to $(\underline{S}, \underline{\mathcal{B}}_{J,\alpha})$ by Lemma 3.1.11. By Lemma 2.23, being a bornological quotient map is preserved by the bornological completion.   □

Next we are going to rewrite the completions $\overline{R_{J,\alpha}}$ using linear growth completions of tube algebras.

DEFINITION 3.1.19. Let J be an ideal in R and let $\alpha \in [0,1]$. The α-*tube algebra* of R around J is

$$\mathcal{T}_\alpha(R, J) \coloneqq \sum_{n=0}^\infty \epsilon^{-\alpha n} \star J^n \subseteq \underline{R}, \tag{6}$$

where the 0th summand is $J^0 \coloneqq R$. This is a V-algebra. We equip it with the bornology generated by the bounded submodules of $J^0 = R$ and by V-submodules of the form $\epsilon^{-\alpha n} \star M^n$ for $n \in \mathbb{N}_{\geq 1}$ and a bounded V-submodule $M \subseteq J$.

Almost by definition, a subset of $\mathcal{T}_\alpha(R, J)$ is bounded if and only if it is contained in $M_0 + \sum_{n=1}^N \epsilon^{-\alpha n} \star M^n$ for some $N \in \mathbb{N}$ and some bounded V-submodules $M_0 \subseteq R$, $M \subseteq J$. The multiplication on $\mathcal{T}_\alpha(R, J)$ is bounded by (4). So it is a bornological V-algebra. The inclusion map $\mathcal{T}_\alpha(R, J) \to \underline{R}$ extends to an isomorphism $\mathcal{T}_\alpha(R, J) \otimes K \cong \underline{R}$ of bornological K-algebras.

LEMMA 3.1.20. *If R carries the fine bornology, then so does $\mathcal{T}_\alpha(R, J)$.*

Proof. If M_0 and M are finitely generated, then so is $M_0 + \sum_{n=1}^N \epsilon^{-\alpha n} \star M^n$.   □

Proposition 3.1.21. *Let $J \lhd R$ be an ideal with $\pi \in J$ and let $\alpha \in [0,1]$. There is an isomorphism of bornological K-algebras*

$$\mathcal{T}_\alpha(R,J)_{\lg} \otimes K \cong (\underline{R}, \underline{\mathcal{B}}_{J,\alpha}).$$

It induces an isomorphism of the completions $\overline{\mathcal{T}_\alpha(R,J)_{\lg}} \otimes K \cong \overline{\underline{R}_{J,\alpha}}$.

Proof. By Lemma 3.1.10, the bornology on $\mathcal{T}_\alpha(R,J)_{\lg} \otimes K$ is the smallest K-vector space bornology where all bounded subsets of $\mathcal{T}_\alpha(R,J)$ have spectral radius at most 1. Equivalently, if $r < 1$, $N \in \mathbb{N}$ and $M_0 \subseteq R$, $M \subseteq J$ are bounded V-submodules, then

$$S_{r,N,M_0,M} := \sum_{n=1}^\infty r^n \star \left(M_0 + \sum_{i=1}^N \epsilon^{-\alpha i} \star M^i \right)^n$$

is bounded. This implies that $\epsilon^{-\lfloor \alpha n \rfloor} \varrho(M)^n = \varrho(\epsilon^{-\alpha n} \star M^n) \le 1$ if $n \in \mathbb{N}_{\ge 1}$ and $M \subseteq J$ is bounded. This is equivalent to $\varrho(M) \le \epsilon^\alpha$ for all bounded $M \subseteq J$ by Lemma 3.1.2. Therefore, all subsets in $\underline{\mathcal{B}}_{J,\alpha}$ are bounded in $\mathcal{T}_\alpha(R,J)_{\lg} \otimes K$.

It remains to prove that for r, N, M_0 and M as above, $S := S_{r,N,M_0,M} \in \underline{\mathcal{B}}_{J,\alpha}$. We may and do assume that $M_0 \ni 1$. Let $\ell \in \mathbb{N}_{\ge 1}$. Consider the following V-submodule of R

$$M' := M_0^{\ell-1} M M_0^{\ell-1} + \pi M_0^\ell$$

Since $\pi \in J$ and $M \subseteq J$ and J is an ideal and $1 \in M_0$, M' is a bounded V-submodule in J which contains $M_0^i M M_0^j$ for all $0 \le i, j \le \ell - 1$. Hence $\varrho(M') \le \epsilon^\alpha$ in the bornology $\underline{\mathcal{B}}_{J,\alpha}$. Thus

$$T_{s,M'} := \sum_{n=1}^\infty s^n \epsilon^{-\alpha n} \star (M')^n$$

is in $\underline{\mathcal{B}}_{J,\alpha}$ for all $s < 1$. We shall show that $S \subseteq T_{s,M'}$ if

$$r^{\ell-1} < \epsilon^{1-\alpha}, \quad 1 > s \ge \max\{r^{\ell-1}/\epsilon^{1-\alpha}, r^{1/N}\}. \tag{7}$$

We may choose ℓ so that this holds.

An element of S is a V-linear combination of products $\pi^h x_1 \cdots x_n$ with $x_1, \ldots, x_n \in M_0 \cup \bigcup_{i=1}^N M^i$; here h is the sum of $\lceil n \cdot \log(r)/\log \epsilon \rceil$ and one summand $-\lfloor \alpha i \rfloor$ for each factor $x_j \in M^i$. In most cases, we may group these factors x_i together so that $x_1 \cdots x_n = x'_1 \cdots x'_{n'}$, where each x'_j consists either of ℓ factors in M_0 or of one factor in M^i surrounded by at most $\ell-1$ factors in M_0 on each side. The only exception is the case where all factors x_i are from M_0; then we allow x'_1 to be a product of less than ℓ factors in M_0. By construction, each factor x'_j belongs to $M_0^\ell \subseteq \pi^{-1} M'$ or to $(M')^i$. For factors x'_j in $M_0^\ell \subseteq \pi^{-1} M'$, we put π^{-1} into the scalar factor π^h; and we split factors x'_j in $(M')^i$ into i factors in M'. This gives $\pi^h x_1 \cdots x_n = \pi^{h-b} x''_1 \cdots x''_{n''}$ with $x''_j \in M'$ for all j, where b is the number of factors x'_j in M_0^ℓ. We claim that $|\pi^{h-b}| = \epsilon^{h-b} \le s^{n''} \epsilon^{-\alpha n''}$, so that $\pi^h x_1 \cdots x_n \in \sum_{m=1}^\infty s^m \epsilon^{-\alpha m} \star (M')^m$ as desired.

We have $n'' - b = \sum_{x_j \in M^i} i$. Hence

$$
\begin{aligned}
h - b &= \lceil n \cdot \log(r)/\log \epsilon \rceil - \sum_{x_j \in M^i} \lfloor \alpha i \rfloor - b \\
&\geq n \cdot \log(r)/\log \epsilon - b - \alpha \sum_{x_j \in M^i} i = n \cdot \log(r)/\log \epsilon - b - \alpha(n'' - b).
\end{aligned}
$$

Each of the b factors $x'_j \in M_0^\ell$ comes from ℓ consecutive factors $x_i \in M_0$, and each factor $x'_j \in (M')^i$ contains at least one factor $x_m \in M^i$. Hence $n' - b \leq n - \ell b$. The $n' - b$ factors $x'_j \in (M')^i$ each produce $i \leq N$ factors in M'. Therefore, $n'' - b \leq N \cdot (n' - b) \leq N \cdot (n - \ell b)$ or, equivalently, $n \geq (\ell - N^{-1}) \cdot b + n''/N$. Using this and $s \geq r^{1/N}$, we estimate

$$
\begin{aligned}
h - b &\geq (\ell - N^{-1}) b \log(r)/\log \epsilon + n'' \log(r^{1/N})/\log \epsilon - b(1 - \alpha) - \alpha n'' \\
&\geq -\alpha n'' + n'' \log(s)/\log \epsilon + (\ell - N^{-1}) b \log(r)/\log \epsilon - b(1 - \alpha).
\end{aligned}
$$

The first two summands are exactly what we need for our estimate. And the sum of the other two is non-negative because $r^{\ell - N^{-1}} < \epsilon^{1-\alpha}$ by (7). The exceptional summand where all factors are in M_0 belongs to $r^n M_0^n \subseteq r^n \pi^{-\lceil n/\ell \rceil} \star (M')^{\lceil n/\ell \rceil}$. Since $n \geq \ell \lceil n/\ell \rceil - 1$, this is contained in $r^{(\ell-1)m} \pi^{-m} \star (M')^m \subseteq \epsilon^{-\alpha m} \star (M')^m$ by (7) with $m = \lceil n/\ell \rceil$. □

EXAMPLE 3.1.22. Let $\alpha = 1$. Then $\mathcal{T}_1(R, J) = \sum_{n=0}^{\infty} \pi^{-n} J^n$. If $\pi \in J$, we have $R \subseteq \pi^{-1} J \subseteq \pi^{-2} J^2 \subseteq \cdots$. So $\mathcal{T}_1(R, J)$ is the union of the increasing chain of V-submodules $\pi^{-n} J^n \subseteq \underline{R}$. A subset of $\mathcal{T}_1(R, J)$ is bounded if and only if it is contained in $\pi^{-n} M^n$ for some bounded V-submodule $M \subseteq J$. In this case, the proof of Proposition 3.1.21 is much easier.

The case $\alpha = 1/m$ for some $m \in \mathbb{N}_{\geq 1}$ is also somewhat easier, at least under a mild hypothesis. We have the following identity of rings:

$$
\mathcal{T}_{1/m}(R, J) = \sum \epsilon^{n/m} \star J^n = \sum \pi^{-n} J^{mn} = \mathcal{T}_1(R, J^m).
$$

The two bornologies are the same as well if we assume that every bounded subset of J^m is contained in M^m for some bounded submodule $M \subseteq J$; in other words, if the map $J^{\otimes m} \to J^m$ is a bornological quotient map for all m. This hypothesis is satisfied, for example, when R carries the fine bornology. Let $\alpha > 0$ be rational, $\alpha = i/m$ with $i, m \in \mathbb{N}_{\geq 1}$, $\gcd(i, m) = 1$. Then

$$
\begin{aligned}
\mathcal{T}_{i/m}(R, J) &= \sum_{\ell=0}^{\infty} \pi^{-\ell} J^{\lceil \ell m/i \rceil} \\
&= \left(R + \pi^{-1} J^{\lceil m/i \rceil} + \pi^{-2} J^{\lceil 2m/i \rceil} + \cdots + \pi^{-(i-1)} J^{\lceil (i-1)m/i \rceil} \right) \cdot \sum_{\ell=0}^{\infty} (\pi^{-i} J^m)^\ell \\
&= \sum_{\ell=0}^{\infty} \left(R + \pi^{-1} J^{\lceil m/i \rceil} + \pi^{-2} J^{\lceil 2m/i \rceil} + \cdots + \pi^{-i} J^{\lceil im/i \rceil} \right)^\ell.
\end{aligned}
$$

If $\alpha = 0$, then $\mathcal{T}_0(R, J) = R$.

Remark 3.1.23. Let R be commutative and finitely generated and equipped with the fine bornology. Let $\pi \in J \lhd R$ and $\alpha \in [0,1]$. Then $\mathcal{T}_\alpha(R,J)$ also carries the fine bornology by Lemma 3.1.20. The tube algebra $\mathcal{T}_1(R,J)$ is described in Example 3.1.22. It is closely related to the rings of functions on tubes used in the construction of rigid cohomology, see Lemma 6.6. Example 3.1.22 also identifies $\mathcal{T}_{1/m}(R,J) \cong \mathcal{T}_1(R,J^m)$, which is the usual tube algebra of R with respect to J^m.

Lemma 3.1.24. *Let R be commutative and finitely generated and equipped with the fine bornology. Let $\pi \in J \lhd R$ and $\alpha \in [0,1]$. If $\alpha \in \mathbb{Q}$, then the tube algebra $\mathcal{T}_\alpha(R,J)$ is finitely generated and in particular Noetherian.*

Proof. If α is rational, $\alpha = m/i$, then Example 3.1.22 shows that $\mathcal{T}_\alpha(R,J)$ is generated as an algebra by $R + \pi^{-1}J^{\lceil m/i \rceil} + \pi^{-2}J^{\lceil 2m/i \rceil} + \cdots + \pi^{-i}J^{\lceil im/i \rceil}$. Since R is Noetherian, there are finite generating sets $S_j \subseteq J^j$ for the ideals $J^j \lhd R$ for $1 \le j \le m$, and R is generated as an algebra by a finite set of generators S_0. Then the finite subset $\bigcup_{l=0}^{i-1} \pi^{-l}S_{\lceil lm/i \rceil}$ generates $\mathcal{T}_\alpha(R,J)$ as an algebra. $\quad\square$

In contrast, we claim that $\mathcal{T}_\alpha(R,J)$ for irrational α is usually not finitely generated, except in trivial cases. If $\alpha' \le \alpha$, then $\mathcal{T}_{\alpha'}(R,J) \subseteq \mathcal{T}_\alpha(R,J)$. Any element of $\mathcal{T}_\alpha(R,J)$ belongs to $\mathcal{T}_{\alpha'}(R,J)$ for some $\alpha' \in \mathbb{Q}$ with $\alpha' \le \alpha$. Thus $\mathcal{T}_\alpha(R,J)$ is the increasing union of its subalgebras $\mathcal{T}_{\alpha'}(R,J)$ for $\alpha' \in \mathbb{Q}$ with $\alpha' \le \alpha$. However, except in trivial cases such as $J = \pi R$ or if this holds up to torsion, we have $\mathcal{T}_{\alpha'}(R,J) \ne \mathcal{T}_\alpha(R,J)$ if $\alpha' < \alpha$.

Proposition 3.1.25. *Let R and S be bornological algebras. Then $(R \otimes S)_{\mathrm{lg}} = R_{\mathrm{lg}} \otimes S_{\mathrm{lg}}$ and hence $\overline{(R \otimes S)_{\mathrm{lg}}} \cong \overline{R_{\mathrm{lg}}} \,\overline{\otimes}\, \overline{S_{\mathrm{lg}}}$.*

Proof. A V-submodule of $R \otimes S$ is bounded if and only if it is contained in the image of $T \otimes U$ for bounded V-submodules $T \subseteq R$, $U \subseteq S$; we may assume $1_R \in T$, $1_S \in U$. Then

$$\sum \pi^{2i}(T \otimes U)^{i+1} \subseteq \left(\sum \pi^i T^{i+1}\right) \otimes \left(\sum \pi^j U^{j+1}\right)$$
$$\subseteq \sum \pi^{i+j} T^{\max\{i,j\}+1} \otimes U^{\max\{i,j\}+1} \subseteq \sum \pi^m (T \otimes U)^{m+1}.$$

By Lemma 3.1.10, this shows that a subset is bounded in $R_{\mathrm{lg}} \otimes S_{\mathrm{lg}}$ if and only if it is bounded in $(R \otimes S)_{\mathrm{lg}}$. The isomorphism $\overline{(R \otimes S)_{\mathrm{lg}}} \cong \overline{R_{\mathrm{lg}}} \,\overline{\otimes}\, \overline{S_{\mathrm{lg}}}$ follows from $(R \otimes S)_{\mathrm{lg}} = R_{\mathrm{lg}} \otimes S_{\mathrm{lg}}$ by taking completions on both sides, see Lemma 2.18. $\quad\square$

Proposition 3.1.26. *Let R and S be commutative bornological V-algebras. Let $I \lhd R$ and $J \lhd S$ be ideals containing π. Let $\alpha \in [0,1]$. Let $I + J$ denote the sum of the images of $I \otimes S$ and $R \otimes J$ in $R \otimes S$; this is an ideal. Assume that any bounded V-submodule of $I + J$ is the sum of the images of $M_I \otimes M_S$ and of $M_R \otimes M_J$ for bounded V-submodules $M_I \subseteq I$, $M_S \subseteq S$, $M_R \subseteq R$, and $M_J \subseteq J$. Then*

$$\overline{\underline{R}_{I,\alpha}} \,\overline{\otimes}\, \overline{\underline{S}_{J,\alpha}} \cong \overline{\underline{R \otimes S}_{I+J,\alpha}}.$$

Proof. We show that $\overline{R_{I,\alpha}} \overline{\otimes} \overline{S_{J,\alpha}}$ has the universal property of $T := \overline{R \otimes S}_{I+J,\alpha}$ in Proposition 3.1.16. So let A be a complete bornological algebra. Bounded unital homomorphisms $T \to A$ are in natural bijection with bounded unital homomorphisms $\varphi \colon R \otimes S \to A$ such that $\varrho(\varphi(M)) \le \epsilon^\alpha$ for all bounded V-submodules $M \subseteq I + J$. The bounded unital homomorphism φ corresponds to a pair of bounded unital homomorphisms $\varphi^R \colon R \to A$ and $\varphi^S \colon S \to A$ with commuting images through $\varphi(r \otimes s) = \varphi^R(r) \cdot \varphi^S(s)$ for all $r \in R$, $s \in S$. By our assumption, any bounded V-submodule $M \subseteq I + J$ is contained in the sum of the images of $M_I \otimes M_S$ and $M_R \otimes M_J$ for bounded V-submodules $M_I \subseteq I$, $M_S \subseteq S$, $M_R \subseteq R$, and $M_J \subseteq J$. Conversely, the sum of the images of $M_I \otimes M_S$ and $M_R \otimes M_J$ as above is clearly a bounded V-submodule of $I + J$ for the tensor product bornology. Hence $\varrho(\varphi(M)) \le \epsilon^\alpha$ for all M as above if and only if the same spectral radius estimate holds for $\varphi^R(M_I) \cdot \varphi^S(M_S) + \varphi^R(M_R) \cdot \varphi^S(M_J)$ for all bounded V-submodules $M_I \subseteq I$, $M_S \subseteq S$, $M_R \subseteq R$, and $M_J \subseteq J$. Since φ^R and φ^S have commuting images,

$$\left(\varphi^R(M_I) \cdot \varphi^S(M_S) + \varphi^R(M_R) \cdot \varphi^S(M_J)\right)^n = \sum_{l=0}^{n} \varphi^R(M_I^l M_R^{n-l}) \cdot \varphi^S(M_J^{n-l} M_S^l),$$

where $M_I^0 := M_R^0 := V \cdot 1_R$ and $M_J^0 := M_S^0 := V \cdot 1_S$. Let $\beta, \gamma > 0$, then

$$\sum_{n=0}^{\infty} (\beta\gamma)^n \star \left(\varphi^R(M_I) \cdot \varphi^S(M_S) + \varphi^R(M_R) \cdot \varphi^S(M_J)\right)^n$$

$$= \sum_{n,m=0}^{\infty} (\beta\gamma)^{n+m} \star \varphi^R(M_I^n) \cdot \varphi^S(M_J^m) \cdot \varphi^R(M_R^m) \cdot \varphi^S(M_S^n).$$

The modules $(\beta\gamma)^{n+m} \star \varphi^R(M_I^n) \cdot \varphi^S(M_J^m) \cdot \varphi^R(M_R^m) \cdot \varphi^S(M_S^n)$ and $(\beta^n \star \varphi^R(M_I^n)) \cdot (\gamma^m \star \varphi^R(M_R^m)) \cdot (\beta^m \star \varphi^S(M_J^m)) \cdot (\gamma^n \star \varphi^S(M_S^n))$ differ at most by four factors of π due to rounding errors when replacing β^n, β^m, γ^n, γ^m and $(\beta\gamma)^{n+m}$ by powers of ϵ, see (4). For the boundedness of the sum, this is irrelevant. We may assume that $1 \in M_R$ and $1 \in M_S$. Then $1 \in \varphi^R(M_R^n)$ and $1 \in \varphi^S(M_S^n)$ for all $n \in \mathbb{N}$. Therefore, if the sum above is bounded for all $\beta < \epsilon^\alpha$ and $\gamma < 1$, then both $\sum_{n=0}^{\infty} \beta^n \star \varphi^R(M_I^n)$ and $\sum_{m=0}^{\infty} \beta^m \star \varphi^S(M_J^m)$ are bounded for all such β. Conversely, assume that the latter are bounded for all such β, M_I and M_J. Here we may also take $M_I = \pi M_R^m$ and $M_J = \pi M_S^m$ for any $m \in \mathbb{N}$. By Lemma 3.1.2, this implies that $\sum_{n=0}^{\infty} \gamma^n \star \varphi^R(M_R^n)$ and $\sum_{m=0}^{\infty} \gamma^m \star \varphi^S(M_S^m)$ are bounded for any $\gamma < 1$. Thus

$$\sum_{n,m=0}^{\infty} (\beta^n \star \varphi^R(M_I^n)) \cdot (\gamma^m \star \varphi^R(M_R^m)) \cdot (\beta^m \star \varphi^S(M_J^m)) \cdot (\gamma^n \star \varphi^S(M_S^n))$$

is bounded if $\beta < \epsilon^\alpha$ and $\gamma < 1$. By the argument above, this is equivalent to

$$\sum_{n=0}^{\infty} (\beta\gamma)^n \star \left(\varphi^R(M_I) \cdot \varphi^S(M_S) + \varphi^R(M_R) \cdot \varphi^S(M_J)\right)^n$$

being bounded. Since $\gamma < 1$ is arbitrary, we see that φ satisfies the condition in Proposition 3.1.16 that characterizes when it extends to a homomorphism on T if and only if both φ^R and φ^S satisfy the corresponding condition to extend to $\overline{\underline{R}_{I,\alpha}}$ and $\overline{\underline{S}_{J,\alpha}}$, respectively. These unique extensions still commute when they exist. Thus the pairs (φ^R, φ^S) as above are in natural bijection with bounded unital homomorphisms $\overline{\underline{R}_{I,\alpha} \overline{\otimes} \underline{S}_{J,\alpha}} \to A$. Putting things together, we have got a natural bijection between bounded unital homomorphisms $\overline{\underline{R}_{I,\alpha} \overline{\otimes} \underline{S}_{J,\alpha}} \to A$ and $\overline{\underline{R \otimes S}_{I+J,\alpha}} \to A$. $\qquad\square$

COROLLARY 3.1.27. *Let R and S be commutative bornological algebras. Let $I \lhd R$ be an ideal containing π. Let $\alpha \in [0,1]$. Let I' denote the image of $I \otimes S$ in $R \otimes S$; this is an ideal. Assume that any bounded V-submodule of I' is the image of $M_I \otimes M_S$ for bounded V-submodules $M_I \subseteq I$ and $M_S \subseteq S$. Then*

$$\overline{\underline{R}_{I,\alpha} \overline{\otimes} S_{\lg}} \cong \overline{\underline{R \otimes S}_{I',\alpha}}.$$

Proof. Apply Proposition 3.1.26 in the special case $J = \pi \cdot S$ and use $\overline{\underline{S}_{\pi S,\alpha}} = \overline{S_{\lg}}$ for all $\alpha \in [0,1]$ by Example 3.1.14. $\qquad\square$

We shall only apply Propositions 3.1.25 and 3.1.26 and Corollary 3.1.27 when R and S carry the fine bornology. Then the technical assumptions in Proposition 3.1.26 and Corollary 3.1.27 about bounded V-submodules of $I + J$ or I' hold automatically.

Remark 3.1.28. If R is not flat over V, then the canonical map $R \to \underline{R}$ is not injective: its kernel is the V-torsion submodule τR. Let $R' := R/\tau R$. Then $\underline{R'} = \underline{R}$. If $J \lhd R$ is an ideal, let $J' \lhd R'$ be the image in R'. Then $\underline{\mathcal{B}}_{J,\alpha} = \underline{\mathcal{B}}_{J',\alpha}$. So it suffices to study the bornology $\underline{\mathcal{B}}_{J,\alpha}$ and the resulting completion of $\underline{R}$ if R is flat over V.

3.2 Dagger completions and linear growth bornologies

Let R be a finitely generated, commutative V-algebra. Monsky–Washnitzer [27] define the *weak completion* $R^\dagger$ of R as the subset of the π-adic completion

$$\widehat{R} = \varprojlim_{S} R/\pi^j R$$

consisting of elements z having representations

$$z = \sum_{j=0}^\infty \pi^j w_j \tag{8}$$

with $w_j \in M^{\kappa_j}$, where $M \ni 1$ is a finitely generated V-submodule of R and $\kappa_j \le c(j+1)$ for some constant $c > 0$ depending on z. We equip $R^\dagger$ with the following bornology: call a subset S bounded if all its elements are of the form $z = \sum_{j=0}^\infty \pi^j w_j$ as in (8) with $w_j \in M^{\kappa_j}$ and $\kappa_j \le c(j+1)$ for a fixed finitely generated V-submodule $M \subseteq R$ and a fixed $c > 0$.

THEOREM 3.2.1. *Let R be a finitely generated, commutative V-algebra, equipped with the fine bornology. The canonical map $R \to R^\dagger$ extends uniquely to an isomorphism of bornological algebras from $\overline{R_{\mathrm{lg}}}$ onto $R^\dagger$.*

The proof of Theorem 3.2.1 will be finished by Proposition 3.2.5. We first treat the case of the full polynomial algebra and then reduce the general case to it.

LEMMA 3.2.2. *Let $P = V[x_1, \ldots, x_n]$. Then the canonical map $\overline{P_{\mathrm{lg}}} \to \widehat{P}$ is an isomorphism of bornological algebras onto the Monsky–Washnitzer completion $P^\dagger$ of P.*

Proof. Let $S_0 = V1 + Vx_1 + \cdots + Vx_n$ be the obvious generating V-submodule for P containing 1. Any finitely generated V-submodule of P is contained in S_0^c for some $c \in \mathbb{N}_{\geq 1}$. By Lemma 3.1.10, a subset of P has linear growth if and only if it is contained in $P_c := \sum_{j=0}^\infty \pi^j S_0^{c(j+1)}$ for some $c \in \mathbb{N}_{\geq 1}$. This is the set of polynomials $\sum b_\alpha x^\alpha$, $b_\alpha \in V$, with $\nu(b_\alpha) + 1 \geq |\alpha|/c$.

Since V is π-adically complete, the π-adic completion $\widehat{P_c}$ of P_c is the set of all formal power series $\sum b_\alpha x^\alpha$ with $b_\alpha \in V$, $\nu(b_\alpha) + 1 \geq |\alpha|/c$ for all α and $\lim_{|\alpha| \to \infty} \nu(b_\alpha) + 1 - |\alpha|/c = +\infty$. In particular, $\widehat{P_c}$ is contained in the π-adic completion $\widehat{P}$. Since $P_{\mathrm{lg}} = \varinjlim P_c$, Proposition 2.15 implies $\overline{P_{\mathrm{lg}}} = \varinjlim \widehat{P_c}$. This is contained in $\widehat{P}$.

If $c' > c$ then $1/c' - 1/c < 0$. So if the series $f = \sum b_\alpha x^\alpha$ satisfies $\nu(b_\alpha) + 1 \geq |\alpha|/c$, then $f \in \widehat{P_{c'}}$ because

$$0 \leq \nu(b_\alpha) + 1 - |\alpha|/c = \nu(b_\alpha) + 1 - |\alpha|/c' + |\alpha|(1/c' - 1/c).$$

Thus $\varinjlim \widehat{P_c} \subseteq \widehat{P}$ is equal to the set of all formal power series $\sum b_\alpha x^\alpha \in \widehat{P}$ such that

$$(\exists \delta > 0)(\forall \alpha) \quad \nu(b_\alpha) + 1 \geq \delta |\alpha|. \tag{9}$$

A subset of $\overline{P_{\mathrm{lg}}}$ is bounded if and only if there is one δ that works for all its elements. We may replace the condition (9) by

$$\liminf_{|\alpha| \to \infty} \frac{\nu(b_\alpha)}{|\alpha|} > 0.$$

This gives the Monsky–Washnitzer completion $P^\dagger$ with the bornology specified above, compare [27, Theorem 2.3]. □

Remark 3.2.3. Let R be a finitely generated, commutative V-algebra. Then $R^\dagger$ is Noetherian by [16]. The ideal $\pi R^\dagger$ is contained in the Jacobson radical because any element of the form $1 - \pi z$ with $z \in R^\dagger$ has an inverse given by $\sum_{j=0}^\infty \pi^j z^j \in R^\dagger$. Krull's Intersection Theorem implies that any finitely generated $R^\dagger$-module is π-adically separated (see [1, Corollary 10.19]). In particular, $R^\dagger/J \cdot R^\dagger$ is π-adically separated for any ideal J in R. Equivalently, $J \cdot R^\dagger$ is π-adically closed in $R^\dagger$.

Lemma 3.2.4. *Let R be a finitely generated, commutative V-algebra and let $J \triangleleft R$ be an ideal. The natural map $R^\dagger/JR^\dagger \to (R/J)^\dagger$ is an isomorphism.*

Proof. By Remark 3.2.3, $R^\dagger/JR^\dagger$ is π-adically separated. As a quotient of a weakly complete algebra, it is then weakly complete (see [27, Theorem 1.3]). Hence the natural map $R/J \to R^\dagger/JR^\dagger$ extends to a map $(R/J)^\dagger \to R^\dagger/JR^\dagger$. This map is inverse to the map in the statement of the lemma. □

Proposition 3.2.5. *Let $P = V[x_1, \ldots, x_n]$, let $J \triangleleft P$ be an ideal, and $R = P/J$. The bornological closure of J in $P^\dagger$, the closure of J in the π-adic topology on $P^\dagger$, and the ideal $J \cdot P^\dagger$ generated by J in $P^\dagger$ are all the same. The resulting quotient $P^\dagger/(J \cdot P^\dagger)$ is isomorphic to both $R^\dagger$ and $\overline{R_{\mathrm{lg}}}$. Thus $R^\dagger \cong \overline{R_{\mathrm{lg}}}$.*

Proof. The multiplication map in $P^\dagger$ is bounded and every element of $P^\dagger$ is in the π-adic closure of a bounded subset of P (with respect to the bornology defined at the beginning of Section 3.2). Thus $J \cdot P^\dagger$ is contained in the bornological closure of $J = J \cdot P$. This is further contained in the π-adic closure of J because any bornologically convergent sequence converges in the π-adic topology on $P^\dagger$ and so a π-adically closed subset is bornologically closed. Remark 3.2.3 applied to P shows that $J \cdot P^\dagger$ is π-adically closed in $P^\dagger$. Thus the bornological and the π-adic closures of J are both equal to $J \cdot P^\dagger$. Lemma 3.2.4 applied to P identifies $P^\dagger/J \cdot P^\dagger$ with $R^\dagger$.

Let T be a complete bornological V-algebra with $\varrho(M) \le 1$ for all bounded $M \subseteq T$. A homomorphism $\overline{R_{\mathrm{lg}}} \to T$ is equivalent to a homomorphism $R \to T$ by Proposition 3.1.15. This is equivalent to a homomorphism $P \to T$ that vanishes on J. By the universal property of $\overline{P_{\mathrm{lg}}}$ in Proposition 3.1.15, this is equivalent to a homomorphism $\overline{P_{\mathrm{lg}}} \to T$ that vanishes on the image of J. Any such homomorphism vanishes on the bornological closure of J and then descends to a homomorphism on the quotient of $\overline{P_{\mathrm{lg}}}$ by this bornological closure. This quotient is complete, and $\varrho(M) \le 1$ for all bounded V-submodules in this quotient. Thus it has the universal property that characterizes $\overline{R_{\mathrm{lg}}}$, that is, it is naturally isomorphic to $\overline{R_{\mathrm{lg}}}$. Since $\overline{P_{\mathrm{lg}}} = P^\dagger$ by Lemma 3.2.2 and since the bornological and π-adic closures of J are the same, we get $R^\dagger \cong \overline{R_{\mathrm{lg}}}$. □

This finishes the proof of Theorem 3.2.1. This theorem allows us to apply results about linear growth completions such as Propositions 3.1.17 and 3.1.25 to dagger completions of finitely generated, commutative V-algebras.

Now we turn to the completions $\underline{\overline{R_{J,\alpha}}}$ for $\alpha \in \mathbb{Q} \cap [0,1]$. We use the tube algebra $\mathcal{T}_\alpha(R, J)$ introduced in Definition 3.1.19 and its Monsky–Washnitzer completion $\mathcal{T}_\alpha(R, J)^\dagger$.

Theorem 3.2.6. *Let R be a finitely generated, commutative V-algebra, let $J \triangleleft R$ be an ideal with $\pi \in J$, and $\alpha \in \mathbb{Q} \cap [0,1]$. The completion $\underline{\overline{R_{J,\alpha}}}$ is $K \otimes \mathcal{T}_\alpha(R, J)^\dagger$.*

Proof. Since $\alpha \in \mathbb{Q}$, the tube algebra $\mathcal{T}_\alpha(R, J)$ is again commutative and finitely generated and carries the fine bornology by Lemmas 3.1.24 and 3.1.20. Proposition 3.1.21 and Theorem 3.2.1 for the tube algebra identify

$$\overline{\underline{R}_{J,\alpha}} \cong K \otimes \overline{\mathcal{T}_\alpha(R, J)_{\mathrm{lg}}} \cong K \otimes \mathcal{T}_\alpha(R, J)^\dagger. \qquad \square$$

4   Homological algebra for completions of commutative algebras

4.1   Hochschild homology

The Hochschild homology of the commutative algebras of functions on smooth algebraic varieties is described by the Hochschild–Kostant–Rosenberg Theorem. Connes proves an analogous result for algebras of smooth functions on smooth manifolds, with Hochschild homology defined using completed tensor products. We shall carry over Connes' result to the completions $\mathrm{HH}_*(\overline{R_{\mathrm{lg}}})$ and $\mathrm{HH}_*(\overline{\underline{R}_{J,\alpha}})$. Our computation uses purely algebraic tools and hence describes $\mathrm{HH}_*(R^\dagger)$ and $\mathrm{HH}_*(\overline{\underline{R}_{J,\alpha}})$ only as V-modules, without their canonical bornologies.

DEFINITION 4.1.1. Let A be a complete bornological V-algebra. The *Hochschild chain complex* $(C(A), b)$ consists of the complete bornological V-modules $C_n(A) := A^{\overline{\otimes} n+1}$ for $n \geq 0$ with the boundary maps $b \colon C_n(A) \to C_{n-1}(A)$,

$$b(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes (a_i \cdot a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_n$$
$$+ (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

The *Hochschild homology* $\mathrm{HH}_*(A)$ is the homology of $(C(A), b)$.

If no bornology on a V-algebra A is specified, then we give it the fine bornology. Then $A^{\otimes n} = A^{\overline{\otimes} n}$, and so the bornological Hochschild chain complex of A as a bornological algebra is the usual, purely algebraic one.

There are many equivalent chain complexes that compute periodic cyclic homology, and all of them would work for our purposes. To be definite, we choose the total complex of the cyclic bicomplex.

DEFINITION 4.1.2 ([22]). A *mixed complex* in an additive category $\mathcal{C}$ is a $\mathbb{Z}$-graded object M of $\mathcal{C}$ together with homogeneous maps $b \colon M \to M[-1]$ and $B \colon M \to M[+1]$ such that $b^2 = B^2 = bB + Bb = 0$; here $[\pm 1]$ denotes a degree shift, so b and B have degree -1 and $+1$, respectively. The *Hochschild complex* of M is the chain complex (M, b). We will only consider mixed complexes that are nonnegatively graded in the sense that $C_n = 0$ for $n < 0$.

Given a nonnegatively graded mixed complex (M, b, B) in a complete, additive category $\mathcal{C}$, let CM be the $\mathbb{Z}/2$-graded chain complex that is $\prod_n M_n$ graded by

parity of n as a $\mathbb{Z}/2$-graded object of $\mathcal{C}$, with the boundary map $b + B$. The homology of CM is called the *periodic cyclic homology* of M.

Let R be a unital, complete bornological V-algebra. Then $(C(R), b, B)$ with Connes' boundary map B is a mixed complex. Its periodic cyclic homology is called the *periodic cyclic homology* of R and denoted $\mathrm{HP}_*(R)$.

Explicitly, $\mathrm{HP}_*(R)$ is the homology of the $\mathbb{Z}/2$-graded chain complex $CC(R) :=$ $CC^{\mathrm{ev}}(R) \oplus CC^{\mathrm{odd}}(R)$ with the boundary map $b + B$, where

$$CC^{\mathrm{ev}}(R) := \prod_{j=0}^{\infty} R^{\overline{\otimes 2j}}, \qquad CC^{\mathrm{odd}}(R) := \prod_{j=0}^{\infty} R^{\overline{\otimes 2j+1}},$$

and $B := (1 - t)sN$ as in [24] with

$$N(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n} (-1)^n a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1},$$
$$s(a_0 \otimes \cdots \otimes a_n) := 1 \otimes a_0 \otimes \cdots \otimes a_n,$$
$$(1 - t)(1 \otimes a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n + (-1)^n a_n \otimes 1 \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

EXAMPLE 4.1.3. Let (Ω, d) be a nonnegatively graded cochain complex. Consider the mixed complex $M(\Omega) := (\Omega, 0, d)$ with zero boundary map $\Omega_* \to \Omega_{*-1}$. The periodic cyclic homology of $M(\Omega)$ is the homology of Ω made periodic; we have

$$\mathrm{HP}_j(M(\Omega)) = \prod_{n \geq 0} H^{2n+j}(\Omega, d) \qquad (j = 0, 1).$$

After these general definitions, we describe these homology theories for the complete bornological algebras introduced above. Let R be a finitely generated, commutative, torsion-free V-algebra. Let $J \lhd R$ be an ideal with $\pi \in J$. Let $\alpha \in [0, 1] \cap \mathbb{Q}$. We shall describe the Hochschild homology of the bornological V-algebras $\overline{R_{\mathrm{lg}}}$ and $\overline{R_{J,\alpha}}$.

Hochschild homology is a special case of the derived tensor product functor Tor on the category of complete bornological bimodules. Hence we need some homological properties of the dagger completion $R^\dagger$, which is identified with $\overline{R_{\mathrm{lg}}}$ in Theorem 3.2.1. Intermediate results in our proof suggest that $R \to R^\dagger$ and $R \to \overline{R_{J,\alpha}}$ for $\alpha \in \mathbb{Q}$ behave like the localisations of Taylor [30].

LEMMA 4.1.4. *Let R be a finitely generated, commutative V-algebra. Then the canonical homomorphism $R \to R^\dagger$ is flat and the map to the π-adic completion $R^\dagger \to \widehat{R}$ is faithfully flat.*

Proof. Since R and $R^\dagger$ are Noetherian, the maps to their π-adic completion $R \to \widehat{R}$ and $R^\dagger \to \widehat{R}$ are flat (see [1, Proposition 10.14]). The ideal $\pi R^\dagger$ is contained in the Jacobson radical of $R^\dagger$ by Remark 3.2.3. Hence the map $R^\dagger \to \widehat{R}$ is faithfully flat. This implies that $R \to R^\dagger$ is flat. □

LEMMA 4.1.5. *Let $f : R \to S$ be a homomorphism of commutative V-algebras of finite type. If f is flat, then so is $f^\dagger : R^\dagger \to S^\dagger$.*

Proof. It is enough to prove that $\widehat{f} \colon \widehat{R} \to \widehat{S}$ is flat. Indeed, if this is the case, then the composition $R^\dagger \xrightarrow{f^\dagger} S^\dagger \to \widehat{S}$, which equals the composition $R^\dagger \to \widehat{R} \xrightarrow{\widehat{f}} \widehat{S}$, is flat. Since $S^\dagger \to \widehat{S}$ is faithfully flat by Lemma 4.1.4 this implies that $f^\dagger$ is flat. To see that $\widehat{f}$ is flat, we use a flatness criterion of Bourbaki [6, Ch. III, §5, Thm. 1]. Let I be any ideal of $\widehat{R}$. Since $\widehat{R}$ is Noetherian, I is finitely generated. Hence $I \otimes_{\widehat{R}} \widehat{S}$ is a finitely generated $\widehat{S}$-module. Since also $\widehat{S}$ is Noetherian, $I \otimes_{\widehat{R}} \widehat{S}$ is π-adically separated. This means that $\widehat{S}$ as an $\widehat{R}$-module is 'idéalement séparé pour πR' in the sense of Bourbaki. Since $R \to S$ is flat, also $\widehat{R}/\pi^n \widehat{R} \to \widehat{S}/\pi^n \widehat{S}$ is flat for every n. Now the flatness criterion implies that $\widehat{S}$ is flat as an $\widehat{R}$-module. $\qquad\square$

LEMMA 4.1.6. *Let R be a finitely generated, commutative V-algebra and M a finitely generated module over R. Then $R^\dagger \otimes_R M$ is a complete bornological V-module, in the bornology induced by the bornology on $R^\dagger$, introduced in Section* 3.2, *and the fine bornology on M.*

Proof. The symmetric bimodule structure on M gives an algebra structure on $M \oplus R$ so that M is an ideal with $M^2 = 0$ and the projection to R is an algebra homomorphism with kernel M. We denote this algebra by $M \rtimes R$. Since M is finitely generated we may write it as $M = \bigoplus_{i=1}^n Rx_i/L$, so

$$M \rtimes R = R[x_1, \ldots, x_n]/(\langle x_1, \ldots, x_n \rangle^2 + \langle L \rangle).$$

If S is of finite type and J is an ideal in S, then $(S/J)^\dagger = S^\dagger/JS^\dagger$ by Lemma 3.2.4. Applying this to the identity above, and using that $(R[X]/\langle X \rangle^2)^\dagger = R^\dagger[X]/\langle X \rangle^2$, we get

$$\begin{aligned}
(M \rtimes R)^\dagger &= R^\dagger[x_1, \ldots, x_n]/(\langle x_1, \ldots, x_n \rangle^2 + \langle L \rangle) \\
&= R^\dagger \otimes_R R[x_1, \ldots, x_n]/(\langle x_1, \ldots, x_n \rangle^2 + \langle L \rangle) = R^\dagger \otimes_R (M \rtimes R).
\end{aligned}$$

Finally, $R^\dagger \otimes_R M$ is the kernel of the map $R^\dagger \otimes_R (M \rtimes R) \to R^\dagger$. Since $R^\dagger$ is bornologically separated, this kernel is bornologically closed and inherits bornological completeness from $(M \rtimes R)^\dagger$ by Lemma 2.13. $\qquad\square$

Let R be a commutative bornological V-algebra of finite type equipped with the fine bornology. The Hochschild complex $C(R)$ is the chain complex associated to a cyclic object in the category of V-modules. The face and degeneracy maps as well as the (unsigned) cyclic permutations of this cyclic object are in fact V-algebra maps.

All algebra maps are bounded for the linear growth bornology associated to the fine bornology. Hence $C(R)_{\mathrm{lg}}$ is a cyclic object in the category of bornological commutative V-algebras. The bornological completion of $C(R)_{\mathrm{lg}}$ is the Monsky–Washnitzer completion $C(R)^\dagger$ by Theorem 3.2.1. Proposition 3.1.25 implies

$$C_n(R)^\dagger \cong \overline{C_n(R)_{\mathrm{lg}}} = \overline{(R^{\otimes n+1})_{\mathrm{lg}}} \cong (\overline{R_{\mathrm{lg}}})^{\overline{\otimes} n+1} = C_n(\overline{R_{\mathrm{lg}}}) \cong C_n(R^\dagger) \qquad (10)$$

by the definition of C_n for complete bornological V-algebras such as $R^\dagger$.

The *bar resolution* of a complete bornological algebra R is the complex $(B_n(R) = R^{\overline{\otimes}\, n+2})_{n \geq 0}$ with boundary map

$$b'(a_0 \otimes \cdots \otimes a_{n+1}) := \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes (a_i \cdot a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}.$$

The bar resolution is an R-bimodule resolution of R. It admits a bounded contraction $s : B_n \to B_{n+1}$ defined by $s(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$. One has $b's + sb' = \mathrm{id}$. Thus $B_n(R)$ even is a split exact resolution. Write $R^e := R \otimes_V R$.

PROPOSITION 4.1.7. *Let R be a finitely generated, torsion-free, commutative V-algebra.*

(a) *Both $(R^e)^\dagger \otimes_{R^e} B(R)$ and $B(R^\dagger)$ are flat $(R^e)^\dagger$-module resolutions of $R^\dagger$. The natural map $C(R) \to C(R^\dagger)$ induces a quasi-isomorphism*

$$R^\dagger \otimes_R (C(R), b) \to (C(R^\dagger), b). \tag{11}$$

The natural map $\mathrm{HH}(R) \to \mathrm{HH}(R^\dagger)$ induces an isomorphism

$$R^\dagger \otimes_R \mathrm{HH}_*(R) \cong \mathrm{HH}_*(R^\dagger). \tag{12}$$

(b) *Let R be equipped with the fine bornology, let $J \lhd R$ be an ideal with $\pi \in J$, and $\alpha \in [0,1]$ rational. The canonical map is an isomorphism*

$$\overline{R_{J,\alpha}} \otimes_{\underline{R}} \mathrm{HH}_*(\underline{R}) \cong \mathrm{HH}_*(\overline{R_{J,\alpha}}).$$

Proof. (a) The bar resolution $B(R)$ is a resolution of R by flat R^e-modules because R is torsion-free. The inclusion $R^e \to (R^e)^\dagger$ is flat by Lemma 4.1.4. Hence $(R^e)^\dagger \otimes_{R^e} B(R)$ is a resolution of $(R^e)^\dagger \otimes_{R^e} R$. The multiplication map $\mu \colon R \otimes R \to R$ is a surjective algebra homomorphism, so $R \cong R^e / \ker \mu$. Hence $(R^e)^\dagger \otimes_{R^e} R \cong (R^e)^\dagger \otimes_{R^e} (R^e / \ker \mu) \cong (R^e)^\dagger / (R^e)^\dagger \cdot (\ker \mu)$. This is isomorphic to $R^\dagger$ by Lemma 3.2.4. The resolution $(R^e)^\dagger \otimes_{R^e} B(R)$ of $R^\dagger$ consists of flat $(R^e)^\dagger$-modules because $Q \otimes_{(R^e)^\dagger} ((R^e)^\dagger \otimes_{R^e} B_n(R)) \cong Q \otimes_{R^e} B_n(R) \cong Q \otimes R^{\otimes n}$ for any $(R^e)^\dagger$-module Q, and $R^{\otimes n}$ is a flat V-module.

Equip $B_n(R) = R^{\otimes n+2}$ with the V-algebra structure as a tensor product of copies of R. Then

$$R^e \to B_n(R), \qquad b_1 \otimes b_2 \mapsto b_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes b_2,$$

is a ring homomorphism. It is flat because R is a flat V-module. Then the induced homomorphism $(R^e)^\dagger \to B_n(R)^\dagger$ is also flat by Lemma 4.1.5. Proposition 3.1.25 and Theorem 3.2.1 imply $B_n(R)^\dagger \cong (R^\dagger)^{\overline{\otimes}\, n+2} = B_n(R^\dagger)$. Thus $B(R^\dagger)$ is a resolution of $R^\dagger$ by flat $(R^e)^\dagger$-modules (even admitting a bounded contraction).

So far, we have shown that both $(R^e)^\dagger \otimes_{R^e} B(R)$ and $B(R^\dagger)$ are resolutions of $R^\dagger$ by flat $(R^e)^\dagger$-modules. They are related by an obvious chain map

$$(R^e)^\dagger \otimes_{R^e} B_n(R) \cong (R^e)^\dagger \otimes R^{\otimes n} \to (R^e)^\dagger \, \overline{\otimes} \, (R^\dagger)^{\overline{\otimes}n} = B_n(R^\dagger). \qquad (13)$$

If Q is an $(R^e)^\dagger$-module, then both $Q \otimes_{(R^e)^\dagger}((R^e)^\dagger \otimes_{R^e} B(R)) \cong Q \otimes_{R^e} B(R)$ and $Q \otimes_{(R^e)^\dagger} B(R^\dagger)$ compute the derived functors $\mathrm{Tor}_n^{(R^e)^\dagger}(Q, R^\dagger)$. Hence the chain map $Q \otimes_{R^e} B(R) \to Q \otimes_{(R^e)^\dagger} B(R^\dagger)$ induced by (13) is a quasi-isomorphism. We claim that this quasi-isomorphism specializes to (11) if $Q = R^\dagger$. We have

$$R^\dagger \otimes_{(R^e)^\dagger}\big((R^e)^\dagger \otimes_{R^e} B(R)\big) \cong R^\dagger \otimes_{R^e} B(R) \cong R^\dagger \otimes_R (R \otimes_{R^e} B(R)) \cong R^\dagger \otimes_R C(R).$$

It remains to identify $R^\dagger \otimes_{(R^e)^\dagger} B_n(R^\dagger)$ with $C_n(R^\dagger)$. Here we take an incomplete tensor product of two completions. This often causes trouble (compare [26, Section 3]), but not here. We have identified $R^\dagger \cong (R^e)^\dagger \otimes_{R^e} R$ above. Hence

$$R^\dagger \otimes_{(R^e)^\dagger} B_n(R^\dagger) \cong R \otimes_{R^e} (R^e)^\dagger \otimes_{(R^e)^\dagger} B_n(R^\dagger) \cong R \otimes_{R^e} B_n(R^\dagger)$$
$$\cong B_n(R^\dagger)/(\ker \mu) \cdot B_n(R^\dagger),$$

where $\ker \mu \triangleleft R^e$ is the kernel of the multiplication homomorphism $\mu \colon R^e \to R$. We have $(\ker \mu) \cdot B_n(R^\dagger) = (B_n(R) \cdot \ker \mu) \cdot B_n(R^\dagger)$ for the ideal $B_n(R) \cdot \ker \mu \triangleleft B_n(R)$ generated by $\ker \mu$ in the finitely generated commutative algebra $B_n(R)$. Lemma 3.2.4 identifies $B_n(R^\dagger)/(B_n(R) \cdot \ker \mu) \cdot B_n(R^\dagger)$ with the dagger completion of the quotient $B_n(R)/B_n(R) \cdot \ker \mu \cong C_n(R)$. Thus $R^\dagger \otimes_{(R^e)^\dagger} B_n(R^\dagger) \cong C_n(R)^\dagger \cong C_n(R^\dagger)$. This finishes the proof of (11).

The homology on the left in (11) is $R^\dagger \otimes_R H_*(C(R), b) = R^\dagger \otimes_R \mathrm{HH}_*(R)$ because $R^\dagger$ is a flat R-module by Lemma 4.1.4. The homology on the right in (11) is $\mathrm{HH}_*(R^\dagger)$ by definition. Thus the quasi-isomorphism (11) implies the isomorphism (12).

(b) Let $T := \mathcal{T}_\alpha(R, J)$ be the α-tube algebra for $J \triangleleft R$. This is a torsion-free, commutative V-algebra. It is finitely generated because α is rational. It carries the fine bornology by Lemma 3.1.20. Proposition 3.1.21 gives an isomorphism $\overline{R_{J,\alpha}} \cong \overline{T_{\mathrm{lg}}} \otimes K$. The functor $R \mapsto C(R)$ commutes with $- \otimes K$. These facts and (10) give

$$C_n(\overline{R_{J,\alpha}}) \cong C_n(\overline{T_{\mathrm{lg}}} \otimes K) \cong C_n(\overline{T_{\mathrm{lg}}}) \otimes K \cong \overline{C_n(T)_{\mathrm{lg}}} \otimes K.$$

Equation (12) says that

$$\overline{T_{\mathrm{lg}}} \otimes_T \mathrm{HH}_*(T) \cong \mathrm{HH}_*(\overline{T_{\mathrm{lg}}}).$$

Taking Hochschild homology commutes with tensoring with K, that is,

$$\mathrm{HH}_*(X) \otimes K \cong \mathrm{HH}_*(X \otimes K)$$

for $X = R, T, \overline{T_{\mathrm{lg}}}$. By construction of T, $\underline{T} \coloneqq K \otimes T$ and $\underline{R} \coloneqq K \otimes R$ are isomorphic. Thus

$$
\begin{aligned}
\overline{R_{J,\alpha}} \otimes_{\underline{R}} \mathrm{HH}_*(\underline{R}) &\cong \overline{T_{\mathrm{lg}}} \otimes K \otimes_{\underline{T}} \mathrm{HH}_*(\underline{T}) \cong \overline{T_{\mathrm{lg}}} \otimes K \otimes_T (\mathrm{HH}_*(T) \otimes K) \\
&\cong \overline{T_{\mathrm{lg}}} \otimes_T \mathrm{HH}_*(T) \otimes K \cong \mathrm{HH}_*(\overline{T_{\mathrm{lg}}}) \otimes K \cong \mathrm{HH}_*(\overline{T_{\mathrm{lg}}} \otimes K) \cong \mathrm{HH}_*(\overline{R_{J,\alpha}}),
\end{aligned}
$$

where the first and last step use $\overline{R_{J,\alpha}} \cong K \otimes \overline{T_{\mathrm{lg}}}$. $\qquad\square$

DEFINITION 4.1.8. Let R be a commutative V-algebra. Then $\Omega^*_{\underline{R}}$ denotes the dg-algebra of Kähler differential forms for $\underline{R}$ (over the ground field K) and $(\Omega^*_{\underline{R}}, d)$ is the de Rham complex for $\underline{R}$. The de Rham complex for $\underline{R}^\dagger$ is $(\underline{R}^\dagger \otimes_{\underline{R}} \Omega^*_{\underline{R}}, d)$. For an ideal J in R and $\alpha \in [0,1]$, the de Rham complex for $\overline{R_{J,\alpha}}$ is $(\overline{R_{J,\alpha}} \otimes_{\underline{R}} \Omega^*_{\underline{R}}, d)$.

Remark 4.1.9. For finitely generated R and rational α, Lemma 4.1.6 (in combination with Proposition 3.1.21) shows that $\underline{R}^\dagger \otimes_{\underline{R}} \Omega^*_{\underline{R}}$ and $\overline{R_{J,\alpha}} \otimes_{\underline{R}} \Omega^*_{\underline{R}}$ are the $\dagger$- and J,α-completions of Ω^*_R tensored by K, respectively. This justifies our definition of the de Rham complexes for $\underline{R}^\dagger$ and $\overline{R_{J,\alpha}}$ above.

We combine Corollary 4.1.7(b) with the Hochschild–Kostant–Rosenberg Theorem for $\underline{R}$:

THEOREM 4.1.10. *Let R be a finitely generated commutative V-algebra with the fine bornology, $J \lhd R$ an ideal with $\pi \in J$, and $\alpha \in [0,1]$ rational. If $\underline{R} \coloneqq R \otimes K$ is smooth over K, then the antisymmetrisation map $\Omega^*_{\underline{R}} \to \mathrm{HH}_*(\underline{R})$ induces an isomorphism*

$$
\overline{R_{J,\alpha}} \otimes_{\underline{R}} \Omega^*_{\underline{R}} \cong \mathrm{HH}_*(\overline{R_{J,\alpha}}).
$$

Proof. The usual Hochschild–Kostant–Rosenberg Theorem for smooth algebras over fields says that the antisymmetrisation map $\Omega^*_{\underline{R}} \to \mathrm{HH}_*(\underline{R})$ is an isomorphism (see [23, Theorem 3.4.4]). This map is $\underline{R}$-linear, so it induces an isomorphism $\overline{R_{J,\alpha}} \otimes_{\underline{R}} \Omega^*_{\underline{R}} \cong \overline{R_{J,\alpha}} \otimes_{\underline{R}} \mathrm{HH}_*(\underline{R})$. This is isomorphic to $\mathrm{HH}_*(\overline{R_{J,\alpha}})$ by Corollary 4.1.7(b). $\qquad\square$

In particular, if $J = \pi R$, then $\overline{R_{J,\alpha}} \cong R^\dagger \otimes K$ by Example 3.1.14. Theorem 4.1.10 shows that the natural map from the mixed complex $(C(\underline{R}^\dagger), b, B)$ to the mixed complex $(\underline{R}^\dagger \otimes_{\underline{R}} \Omega_{\underline{R}}, 0, d)$ (mapping $a_0 \otimes \ldots \otimes a_n$ to $(1/n!)a_0 da_1 \ldots da_n$, see [23, Proposition 1.3.15]) is an isomorphism on 'Hochschild' homology (that is, the b-homology). But then it also is an isomorphism on the associated cyclic homology of the mixed complexes (see [23, Proposition 2.5.15]). So our result specializes to an isomorphism

$$
\mathrm{HP}_j(R^\dagger \otimes K) \cong \bigoplus_{n \geq 0} H_{2n+j}(R^\dagger \otimes_R (\Omega_{\underline{R}}, d)). \tag{14}
$$

If R is smooth, then the right-hand side is, by definition, the Monsky–Washnitzer homology of the quotient $A \coloneqq R/J$ made periodic.

4.2 A short proof of the Feigin–Tsygan Theorem

The Feigin–Tsygan Theorem establishes an important property of cyclic homology: The periodic cyclic homology of the coordinate ring B of an affine algebraic variety in characteristic 0 gives exactly Grothendieck's infinitesimal cohomology of that variety.

The proofs of Proposition 4.1.7(a) and Theorem 4.1.10 can be adapted to give a short proof of the Feigin–Tsygan Theorem. Actually, the proofs substantially simplify in this case since we can replace some of the preparatory lemmas by well-known easy facts from commutative algebra.

In this section we always assume that P is a finitely generated commutative algebra over a field K of characteristic 0. If J is an ideal in P, we denote by $\overline{P_J}$ the J-adic completion. In the following proof we will consider tensor powers $P^{\otimes n}$ (over K) and we will denote by J_n the kernel of the natural map $P^{\otimes n} \to (P/J)^{\otimes n}$.

We denote by $B_n(\overline{P_J})$ the J_{n+2}-adic completion of $B_n(P)$, by $C_n(\overline{P_J})$ the J_{n+1}-adic completion of $C_n(P)$ and by $\mathrm{HH}(\overline{P_J})$ the homology of $C(\overline{P_J})$. These are the usual conventions and the analogue of the †-versions in Section 4. Again $B_n(\overline{P_J})$ is a resolution of $\overline{P_J}$ by $\overline{P_J}$-bimodules, admitting the continuous contraction s.

PROPOSITION 4.2.1. *Let P and J be as above and denote by $P^e := P \otimes P$ the enveloping algebra. Then*

(a) $\overline{P^e_{J_2}} \otimes_{P^e} B(P)$ *is a flat $\overline{P^e_{J_2}}$-module resolution of $\overline{P_J}$.*

(b) $B(\overline{P_J})$ *is a flat $\overline{P^e_{J_2}}$-module resolution of $\overline{P_J}$.*

(c) *One has $\overline{P_J} \otimes_{\overline{P^e_{J_2}}} B(\overline{P_J}) \cong C(\overline{P_J})$.*

As a consequence the natural map $C(P) \to C(\overline{P_J})$ induces a quasi-isomorphism

$$\overline{P_J} \otimes_P (C(P), b) \to (C(\overline{P_J}), b) \tag{15}$$

and the natural map $\mathrm{HH}_*(P) \to \mathrm{HH}_*(\overline{P_J})$ induces an isomorphism

$$\overline{P_J} \otimes_P \mathrm{HH}_*(P) \cong \mathrm{HH}_*(\overline{P_J}) \tag{16}$$

Proof. (a) Since P^e is Noetherian, $P^e \to \overline{P^e_{J_2}}$ is flat and $\overline{P^e_{J_2}} \otimes_{P^e} P = \overline{P_J}$ (see [1, 10.14] and [1, 10.13]). Thus $\overline{P^e_{J_2}} \otimes_{P^e} B(P)$ is a resolution of $\overline{P_J}$. Moreover, $\overline{P^e_{J_2}} \otimes_{P^e} B_n(P) = \overline{P^e_{J_2}} \otimes_K P^{\otimes n}$ is a flat (even free) $\overline{P^e_{J_2}}$-module.

(b) As already mentioned above, the existence of the continuous contraction s shows that $B(\overline{P_J})$ is a resolution of $\overline{P_J}$. Identifying the algebra $P^e \otimes P^{\otimes n}$ with $P \otimes P^{\otimes n} \otimes P = B_n(P)$, we can consider $B_n(\overline{P_J})$ as the completion of the Noetherian algebra $\overline{P^e_{J_2}} \otimes P^{\otimes n}$. Therefore, the maps $\overline{P^e_{J_2}} \to \overline{P^e_{J_2}} \otimes P^{\otimes n} \to B_n(\overline{P_J})$ are flat.

(c) Let I be the kernel of the algebra map $B_n(P) = P \otimes P^{\otimes n} \otimes P \to P \otimes P^{\otimes n} = C_n(P)$ induced by the homomorphism $P^e \to P$. Since $B_n(P)$ is Noetherian, we have $I \cdot \overline{B_n(P)}_{J_{n+2}} = \overline{I}_{J_{n+2}}$ and $\overline{B_n(P)}_{J_{n+2}}/\overline{I}_{J_{n+2}} = \overline{(B_n(P)/I)}_{J_{n+2}}$. Finally, we have

$$C_n(\overline{P_J}) := \overline{C_n(P)}_{J_{n+1}} = \overline{B_n(P)}_{J_{n+2}}/I \cdot \overline{B_n(P)}_{J_{n+2}} = P \otimes_{P^e} \overline{B_n(P)}_{J_{n+2}},$$

whence $\overline{P_J} \otimes_{\overline{P^e_{J_2}}} \overline{B_n(P)}_{J_{n+2}} = P \otimes_{P^e} \overline{P^e_{J_2}} \otimes_{\overline{P^e_{J_2}}} \overline{B_n(P)}_{J_{n+2}} = P \otimes_{P^e} \overline{B_n(P)}_{J_{n+2}} = C_n(\overline{P_J})$.

Now, since $\overline{P_J} \otimes_P (C(P), b)$ and $(C(\overline{P_J}), b)$ both compute the same Tor-functor, we get the asserted quasi-isomorphism. $\qquad\square$

Repeating now verbatim the (short) proof of Theorem 4.1.10 we obtain

THEOREM 4.2.2. *Let P be a smooth commutative algebra over a field K of characteristic 0 and J an ideal in P. Then*

$$\overline{P_J} \otimes_P \Omega_P^* \cong \mathrm{HH}_*(\overline{P_J})$$

As an immediate consequence we get the Feigin–Tsygan Theorem

THEOREM 4.2.3. *Let P and J be as above. Then*

$$\mathrm{HP}_*(P/J) = \mathrm{HP}_*(\overline{P_J}) = HdR_*(\overline{P_J})$$

Here, $HdR_*(\overline{P_J})$ stands for the homology of the complex $\left(\overline{P_J} \otimes_P \Omega_P^*, d\right)$ made $\mathbb{Z}/2$-periodic.

The proof of the second equality in the displayed formula follows again by repeating the discussion after Theorem 4.1.10. In addition, the first equality follows from the invariance of HP_* under nilpotent extensions. Alternatively, it follows directly from the definition of HP_* in the description of cyclic homology in [12].

Remark 4.2.4. Since Ω_P^* is a finitely generated module over the Noetherian algebra P, we see that $\overline{P_J} \otimes_P \Omega_P^*$ is its completion. Thus $HdR_*(\overline{P_J})$ describes the infinitesimal cohomology of P/J in the sense of Grothendieck, see [8, Theorem 6.1].

4.3 HOMOTOPY INVARIANCE

Let R and S be complete bornological V-algebras and let $f_0, f_1 : R \to S$ be bounded unital algebra homomorphisms.

DEFINITION 4.3.1. A (dagger-continuous) *homotopy between f_0 and f_1* is a bounded unital algebra homomorphism $F : R \to S \overline{\otimes} V[x]^\dagger$ with $(\mathrm{id}_S \overline{\otimes} \mathrm{ev}_t) \circ F = f_t$ for $t = 0, 1$.

As we shall see, if such a homotopy exists, then f_0 and f_1 induce the same map in periodic cyclic homology, $\mathrm{HP}_*(f_0) = \mathrm{HP}_*(f_1)$. Put in a nutshell, periodic cyclic homology is invariant under dagger-continuous homotopies. This is well known for polynomial homotopies, that is, $F\colon R \to S \otimes V[x]$. The proof of polynomial homotopy invariance shows, in fact, that a polynomial homotopy induces a bounded chain homotopy between the maps $CC(f_0)$ and $CC(f_1)$. Making this chain homotopy explicit, it can be checked then that the same formulas still define a bounded chain homotopy between the maps $CC(f_0)$ and $CC(f_1)$ when we start from a homotopy that is only dagger-continuous. We shall outline another proof that exhibits more clearly why dagger-continuous homotopies work here (in contrast, continuous homotopies in the usual sense do not work).

Equation (14) implies easily that the evaluation map at 0 induces an isomorphism $\mathrm{HP}_*(K \otimes V[x]^\dagger) \cong K$ if K has characteristic 0. We shall need a stronger statement:

Lemma 4.3.2. *Assume that K has characteristic 0. The kernel of the chain map from $CC(K \otimes V[x]^\dagger)$ to K with zero boundary map induced by evaluation at 0 on $C_0(K \otimes V[x]^\dagger)$ is contractible through a bounded contracting homotopy. Similarly, the kernel of the chain map from $\left((K \otimes V[x]^\dagger) \otimes_{K[x]} \Omega^*_{K[x]}, d\right)$ to K is contractible through a bounded chain homotopy.*

Proof. Let $R := V[x]$ and identify $R \otimes R \cong V[x, y]$. The map

$$R \otimes R \to R \otimes R, \qquad f \mapsto (x - y) \cdot f,$$

is an injective R-bimodule homomorphism, and its cokernel is isomorphic to R with the usual bimodule structure through the map $R \otimes R \to R$, $f \mapsto f(x, x)$. The chain complex $0 \to R \otimes R \to R \otimes R \to R \to 0$ remains exact when we tensor with $V[x, y]^\dagger$. Even more,

$$0 \to V[x, y]^\dagger \to V[x, y]^\dagger \to V[x]^\dagger \to 0 \tag{17}$$

is a short exact sequence of bornological left $V[x]^\dagger$-modules with a bounded linear section. The non-trivial (but easy) observation is that division by $x - y$ is a bounded linear map from the kernel of the map $V[x, y]^\dagger \to V[x]^\dagger$ to $V[x, y]^\dagger$. The bar resolution of $V[x]^\dagger$ is a resolution with bounded linear section as well, and the free bornological bimodules in it are projective with respect to $V[x]^\dagger$-bimodule extensions with a bounded linear section. Hence (17) is homotopy equivalent as a bornological bimodule resolution to the bar resolution of $V[x]^\dagger$. This implies a bounded homotopy equivalence between $(C(V[x]^\dagger), b)$ and the chain complex with $V[x]^\dagger$ in degrees 0 and 1 and 0 in other degrees and with the zero boundary map. This homotopy equivalence intertwines Connes' boundary operator B and the differential $d\colon V[x]^\dagger \to V[x]^\dagger$, $f \mapsto f'$. By a variant of Kassel's perturbation lemma, this implies a chain homotopy equivalence between $(CC(V[x]^\dagger), b + B)$ and $d\colon V[x]^\dagger \to V[x]^\dagger$. All this remains true after tensoring with K.

The absolute integration map $i \colon \sum a_n x^n \mapsto \sum \frac{a_n}{n+1} x^{n+1}$ is a bounded linear map on $K \otimes V[x]^\dagger$. It satisfies $d \circ i = \mathrm{id}$ and $i \circ d = \mathrm{id} - P_0$, where P_0 is defined by $P_0(x^n) = \delta_{n,0} x^n$. Hence $d \colon K \otimes V[x]^\dagger \to K \otimes V[x]^\dagger$ is homotopy equivalent to K concentrated in degree 0. $\square$

PROPOSITION 4.3.3. *Let P and Q be complete bornological K-algebras and let $f_0, f_1 \colon P \rightrightarrows Q$ be bounded unital homomorphisms. Assume that there is a dagger-continuous homotopy between f_0 and f_1 and that K has characteristic 0.*

(a) *Assume that R, S are finitely generated V-algebras, $J \lhd R$ and $I \lhd S$ are ideals, $\alpha \in [0,1]$ is rational and that $P = \overline{R_{J,\alpha}}$, $Q = \overline{S_{I,\alpha}}$. Then the maps induced by f_0, f_1 between the de Rham complexes $(\overline{R_{J,\alpha}} \otimes_{\underline{R}} \Omega^*_{\underline{R}}, d)$ and $(\overline{S_{I,\alpha}} \otimes_{\underline{S}} \Omega^*_{\underline{S}}, d)$ are homotopic with a bounded K-linear chain homotopy (for the bornology explained in Remark 4.1.9).*

(b) *For general complete bornological K-algebras P and Q, there is a bounded V-linear chain homotopy between the induced chain maps*

$$CC(f_0), CC(f_1) \colon (CC(P), B + b) \rightrightarrows (CC(Q), B + b).$$

Hence $\mathrm{HP}_*(f_0) = \mathrm{HP}_*(f_1)$.

Proof. (a) The de Rham complex for $\overline{S_{I,\alpha}} \,\overline{\otimes}\, V[x]^\dagger$ is $(\overline{S_{I,\alpha}} \otimes_{\underline{S}} \Omega^*_{\underline{S}}) \wedge (V[x]^\dagger \otimes_{V[x]} \Omega^*_{K[x]})$, where $\wedge$ stands for the completed antisymmetric tensor product. The assertion then follows from Lemma 4.3.2.

(b) We are working in the symmetric monoidal category of complete bornological K-vector spaces with the tensor product $\overline{\otimes}$. The definitions of Hochschild and periodic cyclic homology above are the standard ones for algebras in this symmetric monoidal category. Thus by [25, Theorem 4.74] (or by [28])

$$(CC(Q \,\overline{\otimes}\, (K \,\overline{\otimes}\, V[x]^\dagger)), B + b) \simeq (CC(Q), B + b) \,\overline{\otimes}\, (CC(K \,\overline{\otimes}\, V[x]^\dagger), B + b),$$

where $\simeq$ means a bounded homotopy equivalence as chain complexes of complete bornological vector spaces. Now Lemma 4.3.2 says that $(CC(K \,\overline{\otimes}\, V[x]^\dagger), B + b) \simeq K$ concentrated in degree 0, so that

$$(CC(Q \,\overline{\otimes}\, (K \,\overline{\otimes}\, V[x]^\dagger)), B + b) \simeq (CC(Q), B + b).$$

Thus evaluation at 0 and at 1 induce chain homotopic maps $CC(Q \,\overline{\otimes}\, V[x]^\dagger) \to CC(Q)$ and $CC(f_0)$ and $CC(f_1)$ are chain homotopic as asserted. Then the induced maps $\mathrm{HP}_*(f_0)$ and $\mathrm{HP}_*(f_1)$ on homology must be equal. $\square$

5 NATURAL CHAIN COMPLEXES FOR COMMUTATIVE ALGEBRAS OVER THE RESIDUE FIELD

We are going to associate some natural chain complexes to commutative algebras over the residue field $k \coloneqq V/\pi V$. For two of them, we show in Section 6

that they compute rigid cohomology. From now on, we will always assume that K has characteristic 0.

First we fix some $\alpha \in [0, 1]$. Let A be a commutative k-algebra. We may view A as a V-algebra with $\pi \cdot A = 0$. Let $S \subseteq A$ be a set of generators for A. Let $V[S]$ be the free commutative algebra on the set S. The inclusion map $S \to A$ defines a unital homomorphism $p \colon V[S] \to A$ because A is commutative. This is surjective because S generates A by assumption. Equip $V[S]$ with the fine bornology. Let $J := \ker p$; this contains π because $\pi \cdot A = 0$. We shall be interested in the completion $\overline{V[S]}_{J,\alpha}$, which is a complete bornological V-algebra, and in the chain complex $(CC(\overline{V[S]}_{J,\alpha}), B + b)$ that computes its periodic cyclic homology. To obtain a natural chain complex, we prefer a natural choice for the set of generators, namely, $S := A$. This may be infinite even if A is finitely generated, but this shall not bother us: it is the price to pay for a natural construction. Let

$$\mathsf{cc}^\alpha(A) := (CC(\overline{V[A]}_{J,\alpha}), B + b)$$

be the chain complex constructed above for the generating set $S = A$. We show that this chain complex is natural. A unital algebra homomorphism $f \colon A_1 \to A_2$ induces a bounded unital algebra homomorphism $V[f] \colon V[A_1] \to V[A_2]$. It is compatible with the two projections $p_i \colon V[A_i] \to A_i$ in the following sense: $p_2 \circ V[f] = f \circ p_1$. Hence $V[f]$ maps the kernel $J_1 := \ker p_1$ into the kernel $J_2 := \ker p_2$. By assumption, it extends uniquely to a bounded unital algebra homomorphism $\overline{V[A_1]}_{J_1,\alpha} \to \overline{V[A_2]}_{J_2,\alpha}$, which in turn induces a bounded chain map

$$\mathsf{cc}^\alpha(f) \colon \mathsf{cc}^\alpha(A_1) \to \mathsf{cc}^\alpha(A_2)$$

Thus we have constructed a functor cc^α from the category of commutative k-algebras to the category of chain complexes of bornological K-vector spaces. We may forget the bornology on cc^α because we shall not use it. Let $\mathsf{hc}^\alpha(A)$ be the homology of the chain complex $\mathsf{cc}^\alpha(A)$. This is also functorial for unital algebra homomorphisms.

The identity map on $V[A]$ extends uniquely to a bounded algebra homomorphism $\overline{V[A]}_{J,\alpha} \to \overline{V[A]}_{J,\beta}$ for $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$. This induces a chain map

$$\sigma_{\alpha\beta} \colon \mathsf{cc}^\alpha(A) \to \mathsf{cc}^\beta(A),$$

which in turn induces a K-linear map

$$\sigma_{\alpha\beta} \colon \mathsf{hc}^\alpha(A) \to \mathsf{hc}^\beta(A).$$

Exactly the same considerations apply to the α-versions of the de Rham complexes associated to A. We set

$$\mathsf{cdR}^\alpha(A) := (\overline{V[A]}_{J,\alpha} \otimes_{V[A]} \Omega^*_{V[A]}, d)$$

and write $\mathsf{hdR}^\alpha(A)$ for the homology of this $\mathbb{N}$-graded complex. Again, for $\alpha \le \beta$, we obtain a natural chain map

$$\sigma_{\alpha\beta} \colon \mathsf{cdR}^\alpha(A) \to \mathsf{cdR}^\beta(A).$$

We shall be most interested in the *homotopy limits* of the projective systems of complexes that we obtain that way. To be specific let us start again with the projective system of cyclic complexes defined by the maps

$$\sigma_{1/(m+1),1/m} \colon \mathsf{cc}^{1/(m+1)}(A) \to \mathsf{cc}^{1/m}(A).$$

The homotopy limit is defined as the mapping cone of the bounded chain map

$$\prod_{m=1}^\infty \mathsf{cc}^{1/m}(A) \to \prod_{m=1}^\infty \mathsf{cc}^{1/m}(A), \qquad (x_m) \mapsto (x_m - \sigma_{1/(m+1),1/m}(x_{m+1})), \quad (18)$$

shifted by 1.

DEFINITION 5.1. We denote the chain complexes obtained as homotopy limits of the projective systems

$$\left(\mathsf{cc}^{1/m}(A)\right), \qquad \left(\mathsf{cdR}^{1/m}(A)\right)$$

by $\mathsf{cc}^{\mathrm{rig}}(A)$ and $\mathsf{cdR}^{\mathrm{rig}}(A)$, respectively. Their homology is denoted by $\mathsf{hc}^{\mathrm{rig}}(A)$ and $\mathsf{hdR}^{\mathrm{rig}}(A)$.

The long exact sequence for the homology of a mapping cone implies a short exact sequence

$$0 \to \lim_m{}^1 \mathsf{hc}_{*+1}^{1/m}(A) \to \mathsf{hc}_*^{\mathrm{rig}}(A) \to \lim_m \mathsf{hc}_*^{1/m}(A) \to 0.$$

Clearly, $\mathsf{cc}^{\mathrm{rig}}$, $\mathsf{hc}^{\mathrm{rig}}$ and $\mathsf{cdR}^{\mathrm{rig}}$, $\mathsf{hdR}^{\mathrm{rig}}$ inherit functoriality for unital k-algebra homomorphisms from the corresponding α-versions.

Remark 5.2. By definition, $\mathsf{hc}_*^\alpha(A) = \mathrm{HP}_*(\overline{V[A]}_{J,\alpha})$ is the periodic cyclic homology of a certain complete bornological K-algebra. Thus $\mathsf{hc}_*^{\mathrm{rig}}(A)$ is exactly the periodic cyclic homology, as defined in [7], of the projective system of complete bornological K-algebras

$$\left(\overline{V[A]}_{J,1/m}\right), \qquad m \in \mathbb{N}_{\ge 1}.$$

The free presentation $V[A]$ of A used to define $\mathsf{cc}^\alpha(A)$ is natural but possibly very large. For computations, we want to use smaller generating sets. These give homotopy equivalent chain complexes:

PROPOSITION 5.3. *Let A be a k-algebra and let $S \subseteq A$ be a generating set. Let $\alpha \in [0,1]$ and let $J^S \lhd V[S]$ be the kernel of the canonical homomorphism $p^S \colon V[S] \to A$.*

There are bounded maps $f \colon \overline{V[S]}_{J^S,\alpha} \to \overline{V[A]}_{J,\alpha}$ and $g \colon \overline{V[A]}_{J,\alpha} \to \overline{V[S]}_{J^S,\alpha}$, such that $f \circ g$ and $g \circ f$ are homotopic to the identity through a dagger-continuous homotopy.

Proof. The inclusion map $S \to A$ induces a unital homomorphism $f: V[S] \to V[A]$. It maps the kernel J^S of $p^S: V[S] \to A$ into the kernel J of $p: V[A] \to A$ because $p \circ f = p^S$. Hence it extends uniquely to a bounded unital algebra homomorphism $f: \overline{V[S]}_{J^S,\alpha} \to \overline{V[A]}_{J,\alpha}$ by Proposition 3.1.18. Since S generates A, the homomorphism $p^S: V[S] \to A$ is surjective. For each $a \in A$, choose some $g(a) \in V[S]$ with $p^S(g(a)) = a$; we may assume $g(a) = a$ for all $a \in S$. These choices define a unital homomorphism $g: V[A] \to V[S]$ because $V[S]$ is commutative. By construction, $p^S \circ g(a) = p(a)$ for all $a \in A$. This implies $p^S \circ g = p$ and hence $g(J) \subseteq J^S$. Hence g extends uniquely to a bounded unital algebra homomorphism $g: \overline{V[A]}_{J,\alpha} \to \overline{V[S]}_{J^S,\alpha}$ by Proposition 3.1.18. We have $g \circ f = \mathrm{id}_{V[S]}$ because $g(a) = a$ for all $a \in S$.

The homomorphism $f \circ g: V[A] \to V[A]$ is homotopic to the identity map through the homotopy $H: V[A] \to V[A,t]$ defined by $H(a) := t \cdot a + (1-t)fg(a)$ for all $a \in A$. Since $p \circ fg = p$, H maps $J := \ker(p) \triangleleft V[A]$ into $J \otimes V[t]$, the kernel of $p \otimes \mathrm{id}_{V[t]}: V[A,t] = V[A] \otimes V[t] \to A \otimes V[t]$. Now Corollary 3.1.27 and Proposition 3.1.18 show that H extends uniquely to a bounded unital algebra homomorphism $\overline{V[A]}_{J,\alpha} \to \overline{V[A]}_{J,\alpha} \hat{\otimes} V[t]_{\mathrm{lg}}$. Here we may identify $\overline{V[t]_{\mathrm{lg}}} \cong V[t]^{\dagger}$ by Theorem 3.2.1. Thus H is a dagger-continuous homotopy between $f \circ g$ and the identity map. $\square$

COROLLARY 5.4. *Let A and S be as in Proposition* 5.3. *Then* $\mathsf{cc}^{\alpha}(A)$ *is naturally chain homotopy equivalent to* $CC(\overline{V[S]}_{J^S,\alpha}, B+b)$. *And* $\mathsf{cc}^{\mathrm{rig}}(A)$ *is naturally chain homotopy equivalent to the homotopy limit of the chain complexes* $CC(\overline{V[S]}_{J^S,\alpha}, B+b)$ *for* $\alpha = 1/m$, $m \to \infty$. *Analogous statements hold for the de Rham complexes* $\mathsf{cdR}^{\alpha}(A)$ *and* $\mathsf{cdR}^{\mathrm{rig}}(A)$.

Proof. The first assertion follows from Proposition 5.3 and Proposition 4.3.3.(b). The maps $CC(f)$ and $CC(g)$ that we get from Proposition 5.3 and the chain homotopy equivalence are compatible for different α and hence induce a chain homotopy equivalence between the homotopy projective limits, giving the second statement. The proof for the de Rham version is the same. $\square$

Roughly speaking, we get chain complexes homotopy equivalent to $\mathsf{cc}^{\alpha}(A)$ and $\mathsf{cc}^{\mathrm{rig}}(A)$ when we replace $p: V[A] \to A$ by any presentation of A by a free commutative algebra over V: a surjective homomorphism $V[S] \to A$ is equivalent to a map $S \to A$ whose image generates A. Proposition 5.3 remains true if the map $S \to A$ is not injective.

THEOREM 5.5. *Let A be a finitely generated k-algebra. The complexes* $\mathsf{cc}^{\mathrm{rig}}(A)$ *and* $\mathsf{cdR}^{\mathrm{rig}}(A)$ *made 2-periodic are quasi-isomorphic.*

Proof. By Corollary 5.4, the complexes $\mathsf{cc}^{\mathrm{rig}}(A)$ and $\mathsf{cdR}^{\mathrm{rig}}(A)$ are quasi-

isomorphic to the homotopy limits of the complexes

$$CC(\overline{V[S]}_{J^S,1/m}, B+b) \quad \text{and} \quad (\overline{V[S]}_{J^S,1/m} \otimes \Omega^*_{V[S]}, d),$$

respectively, for a set S of generators. We may assume that S is finite. Then, by (14) $CC(\overline{V[S]}_{J^S,1/m}, B+b)$ is quasi-isomorphic to the de Rham complex $(\overline{V[S]}_{J^S,1/m} \otimes \Omega^*_{V[S]}, d)$ made 2-periodic. Homotopy invariance (Proposition 4.3.3.(a)) shows that the de Rham complex for $\overline{V[S]}_{J^S,1/m}$ is homotopy equivalent to the one for $\overline{V[A]}_{J,1/m}$. The homotopy limit respects quasi-isomorphisms. This implies the assertion. □

Our construction is invariant under polynomial homotopies of the underlying k-algebras:

PROPOSITION 5.6. *Let A_1, A_2 be finitely generated commutative k-algebras and $\alpha \in [0,1]$ rational. Let $f_0, f_1 \colon A_1 \rightrightarrows A_2$ be unital homomorphisms that are polynomially homotopic, that is, there is a unital homomorphism $F \colon A_1 \to A_2[t]$ with $\mathrm{ev}_t \circ F = f_t$ for $t = 0, 1$. Then the induced bounded chain maps $\mathsf{cc}^\alpha(f_0)$ and $\mathsf{cc}^\alpha(f_1)$ are chain homotopic. So are $\mathsf{cc}^{\mathrm{rig}}(f_0)$ and $\mathsf{cc}^{\mathrm{rig}}(f_1)$. Thus $\mathsf{hc}^\alpha(f_0) = \mathsf{hc}^\alpha(f_1)$ and $\mathsf{hc}^{\mathrm{rig}}(f_0) = \mathsf{hc}^{\mathrm{rig}}(f_1)$, and similarly for hdR.*

Proof. For $i = 1, 2$, let $p_i \colon V[A_i] \to A_i$ be the canonical homomorphisms and let $J_i := \ker(p_i) \lhd A_i$. The homomorphism $F \colon A_1 \to A_2[t]$ induces a unital V-algebra homomorphism $F_* \colon V[A_1] \to V[A_2][t] = V[A_2] \otimes V[t]$, which maps the generator $a \in A_1$ to $\sum b_n \otimes t^n$ with $b_n \in A_2 \subseteq V[A_2]$ if $F(a) = \sum b_n t^n \in A_2[t]$. We have $(p_2 \otimes \mathrm{id}_{V[t]}) \circ F_* = F \circ p_1$ because this holds on all generators $a \in A_1$. Hence $F(J_1) \subseteq J_2 \otimes V[t]$. Now Corollary 3.1.27 and Proposition 3.1.18 show that F extends uniquely to a bounded unital algebra homomorphism $\bar{F}_* \colon \overline{V[A_1]}_{J,\alpha} \to \overline{V[A_2]}_{J,\alpha} \bar\otimes V[t]^\dagger$. Thus $\bar{F}_*$ is a dagger-continuous homotopy between the homomorphisms induced by f_0 and f_1. Now Proposition 4.3.3 shows that the induced maps $\mathsf{cc}^\alpha(f_0)$ and $\mathsf{cc}^\alpha(f_1)$ are chain homotopic, and similarly for cdR^α. Since this holds for all α the induced maps $\mathsf{cc}^{\mathrm{rig}}(f_0)$ and $\mathsf{cc}^{\mathrm{rig}}(f_1)$ are chain homotopic as well, and similarly for $\mathsf{cdR}^{\mathrm{rig}}$. Then the induced maps on homology are equal. □

6 Comparison to rigid cohomology

Let A be a finitely generated, commutative k-algebra. We are going to identify $\mathsf{hdR}^{\mathrm{rig}}(A)$ with Berthelot's rigid cohomology of A, as our notation already suggests. Berthelot defines rigid cohomology of A, or more generally of separated k-schemes of finite type, in [4]. His construction is quite involved. Even in the affine case, it explicitly uses non-affine schemes. Große-Klönne's theory of dagger spaces (or rigid analytic spaces with overconvergent structure sheaf) [17] simplifies Berthelot's construction a little bit. In the following, we describe this

simplified construction. We begin by recalling the relevant things from dagger geometry.

For $n \in \mathbb{N}$ the *Washnitzer algebra* over K is

$$W_n := V[x_1, \ldots, x_n]^\dagger \otimes_V K.$$

We equip $V[x_1, \ldots, x_n]^\dagger$ with the bornology as at the beginning of Section 3.2 and W_n with the induced bornology. A *dagger K-algebra* is a quotient of some W_n by an ideal.

Let L be a dagger K-algebra. We denote the set of maximal ideals of L by $\mathrm{Sp}(L)$. We denote an element in $\mathrm{Sp}(L)$ either by x or by $\mathfrak{m}_x$ depending on whether we think of it as a point x in the set $\mathrm{Sp}(L)$ or as a maximal ideal $\mathfrak{m}_x \lhd L$.

Any morphism of dagger algebras $g\colon L_1 \to L_2$ induces a map $\tilde{g}\colon \mathrm{Sp}(L_2) \to \mathrm{Sp}(L_1)$ given by $\tilde{g}(x) = g^{-1}(\mathfrak{m}_x)$.

Consider an element f of the dagger algebra L and $x \in \mathrm{Sp}(L)$. The *residue field* $K(x) := L/\mathfrak{m}_x$ is a finite field extension of K. Hence the absolute value $|\ |$ on K extends uniquely to an absolute value $|\ |\colon K(x) \to \mathbb{R}_{\geq 0}$. The image of f under the composition $L \to K(x) \to \mathbb{R}_{\geq 0}$ will be denoted by $|f(x)|$.

A subset $T \subseteq X = \mathrm{Sp}(L)$ is called *special* if there are elements $f_0, \ldots, f_r \in L$ which generate the unit ideal and such that

$$T = \{x \in X : |f_i(x)| \leq |f_0(x)| \text{ for all } i = 1, \ldots, r\}. \tag{19}$$

Special subsets are stable under finite intersections. A subset $U \subseteq X$ is called *admissible open* if there is a covering $(T_i)_i$ of U by special subsets satisfying the following condition. For every morphism of dagger algebras $g\colon L \to L_2$ such that the induced map $\tilde{g}\colon \mathrm{Sp}(L_2) \to \mathrm{Sp}(L) = X$ factors through U, the induced covering $(\tilde{g}^{-1}(T_i))_i$ of $\mathrm{Sp}(L_2)$ admits a finite subcovering. A covering $(U_i)_i$ of the admissible open subset U by admissible open subsets U_i is called *admissible*, if for every $g\colon L \to L_2$ as above, the induced covering $(\tilde{g}^{-1}(U_i))_i$ of $\mathrm{Sp}(L_2)$ admits a finite refinement by special subsets. The admissible open subsets with the admissible coverings form a Grothendieck pretopology. By a *sheaf* on X we mean a sheaf for this pretopology.

To a special subset T as in (19) one associates the dagger algebra

$$\Gamma(T, \mathcal{O}_X) := L\langle x_1, \ldots, x_r\rangle^\dagger / \langle f_i - f_0 x_i, \ i = 1, \ldots, r\rangle. \tag{20}$$

Here $L\langle x\rangle^\dagger := L \,\overline{\otimes}\, V[x]^\dagger$. The natural map $L \to \Gamma(T, \mathcal{O}_X)$ induces a bijection $\mathrm{Sp}(\Gamma(T, \mathcal{O}_X)) \xrightarrow{\cong} T \subseteq X$. Tate's Acyclicity Theorem implies that the construction $T \mapsto \Gamma(T, \mathcal{O}_X)$ extends uniquely to a sheaf of rings $\mathcal{O}_X$ on X, called the *structure sheaf*. The pair $(X, \mathcal{O}_X)$ is called an *affinoid dagger space*. One may glue affinoid dagger spaces along admissible open subsets to get the notion of a *dagger space*. For example, every admissible open subset $U \subseteq X$ with structure sheaf given by the restriction of $\mathcal{O}_X$ is a dagger space, which is in general not affinoid.

Let L be a dagger K-algebra. There is a universal K-derivation from L to a finite L-module, denoted by $d\colon L \to \Omega^{\dagger,1}_{L/K}$. This construction extends uniquely to a coherent sheaf $\Omega^{\dagger,1}_{X/K}$ on any dagger space X over K. In the usual way, we obtain from this a de Rham complex $\Omega^{\dagger}_{X/K}$ of coherent sheaves on X. Explicitly, if $L = W_n/\langle f_1, \ldots, f_r \rangle$, then

$$\Omega^{\dagger,1}_{L/K} \cong (L \otimes_{V[x_1,\ldots,x_n]} \Omega^1_{V[x_1,\ldots,x_n]/V})/\langle df_1, \ldots, df_r \rangle.$$

Remark 6.1. If R is a finitely generated commutative V-algebra, then $\underline{R}^{\dagger}$ is a dagger K-algebra and the de Rham complex $\Omega^{\dagger,\star}_{\underline{R}^{\dagger}/K}$, just introduced, is the one of Definition 4.1.8.

Rigid cohomology is defined as the de Rham cohomology of certain dagger spaces that arise as tubes, which are defined by means of specialisation maps. We begin by recalling the latter. These can be defined in a more general setting, but to avoid technicalities, we restrict to the situation we need. Let R be a finitely generated commutative V-algebra. Then $\underline{R}^{\dagger} := R^{\dagger} \otimes_V K$ is a dagger K-algebra, and we write $X = \mathrm{Sp}(\underline{R}^{\dagger})$. The *specialisation map* is a map of sets

$$\mathrm{sp}\colon X \to \mathrm{Spec}(R/\pi R),$$

which is continuous for the Grothendieck pretopology on X and the Zariski topology on $\mathrm{Spec}(R/\pi R)$. It is constructed as follows. The residue field $K(x)$ of a point $x \in X$ is a finite extension of K. We denote its valuation ring by $V(x) \subseteq K(x)$ and its residue field by $k(x)$. Since the elements of $R^{\dagger}$ are power bounded, their images in $K(x)$ lie in $V(x)$. Now the kernel of the composite map $R \to V(x) \to k(x)$ is a prime ideal of R containing πR, hence corresponds to a unique prime ideal of $R/\pi R$. This is the desired point $\mathrm{sp}(x) \in \mathrm{Spec}(R/\pi R)$. We are now ready to discuss tubes. In the situation of the preceding paragraph, let Z be a closed subset of $\mathrm{Spec}(R/\pi R)$. The *tube of Z* in $X = \mathrm{Sp}(\underline{R}^{\dagger})$ is the subset

$$]Z[:= \mathrm{sp}^{-1}(Z) \subseteq X. \tag{21}$$

It is an admissible open subset of X (see [3, Prop. 1.1.1]). Hence it inherits the structure of a dagger space.

Remark 6.2. We can describe tubes more explicitly. Let $f_1, \ldots, f_r \in R$ be elements whose images in $R/\pi R$ define the closed subset $Z \subseteq \mathrm{Spec}(R/\pi R)$. We may consider the f_i as elements of $\underline{R}^{\dagger}$. From the construction of the specialisation map, we see that a point $x \in X$ belongs to the tube $]Z[$ if and only if $|f_i(x)| < 1$ for all $i = 1, \ldots, r$, that is,

$$]Z[= \{x \in X : |f_i(x)| < 1 \text{ for } i = 1, \ldots, r\}.$$

Hence $]Z[$ is the increasing union $]Z[= \bigcup_{n \geq 1} [Z]_{\epsilon^{1/n}}$, where $[Z]_{\epsilon^{1/n}}$ denotes the special subset

$$[Z]_{\epsilon^{1/n}} := \{x \in X : |f_i(x)| \leq \epsilon^{1/n} \text{ for } i = 1, \ldots, r\}$$
$$= \{x \in X : |f_i^n(x)| \leq \epsilon \text{ for } i = 1, \ldots, r\}.$$

This gives an admissible covering of $]Z[$ (see [3, Prop. 1.1.9]). Using (20) and the remarks there, we get

$$[Z]_{\epsilon^{1/n}} \cong \mathrm{Sp}(\underline{R[x_1,\ldots,x_r]}^\dagger/\langle f_i^n - \pi x_i, i = 1,\ldots,r\rangle).$$

Finally, we can define rigid cohomology. Let A be a finitely generated commutative k-algebra. Choose a smooth commutative V-algebra R together with a surjection $p\colon R \to A$. As before, we let X be $\mathrm{Sp}(\underline{R}^\dagger)$. Via p we identify $\mathrm{Spec}(A)$ with a closed subscheme Z of $\mathrm{Spec}(R/\pi R)$ and consider its tube $]Z[$ in X (by construction, $]Z[$ only depends on the underlying set of the scheme Z).

DEFINITION 6.3. The *rigid cohomology* of A with coefficients in K is the de Rham cohomology of the tube $]Z[$,

$$H^*_{\mathrm{rig}}(A, K) := \mathbb{H}^*(]Z[, \Omega^\dagger_{]Z[/K}).$$

Remark 6.4. Berthelot's original definition is [4, (1.3.1)]. It depends on the additional choice of a compactification $\overline{Z}$ of Z. This compactification is used to define a functor $j^\dagger$ which associates to a sheaf on the tube $]\overline{Z}[$ the sheaf of its overconvergent sections. This is necessary because Berthelot works with rigid spaces. In contrast, in Große-Klönne's theory of dagger spaces, the overconvergence is already built in. This is why the compactification or the functor $j^\dagger$ do not appear in the formula in Definition 6.3. Berthelot proves the independence of choices up to isomorphism in [4, §1].

By [18, Proposition 3.6], the rigid cohomology groups defined in Definition 6.3 are canonically isomorphic to the groups defined by Berthelot. The isomorphism is functorial in the following sense. Suppose that $f\colon A_1 \to A_2$ is a morphism of finitely generated commutative k-algebras, R_1, R_2 are smooth commutative V-algebras with surjections $p_i\colon R_i \to A_i$, $i = 1, 2$, and there is a homomorphism $F\colon R_1 \to R_2$ such that the diagram

$$\begin{array}{ccc}
R_1 & \xrightarrow{\ F\ } & R_2 \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\
A_1 & \xrightarrow{\ f\ } & A_2
\end{array}$$

commutes. Then F induces a morphism between the tubes $]\mathrm{Spec}(A_2)[\to]\mathrm{Spec}(A_1)[$, which in turn induces a homomorphism $H^*_{\mathrm{rig}}(A_1, K) \to H^*_{\mathrm{rig}}(A_2, K)$. This homomorphism coincides with the one constructed by Berthelot.

The main result of this section is the following.

THEOREM 6.5. *Let A be a finitely generated, commutative k-algebra. There are natural isomorphisms* $\mathsf{hdR}^{\mathrm{rig}}_j(A) \cong H^j_{\mathrm{rig}}(A, K)$. *Hence*

$$\mathsf{hc}^{\mathrm{rig}}_j(A) \cong \bigoplus_{n \geq 0} H^{2n+j}_{\mathrm{rig}}(A, K).$$

Proof. By Theorem 5.5, the complexes $\mathsf{cc}^{\mathrm{rig}}(A)$ and $\mathsf{cdR}^{\mathrm{rig}}(A)$ made 2-periodic are quasi-isomorphic. So the statement about $\mathsf{hdR}^{\mathrm{rig}}_j(A)$ implies the one about $\mathsf{hc}^{\mathrm{rig}}_j(A)$.

Let $S \subseteq A$ be a finite set generating A as k-algebra. Let $R := V[S]$, let $p \colon R \to A$ be the homomorphism induced by the inclusion of S into A, and let $J \lhd R$ be the kernel of p (the ideal J was denoted by J^S in Section 5). Since R is smooth over V, we may use it to compute rigid cohomology. The proof of Theorem 5.5 shows that $\mathsf{cdR}^{\mathrm{rig}}(A)$ is quasi-isomorphic to the homotopy limit of the complexes $(\overline{V[S]}_{J,1/m} \otimes \Omega^*_{V[S]}, d)$ for $m \in \mathbb{N}$.

As before, we write $X = \mathrm{Sp}(\underline{R}^\dagger)$ and $Z = \mathrm{Spec}(A) \subseteq \mathrm{Spec}(R/\pi R)$. Before continuing with the proof of 6.5 we relate the tube (21) to the bornological completions $\overline{\underline{R}_{J,\alpha}}$ using the tube algebras of Definition 3.1.19.

Lemma 6.6. *For $m \in \mathbb{N}_{\geq 1}$ the algebra $\overline{\underline{R}_{J,1/m}}$ is a dagger K-algebra. The affinoid dagger space $\mathrm{Sp}(\overline{\underline{R}_{J,1/m}})$ is an admissible open subset of X contained in $]Z[$, and*

$$]Z[= \bigcup_{m \geq 1} \mathrm{Sp}(\overline{\underline{R}_{J,1/m}})$$

is an admissible covering.

Proof. Choose generators $g_1, \ldots, g_s$ of the ideal $J^m \lhd R$ and set

$$S_m := R[y_1, \ldots, y_s]/\langle g_i - \pi y_i, i = 1, \ldots, s\rangle.$$

Then $K \otimes S_m^\dagger$ is a dagger K-algebra, and by (20) and the remarks there, the affinoid dagger space $\mathrm{Sp}(K \otimes S_m^\dagger)$ is the special open subset

$$[Z]'_{\epsilon^{1/m}} := \{x \in X : |g_i(x)| \leq \epsilon \text{ for all } i = 1, \ldots, s\}$$

of X. Let $f_1, \ldots, f_r$ be generators of the ideal J. We use the notation introduced in Remark 6.2. From the inclusions of ideals $J^{mr} \subseteq \langle f_1^m, \ldots, f_r^m \rangle \subseteq J^m$ we deduce that $[Z]'_{\epsilon^{1/m}} \subseteq [Z]_{\epsilon^{1/m}} \subseteq [Z]'_{\epsilon^{1/mr}}$. Since the $[Z]_{\epsilon^{1/n}}$, $n \in \mathbb{N}_{\geq 1}$, form an admissible covering of $]Z[$ by Remark 6.2, it follows formally that also the $[Z]'_{\epsilon^{1/m}}$, $m \in \mathbb{N}_{\geq 1}$, form an admissible covering of the tube $]Z[\subseteq X$.

Hence it is enough to show that $K \otimes S_m^\dagger \cong \overline{\underline{R}_{J,1/m}}$. By Theorem 3.2.6 and Example 3.1.22 we have isomorphisms

$$\overline{\underline{R}_{J,1/m}} \cong K \otimes \mathcal{T}_{1/m}(R, J)^\dagger = K \otimes \mathcal{T}_1(R, J^m)^\dagger.$$

Moreover, we have a surjective homomorphism $S_m \to \mathcal{T}_1(R, J^m) \subseteq \underline{R}$ induced by $y_i \mapsto \pi^{-1}g_i$, $i = 1, \ldots, s$. It induces an isomorphism after tensoring with K. Hence $\mathcal{T}_1(R, J^m) \cong S_m/I$, where $I \lhd S_m$ is the ideal of V-torsion elements. Lemma 3.2.4 implies $S_m^\dagger/IS_m^\dagger \cong \mathcal{T}_1(R, J^m)^\dagger$ and hence $K \otimes S_m^\dagger \cong K \otimes \mathcal{T}_1(R, J^m)^\dagger$, as desired. $\square$

Recall that the rigid cohomology of A is defined as the cohomology of the de Rham complex $\Omega^\dagger_{]Z[/K}$ on the tube $]Z[$. We use the admissible covering of $]Z[$ from Lemma 6.6 to compute this cohomology. To simplify notation, we set $U_m = \mathrm{Sp}(\overline{\underline{R}_{J,\,^1\!/_m}}) \subseteq]Z[$. Explicitly, we have $\Gamma(U_m, \Omega^{\dagger,l}_{]Z[/K}) \cong \overline{\underline{R}_{J,\,^1\!/_m}} \otimes_{\underline{R}} \Omega^l_{\underline{R}}$.

Let $\Omega^\dagger_{]Z[/K} \to \mathcal{I}$ be a Cartan–Eilenberg injective resolution in $\mathcal{O}_{]Z[}$-Mod. Then $H^*(]Z[, \Omega^\dagger_{]Z[/K})$ is the cohomology of the complex of global sections $\Gamma(]Z[, \mathcal{I})$.

Let $\mathrm{Vect}_K^{\mathbb{N}^{\mathrm{op}}}$ be the category of contravariant functors $\mathbb{N} \to \mathrm{Vect}_K$ with natural transformations as morphisms. The section functor $\Gamma(]Z[, -): \mathcal{O}_{]Z[}$-Mod $\to$ $\mathrm{Vect}(K)$ from sheaves to vector spaces factors as the composite of the functor

$$\Gamma(U_\bullet, -): \mathcal{O}_{]Z[}\text{-Mod} \to \mathrm{Vect}(K)^{\mathbb{N}^{\mathrm{op}}}, \qquad \mathcal{S} \mapsto \{\Gamma(U_m, \mathcal{S}_{|U_m})\}_m,$$

followed by

$$\varprojlim: \mathrm{Vect}(K)^{\mathbb{N}^{\mathrm{op}}} \to \mathrm{Vect}(K).$$

Since the U_m are affinoid and the higher cohomology of coherent sheaves on affinoid dagger spaces vanishes (see [17, Proposition 3.1]), the map $\Gamma(U_\bullet, \Omega^\dagger_{]Z[/K}) \to \Gamma(U_\bullet, \mathcal{I})$ is a levelwise quasi-isomorphism.

Write holim for the homotopy limit construction of a projective system of complexes as explained around (18). As already mentioned, it respects levelwise quasi-isomorphisms. Collecting what we have done so far, we get quasi-isomorphisms

$$\begin{aligned}
\mathsf{cdR}^{\mathrm{rig}}(A) &\simeq \operatorname*{holim}_m \overline{\underline{R}_{J,\,^1\!/_m}} \otimes_{\underline{R}} (\Omega^*_{\underline{R}}, d) \\
&\cong \operatorname*{holim}_m \Gamma(U_m, \Omega^\dagger_{]Z[/K}) \\
&\simeq \operatorname*{holim}_m \Gamma(U_m, \mathcal{I}).
\end{aligned}$$

Since each $\mathcal{I}^l$ is injective, the maps $\Gamma(U_{m+1}, \mathcal{I}^l) \to \Gamma(U_m, \mathcal{I}^l)$ are surjective. This implies that $\operatorname{holim}_m \Gamma(U_m, \mathcal{I})$ is quasi-isomorphic to $\varprojlim_m \Gamma(U_m, \mathcal{I}) \cong \Gamma(]Z[, \mathcal{I})$; taking cohomology, we get the theorem. $\qquad\square$

7   A quick route to the construction of complexes computing rigid cohomology

In Sections 2 and 3, we have developed a general conceptual framework for treating the kind of generalized weak completions that are relevant to our approach to rigid cohomology. It should be noted however that much of the material in these sections can be avoided if one is interested only in a natural description of a complex computing rigid cohomology. Also, only part of the results in Sections 4 and 5 are needed for that purpose. We now sketch a more direct route to the main result in Section 6.

DEFINITION 7.1. Let R be a V-algebra and J an ideal in R that contains π^k for some $k \in \mathbb{N}$. We define the J-adic bornology on $\underline{R}$ as the bornology generated

by subsets S of the form

$$S = C \sum_{n \geq 0} \lambda_n M^n,$$

where $C \in K$, M is a finitely generated V-submodule of J and $\lambda_n \in K$ are such that $|\lambda_n| \leq \epsilon^{-\alpha n}$ for some $\alpha < 1$.

Let $\overline{\underline{R}_J}$ be the bornological completion of $\underline{R}$ with respect to this bornology. The easy fact that this completion coincides with $\overline{\underline{R}_{J,1}}$, as defined in Definition 3.1.7 (for the fine bornology on R), is explained in Lemma 3.1.11.

Also the following is not difficult to see (compare Proposition 3.2.5).

PROPOSITION 7.2. *If R is finitely generated and $J = \pi R$, then $\overline{\underline{R}_J} = \underline{R}^{\dagger}$.*

Now let A be a finitely generated, commutative k-algebra. We choose a presentation $J \to R \to A$ by a free commutative V-algebra R. We obtain a projective system

$$\overline{\underline{R}_{J^{n+1}}} \to \overline{\underline{R}_{J^n}}$$

This defines a bornological pro-K-algebra $\left(\overline{\underline{R}_{J^n}} \right)$.

DEFINITION 7.3. We define

$$\mathrm{HP}_*(A) = \mathrm{HP}_*\left(\left(\overline{\underline{R}_{J^n}} \right) \right) \tag{22}$$

Here HP_* for a pro-algebra is defined as the homology of the homotopy limit of the projective system of cyclic $B-b$-complexes as in Definition 5.1.

We briefly explain why this definition of $\mathrm{HP}_*(A)$ gives the same result as $\mathsf{hc}^{\mathrm{rig}}(A)$ in Definition 5.1, and thus also describes $H^{\mathrm{rig}}_*(A, K)$ if A is finitely generated.

First, by definition, one trivially has $\overline{\underline{R}_{J^n,1}} = \overline{\underline{R}_{J,1/n}}$. The less trivial result proved in Proposition 3.1.21, is that $\overline{\underline{R}_{J^n}} = \mathcal{T}_{1/n}(R, J)^{\dagger} \otimes K$. The proof of Proposition 3.1.21 also slightly simplifies for the case $\alpha = 1/n$ that we need here. This identification of $\overline{\underline{R}_{J^n}}$ with the completed tube algebra yields an isomorphism of $\mathrm{HP}_*(\overline{\underline{R}_J})$ with the homology of the de Rham complex $\left(\overline{\underline{R}_J} \otimes_{\underline{R}} \Omega^*_{\underline{R}}, d \right)$ as in (14).

Finally, the discussion in Section 6 remains valid – we simply have to replace $\overline{\underline{R}_{J,1/m}}$ there by $\overline{\underline{R}_{J^m,1}} = \overline{\underline{R}_{J^m}}$. We obtain that $\mathrm{HP}_*(A)$, as defined in (22), describes the rigid cohomology of A made 2-periodic.

Of course, in this discussion, one could also directly replace the homotopy limit of the cyclic complexes by the homotopy limit of the corresponding de Rham complexes and thus avoid using many of the results in Sections 4 and 5.

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Guillermo Cortiñas
Dep. Matemática-IMAS
FCEyN-Universidad de
 Buenos Aires
Ciudad Universitaria
C1428EGA Buenos Aires
Argentina
gcorti@dm.uba.ar

Joachim Cuntz
Mathematisches Institut
Westfälische Wilhelms
 Universität Münster
Einsteinstr. 62
48149 Münster
Germany
cuntz@uni-muenster.de

Ralf Meyer
Mathematisches Institut
Georg-August Universität
 Göttingen
Bunsenstraße 3–5
37073 Göttingen
Germany
rmeyer2@uni-goettingen.de

Georg Tamme
Universität Regensburg
Fakultät für Mathematik
93040 Regensburg
Germany
georg.tamme@ur.de

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