Solvable Lattice Models and the Geometry of Flag Varieties

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Dedication

To my family.

Abstract

Geometric constructions of quantum groups and their associated R-matrices arose in the early 90's and have been generalized further in recent works of Aganagic, Maulik, and Okounkov [1, 2], creating a bridge between geometry and solvable lattice models. One nice aspect of this bridge is that the "hard" basis of one theory corresponds to the "easy" basis of the other. In this thesis, we explore various lattice models using this perspective as guidance. We first describe how both the torus fixed point basis and the basis of Schubert classes in the equivariant cohomology of the flag variety are manifest in the "Frozen Pipes" lattice model of [3]. This analysis is a straightforward generalization of results due to Gorbunov, Korff, and Stroppel [4] (see also the notes [5] of Zinn-Justin) for the Grassmannian.

Then we describe how the fixed point basis and the basis of motivic Chern classes in the equivariant K-theory of the cotangent bundle of the flag variety appear (in a more novel way) in the Tokuyama model of [6] and colored Iwahori Whittaker model of [7]. The recent work [8] of Aluffi, Mihalcea, Schürmann, and Su identifies these geometric bases with the Casselman and standard bases, respectively, of the Iwahori fixed vectors in the principal series representation, so this perspective allows us to make contact with formulas from p-adic representation theory, such as the Langlands-Gindikin-Karpelevich formula.

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Chapter 1

Introduction

Solvable lattice models, while initially created in the context of statistical mechanics, have proven useful in the study of many functions from representation theory and Schubert calculus. In statistical mechanics, one is interested in determining the global properties of a lattice of particles via the local particle interactions. The local interactions are encoded in functions called *Boltzmann weights* assigned to each particle, and the global *partition function* of the model is the generating function made from summing over the weights of potential particle configurations. A lattice model is considered *solvable* (or *integrable*) if there exists a system of solutions to the (quantum) Yang-Baxter equation that allow one to *solve the model*: Baxter [10] recognized that, given a solution to the Yang-Baxter equation, one is able to prove recursive formulas for the partition function and thus potentially get a handle on its closed-form solution.

Being able to express a function as the partition function of a solvable lattice model then naturally leads to other interesting combinatorial properties, such as branching rules, exchange rules under the action of Hecke divided-difference operators, and Cauchy- and Pieri-type identities. Remarkably, Zinn-Justin [11], Wheeler and Zinn-Justin [12], and Knutson and Zinn-Justin [13, 14] have also used solvability to prove the Littlewood-Richardson rule and some of its generalizations, long considered the most important and most difficult formulas in Schubert calculus. The fact that lattice models encode lots of algebraic structure simultaneously is one of the main sources of their appeal. Moreover, since quantum groups are a natural source of solutions to the Yang-Baxter equation, there is a strong link between solvable lattice models and quantum groups.

Going one step further, quantum groups and lattice models can often be understood as arising via a *geometric* construction of solutions to the quantum YBE. This construction allows one to identify certain (generalized) cohomology rings with Bethe algebras of quantum groups, as summarized in Figure 4.2, reproduced from the paper [9]; also shown in Figure 1.1.

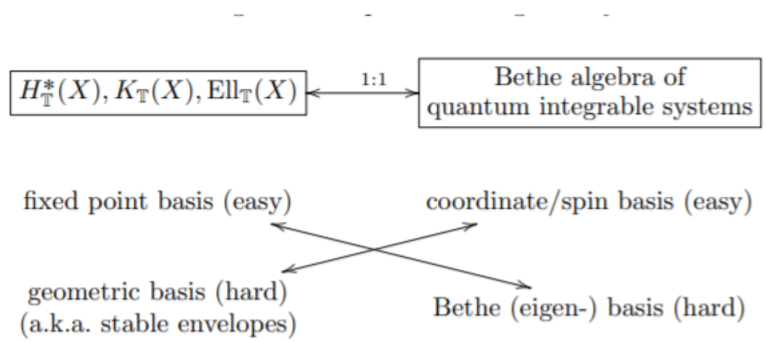


Figure 1.1: From [9].

This thesis aims to concretely flesh out these connections for certain existing lattice models, namely, the Frozen Pipes model of [3] and the Tokuyama and Iwahori models of [6, 7], whose partition functions are (β)-Grothendieck polynomials, spherical Whittaker functions, and Iwahori Whittaker functions, respectively.

We now outline the structure of this thesis and summarize our main results. Chapter 2 gives an overview of the theory of both spherical and Iwahori Whittaker functions in the context of p-adic representation theory. Chapter 3 reviews the general mechanics of solvable lattice models and briefly introduces the specifics of the models in [3, 6, 7]. Chapter 4 introduces the necessary background on Schubert calculus and its generalizations. Chapter 5 details our first concrete connection between the material in Chapters 3 and 4: in Proposition 5 we derive the R-matrix (the solution to the YBE) of the Frozen Pipes model as a change of basis matrix in equivariant cohomology, and in Propositions 5.6 and 5.7, we employ the algebraic Bethe ansatz to identify the partition function with the geometric Poincaré pairing between a Schubert class and the class of a torus fixed point; consequently, we are able to easily calculate the coefficients of the expansion of the fixed point basis into the Schubert basis. These results are likely well-known to experts, as they generalize those of [4, 5].

In the same vein, but with notable differences in technique, Chapter 6 explores the geometry of the Tokuyama and Iwahori models. Theorem 6.2 gives an expression for the Tokuyama partition function (and hence the spherical Whittaker function) using a method similar to the algebraic Bethe ansatz. Alternatively, it is known that the spherical Whittaker function can be written as a sum of Iwahori Whittaker functions. Equating this sum with the expression in Theorem 6.2, and using geometric results of Aluffi, Mihalcea, Schürmann, and Su [8], we obtain Theorem 6.6 and Corollary 6.1, which results in the expansion of a (modified) motivic Chern class–a generalization of a Schubert class–into the fixed point basis of the equivariant K-theory ring of the flag variety, as well as interpretations of the partition functions as K-theoretic Poincaré pairings with line bundles. In p-adic representation theory, this expansion is closely related to the Langlands-Gindikin-Karpelevich formula. We then end by comparing and contrasting the results of Chapter 5 and Chapter 6.

Chapter 2

Background on Whittaker functions

We review the construction of spherical Whittaker functions and Iwahori Whittaker functions, mostly following the presentation in [7]. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$. (Common examples are finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers, or the field of Laurent series over $\mathbb{F}_p$.) Let $\mathfrak{p} = \langle \varpi \rangle$ be the maximal ideal of $\mathfrak{o}$ with uniformizer ϖ. Then the residue field $\mathfrak{o}/\mathfrak{p}$ is isomorphic to a finite field, whose cardinality we denote by q.

Let $\widehat{G}$ be a split reductive Chevalley group, i.e., an affine algebraic group scheme over $\mathbb{Z}$ with a fixed Chevalley basis for its Lie algebra $\mathfrak{g}_{\mathbb{Z}}$. (We use the notation $\widehat{G}$, instead of G, since we will primarily be working with the Langlands dual group $\widehat{\widehat{G}} = G$. In addition, when working with lattice models, we will only be considering the case $\widehat{G} = GL_n$, but it is worth noting that many of the following definitions and results hold in this most general case.) Let T be the maximal split torus of $\widehat{G}$ obtained from our choice of Chevalley basis, and let N be the maximal unipotent subgroup of $\widehat{G}$ whose Lie algebra is the union of the positive root spaces. These form the standard Borel subgroup $B = TN$. Let $W = N_{\widehat{G}}(T)/T$ be the Weyl group of G, where $N_{\widehat{G}}(T)$ denotes the normalizer of T in $\widehat{G}$. Since $\widehat{G}$ is defined over $\mathbb{Z}$, we can consider the group $\widehat{G}(F)$ of F-points of $\widehat{G}$. The Iwahori subgroup J_- of $G(F)$ is the subgroup of $K = \widehat{G}(\mathfrak{o})$ defined as the preimage of $B_- := w_0 B w_0$ in the mod $\mathfrak{p}$ reduction map from K to $\widehat{G}(\mathbb{F}_q)$.

For GL_n, B is the subgroup of upper triangular matrices, T is the diagonal matrices, $W \cong S_n$ is the group of permutation matrices, and J_- consists of matrices in $GL_n(\mathfrak{o})$ which are lower triangular mod $\mathfrak{p}$.

We will also need to consider the Langlands dual group $\widehat{\widehat{G}} =: G$ of $\widehat{G}$, whose root system is dual to that of $\widehat{G}$. We denote the root system of G by Δ and the root system of $\widehat{G}$ by $\Delta^\vee$. Denote the simple roots of Δ by $\alpha_1, \ldots, \alpha_n$, and the sets of positive and negative roots by $\Delta+$ and $\Delta-$, respectively.

Consider an unramified character τ, i.e., a character $\tau : T(F) \to F^\times$ that is trivial on $T(\mathfrak{o})$. The group of such characters is isomorphic to $\widehat{T}(\mathbb{C}) \cong (\mathbb{C}^\times)^n$. Indeed, for each $\boldsymbol{z} \in \widehat{T}(\mathbb{C})$, we will now describe how to obtain the corresponding unramified character $\tau_{\boldsymbol{z}}$. First, we note that the group $X_*(T)$ of rational cocharacters of T is isomorphic to $T(F)/T(\mathfrak{o})$: to a cocharacter $\lambda : F^\times \to T(F)$ we associate the coset $\varpi^\lambda T(\mathfrak{o})$, where ϖ^λ is the image of ϖ under λ. On the other hand, by the definition of dual root systems, $X_*(T)$ is isomorphic to the weight lattice $\Lambda := X^*(\widehat{T})$. Considering λ now as an element of Λ, let $\boldsymbol{z}^\lambda \in F^\times$ denote the image of $\boldsymbol{z} \in \widehat{T}(\mathbb{C})$ under λ. Then we define $\tau_{\boldsymbol{z}}$ by $\tau_{\boldsymbol{z}}(t) = \boldsymbol{z}^\lambda$ when $t \in \varpi^\lambda T(\mathfrak{o})$.

For $\widehat{G} = GL_n$, with $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \cong \Lambda$ and $\boldsymbol{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n \cong \widehat{T}(\mathbb{C})$, we have

$$\varpi^\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & & \\ & \varpi^{\lambda_2} & & \\ & & \ddots & \\ & & & \varpi^{\lambda_n} \end{pmatrix} \in GL_n(F) \qquad \text{and} \qquad \tau_{\boldsymbol{z}}(\varpi^\lambda) = \boldsymbol{z}^\lambda = \prod_{i=1}^n z_i^{\lambda_i}.$$

We can trivially extend $\tau_{\boldsymbol{z}}$ to $B(F)$. Then the **principal series representation** $(\pi, I(\boldsymbol{z}))$ is the induced representation

$$I(\boldsymbol{z}) := \mathrm{Ind}_B^{\widehat{G}}(\delta^{1/2}\tau_{\boldsymbol{z}})$$

under the the right-regular action π of $\widehat{G}(F)$, where $\delta^{1/2}$ is the modular quasicharacter of B. In other words, $I(\boldsymbol{z})$ consists of locally constant functions f on $\widehat{G}(F)$ such that $f(bg) = \delta^{1/2}(b)\tau_z(b)f(g)$ for all $b \in B$.

The space $I(\boldsymbol{z})^{J_-}$ of Iwahori fixed vectors generates $I(\boldsymbol{z})$ as a $\widehat{G}(F)$-module [15] and has dimension $\dim I(\boldsymbol{z}) = |W|$. There are two well-known bases of $I(\boldsymbol{z})^{J_-}$ that we will be interested in. The first, the **standard basis** $\{\Phi_w^{\boldsymbol{z}}\}_{w\in W}$, consists of the characteristic functions on the double cosets in the decomposition

$$\widehat{G}(F) = \bigsqcup_{w\in W} B(F)wJ_-,$$

obtained by combining the Bruhat decomposition $\widehat{G}(F) = \bigsqcup_{w\in W} B(F)wB(F)$ and the Iwahori factorization $J_- = N(\mathfrak{p})T(\mathfrak{o})N_-(\mathfrak{o})$ (where N_- is the opposite maximal unipotent subgroup). In other words, for $b \in B(F), w' \in W$, and $k \in J_-$,

$$\Phi_w^{\boldsymbol{z}}(bw'k) = \begin{cases} \delta^{1/2}\tau_{\boldsymbol{z}}(b) & \text{if } w' = w \\ 0 & \text{otherwise.} \end{cases}$$

The second, called **Casselman's basis**, is defined in terms of duality with the standard intertwining operators $\mathcal{A}_w^{\boldsymbol{z}} : I(\boldsymbol{z}) \to I(w\boldsymbol{z})$, defined below. To avoid technical issues with the poles and zeroes of $\mathcal{A}_w^{\boldsymbol{z}}$, we will assume that $\boldsymbol{z}^{\alpha^\vee} \neq 1, q^{\pm 1}$ for all α. Then:

$$\mathcal{A}_w^{\boldsymbol{z}}\Phi(\boldsymbol{z}) = \int_{N(F)\cap wN_-(F)w^{-1}} \Phi(w^{-1}ng)dn.$$

This integral converges for $|\boldsymbol{z}^\alpha| < 1$ for $\alpha \in \Delta^+$, and can be meromorphically continued to arbitrary $\boldsymbol{z}$. It defines an induced map $\mathcal{A}_w^{\boldsymbol{z}} : I(\boldsymbol{z})^{J_-} \to I(w\boldsymbol{z})^{J_-}$ on the Iwahori fixed vectors, and Casselman's basis is defined by the condition that:

$$\mathcal{A}_w^{\boldsymbol{z}}(f_v)(1) = \delta_{v,w}.$$

These functions are difficult to calculate explicitly, but have proven to be very useful regardless. They were first employed in Casselman [15] and Casselman and Shalika [16] to calculate Macdonald's formula for the zonal spherical function and the Casselman-Shalika formula for the spherical Whittaker function, respectively. For their methods, one only needs to know f_{w_0} explicitly, and in [15], Proposition 3.7 it is shown that:

$$f_{w_0} = \Phi_{w_0}.$$

The rest of their calculations then follow from functional equations. The action of $A_w^{\boldsymbol{z}}$ on the standard basis can be calculated explicitly; see Proposition 3 in [17], which is equivalent to Theorem 3.4 in [15].

We now define the *Iwahori Whittaker functions*, which are obtained by applying the Whittaker functional to right translates of standard basis elements. First, we need some setup in order to define this functional. Let $\alpha \in \Delta$ be a root of G, and let $x_\alpha : \mathbb{G}_a \to \widehat{G}$ be the one-parameter subgroup of $\widehat{G}$ associated to $\alpha^\vee$, i.e., $x_\alpha(t) = \exp(tX_\alpha)$, where X_α is a Chevalley basis element of the Lie algebra. Fix a unitary character ψ on $N_-(F)$ such that, for any $\alpha^\vee$, $\psi \circ x_\alpha : F \to \mathrm{c}^\times$ is trivial on o but no larger fractional ideal. Then the space of **Whittaker functionals** consists of linear maps $\Omega_{\boldsymbol{z}} : I(\boldsymbol{z}) \to \mathrm{c}$ satisfying $\Omega_{\boldsymbol{z}}(\pi(n_-)f) = \psi(n_-)\Omega_{\boldsymbol{z}}(f)$ for $n_- \in N_-(F)$. It is well-known that this space is one-dimensional [18, 19], so we need only consider the following explicit Whittaker functional:

$$\Omega_{\boldsymbol{z}}(f) := \int_{N_-(F)} f(n)\psi(n)^{-1}\, dn, \qquad f \in I(\boldsymbol{z}),$$

which is convergent if $|\boldsymbol{z}^\alpha| < 1$ for positive roots α, and can be analytically continued to all $\boldsymbol{z}$.

Then the Iwahori Whittaker functions $\phi_w : \widehat{G}(F) \to \mathrm{c}^\times$, up to conventions and normalizations used for convenience, are defined by

$$\phi_w(\boldsymbol{z}; g) := \delta^{-1/2}(g)\Omega_{\boldsymbol{z}^{-1}}(\pi(g)\Phi_w^{\boldsymbol{z}^{-1}}). \tag{2.1}$$

Since $\Omega_{\boldsymbol{z}^{-1}}$ is left $N_-(F)$-invariant and $\Phi_w^{\boldsymbol{z}^{-1}}$ is right J-invariant, these functions are determined by their values at $\varpi^{-\lambda}w$ for $\varpi^{-\lambda} \in T(F)/T(\mathfrak{o})$ and $w \in W$. Similarly, the *spherical Whittaker function* W is defined by applying the Whittaker functional to right translates of the (unique) $K = \widehat{G}(\mathfrak{o})$-fixed vector $\Phi \in I(\boldsymbol{z})^K$. Unlike the Iwahori Whittaker functions, the spherical Whittaker functions are determined solely by their values at torus elements $\varpi^{-\lambda}$:

$$W(\boldsymbol{z}; \varpi^\lambda) := \delta^{-1/2}(\varpi^{-\lambda})\Omega_{\boldsymbol{z}^{-1}}(\pi(\varpi^{-\lambda}\Phi))s.$$

(And in fact, W vanishes unless λ is a dominant weight.) The spherical and Iwahori

Whittaker functions are related via:

$$W(\boldsymbol{z};\varpi^{-\lambda}) = \sum_{w\in W} \phi_w(\boldsymbol{z};\varpi^{-\lambda}) \tag{2.2}$$

(see, for example, [7] Proposition 4.8). This is a deformation of the Demazure character formula for Schur functions.

Brubaker, Bump, Buciumas, and Gustafsson, in §2 of [7] show, inspired by methods in [16], that one can compute $\phi_{w_1}(\boldsymbol{z};\varpi^{\lambda}w_2)$ recursively via what they call *Demazure-Whittaker operators*, generalizing the results from [17] which calculated the values of ϕ_w on $\varpi^{-\lambda}$ only. The idea is to prove the following base case:

$$\phi_w(\boldsymbol{z};w^{-\lambda}w) = q^{-l(w)}\boldsymbol{z}^{\lambda}$$

and then develop the recursion by computing $\Omega_{\boldsymbol{z}}(\mathcal{A}_{s_i}\cdot\Phi_w)$ in two ways and comparing the results.

We will omit the details of these calculations, but now describe the resulting operators. Let $\mathcal{O}$ be the ring of polynomial functions on $\widehat{T} = (\mathrm{c}^{\times})^n$, which is isomorphic to the group algebra of $\mathbb{Z}^n$ and is spanned by the functions $\boldsymbol{z}^{\lambda}$. Let $v \in \mathrm{c}^{\times}$ and $f \in \mathcal{O}$. The **Demazure-Whittaker operators** $\mathfrak{T}_{i,v} = \mathfrak{T}_i$ are defined by:

$$\mathfrak{T}_{\mathrm{i}} f(\boldsymbol{z}) = \frac{(1-v)}{\boldsymbol{z}^{\alpha_i}-1}f(\boldsymbol{z}) - \frac{(1-v\boldsymbol{z}^{\alpha_i})}{\boldsymbol{z}^{\alpha_i}-1}f(s_i\boldsymbol{z}).$$

These operators satisfy the braid relations, as well as the quadratic relation

$$\mathfrak{T}_i^2 = (v-1)\mathfrak{T}_i + v,$$

which implies that $\mathfrak{T}_i$ is invertible. Its inverse is:

$$\mathfrak{T}_i^{-1} f(\boldsymbol{z}) = \frac{\boldsymbol{z}^{\alpha_i}(v^{-1}-1)}{\boldsymbol{z}^{\alpha_i}-1}f(\boldsymbol{z}) + \frac{(v^{-1}-\boldsymbol{z}^{-\alpha_i})}{\boldsymbol{z}^{\alpha_i}-1}f(s_i\boldsymbol{z}).$$

Thus, these operators generate a finite Iwahori Hecke algebra. See the published version of [17] for proofs of these facts and further investigation of the Hecke algebra structure.

We have the following recursion:

Proposition 1 ([17] Theorem 2; [7] Proposition 2.4) *For any $w \in W$, simple reflection s_i, and with $v = q^{-1}$,*

$$\phi_{s_i w}(\boldsymbol{z}; g) = \begin{cases} \mathfrak{T}_i \cdot \phi_w(\boldsymbol{z}; g) & \text{if } l(s_i w) > l(w) \\ \mathfrak{T}_i^{-1} \cdot \phi_w(\boldsymbol{z}; g) & \text{if } l(s_i w) < l(w) \end{cases} \tag{2.3}$$

Note that since the $\mathfrak{T}_i$ satisfy the braid relations, this recursion does not depend on the choice of reduced word for w.

Later, we will also make use of the well-known **Demazure-Lusztig operators** $\mathfrak{L}_i$ of [20], which are very similar and also arose in the context of Hecke algebra representations:

$$\mathfrak{L}_i f(\boldsymbol{z}) = \frac{1 - v}{\boldsymbol{z}^{\alpha_i} - 1} f(\boldsymbol{z}) + \frac{v \boldsymbol{z}^{\alpha_i} - 1}{\boldsymbol{z}^{\alpha_i} - 1} f(s_i \boldsymbol{z}).$$

Chapter 3

Brief introduction to lattice models

In this section, we will illustrate the general setup of solvable lattice models by briefly describing the six-vertex "Tokuyama" model from [6], whose partition function is (up to a factor) the spherical Whittaker function described in §2, and whose degeneration to a five-vertex model gives the Schur polynomial. We will then introduce two models obtained by modifying the Tokuyama model: the six vertex Iwahori model of [7], whose partition function is an Iwahori Whittaker function, and the five vertex Frozen Pipes model of [3], whose partition function is a β-Grothendieck polynomial. The results about these models in subsequent sections should be viewed as supplementary to those described in the papers [7, 6, 3].

Consider a finite two-dimensional square grid as depicted below in Figure 3.1. To each edge we assign a value of $+$ or $-$, called a *spin*. For a specific system, boundary spins are fixed, and a *state* of the system is a specification of interior spins. In statistical mechanics, one is interested in determining the global properties of this system, such as its free energy and the probability of it being in a certain state, via the local interactions at each vertex. The local information is contained in certain *Boltzmann weights* that are assigned to each type of vertex, and the global information can be ascertained from the *partition function*:

$$\sum_{\mathfrak{s}} \prod_{v \in \mathfrak{s}} w(v),$$

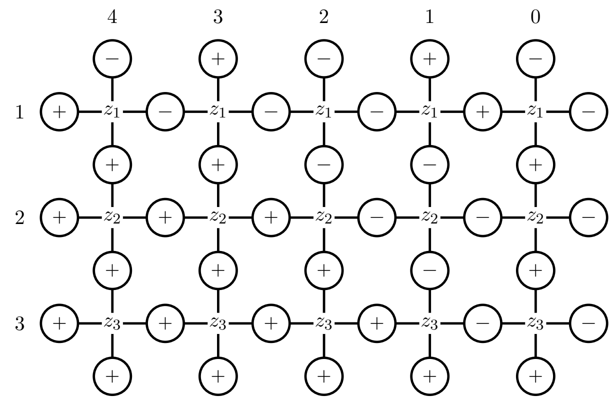


Figure 3.1: A state of a lattice model.

where $\mathfrak{s}$ denotes an admissible state (i.e. one whose weights are nonzero) of the model, v a vertex in the state $\mathfrak{s}$, and $w(v)$ its corresponding Boltzmann weight. We also call the product $\prod_{v\in\mathfrak{s}} w(v)$ the Boltzmann weight of $\mathfrak{s}$ and denote it $\mathfrak{w}(\mathfrak{s})$. The labels z_i associated to the rows of the lattice are called *spectral parameters*, and the weights of the vertices often depend on these parameters.

As mentioned in the introduction, a lattice model is considered *solvable* or *integrable* if the Boltzmann weights satisfy a Yang-Baxter equation (YBE). The YBE can then be used to prove functional equations for the partition function. We will now describe the Boltzmann weights, boundary conditions, YBEs, and resulting functional equations for each of our specific models.

3.0.1 The Tokuyama model

Recall that for $\widehat{G} = GL_n$, the nonzero spherical Whittaker functions $W_\lambda(\boldsymbol{z}) := W(\boldsymbol{z};\varpi^{-\lambda})$ are indexed by dominant weights, i.e., partitions λ of n. For a fixed k, let $\rho = (k-1, k-2, \ldots, 1, 0)$. Then for a given $\lambda = (\lambda_1, \ldots, \lambda_k)$, the Tokuyama model has k rows, labeled 1 through k from top to bottom, and $\lambda_1 + k$ columns, labeled 0 to $\lambda_1 + k - 1$ from right to left. The right boundary is filled with $-$ spins, the bottom and left boundaries with $+$ spins, and the top boundary with $-$ spins in the columns labeled by the parts of $\lambda+\rho$ and $+$ spins elsewhere. Recall that q denotes the cardinality of the residue field $\mathfrak{o}/\mathfrak{p}$. The Boltzmann weights are given by Figure 3.2. As a useful perspective, one can interpret the entries of Figure 3.2 as encoding matrix coefficients of an endomorphism of $V \otimes V$ for a two-dimensional vector space $V = \mathrm{span}\{v_+, v_-\}$.

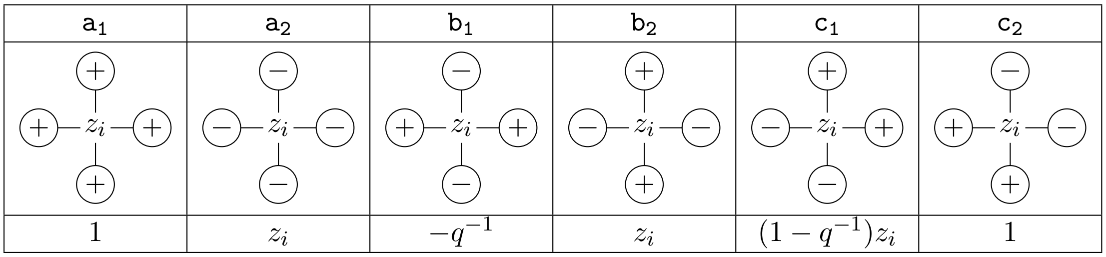


Figure 3.2: (Figure 7 of [7], arXiv v2, see also [6].) Boltzmann weights for vertices in row i of the Tokuyama model. If a vertex configuration does not appear in this table, its Boltzmann weight is zero.

For the Yang-Baxter equation, we consider the following set of rotated vertices (Figure 3.3, which we call R-vertices. Indeed, the R-vertices are the matrix coefficients of (a Drinfeld twist of) the R-matrix for the quantum group $U_q(\widehat{\mathfrak{gl}}(1|1))$ [7].

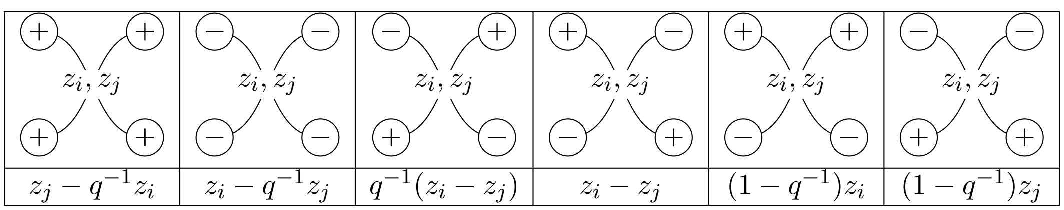
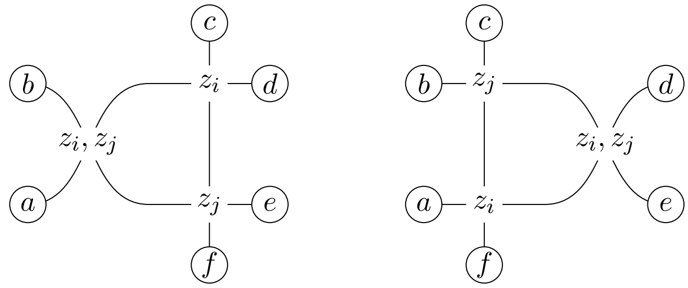


Figure 3.3: (Figure 8 in [7], see also BBF) The R-vertex weights for the Tokuyama model.

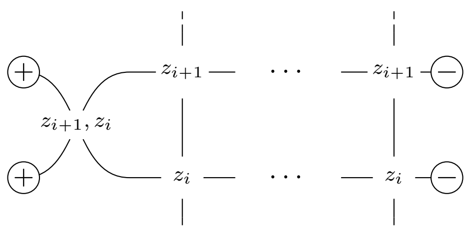
With this choice of R-vertex weights, the Tokuyama model satisfies the Yang-Baxter equation (sometimes also called the star-triangle relation, or RTT relation, in this context), meaning that for any choice of boundary conditions (a, b, c, d, e, f), we have the following equality of partition functions.



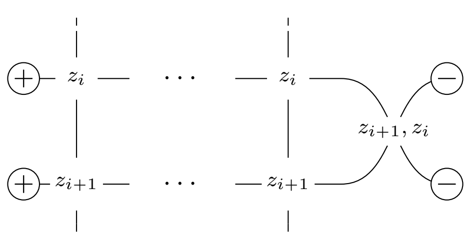
(3.1)

As mentioned above, the Yang-Baxter equation allows one to prove functional equations for the partition function. Let $\mathfrak{S}_\lambda(\boldsymbol{z})$ denote the Tokuyama lattice system (i.e., the choice of the lattice shape, Boltzmann weights, and boundary conditions) and $Z(\mathfrak{S}_\lambda(\boldsymbol{z}))$ the partition function of the system. In this case, we can apply the YBE to show that $(z_{i+1} - qz_i)Z(\mathfrak{S}_\lambda(\boldsymbol{z}))$ is symmetric as follows ([6], Lemma 4).

We modify the vertex model for $Z(\mathfrak{S}_\lambda(\boldsymbol{z}))$ by interchanging the spectral parameters z_i and z_{i+1} and attaching an R vertex to the left:



Consulting Figure 3.3, we see that the only admissible possibility for the R vertex is the one in which all spins are $+$, and hence the partition function for this model is $(z_i - qz_{i+1})Z(\mathfrak{S}_\lambda(s_i\boldsymbol{z}))$, where $Z(\mathfrak{S}_\lambda(s_i\boldsymbol{z}))$ denotes $(z_i - qz_{i+1})Z(\mathfrak{S}_\lambda(\boldsymbol{z}))$ with the variables z_i and z_{i+1} switched. By repeatedly applying the YBE 3.1, we see that this configuration is equal to:



which has partition function $(z_{i+1} - qz_i)Z(\mathfrak{S}_\lambda(\boldsymbol{z}))$. Thus,

$$(z_i - qz_{i+1})Z(\mathfrak{S}_\lambda(s_i\boldsymbol{z})) = (z_{i+1} - qz_i)Z(\mathfrak{S}_\lambda(\boldsymbol{z}))$$

and hence $(z_{i+1} - qz_i)Z(\mathfrak{S}_\lambda(\boldsymbol{z}))$ is symmetric.

This method of proof is standard and often referred to as the "train argument."

3.0.2 The Iwahori model

The Iwahori model is obtained by adding an extra piece of data–color–to each edge in the Tokuyama model. Specifically, we replace the $-$ spins by n distinct ordered colors, which we identify with the integers $1 < \cdots < n$. In addition, vertical edges are allowed to carry multiple colors, via a process similar to fusion for quantum groups (see [7] §5). The Boltzmann weights are shown in Figure 3.4 below, where $v = q^{-1}$.

The boundary conditions now also depend on an additional piece of data, namely, a permutation $w \in S_n$. (In fact, [7] define a more general lattice model that depends on two permutations w_1 and w_2, but we will consider only $w_2 = id$ here for simplicity.) The top boundary is still dictated by the partition $\lambda + \rho$, as for the Tokuyama model, but now the $-$ spins (in the unfused model) are instead labeled by the colors 1 through n in order from left to right. The right boundary is labeled by the colors $n+1-w^{-1}(1)$ through $n+1-w^{-1}(n)$ from top to bottom, and the left and bottom boundaries still consist of $+$ spins.

There exist colored R-vertices and a corresponding YBE (with some additional, but manageable, subtleties due to the fusion construction) for the Iwahori model; see §6 of [7]. In contrast to that of the Tokuyama model, the partition function of the Iwahori model can actually be calculated via the train argument, using induction on the length of the permutation w. Let $\mathfrak{S}_{\lambda,w}(\boldsymbol{z})$ denote the lattice system with fixed boundary according to λ and w, and $Z(\mathfrak{S}_{\lambda,w}(\boldsymbol{z}))$ its partition function. When we apply the train argument to $Z(\mathfrak{S}_{\lambda,w}(\boldsymbol{z}))$, there are now two possibilities for the R vertex on one side of the "train," and upon rearranging the resulting equation, we obtain an expression for $Z(\mathfrak{S}_{\lambda,s_iw}(\boldsymbol{z}))$ in terms of the conjugated Demazure-Whittaker operator $\boldsymbol{z}^\rho\mathcal{T}_i\boldsymbol{z}^{-\rho}$ acting on $Z(\mathfrak{S}_{\lambda,w}(\boldsymbol{z}))$. Hence:

Proposition 2 $Z(\mathfrak{S}_{\lambda,w}(\boldsymbol{z})) = \boldsymbol{z}^\rho\phi_w(\boldsymbol{z};\varpi^{-\lambda})$.

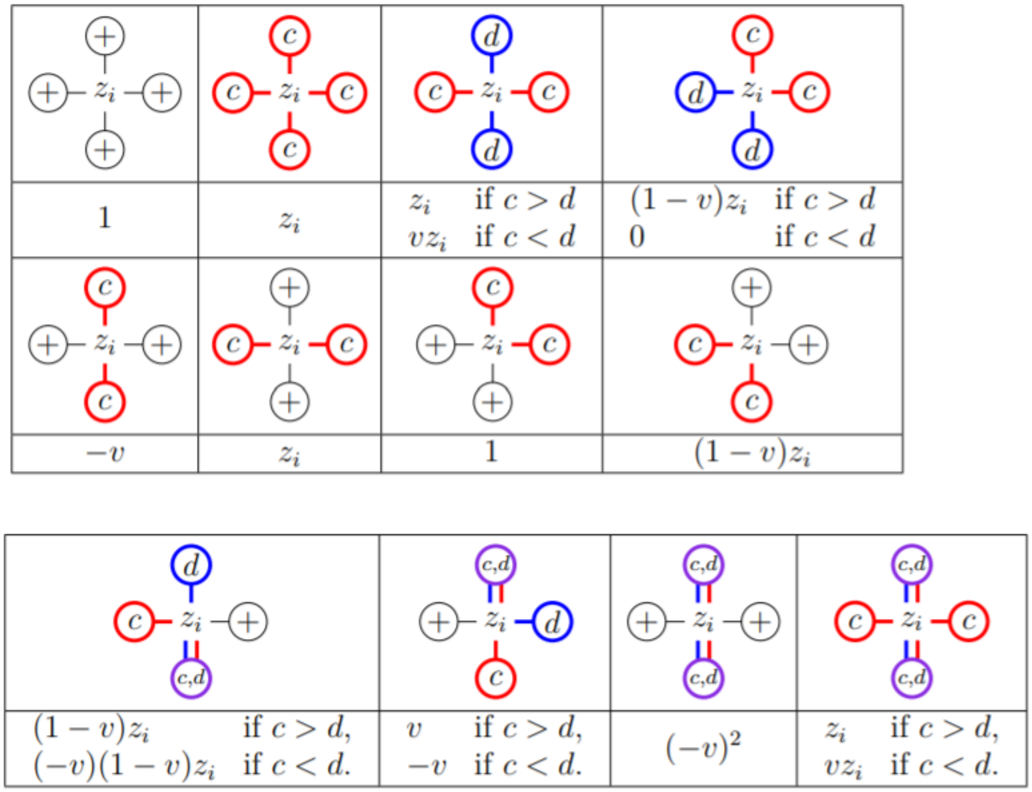


Figure 3.4: The Boltzmann weights for row i of the Iwahori model.

See Proposition 7.1 and Theorem 7.2 of [7] for more details on the proof, and for its statement in full generality.

In this way, we think of color as "refining" the Tokuyama model, since we can write the Tokuyama model associated to λ as a sum over the Iwahori models $Z(\mathfrak{S}_{\lambda,w}(\boldsymbol{z}))$ for $w \in S_n$, and, moreover, it is possible to identify each colored state with a unique uncolored state (see [7], arXiv v2, Proposition 6.3).

3.0.3 The Frozen Pipes model

The Frozen Pipes model of [3] is also related to the Tokuyama model via the addition of color to the spin set (as well as a horizontal flip of and enlargement of the lattice). However, instead of being a refinement of the Tokuyama model, the Frozen Pipes model utilizes color in order to represent a wider class of functions. Specifically, recall that when $q = 0$, the partition function of the Tokuyama model returns $\boldsymbol{z}^{\rho}$ times a Schur polynomial $s_{\lambda}(\boldsymbol{z})$. Schur polynomials are the same as Schubert polynomials associated to *Grassmannian permutations*, i.e., permutations with a single descent. The partition functions of the five-vertex Frozen Pipes model (after setting the parameter $\beta = 0$), on the other hand, calculate Schubert polynomials for arbitrary permutations. The model was inspired by the combinatorial realization of Schubert polynomials in terms of diagrams called pipe dreams. Further, for $\beta \neq 0$, we leave the connection to the Tokuyama model behind, and the partition function is a β-Grothendieck polynomial.[1] See section 4 for the initial geometric definitions of Schubert/Grothendieck polynomials, and [3] for more discussion on and references for pipe dreams. There is also discussion in [3] Remark 3.3 about how to obtain the lattices for Schur and Grassmannian Grothendieck polynomials from the Frozen Pipes lattice.

The Frozen Pipes model is defined on an $n \times n$ square lattice, and depends on two sets of spectral parameters: $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ and $\boldsymbol{y} = \{y_1, \ldots, y_n\}$, which we choose to think of as being associated to the rows and columns, respectively, of the model. We will work primarily with what is called the "pipe model" in [3], in which the left boundary is labeled with the colors 1 to n from top to bottom, the top boundary is labeled with $w^{-1}(1)$ to $w^{-1}(n)$ from left to right for a permutation $w \in S_n$, the right and bottom boundaries are labeled with $+$, and the Boltzmann weights are listed below.

$\mathtt{a}_1$	$\mathtt{a}_2^{\dagger}$	$\mathtt{a}_2^{\sharp}$	$\mathtt{a}_2^{\flat}$	$\mathtt{a}_2^{\circ}$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
1	$x_i \oplus y_j$	$1 + \beta(x_i \oplus y_j)$	1	1	$x_i \oplus y_j$	$1 + \beta(x_i \oplus y_j)$	1

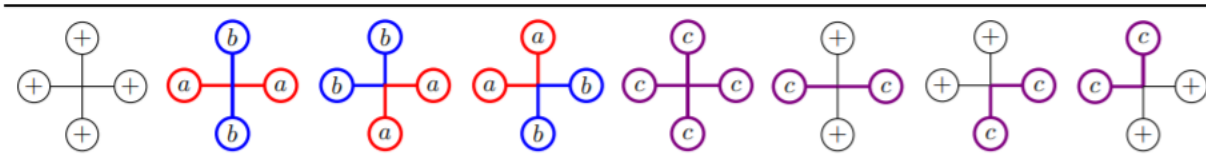


Figure 3.5: The Frozen Pipes pipe model Boltzmann weights for vertices in row i, column j, where $b > a$ and the notation $x \oplus y = x + y + \beta xy$ denotes the multiplicative formal group law. Note that $+$ behaves like the largest color: $1 < \cdots < n < +$.

[1] There is also a six-vertex generalization of the Frozen Pipes model (in preparation), very similar to that of [14], and which we expect to represent a variant of the motivic Chern class of a Schubert variety, but we will not discuss this version of the model here.

There again exists a row YBE for this set of weights, and, with the addition of the column parameters $\boldsymbol{y}$, we also obtain a column YBE, which gives the equivalence of the following partition functions for any fixed choice of boundary conditions.

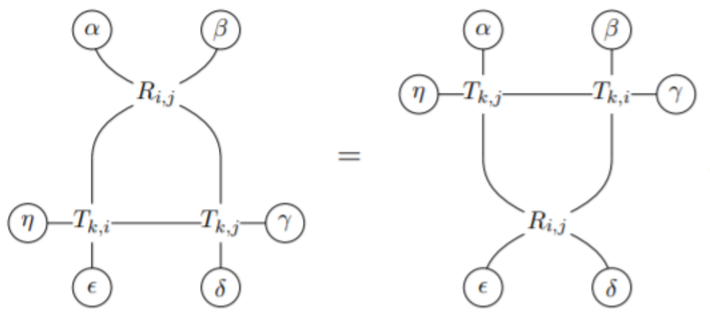
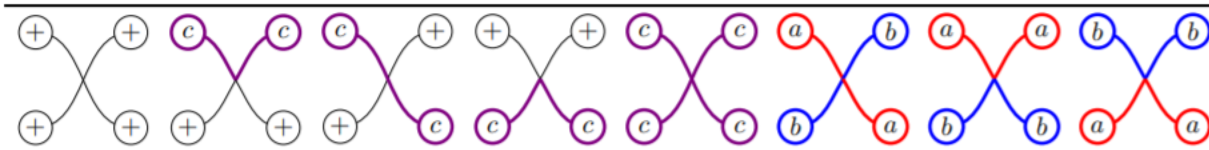


Figure 3.6: The column Yang-Baxter equation.

As with the Iwahori model, we can use the train argument inductively–on the columns, in this case–and recognize the resulting equations in terms of the operators defining the β-Grothendieck polynomials. The Schubert polynomials are obtained by setting $\beta = 0$ and $\boldsymbol{y} = -\boldsymbol{y}$, and the (ordinary) Grothendieck polynomials by setting $\beta = -1$. See [3], §5.

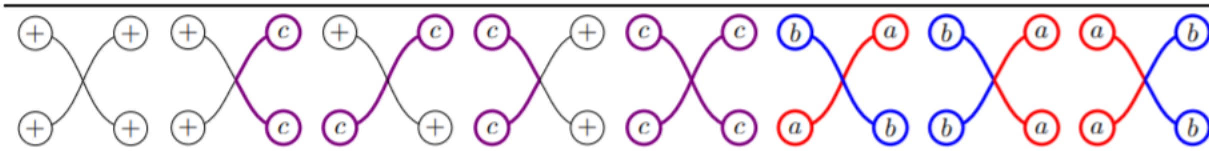
Pipe Model R-vertex weights:

Row Yang-Baxter equation R-vertex weights:



$\mathtt{a}_1$	$\mathtt{c}_2$	$\mathtt{b}_1$	$\mathtt{c}_1$	$\mathtt{a}_2^{\circ}$	$\mathtt{a}_2^{\dagger}$	$\mathtt{a}_2^{\flat}$	$\mathtt{a}_2^{\sharp}$
$1+\beta x_i$	$1+\beta x_i$	$x_j - x_i$	$1+\beta x_j$	$1+\beta x_i$	$x_j - x_i$	$1+\beta x_i$	$1+\beta x_j$

Column Yang-Baxter equation R-vertex weights:



$\mathtt{a}_1$	$\mathtt{c}_1$	$\mathtt{b}_2$	$\mathtt{c}_2$	$\mathtt{a}_2^{\circ}$	$\mathtt{a}_2^{\dagger}$	$\mathtt{a}_2^{\sharp}$	$\mathtt{a}_2^{\flat}$
$1+\beta y_i$	$1+\beta y_j$	$y_j - y_i$	$1+\beta y_i$	$1+\beta y_i$	$y_j - y_i$	$1+\beta y_j$	$1+\beta y_i$

Chapter 4

Schubert calculus background

We now describe the relevant background on Schubert calculus, following primarily the sources [21, 22, 23, 24, 5]

4.0.1 Flag varieties and Schubert classes

Let $X_n(d_1, \ldots d_k)$ denote the set of flags in $\mathbb{C}^n$ of type $(d_1, \ldots, d_k)$, i.e.,

$$X_n(d_1, \ldots d_k) := \{0 \subset V_1 \subset \cdots \subset V_{k-1} \subset \mathbb{C}^n : \dim(V_i) = \sum_{j=1}^{i} d_j, \sum_{j=1}^{n} d_j = n\}.$$

For example, $X_n(d, n-d)$ is the Grassmannian $Gr(d,n)$ of d-dimensional subspaces of $\mathbb{C}^n$. Let $G := GL_n(\mathbb{C})$, and fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$. Then G acts transitively on $X_n(d_1, \ldots, d_k)$ with stabilizer $P(d_1, \ldots, d_k)$, the parabolic subgroup with diagonal blocks of sizes $d_1, \ldots, d_k$, and hence $X_n(d_1, \ldots d_k) \cong G/P(d_1, \ldots, d_k)$. When the sequence $d_1, \ldots, d_k$ is clear from context, we will refer to the relevant parabolic subgroup simply as P. By embedding G/P in a suitable projective space, one finds that it is a smooth variety, which we refer to as a **partial flag variety**. We will often switch between the two perspectives on flag varieties (in terms of matrices or in terms of the flags themselves) depending on context.

Flags in $X_n(d_1, \ldots, d_k)$ are parametrized by cosets in $W/W_P := S_n/(S_{d_1} \times \cdots \times S_{d_k})$. Each coset has a *minimal representative*: a unique permutation w such that $w(1) < \cdots < w(d_1)$, $w(d_1+1) < \cdots < w(d_1+d_2)$, and so forth. Following Brion, we denote the set of minimal representatives as W^P. Let F_P denote the flag

$$F_P : 0 \subset \langle e_1, \ldots, e_{d_1} \rangle \subset \cdots \subset \langle e_1, \ldots, e_{d_{k-1}} \rangle \subset \mathbb{C}^n.$$

Then, indeed, the flags in $X_n(d_1, \ldots, d_k)$ are exactly $F_{wP} := wF_P \cong wP/P$ for $w \in W^P$. For example, if $w = 2341$ in one-line notation, the flag F_{wP} in $X_4(1,2,1)$ is:

$$0 \subset \langle e_2 \rangle \subset \langle e_2, e_3, e_4 \rangle \subset \mathbb{C}^4.$$

Alternatively, we can represent a coset in W/W_P as a word w in the alphabet $\{1, \ldots, k\}$ such that i appears d_i times: $w(j) = i$ indicates that the basis vector e_i first appears in the jth flag. For example, the flag above corresponds to $w = 3122$.

In particular, the **full flag variety** $X_n(1, \ldots, 1)$ is parametrized by permutations in $W = S_n$ and is isomorphic to G/B, where B is the Borel subgroup of upper triangular matrices. By sending a flag to the corresponding partial flag of type $d_1, \ldots, d_k$, we obtain a G-equivariant fibration

$$G/B \to G/P$$

with fiber P/B over B/B. Due to the existence of this map, one can address many questions about partial flag varieties by reducing to the case of the full flag variety.

We define **Schubert cells** C_{wP} $(w \in W^P)$ in terms of intersections with the standard flag $F_B \in X_n(1, \ldots, 1)$ as follows:

$$\begin{aligned} C_{wP} = &\{0 \subset V_{d_1} \subset \cdots \subset V_{d_k} \subset \mathbb{C}^n : \dim(V_{d_i} \cap \mathbb{C}^m) = \#\{j \leqslant d_i : w(j) \leqslant m\} \\ &\text{for } 1 \leqslant i \leqslant k \text{ and } 1 \leqslant m \leqslant n\}. \end{aligned}$$

To obtain the corresponding **Schubert varieties** X_{wP}, replace "=" with "$\geqslant$" in the definition above. In terms of matrices, C_{wP} is the Borel orbit $BF_{wP} \cong BwP/P$, and the **Schubert variety** X_{wP} is its Zariski closure $\overline{BF_{wP}}$.

From now on, we will work solely with the full flag variety G/B, and denote the Schubert classes/varieties simply by C_w and X_w. We can make the definition of Schubert cells and varieties in terms of matrices more explicit: a flag $F \in C_w$ is represented–in the sense that the first k rows span the k-th subspace of F–by a unique matrix $(x_{ij})_{1\leqslant i,j\leqslant n}$ such that

$$x_{i,w(i)} = 1, \text{ and } x_{i,j} = 0 \text{ if } j > w(i) \text{ or } i > w^{-1}(j),$$

i.e., we place 1's in the $w(i)$-th entry of the i-th row, and zeros to the right of and down from each 1. For example, if $w = 3142 \in S_4$, C_w consists of matrices of the form

$$\begin{pmatrix} * & * & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & * & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

One can verify that the number of free entries is always $l(w)$, where $l(w) = \#\{i < j : w(i) > w(j)\}$ is the the length of w, and thus $C_w \cong \mathbb{C}^{l(w)}$. Moreover,

$$X_w = \bigsqcup_{v\leqslant w} C_v \text{ and } G/B = \bigsqcup_{w\in W} C_w$$

(see e.g., [22] §3.6 for proof). Thus, by general facts from cohomology, the **Schubert classes** S^w in $H^*(G/B)$ form an additive basis of the cohomology ring $H^*(G/B;\mathbb{Z})$. On the other hand, there is an algebraic presentation of $H^*(G/B)$ in terms of generators $x_1,\ldots,x_n \in H^2(G/B)$. (These generators are the first Chern classes of some natural line bundles on G/B; see Fulton [23] Chapter 10 for more details.) The presentation is as follows:

Proposition 3 *Let $e_k(\boldsymbol{x})$ denote the k-th elementary symmetric polynomial in the variables $\boldsymbol{x} = \{x_1,\ldots,x_n\}$. Then*

$$H^*(G/B;\mathbb{Z}) \cong \mathbb{Z}[\boldsymbol{x}]/(e_1(\boldsymbol{x}),\ldots,e_n(\boldsymbol{x})).$$

The monomials $x_1^{i_1}\cdots x_n^{i_n}$, with exponents $i_j \leqslant n-j$, form an additive basis for $H^*(G/B;\mathbb{Z})$.

Thus, we should be able to express the basis of Schubert classes in terms of elements in this quotient ring. Moreover, we would like the representatives we pick to satisfy the following *stability property*: since $w \in S_k$ can be regarded as an element of S_{k+1} fixing $k+1$, and there is a natural embedding $X_k(1,\ldots,1) \hookrightarrow X_{k+1}(1,\ldots,1)$ inducing a pullback $H^*(X_{k+1}(1,\ldots,1)) \to H^*(X_k(1,\ldots,1))$ that maps the Schubert class S^w to itself, an ideal polynomial representative would respect this inclusion. Indeed:

Proposition 4 ([23], §10.2 **Proposition** 3**)** *For $w \in S_k$, let S_k^w denote the Schubert class S^w considered as an element of $H^*(X_k(1,\ldots,1))$. Then there is a unique homogeneous polynomial of degree $l(w)$, the* ***Schubert polynomial*** *$\mathfrak{S}_w$ corresponding to w, in $\mathbb{Z}[x_1,\ldots,x_k]$ that maps to $S_k^{w_0w}$ in $H^*(X_k(1,\ldots,1))$.*

4.0.2 Generalized Schubert Calculus

Before discussing how to actually calculate the Schubert polynomials, we will introduce some of their natural generalizations so that we can treat them simultaneously. Indeed, the classical Schubert calculus story can be generalized in many different directions. The three "orthogonal" directions that are relevant for this thesis are nicely encapsulated by the following figure from [9].

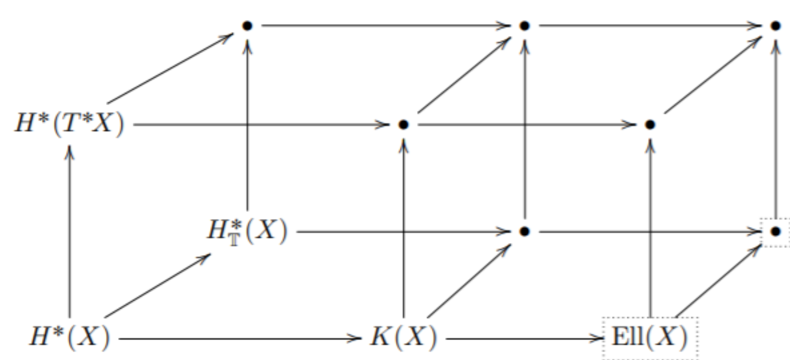


Figure 4.1: [9], Figure 1: three "orthogonal" directions to generalize classical Schubert calculus.

For X a flag variety, the lower left corner corresponds to the classical Schubert calculus story described above. Traveling to the right, we obtain increasingly more general cohomology theories: K-theory $K(X)$ and elliptic cohomology $\mathrm{Ell}(X)$. We single out these theories in particular because they (along with their "universal" version, complex cobordism) are *oriented cohomology theories*: certain cohomology theories that are classified by 1-dimensional formal group laws. Quillen described how to extract a formal group law from an oriented cohomology theory in his landmark 1969 paper [25]:

such theories are equipped with Chern classes (of line bundles over the base space X), and the formal group law F describes the first Chern class c_1 of the tensor product of two line bundles:

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2)).$$

The significance of this for our purposes is the fact that solutions to the quantum Yang-Baxter equation (YBE) are also classified by these formal group laws, and hence we might expect there to be integrable systems associated to each of the points on the cube. The recent work of Aganagic, Maulik, and Okounkov [1, 2] has begun to flesh out the connection between quantum integrable systems and the cohomology of Nakajima quiver varieties, a broad class of varieties that includes both flag varieties and their cotangent bundles, by identifying the (oriented) cohomology ring with the Bethe algebra of commuting transfer matrices of the integrable system. An especially nice feature of this correspondence is that the "easy" basis of one theory matches up with the "hard" basis of the other, as summarized in [9] and reproduced in Figure 4.2. This correspondence is what initially inspired the investigation in §6.0.1.

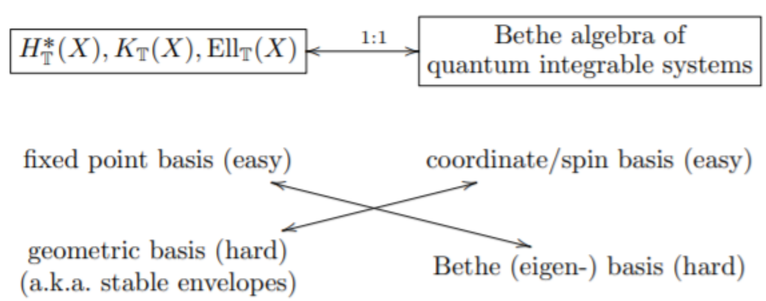


Figure 4.2: From [9].

Traveling up to the top face of the box in Figure 4.1 corresponds to passing from X to T^*X, and traveling to the back face of the diagram corresponds to passing from ordinary (oriented) cohomology to equivariant (oriented) cohomology, which we will discuss in more detail in 4.0.4. We also note that here we will really only focus on the left half of the cube, as well as on *connective* K-theory CK, a theory that interpolates between H^* and K. Elliptic cohomology is often singled out due to its relative complexity in the hierarchy: while H^*, K, and CK have naturally defined Schubert classes that do not depend on the choice of a Bott-Samelson resolution, the classes in Ell do depend on choices. While we will not be discussing elliptic cohomology in detail, we mention that there has been recent progress in this area. In particular, Kumar, Rimányi, and Weber [26, 27] have defined an "h-deformed Schubert class" in *equivariant* elliptic cohomology $\mathrm{Ell}_T(X)$ that does *not* depend on choices (though its specialization to the non-equivariant theory has a singularity, again reflecting the need for choices in the non-equivariant case), and have shown that these classes can be represented by "elliptic weight functions," functions which already existed in the literature [28]. Motegi [29] has begun analysis of certain nested lattice models whose partition functions give these elliptic weight functions.

4.0.3 K-theory and connective K-theory

The **K-theory ring** $K(X)$ is the Grothendieck ring generated by symbols $[E]$ for vector bundles $E \to X$, modulo the relations $[E] = [E_1] + [E_2]$ whenever there exists a short exact sequence $0 \to E_1 \to E \to E_2$ of vector bundles. Addition and multiplication in $K(X)$ are given by direct sums and tensor products, respectively, of vector bundles. When X is smooth, as in the case of $X = G/B$, this definition of $K(X)$ coincides with the Grothendieck group of coherent sheaves on X, since every coherent sheaf has a finite resolution by vector bundles. There is a topological filtration $F_0 \supset F_1 \supset \cdots$ of $K(X)$, where F_j consists of coherent sheaves with codimension at least j, and with respect to this filtration, the lowest graded piece of the associated graded ring of $K(X)$ is isomorphic to the cohomology ring $H^*(X)$. This is the sense in which K-theory is considered a generalization of cohomology.

Connective K-theory, which we denote $CK(X)$, is a graded version of $K(X)$, obtained by tensoring $K(X)$ with $\mathbb{Z}[\beta, \beta^{-1}]$, for $\deg(\beta) = 1$, and modifying the push-forward and pullback maps to include powers of β. When $\beta = 0$, we recover $H^*(X)$, and when $\beta = -1$, we recover $K(X)$. For a nice summary of connective K-theory in the context of oriented cohomology theories and formal group laws (from the coherent sheaf point of view), we like the reference [30]. Both $K(G/B)$ and $CK(G/B)$ have additive bases consisting of Schubert classes $[\mathcal{O}_{X_w}]$, where $\mathcal{O}_{X_w}$ is the structure sheaf of X_w. We denote the polynomial representatives of these Schubert classes in $K(G/B)$ and

$CK(G/B)$ by $\mathcal{G}_w$ and $\mathcal{G}_w^\beta$, respectively; these are the (β)-**Grothendieck polynomials**.

Schubert and (β)-Grothendieck polynomials via divided difference operators

In this subsection, we will work primarily with β-Grothendieck polynomials, since they specialize both to Schubert polynomials (when $\beta = 0$) and to Grothendieck polynomials (when $\beta = -1$). We now describe one method for calculating these polynomials: via divided-difference operators, which turn out to correspond to certain push-pull operators in geometry, as we will describe. Schubert polynomials were initially defined by Lascoux and Schutzenberger in [31], and Fomin and Kirillov defined β-Grothendieck polynomials in [32]. Fomin and Kirillov were motivated, in part, by attempts to classify exponential solutions to the Yang-Baxter equation; the connection to connective K-theory came much later, in [30]. See the appendix to [30] for a nice exposition of the relation between Fomin-Kirillov's definition and the divided difference operator definition.

Let ∂_i be the ith divided difference operator, which acts on functions $f = f(x_1, \ldots, x_n)$ by:

$$\partial_i(f) = \frac{f - f^{s_i}}{x_i - x_{i+1}},$$

where $f^{s_i} = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$. The β-Grothendieck polynomials $\mathcal{G}_w^{(\beta)}(\boldsymbol{x})$ are built from the modified operators

$$\pi_i^{(\beta)} := \partial_i \circ (1 + \beta x_{i+1})$$

recursively with respect to length in the symmetric group, starting with the longest word w_0. Specifically, let $\rho = (n-1, n-2, \ldots, 2, 1)$. Then we define

$$\mathcal{G}_{w_0}^\beta = \boldsymbol{x}^\rho$$

and for $w \in S_n$ and $s_i = (i \;\; i+1) \in S_n$, if $l(ws_i) = l(w) - 1$,

$$\mathcal{G}_{ws_i} = \pi_i^{(\beta)} \mathcal{G}_w^{(\beta)}.$$

Note that these polynomials are well-defined since the operators $\pi_i^{(\beta)}$ satisfy the braid relations $\pi_i^{(\beta)} \pi_{i+1}^{(\beta)} \pi_i^{(\beta)} = \pi_{i+1}^{(\beta)} \pi_i^{(\beta)} \pi_{i+1}^{(\beta)}$.

To interpret these operators geometrically, we first need to define **Bott-Samelson varieties**. In general, Schubert varieties X_w can be singular; the corresponding Bott-Samelson varieties $\Gamma_{\underline{w}}$ provide a resolution of singularities $r_{\underline{w}} : \Gamma_{\underline{w}} \to X_w$. Let P_i denote the minimal parabolic subgroup generated by B and s_i, i.e.,

$$P_i = \left\{ \begin{array}{cc} & \begin{array}{ccccc} & & i & & \end{array} \\ \begin{array}{c} \\ \\ \\ i+1 \\ \\ \\ \end{array} & \begin{pmatrix} * & & \cdots & & * \\ & \ddots & & & \\ & & * & & \vdots \\ & & * & * & \\ & & & \ddots & \\ & & & & * \end{pmatrix} \end{array} \right\}.$$

Note that if $\underline{w} = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition of w, then $X_w = P_{i_1} X_{s_{i_1} w} = P_{i_1} \cdots P_{i_k}/B$. Letting $v := s_{i_1} w$, we see that $l(v) = l(w) - 1$. Then the Bott-Samelson varieties $\Gamma_{\underline{w}}$ are defined by:

$$\Gamma_{\underline{id}} = X_{id}$$

and

$$\Gamma_{\underline{w} s_i} = P_{i_1} \times_B \Gamma_{\underline{v}},$$

where the action of B is by conjugation. We will denote the **Bott-Samelson classes** $(r_{\underline{w}})_* [\Gamma_{\underline{w}}]$ in $CK(G/B)$ by $Z^{\underline{w}}$. Consider the following pullback square, where ϕ and p_{i_k} are the natural projection maps:

$$\begin{tikzcd} \Gamma_{s_{i_1} \cdots s_{i_k}} \arrow[rr, "r_{s_{i_1} \cdots s_{i_k}}"] \arrow[d, "\phi"] & & G/B \arrow[d, "p_{i_k}"] \\ \Gamma_{s_{i_1} \cdots s_{i_{k-1}}} \arrow[r, "r_{s_{i_1} \cdots s_{i_{k-1}}}"] & G/B \arrow[r, "p_{i_k}"] & G/P_{i_k}. \end{tikzcd}$$

Then:

Theorem 1 ([33]) *Let $A_i = (p_i)^* (p_i)_*$, and let $\underline{w} = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of w. Then:*

$$Z^{\underline{w}} = A_{i_1} \cdots A_{i_k} (Z^{\underline{id}}).$$

The A_i satisfy the braid relations, and hence the class $Z^{\underline{w}}$ does not depend on the choice of reduced decomposition for w. Moreover, $Z^{\underline{w}}$ is equal to the Schubert class $S^{w_0 w}$, and the image of the push-pull operator $(p_i)^*(p_i)_*$ in the realization of $CK(G/B))$ as a polynomial ring is $\pi_i^{(\beta)}$. Thus, the β-Grothendieck polynomials are indeed polynomial representatives of the Schubert classes in $CK(G/B)$.

(Note: Bressler and Evans demonstrated that connective K-theory is essentially the last oriented cohomology theory in the hierarchy for which the A_i satisfy the braid relations, and hence the last theory in which we can identify the Schubert classes with the Bott-Samelson classes.)

4.0.4 Equivariant Cohomology and Localization

For a topological space X equipped with an action of a group Γ, one can exploit the symmetry imparted by this action to define the **Γ-equivariant cohomology ring** $H^*_\Gamma(X)$. A significant advantage to working with equivariant cohomology is the fact that, in many situations, all important information about $H^*_\Gamma(X)$ is contained in $H^*_\Gamma(X^\Gamma)$, the cohomology of the fixed point locus. This fact can be a huge aid in calculations, particularly in the case when X has finitely many Γ-fixed points.

To define $H^*_\Gamma(X)$, first consider the case where Γ acts freely, in which $H^*_\Gamma(X)$ is defined very naturally as:

$$H^*_\Gamma(X) := H^*(X/\Gamma).$$

In general, we can always find a contractible space $\mathrm{E}\Gamma$ with a free Γ-action, and then

$$H^*_\Gamma(X) := H^*((X \times \mathrm{E}\Gamma)/\Gamma).$$

Note that the equivariant cohomology of a point, $H^*_\Gamma(pt) = H^*_\Gamma(\mathrm{E}\Gamma/\Gamma)$, can now be nontrivial–this is a key feature of the theory. Every Γ-invariant subvariety Y of X has a fundamental class $[Y]_\Gamma \in H^*_\Gamma(X)$. Consider the map $\pi : X \to \{pt\}$. It induces a graded ring map $\pi^* : H^*_\Gamma(pt) \to H^*_\Gamma(X)$ on cohomology, giving $H^*_\Gamma(X)$ the structure of an $H^*_\Gamma(pt)$-module. Moreover, employing the Poincaré isomorphism to identify the homology group $H_i^\Gamma(X)$ with $H_\Gamma^{n-i}(X)$, where $n = \dim_{\mathbb{R}}(X)$, we obtain the pushforward

$\pi_* : H^*_\Gamma(X) \to H^*_\Gamma(pt)$ and (for torsion-free X) the symmetric **Poincaré pairing**

$$\langle a|b\rangle = \pi_*(a \cup b),$$

which extracts the coefficient of $[pt]$ in the cup product $a \cup b$.

Similarly, (connective) Γ-equivariant K-theory is defined analogously to ordinary (connective) K-theory, but with generators given by Γ-equivariant vector bundles. The pushforward to a point, $\pi_* : CK_\Gamma(X) \to CK_\Gamma(pt)$, is now the graded Euler characteristic

$$\pi_*([E]) = \chi(X, E) = \sum_i \beta^i H^i(X; E)$$

so the Poincaré pairing is

$$\langle [E], [F]\rangle := \chi(X, E \otimes F) = \sum_i \beta^i H^i(X; E \otimes F).$$

In our case, we are interested in $H^*_T(G/B)$, $K_T(G/B)$, and $CK_T(G/B)$ where $T \cong (\mathrm{c}^\times)^n$ is the torus of diagonal matrices, acting on G via conjugation and hence on G/B by left multiplication. To understand the structure of $H^*_T(pt)$, note that we can take

$$ET = \{(z_i)_{i>0} | z_i \in \mathrm{c}^n,\ \text{finitely many } z_i \neq 0\}$$

so

$$ET/T \cong (\mathrm{c}\mathbb{P}^\infty)^n$$

and thus $H_T(pt) \cong \mathbb{Z}[y_1, \ldots, y_n]$.

For $K_T(pt)$, since equivariant vector bundles $E \to pt$ are equivalent to representations of T, $K_T(pt)$ is isomorphic to the representation ring $R(T)$ of T. Alternatively, we can view it as $\mathbb{Z}[e^{\pm t_1}, \ldots, e^{\pm t_n}]$, where e^{t_i} are characters corresponding to a basis of the Lie algebra of T.

Recall from §4.0.1 that the Schubert varieties are defined by incidence conditions with respect to the standard flag F_B in the full flag variety. The flags $F_w := F_{wB}$, $w \in W$, are T-invariant, and in fact are exactly the T-fixed points in G/B. Hence the equivariant Schubert classes defined instead with respect to F_{wB} are T-invariant as well.

Thus, for each w, we obtain another basis of **w-twisted Schubert classes** $\{S^v_{(w)}\}_{v\in S_n}$ for each of our equivariant theories. Explicitly, denote the ith subspace in F_w by $(F_w)_i$. Then the w-twisted Schubert cell is

$$\begin{aligned} C_v^{(w)} =& \{0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n | \dim(V_i \cap (F_w)_i) = \#\{j \leqslant i : w(v(j)) \leqslant m\} \\ & \text{for } 1 \leqslant i \leqslant k \text{ and } 1 \leqslant m \leqslant n\}. \end{aligned}$$

and $X_v^{(w)} = \overline{C_v^{(w)}}$. In the matrix formulation, representing w as a permutation matrix, we have $X_v^{(w)} = w \cdot X_v$. The classes ${S^w_{(w_0)}}^{-1}$ are dual to the classes S^w with respect to the Poincaré pairings.

We now describe the key localization results that facilitate computations in H_T^* (and K, and CK). Let $\widetilde{H}_T^*(G/B)$ denote the cohomology ring $H_T^*(G/B)$ localized at $H_T(pt)$, and let ι_w denote the class of the fixed point F_w in $H_T^*(G/B)$. Applying the pullback i_w^* of the inclusion $i_w : \{F_w\} \to G/B$ is commonly called *restricting to the fixed point* F_w and denoted by $|_w$. The localization theorem states that $\widetilde{H}_T^*(G/B)$ is generated (as a vector space over $\widetilde{H}_T^*(pt)$) by the classes ι_w. For practical purposes, however, we would like to know how to actually decompose in this basis. To that end, we have the following general theorem (due to Atiyah, Bott, Berline, and Vergne in the cohomology case):

Theorem 2 *Let X be a compact, nonsingular variety equipped with an action of a torus $T \cong (\mathrm{c}^\times)^n$, and let E_p denote the product of weights of T acting on the tangent space T_pX of X at the fixed point p. Then:*

$$[X] = \sum_{p\in X^T} \frac{[p]}{E_p} \in \widetilde{H}_T^*(X),$$

and in K-theory and connective K-theory, respectively, we have:

$$[X] = \sum_{p\in X^T} \frac{[p]}{\lambda_{-1}(T_p^*X)} \in \widetilde{K}_T(X)$$

and

$$[X] = \sum_{p\in X^T} \frac{[p]}{\lambda_\beta(T_p^*X)} \in \widetilde{CK}_T(X)$$

where if V is a complex vector space with T-action and weight decomposition $V = \oplus_i V_{\mu_i}$, $\lambda_y(V) = \prod_i (1 + ye^{\mu_i})$.

(The Atiyah-Bott-Berline-Vergne theorem is really a slightly more general version of this theorem, which expands $\pi_* \alpha \in H_T^*(pt)$ into fixed points for any class $\alpha \in H_T^*(X)$– see [24]. The above theorem follows from this general version by taking $\alpha = X$ and applying the pushforward $(i_p)_* : H_T^*(\{p\}) \to H_T^*(X)$.)

This theorem will first come into play for us in the next section, when we derive the Boltzmann weights for the (five-vertex) Frozen Pipes model.

Chapter 5

Deriving the Frozen Pipes model via geometry

To connect our discussion of Schubert calculus to the lattice model world, we consider the Hilbert space:

$$\mathcal{H} := \bigoplus_{\substack{k=1 \\ d_1+\cdots+d_k=n}}^{n} H_T^*(X_n(d_1,\ldots,d_k)),$$

We can also think of $\mathcal{H}$ as a tensor product $\bigotimes_{i=1}^n V[y_i]$, where V is an n-dimensional vector space whose basis elements we denote by the "colors" $c_1,\ldots,c_n$. Then there is a natural basis of $\mathcal{H}$ indexed by length n words in the alphabet $\{c_1,\ldots,c_n\}$, i.e., by elements of W/W_P for some parabolic P, as described in §4.0.1, where we saw that the Schubert classes in the $H_T^*(X_n(d_1,\ldots,d_k))$ are also classified by such words. For each $P = P(d_1,\ldots,d_k)$, fix a color vector

$$\boldsymbol{c}_P = \underbrace{c_1\cdots c_1}_{d_1}\cdots\underbrace{c_k\cdots c_k}_{d_k}.$$

For $w \in W/W_P$, we identify the basis vector $|w\boldsymbol{c}_P\rangle$ with the Schubert class $S^w \in H_T^*(G/P)$, and $\langle w^{-1}\boldsymbol{c}_P|$ with its Poincaré dual $S_{(w_0)}^{w^{-1}}$. Then the twisted Schubert class $S_{(v)}^w$ can be identified with $|v(w\boldsymbol{c}_P)v^{-1}\rangle$ (or $\langle v^{-1}(w^{-1}\boldsymbol{c}_P)v|$).

We now show that the change of basis matrix between $\{S_{(s_i)}^v\}$ and $\{S^v\}$ is exactly the matrix of Boltzmann weights for the (Schubert) Frozen Pipes model. This is a straightforward generalization of the argument presented in Zinn-Justin's notes [5] in the case where X is a Grassmannian. In fact, the arguments in this section can be modified easily to obtain analogous results for the (β-)Grothendieck model–provided we replace y_i with $\ominus y_i$, where $\ominus$ denotes the inverse of the multiplicative formal group law: $\ominus y_i = \frac{-y_i}{1+\beta y_i}$.

For $w \in W$, let R_w be the change of basis matrix defined by

$$S^v = \sum_{u \in W} (R_w)_{uv} S^u_{(w)}.$$

Then we claim that

Proposition 5 *The change of basis matrix R_{s_i} associated to the elementary transposition $s_i = (i, i+1)$ acts nontrivially only on the i-th and $(i+1)$-st tensor factors of $\mathcal{H}$; in particular, for S^v, it acts nontrivially only on the colors $c_{v(i)}$ and $c_{v(i+1)}$. The piece of R_{s_i} acting on each pair of colors ${\color{blue}b} < {\color{red}r}$ is of the form:*

$$\begin{array}{c c}
 & \begin{array}{cccc} {\color{red}\bullet\bullet} & {\color{red}\bullet}{\color{blue}\bullet} & {\color{blue}\bullet}{\color{red}\bullet} & {\color{blue}\bullet\bullet} \end{array} \\
\begin{array}{c} {\color{red}\bullet\bullet} \\ {\color{red}\bullet}{\color{blue}\bullet} \\ {\color{blue}\bullet}{\color{red}\bullet} \\ {\color{blue}\bullet\bullet} \end{array} &
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y_{i+1} - y_i & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}$$

Recall that P_i is the minimal parabolic subgroup generated by B and s_i. Consider the quotient $X_w \times_B P_i$ (with B acting on the left of both factors), which comes with two natural projection maps:

$$\begin{tikzcd}
 & X_w \times_B P_i \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X_w & & B\backslash P_i \cong \mathbb{P}^1.
\end{tikzcd}$$

$B\backslash P_i$ has two T-fixed points, $[1] := B$ and $[s_i] := Bs_i$, so by the localization formula:

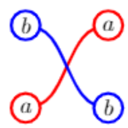
$$[\mathbb{P}^1] = \frac{[1]}{y_{i+1} - y_i} + \frac{[s_i]}{y_i - y_{i+1}}.$$

Applying $f_* g^*$ to this equation and rearranging, we obtain

$$[X_w] = [s_i \cdot X_w] + (y_{i+1} - y_i)[P_i \cdot X_w]\delta_{\dim(P_i \cdot X_w),\dim(X_w)+1}.$$

In terms of Schubert classes, we have $[X_w] = S_w$, $[s_i \cdot X_w] = S^w_{(s_i)}$, and $[P_i \cdot X_w]\delta_{\dim(P_i \cdot X_w),\dim(X_w)+1} = S^{s_i w}_{(s_i)}$ if $wc_i < wc_{i+1}$ (and $[P_i \cdot X_w]\delta_{\dim(P_i \cdot X_w),\dim(X_w)+1} = 0$ otherwise), giving us the desired formula for R_{s_i}.

In terms of the Frozen Pipes lattice, this matrix describes the weights of the column R-matrix acting on the vectors $\langle v^{-1}|$ and $|v\rangle$ associated to the Schubert classes, where y_i is the the spectral parameter in the ith column. (Recall that we treat $+$ as the largest color.) For instance, the entry in the second row and second column corresponds to the vertex:



.

We can think of the spectral parameters y_i and y_{i+1} as being attached to the strands of the R-vertex. Then, by rotating the R-vertices and adjusting the associated spectral parameters, we obtain the matrix of Boltzmann weights and the row R-matrix from section 3.0.3. (Note: In the $(\beta)-$Grothendieck case, the column and row R-matrices are not simply rotations of one another. They are, however, related by duality.)

We can break up the matrix of Boltzmann weights into blocks, one for each fixed pair of left and right boundary conditions. In other words, each block corresponds to an endomorphism of a column module $V[y_i]$. (We will interpret the endomorphism diagrammatically as "flowing" from the bottom to the top of a vertex, i.e., the bottom edge is the input and the top edge is the output.) More generally, we can consider a single lattice row of any length with fixed left and right boundary; then the matrix blocks correspond to endomorphisms of $\mathcal{H}$. We break these endomorphisms into four types: $A(x)$, which has $+$ on both the left and the right; $B_i(x)$, which has color i on the left and $+$ on the right; $C_i(x)$, which has $+$ on the left and i on the right; and $D_{ij}(x)$, which has i on the left and j on the right. Thus, we obtain a block matrix of the form

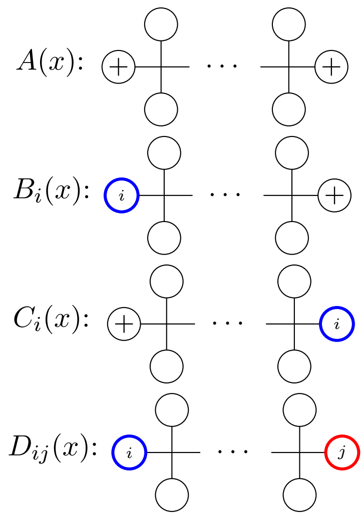


Figure 5.1: The monodromy operators.

$$\begin{pmatrix} A(x) & B_1(x) & B_2(x) & \cdots & B_n(x) \\ C_1(x) & D_{11}(x) & D_{12}(x) & \cdots & D_{1n}(x) \\ \cdot & D_{21}(x) & \cdot & & \\ \cdot & \cdot & & \cdot & \\ C_n(x) & D_{n1}(x) & D_{n2}(x) & \cdots & D_{nn}(x) \end{pmatrix},$$

called the **monodromy matrix**. The Yang-Baxter equation leads to commutation relations between these operators, and we will see some of them put to use in §5.0.1. Further, the algebra generated by the entries of the monodromy matrix modulo these commutation relations, sometimes called a **Yang-Baxter algebra**, is a degeneration of (a Drinfeld twist of) the Yangian $Y(\mathfrak{sl}_n)$ (in the Schubert case) and the quantum group $U_q(sl_n)$ (in the (β-)Grothendieck case) via an "RTT" construction [34]. This is a special case of the general construction employed by Maulik and Okounkov [1].

5.0.1 The algebraic Bethe ansatz

The algebraic Bethe ansatz is a standard method for finding the eigenvectors and eigenvalues of, and thus diagonalizing, the transfer matrix of an integrable system. For the

Frozen Pipes model, the transfer matrix is defined to be

$$T(u) = \sum_{i=1}^{n} D_{ii}(u),$$

and as the "ansatz", we start by assuming that the (right) eigenvectors of $T(u)$ are of the form $B_1(x_1)\cdots B_n(x_n)|\varnothing\rangle$, where $|\varnothing\rangle$ is the vacuum vector consisting of all + spins. We then use the Yang-Baxter commutation relations to obtain conditions (called Bethe equations) on $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ to ensure that these vectors are indeed eigenvectors of $T(u)$. In general, one usually needs to employ the *nested* algebraic Bethe ansatz for higher rank models, but the analysis simplifies nicely in our five-vertex case, and will serve as a good illustration for our purposes.

Thus, we start by examining

$$\sum_{i=1}^{n} D_{ii}(u)B_1(x_1)\cdots B_n(x_n)|\varnothing\rangle.$$

Note that, based on our set of admissible vertices, the empty vector is an eigenvector for $D_{ii}(u)$ with eigenvalue $\prod_{j=1}^{n} u - y_j$. Hence, if we commute the $D_{ii}(u)$'s through the $B_j(x_j)$'s in each summand, we will obtain (up to the factors obtained by applying the commutation relations) a multiple of $B_1(x_1)\cdots B_n(x_n)|\varnothing\rangle$, and setting the unwanted factors equal to zero will determine the Bethe equations. Let $i < j$. By applying the usual YBE train argument, we have the following commutation relation:

$$(u - x_j)D_{ii}(u)B_j(x_j) + B_i(u)D_{ij}(x_j) = B_i(x_j)D_{ij}(u) + (u - x_j)B_j(x_j)D_{ii}(u),$$

so, rearranging:

$$D_{ii}(u)B_j(x_j) = B_j(x_j)D_{ii}(u) - \frac{1}{u - x_j}B_i(u)D_{ij}(x_j) + \frac{1}{u - x_j}B_i(x_j)D_{ij}(u) \qquad (5.1)$$

and, similarly,

$$D_{jj}(u)B_i(x_i) = -\frac{1}{u - x_i}B_j(u)D_{ji}(x_i) + \frac{1}{u - x_i}B_j(x_i)D_{ji}(u). \tag{5.2}$$

In the case where $i = j$, we have:

$$D_{ii}(u)B_i(x_i) = -\frac{1}{u - x_i}B_i(u)D_{ii}(x_i) + \frac{1}{u - x_i}B_i(x_i)D_{ii}(u) + \frac{1}{u - x_i}B_i(x_i)D_{ii}(u) \tag{5.3}$$

Thus, we need the last two terms in each of equations (5.1), (5.2), and (5.3) to be zero. Note that then only $D_{11}(u)$ will be able to commute past all of the B_j's. Examining the possible partition functions of $B_i(u)D_{ij}(u), B_i(x_j)D_{ij}(u), B_j(u)D_{ji}(x_i), B_j(x_i)D_{ji}(u), B_i(x_i)D_{ii}(u)$, and $B_i(u)D_{ii}(x_i)$–which is not too cumbersome, as we only care about the possible placements of the crossing vertices $a_2^\dagger$ and b_2–we see that in order for these terms to disappear, we need the x_i to be distinct and to satisfy

$$\prod_{j=1}^{n}(x_i - y_j) = 0. \tag{5.4}$$

This is only possible if the x_i's are a permutation of the y_j's. By similar calculations, these are also the conditions required for $\langle\varnothing|C_1(x_1)\cdots C_n(x_n)$ to be a (left) eigenvector of $T(u)$. Thus, we have shown:

Proposition 6 *Let $w \in S_n$. Then $B_1(y_{w(1)})\cdots B_n(y_{w(n)})|\varnothing\rangle$ and $\langle\varnothing|C_1(y_{w(1)})\cdots C_n(y_{w(n)})$ are right and left eigenvectors, respectively, of $T(u)$, both with eigenvalue*

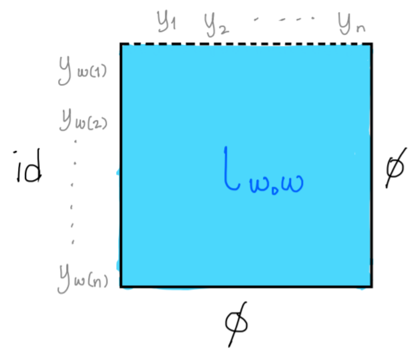
$$\frac{\prod_{i=1}^{n} u - y_i}{u - y_{w(1)}} = \prod_{i \neq w(1)} u - y_i.$$

From now on, we will use the suggestive notation:

$$|\iota_{w_0 w}\rangle := B_1(y_{w(1)})\cdots B_n(y_{w(n)})|\varnothing\rangle,$$

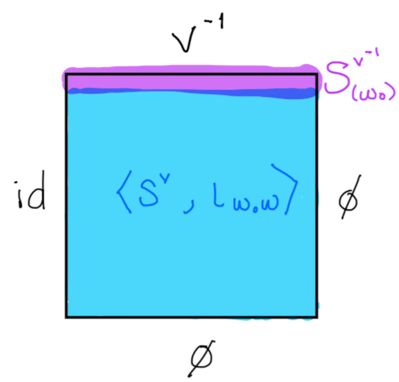
and we now show that $|\iota_{w_0 w}\rangle$ is indeed the class $\iota_{w_0 w}$ of the fixed point $F_{w_0 w}$.

Proposition 7 *$|\iota_{w_0 w}\rangle$ is the class $\iota_{w_0 w}$ of the fixed point $F_{w_0 w}$. In other words, the partition function of the following model, where the dotted line indicates that the top* *boundary is unspecified and hence can range over all permutations, represents the expansion of $\iota_{w_0 w}$ into the Schubert classes $\langle v^{-1}| = S^{v^{-1}}_{(w_0)}$:*



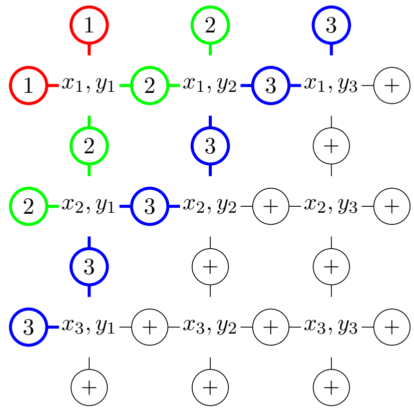
,

and the partition function of the model obtained by specifying a top boundary according to $v^{-1} \in S_n$:

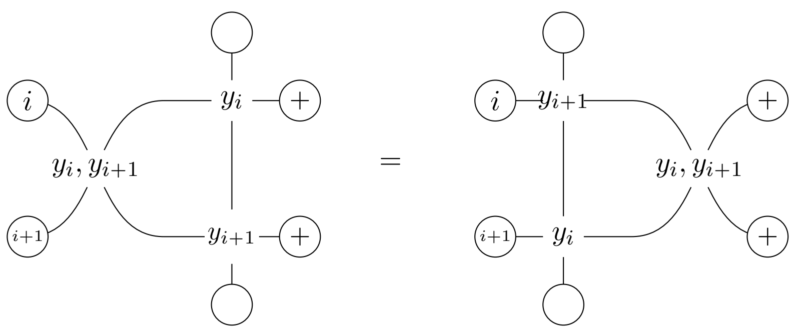


is the Poincaré pairing $\langle S_v, \iota_{w_0 w}\rangle$, i.e., S_v restricted to the fixed point $F_{w_0 w}$. Since equivariant Schubert classes are determined by their restrictions to fixed points, this gives another proof that $\langle v^{-1}|B_1(x_1)\cdots B_n(x_n)|\varnothing\rangle$ is the double Schubert polynomial associated to v.

In the case where $w = id$, $\iota_{w_0 w} = \iota_{w_0}$ coincides with the Schubert class $S^{id}_{(w_0)}$. And indeed, since a crossing at the ith diagonal vertex of $|\iota_{w_0}\rangle$ has a weight of $y_i - y_i = 0$, $|\iota_{id}\rangle$ pairs non-trivially only with the ket $\langle id| = S^{id}_{(w_0)}$, with weight 1 ($n = 3$ case shown below).



Hence the proposition is true in this base case. The other cases then follow via the action of S_n: the matrix R_w acts on $|\iota_{w_0}\rangle$ by permuting the row variables according to w, as we can see in the case of the simple reflection $w = s_i$ from the train argument below (where we only record row variables):



since the only possibility for the R-vertex on the left side is $a_2^\flat$, with weight 1, and the only possibility for the R-vertex on the right is a_1, also with weight 1. This matches the geometric action of w on the fixed point ι_{w_0}.

(We note that choosing to use the pipe model boundary conditions instead of the "Demazure model" ones was arbitrary–the same process works with the Demazure model if we use single-column monodromy operators rather than single-row operators.)

Chapter 6

The Tokuyama partition function: Deformation of the Weyl character formula and geometric interpretation

We now turn to a geometric analysis of the Tokuyama model. We employ solvability in two ways, each resulting in different expressions for the partition function. First, we can immediately take advantage of equation (2.2) to identify the Tokuyama model with a sum of Iwahori models, which we calculated via the standard train argument in §3.0.2:

$$Z(\mathfrak{S}_{\lambda,\boldsymbol{z}}) = \sum_{w \in W} (\boldsymbol{z}^{\rho} \mathcal{T}_i \boldsymbol{z}^{-\rho})(\boldsymbol{z}^{\lambda+\rho}) = \sum_{w \in W} \boldsymbol{z}^{\rho} \phi_w(\boldsymbol{z}; \varpi^{-\lambda}) \tag{6.1}$$

The second method, inspired by the algebraic Bethe ansatz and similar to that in [35], results in the well-known expression of $W_{\boldsymbol{\lambda}}(\boldsymbol{z})$ in terms of the Schur function (by way of the Weyl character formula), due originally to Casselman and Shalika.

In §6.0.1, we state the geometric meanings of these two expressions for the spherical Whittaker function. Moreover, we show that by equating them, one can deduce a variant of the Langlands-Gindikin-Karpelevich formula (and its geometric analogue) evaluating certain structure constants. In the p-adic formulation, the structure constants express the spherical vector in terms of Casselman's basis for the space of Iwahori-fixed vectors. In the geometric formulation, they express the motivic Chern class of the full flag variety in terms of the fixed point basis in equivariant cohomology. Once we have the two expressions, the structure constant formulae can be deduced without recourse to the lattice model; however, it seems significant that the expressions themselves result purely from applying solvable lattice model techniques.

6.0.1 Bethe ansatz-type calculation

We saw in section 5.0.1 an example of the Bethe ansatz, in which we determined conditions on the spectral parameters for the Frozen Pipes model such that $B(x_1)\cdots B(x_n)|\varnothing\rangle$ is an eigenvector of the transfer matrix. A similar method can also be used to compute an explicit symmetrization formula for the Bethe vectors $B(z_1)\cdots B(z_M)|\varnothing\rangle$ of the Tokuyama model with arbitrary $z_1,\ldots,z_M$ (these are sometimes called "off-shell" Bethe vectors, in contrast with the "on-shell" Bethe vectors whose parameters are required to satisfy the Bethe equations). Here, $A(z), B(z), C(z)$, and $D(z)$ denote the monodromy operators depicted in Figure 6.1.

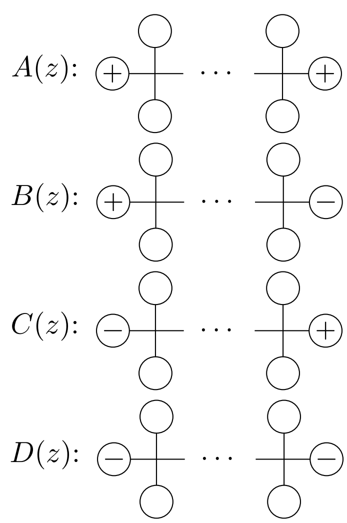


Figure 6.1: The monodromy operators.

Borodin and Petrov ([35], Theorem 4.14), inspired by arguments of Felder and Varchenko [36], apply this method for the higher spin six-vertex model. Here we apply a modified version of their argument to calculate the partition function of the Tokuyama model, obtaining the following:

Theorem 3

$$Z(\mathfrak{S}_{\lambda,\boldsymbol{z}}) = \prod_{i<j} \frac{z_i - q^{-1} z_j}{z_i - z_j} \sum_{w \in S_M} (-1)^{l(w)} \mathbf{z}^{w(\lambda+\rho)}. \tag{6.2}$$

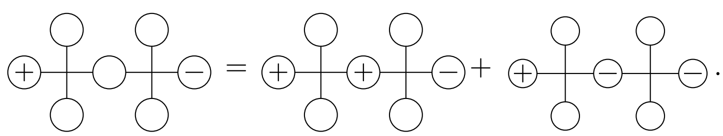
We start by considering the effect of $B(z_M) \cdots B(z_1)$ on two tensor factors. If

$$\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$$

denotes the monodromy matrix for a single vertex acting on the vector space V_i, we obtain the following monodromy matrix for the action on the vector space $V_1 \otimes V_2$:

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

and thus $B(z) = A_2(z)B_1(z) + B_2(z)D_1(z)$. Thinking about this pictorially, we are simply summing over the two possible spins for the middle edge in a one row, two column lattice with boundary conditions prescribed by B:



In our case, $V_1 = V_2 = V$, so the subscripts only matter insofar as they keep track of which column the operators are acting on. Note that operators acting on different columns commute with one another.

Thus, an expression for the composition $B(z_M) \cdots B(z_1)$ in terms of single-vertex operators is given by expanding

$$(B_1(z_1)A_2(z_1) + D_1(z_1)B_2(z_1)) \cdots (A_2(z_M)B_1(z_M) + B_2(z_M)D_1(z_M)),$$

and acting on $e_0 \otimes e_0$. The terms that survive, i.e., those that correspond to nonzero partition functions with our choice of admissible vertices, consist of all possible combinations of factors B_1 and D_1 acting on the first column and of B_2 and A_2 acting on the second. Since e_0 is an eigenvector for the A and D operators, we would like to be able to commute the A's and D's past the B's in each term. We will see that the Yang-Baxter equation allows us to do so, at the expense of multiplying by certain factors. Thus, we can write the partition function as a linear combination of terms of the form

$$B_1(z_{k_1})\cdots B_1(z_{k_{M-s}})D_1(z_{l_1})\cdots D_1(z_{l_s})e_0 \otimes B_2(z_{i_1})\cdots B_2(z_{i_s})A_2(z_{j_1})\cdots A_2(z_{j_{M-s}})e_0$$

where

$$\begin{aligned} &\mathcal{I} = \{i_1 < \cdots < i_s\}, && \mathcal{J} = \{j_1 < \cdots < j_{M-s}\}, && \mathcal{I} \sqcup \mathcal{J} = \{1, \ldots, M\} \\ &\mathcal{K} = \{k_1 < \cdots < k_{M-s}\}, && \mathcal{L} = \{l_1 < \cdots < l_s\}, && \mathcal{K} \sqcup \mathcal{L} = \{1, \ldots, M\}. \end{aligned}$$

Note that $\mathcal{I} \cap \mathcal{K} = \varnothing$. We see that the action of the desired product of B operators on $e_0 \otimes e_0$ has the form

$$\sum_{\mathcal{K} \subseteq \{1,\ldots,M\}} C_{\mathcal{K}} \left(\prod_{k \in \mathcal{K}} B_1(z_k) \prod_{l \notin \mathcal{K}} D_1(z_l) \right) e_0 \otimes \left(\prod_{l \notin \mathcal{K}} B_2(z_l) \prod_{k \in \mathcal{K}} A_2(z_k) \right) e_0 \tag{6.3}$$

for some unique coefficients $C_{\mathcal{K}}(z_1, \ldots, z_M)$. Since their A, B, and D operators all commute among themselves, Borodin and Petrov are able to use symmetry to reduce to computing only those coefficients $C_{\mathcal{K}}$ with $\mathcal{K} = \{1, \ldots, r\}$, $1 \leqslant r \leqslant M$. In our case, however, the B operators only commute up to a factor. So instead, we calculate each $C_{\mathcal{K}}$ individually by analyzing the necessary commutations between the B's and D's and those between the B's and A's.

To aid our calculations, note that in our model, composing more than one B operator and acting on a single tensor factor results in an inadmissible state. Thus, without loss of generality, we can consider only the case where $|\mathcal{K}| \leqslant 1$. However, since our goal is a formula for an arbitrary number of tensor factors, we will not yet impose any restrictions on the size of $\mathcal{K}^c := \{1, \ldots, M\} \backslash \mathcal{K}$. This will allow us to iteratively apply our formula for acting on two tensor factors and eventually "share out" the B operators so that each is acting on a single tensor factor.

Indeed, suppose we would like to compute the action on N tensor factors, where $N \geqslant M$. (In other words, we have an $M \times N$ lattice.) Recall that e_0 is an eigenvector for the operators $A_i(z)$ and $D_i(z)$ (for any i), and let $a_i(z)$ and $d_i(z)$ denote the corresponding eigenvalues. In each term of (6.3), once we factor out these eigenvalues, the second tensor factor involves a sequence of B operators acting on e_0. By replacing e_0 with $e_0 \otimes e_0$, we can again apply our method for computing the action of B operators on two tensor factors, repeating until we have chosen $\mathcal{K}_1, \dots, \mathcal{K}_N \subseteq \{1, \dots, M\}$ such that $\mathcal{K}_1 \sqcup \cdots \sqcup \mathcal{K}_N = \{1, \dots, M\}$ and $|K_i| \leqslant 1$. The coefficient of the term corresponding to a particular sequence $\mathcal{K}_1, \dots, \mathcal{K}_N$ of subsets will then be the product $C_{\mathcal{K}_1,\dots,\mathcal{K}_N} := C_{\mathcal{K}_1} C_{\mathcal{K}_2} \cdots C_{\mathcal{K}_N}$ of the coefficients from each step in this process, and by pulling out eigenvalues at each stage, we recover the factor

$$\prod_{i<j} d_i(\mathcal{K}_j) a_j(\mathcal{K}_i),$$

where we have used the notation $f(\mathcal{K}) = \prod_{k \in \mathcal{K}} f(z_k)$.

We now go through the process of calculating the coefficients $C_{\mathcal{K}_1,\dots,\mathcal{K}_N}$. Via the usual train argument, we obtain the following relation between the A and B operators:

$$(z_j - q^{-1} z_i) B(z_i) A(z_j) = (z_i - z_j) A(z_j) B(z_i) + (1 - q^{-1}) z_j B(z_j) A(z_i)$$

and between the B and D operators:

$$(z_j - q^{-1} z_i) B(z_i) D(z_j) = (z_j - z_i) D(z_j) B(z_i) + (1 - q^{-1}) z_j B(z_j) D(z_i)$$

and therefore

$$A(z_j) B(z_i) = \frac{z_j - q^{-1} z_i}{z_i - z_j} B(z_i) A(z_j) + \frac{(1 - q^{-1}) z_j}{q(z_j - z_i)} B(z_j) A(z_i). \tag{6.4}$$

and

$$D(z_j) B(z_i) = \frac{z_j - q^{-1} z_i}{z_j - z_i} B(z_i) D(z_j) + \frac{(1 - q^{-1}) z_j}{z_j - z_i} B(z_j) D(z_i). \tag{6.5}$$

We are interested in swapping the operators without swapping spectral parameters, and thus are only concerned with the first summands appearing in (6.4) and (6.5).

With these relations in hand, we now turn again to analyzing the expansion of

$$(B_1(z_1)A_2(z_1) + D_1(z_1)B_2(z_1)) \cdots (A_2(z_M)B_1(z_M) + B_2(z_M)D_1(z_M)) \tag{6.6}$$

in order to determine which commutation relations are needed. Note that for each term in the expansion, the indices of the spectral parameters will always appear in increasing order. Thus, we will always commute the A and D operators past those B operators that have larger spectral parameter indices. In the case where $\mathcal{K}_1$ is empty, no commutations are needed, and so $C_{\mathcal{K}_1} = 1$. In the case where $\mathcal{K}_1 = \{j_1\}$, we obtain a factor of

$$\prod_{i<j_1} \frac{z_i - q^{-1}z_{j_1}}{z_i - z_{j_1}}$$

from commuting $B(z_{j_1})$ with $D(i)_{i<j_1}$, and a factor of

$$\prod_{m>j_1} \frac{z_{j_1} - q^{-1}z_m}{z_m - z_{j_1}}$$

from commuting $A(z_{j_1})$ with $B(z_m)_{m>j_1}$. From the next non-empty subset $\mathcal{K} = \{j_2\}$, we obtain

$$\prod_{\substack{i<j_2 \\ i\neq j_1}} \frac{z_i - q^{-1}z_{j_2}}{z_i - z_{j_2}}$$

from commuting B's and D's, and

$$\prod_{\substack{m>j_2 \\ m\neq j_1}} \frac{z_{j_2} - q^{-1}z_m}{z_m - z_{j_2}}$$

from commuting B's and A's. Continuing in this way, we obtain, for every pair (i, j) with $i < j$, either the factor $\frac{z_i - q^{-1}z_j}{z_i - z_j}$ (from B and D commutations) or $\frac{z_i - q^{-1}z_j}{z_j - z_i}$ (from B and A commutations), which differ by a factor of -1. Note that we can associate to each partition $\mathcal{K}_1 \sqcup \mathcal{K}_2 \sqcup \cdots \mathcal{K}_N$ into subsets a permutation $w \in S_M$ by listing the elements of $\{1, 2, \ldots M\}$ in the order in which they appear in the sequence of $\mathcal{K}$'s. (Multiple subset partitions will correspond to the same w, due to the presence of empty sets.) Factoring (-1) out from each B and A commutation, we obtain one for each ascent of w. In other words, we have:

$$C_{\mathcal{K}_1,\ldots,\mathcal{K}_N} = (-1)^{l(w_0 w)} \prod_{i<j} \frac{z_i - q^{-1} z_j}{z_i - z_j}.$$

Now all that remains is to recall the effects of the operators A,B, and D on a single tensor factor (i.e., look at the vertex weights) and to pick off the coefficient of e_λ. From §3.0.1, we have:

$$B_i(z)e_0 = e_1, \qquad d_i(z)e_0 = ze_0, \qquad a_i(z)e_0 = e_0.$$

Thus, to read off the coefficient of e_λ, we examine the summands for which the non-empty subsets are exactly $K_{\lambda_i+\rho_i}$. Each of these summands now corresponds to a unique permutation w, and for each w it is straightforward to see that

$$\prod_{i<j} d_i(\mathcal{K}_j) a_j(\mathcal{K}_i) = \mathbf{z}^{w_0 w(\lambda+\rho)}.$$

Finally, putting everything together, the coefficient of e_λ is:

$$\prod_{i<j} \frac{z_i - q^{-1} z_j}{z_i - z_j} \sum_{w \in S_M} (-1)^{l(w_0 w)} \mathbf{z}^{w_0 w(\lambda+\rho)} = \prod_{i<j} \frac{z_i - q^{-1} z_j}{z_i - z_j} \sum_{w \in S_M} (-1)^{l(w)} \mathbf{z}^{w(\lambda+\rho)} \tag{6.7}$$

and thus our theorem is proved.

Geometric bases equal p-adic bases

The first connection between p-adic representation theory and equivariant K-theory arose in the work [20] of Lusztig, in which he used the operators $\mathfrak{L}_i$ to define an affine Hecke algebra representation on the equivariant K-theory ring of a flag variety. These results were subsequently used to prove the Deligne-Lusztig conjecture in [37]. Recent work of Aluffi, Mihalcea, Schürmann, and Su [8] and Mihalcea and Su [38] has pushed this connection further, showing that both $\mathfrak{L}_i$ and $\mathfrak{T}_i$ arise naturally in the study of motivic Chern classes of generalized flag varieties G/B, where G is any complex simple Lie group. In fact, these operators are adjoint to one another with respect to the equivariant K-theoretic Poincaré pairing. Moreover, the natural bases for the space of Iwahori fixed vectors–the standard basis and Casselman basis, as defined in § 2–are identified with the natural bases on the K-theory side. We now describe some of these results in more detail.

The **equivariant motivic Chern (MC_y) class** is a deformation of the fundamental class in equivariant K-theory with respect to a parameter y. It lives in the ring $K_T(G/B)[y^{\pm}]$ and generalizes the Chern-Schwartz-MacPherson (CSM) class in equivariant homology, which was originally defined in order to generalize the notion of the total Chern class of a tangent bundle to singular varieties. It was shown (by different methods) in both [8] and [39] that the motivic Chern classes of Schubert cells are equivalent to the K-theoretic stable envelopes defined by Okounkov and Aganagic in [2]. Via this equivalence, we can think of these classes as analogues of Schubert classes in the equivariant K-theory of the cotangent bundle of G/B, i.e., roughly, the passage from Schubert classes in $K_T(G/B)$ to motivic Chern classes corresponds to moving from the bottom to the top of the cube in Figure 4.1. It is not necessary for our purposes to use the original construction of motivic Chern classes, as [8] show that they can be calculated recursively using (almost) the same operators as the Iwahori Whittaker functions, starting from the class of a point. More details on the precise original definition can be found in the works cited above.

Let λ be a weight of G and let $\mathcal{L}_\lambda := G \times^B \mathrm{c}_\lambda \in K_T(G/B)$ be the line bundle over G/B with fiber of weight λ over the coset $1.B$. Then we define the following operators on $K_T(G/B)[y^{\pm}]$:

$$\mathcal{T}_i := (1 + y\mathcal{L}_{\alpha_i})\mathcal{D}_i - id, \qquad \mathcal{T}_i^\vee := \mathcal{D}_i(1 + y\mathcal{L}_{\alpha_i}) - id,$$

where $\mathcal{D}_i$ is the Demazure operator in equivariant K-theory. By [8] Lemma 3.3, $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are adjoint, i.e., $\langle \mathcal{T}_i(a), b\rangle = \langle a, \mathcal{T}_i^\vee(b)\rangle$. When we localize at $1.B$, we obtain the following algebraic versions of the above operators on $K_T(pt)[y^{\pm}]$:

$$\widetilde{\mathcal{T}}_i := (1 + ye^{\alpha_i})\widetilde{\mathcal{D}}_i - id, \qquad \widetilde{\mathcal{T}}_i^\vee := \widetilde{\mathcal{D}}_i(1 + y\mathcal{L}_{\alpha_i}) - id,$$

where, in terms of the divided difference operator ∂_i defined in § 4.0.3, $\widetilde{\mathcal{D}}_i = \partial_i e^{\alpha_i}$. Like $\mathfrak{T}_i$ and $\mathfrak{L}_i$, $\widetilde{\mathcal{T}}_i$ and $\widetilde{\mathcal{T}}_i^\vee$ generate a finite Iwahori Hecke algebra acting on $K_T(G/B)$.

One of the main results in [8] is the following.

Theorem 4 ([8], Theorem 4.5) *Let $w \in S_n$ and let s_i be a simple reflection such that $l(ws_i) > l(w)$. Then*

$$MC_y(C_{ws_i}) = \mathcal{T}_i MC_y(C_w).$$

Together with the base case $MC_y(C_{id}) = \mathcal{O}_{id}$, this gives a recursion for calculating the MC_y classes of Schubert cells. [8] also define **dual motivic Chern classes** $MC_y^\vee$ via:

$$MC_y^\vee(C^w) = (\mathcal{T}_{w_0 w}^\vee)^{-1}(C^{w_0}),$$

where C^w is the *opposite Schubert cell* $C^w := B^- wB/B = w_0 C_{w_0 w}$. These classes are so named because the classes $MC_y^\vee(C^w)$ are (up to a factor) dual to $MC_y(C_w)$ with respect to the Poincaré pairing (see [8], Theorem 5.2).

Before we state the results identifying these classes with their p-adic counterparts, we detail how to translate the notational conventions of [7] into those of [8] and [38]. First, in [7], the Iwahori subgroup is defined as the preimage in K of $B_- = w_0 B w_0$ modulo $\mathfrak{p}$, whereas [8] and [38] define it as the preimage of B. Consequently, the standard basis element $\Phi_w^{\boldsymbol{z}}$ in [7] corresponds to $\pi(w_0)\varphi_{ww_0}$ in [8], as we justify below. Let $b \in B, w' \in W, k \in J$, and $k' = w_0 k w_0 \in J_-$. Then:

$$\begin{aligned}
\Phi_w^{\boldsymbol{z}}(bw'k) &= \Phi_w^{\boldsymbol{z}}(bw'w_0k'w_0) \\
&= \pi(w_0)\Phi_w^{\boldsymbol{z}}(bw'w_0k') \\
&= \pi(w_0)\tau_{\boldsymbol{z}}(b) \text{ if } w = w'w_0; \;\; 0 \text{ otherwise} \\
&= \pi(w_0)\varphi_{ww_0}.
\end{aligned}$$

Note that the involution $w \mapsto ww_0$ is inclusion reversing with respect to the Bruhat order. For consistency, since the definition (2.1) of Iwahori Whittaker functions involves evaluating the Whittaker functional on the standard basis, we will adjust the notation for the Whittaker functions as well. Specifically, we replace ϕ_w by ϕ_{ww_0} to indicate that ϕ_{ww_0} is the Iwahori Whittaker function evaluated at $\pi(w_0)\varphi_{ww_0}$.[1]

[1] Unfortunately, [8] use the notation ϕ instead for $\sum \varphi$–to avoid confusion, we will use ϕ exclusively for the Whittaker function.

Next, note that for τ a character of T and α a root of $\widehat{G}$ (aka a coroot of G), [8] and [38] define $e^{\alpha}(\tau)$ by $e^{\alpha}(\tau) = \tau(h_{\alpha}(\varpi))$, where $h_{\alpha} : F^{\times} \to T(F)$ is the one parameter subgroup corresponding to α. Since [7] work with the contragredient representation $I(\boldsymbol{z}^{-1})$ of the principal series, we will usually take $\tau = \tau_{\boldsymbol{z}^{-1}}$, in which case $e^{\alpha}(\tau)$ corresponds to our earlier notation $\boldsymbol{z}^{-\alpha}$ from [7]. The notation $e^{\alpha}(\tau)$ is deliberately similar to that for the generators of $K_T(pt)$, due to the isomorphism we describe in the next section.

Geometric versions of Tokuyama calculations

Keeping the above notational dictionary in mind, we now quote another main result of [8] that will allow us to geometrically interpret the formulas for Whittaker functions that were derived above using lattice model methods. Consider $K_T(pt) \cong R[T]$ as a subring of $\mathcal{O}$, let τ be an unramified character, and let c_{τ} be the one-dimensional $K_T(pt)$-module induced by evaluation at τ. Let $\iota_w \in K_T(G/B)$ denote the fixed point basis, let b_w be the multiple of ι_w defined by the condition $b_w|_w = MC_y^{\vee}(C^w)|_w$, and let $\widetilde{b}_w = b_w \otimes 1 \in \widetilde{K}_T(G/B)[y^{-1}]$. The localization $MC_y^{\vee}(C^w)|_w$ is calculated explicitly in [8] Proposition 7.3, and the resulting explicit formula for $\widetilde{b}_w$ is:

$$\widetilde{b}_w = (-1)^{\dim(G/B) - l(w)} \prod_{\alpha \in \Delta^+,\ w\alpha \in \Delta^+} \frac{y^{-1} + e^{-w\alpha}}{1 - e^{w\alpha}} \iota_w \otimes 1$$

Theorem 5 (See [8], Theorem 10.2) *Define a $K_T(pt)$-module homomorphism*

$$\Psi : K_T(G/B)[y^{\pm}] \otimes_{K_T(pt)[y^{\pm}]} \mathrm{c}_{\tau} \to I(\tau)^J$$

via $\Psi(MC_y^{\vee}(C^w) \otimes 1) = \varphi_w$ and $y \mapsto -q^{-1}$.[2] *Then:*

1. *Ψ is an isomorphism of Hecke algebra modules, where the Hecke actions are that of $\mathcal{T}_i^{\vee}$ on $K_T(G/B)$ and of $\mathfrak{T}_i$ on $I(\tau_{\boldsymbol{z}^{-1}})^J$.*
2. *We have the equality $\Psi(\widetilde{b}_w) = f_w$.*

So, under this isomorphism, $\mathfrak{T}_i$ maps to (the algebraic version of) $\mathcal{T}_i^{\vee}|_{\tau = \tau_{\boldsymbol{z}^{-1}}}^{y \mapsto -q^{-1}}$, and thus the operator $\boldsymbol{z}^{\rho}\mathfrak{T}_i\boldsymbol{z}^{-\rho}$ associated to the lattice model in [7] corresponds to the

[2] In [8], $y \mapsto -q$ rather than $y \mapsto -q^{-1}$; we have adjusted this to account for the slight difference in notation in [7].

operator $(\mathcal{L}_{-\rho} \circ \mathcal{T}_i^\vee \circ \mathcal{L}_\rho)\,|^{y\mapsto -q^{-1}}_{\tau=\tau_{\boldsymbol{z}^{-1}}}$ in [8], or, equivalently, to the operator $(\mathcal{L}_{-\rho} \circ \mathcal{D} \circ \mathcal{T}_i^\vee \circ \mathcal{D} \circ \mathcal{L}_\rho)|^{y\mapsto -q}_{\tau=\tau_{\boldsymbol{z}}}$, where $\mathcal{D}$ is the Grothendieck-Serre duality operator (see [38] §3). We will also need the following fact: for $w \in W$ and $\lambda \in \Lambda$, we have $e^{w\lambda} = \mathcal{L}_\lambda|_w = \langle \mathcal{L}_\lambda, \iota_w \rangle$ in $K_T(pt)$.

Theorem 6 *Applying the isomorphism Ψ to the equality*

$$\sum_{w\in W} \boldsymbol{z}^\rho \phi_w(z; \varpi^{-\lambda}) = \sum_{w\in W} \prod_{i<j} \frac{z_i - q^{-1} z_j}{z_i - z_j} (-1)^{l(w)} \boldsymbol{z}^{w(+\rho)} \tag{6.8}$$

gives

$$\sum_{w\in W} \left\langle \mathcal{L}_{\lambda+\rho}, (-q)^{-l(w)} MC^\vee_{-q^{-1}}(C^{ww_0}) \right\rangle = \sum_{w\in W} \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \frac{e^\alpha - q^{-1}}{1-e^\alpha} \langle \mathcal{L}_{\lambda+\rho}, (w_0 \cdot b_w) \rangle. \tag{6.9}$$

We have the following immediate corollary:

Corollary 1 *Since the $\mathcal{L}_{\lambda+\rho}$ form a generating set for $K_T(G/B)$, it follows from Theorem 6 that*

$$\sum_{w\in W} (-q)^{-l(w)} MC^\vee_{-q^{-1}}(C^{ww_0}) = \sum_{w\in W} \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \frac{e^\alpha - q^{-1}}{1-e^\alpha} (w_0 \cdot b_w). \tag{6.10}$$

The sum $\sum_{w\in W} (-q)^{-l(w)} MC^\vee_{-q^{-1}}(C^{ww_0})$ can be identified with w_0 acting on the modified spherical vector Φ_- of Reeder [40]. Hence we have the decomposition

$$w_0 \cdot \Phi_- = \sum_{w\in W} \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \frac{e^\alpha - q^{-1}}{1-e^\alpha} w_0 \cdot f_w, \tag{6.11}$$

which is a modified version of the Langlands-Gindikin-Karpelevich formula expressing Φ in terms of the f_w.

[proof of Theorem 6] First, consider the left hand side, which, via Proposition 1 and our notational dictionary equals:

$$\sum_{w\in W} \left\langle \left(\mathcal{L}_{-\rho} \circ \mathcal{D} \circ \mathcal{T}^\vee_{w^{-1}} \circ \mathcal{D} \circ \mathcal{L}_\rho\right) |^{y\mapsto -q}_{\tau=\tau(\boldsymbol{z})} \left(\mathcal{L}_{\lambda+\rho}\right), \iota_{id} \right\rangle.$$

Since $\mathcal{T}_w^\vee$ and $\mathcal{T}_{w^{-1}}$ are adjoint with respect to the Poincaré pairing, this is equal to

$$\sum_{w\in W}\left\langle \mathcal{L}_{\lambda+\rho}, (\mathcal{L}_{-\rho}\circ\mathcal{D}\circ\mathcal{T}_w\circ\mathcal{D}\circ\mathcal{L}_\rho)\,|_{\tau=\tau(\boldsymbol{z})}^{y\mapsto -q}\,(\iota_{id})\right\rangle.$$

By Proposition 3.2 of [38], we can replace $\mathcal{D}\circ\mathcal{T}_w\circ\mathcal{D}$ with $(\mathcal{T}_{w^{-1}})^{-1}$:

$$\sum_{w\in W}\left\langle \mathcal{L}_{\lambda+\rho}, \left(\mathcal{L}_{-\rho}\circ(\mathcal{T}_{w^{-1}})^{-1}\circ\mathcal{L}_\rho\right)|_{\tau=\tau(\boldsymbol{z})}^{y\mapsto -q}\,(\iota_{id})\right\rangle$$

and then, by Remark 3.3 of loc. cit., this equals

$$\sum_{w\in W}\left\langle \mathcal{L}_{\lambda+\rho}, \left(\mathcal{L}_{-\rho}\circ\left(y^{l(w^{-1})}\mathcal{L}_\rho\circ\mathcal{T}_{w^{-1}}^\vee\circ\mathcal{L}_{-\rho}\right)^{-1}\circ\mathcal{L}_\rho\right)|_{\tau=\tau(\boldsymbol{z})}^{y\mapsto -q^{-1}}\,(\iota_{ww_0})\right\rangle.$$

Finally, cancelling $\mathcal{L}_\rho$ with $\mathcal{L}_{-\rho}$, noting that $l(w^{-1})=l(w)$, and recalling the definition of $MC_y^\vee(C^w)$, this becomes:

$$\begin{aligned}&\sum_{w\in W}\left\langle \mathcal{L}_{\lambda+\rho}, (-q)^{-l(w)}(\mathcal{T}_{w^{-1}}^\vee)^{-1}|_{\tau=\tau(\boldsymbol{z})}^{y\mapsto -q^{-1}}\,(\iota_{id})\right\rangle\\ =&\sum_{w\in W}\left\langle \mathcal{L}_{\lambda+\rho}, (-q)^{-l(w)}MC_{-q^{-1}}^\vee(C^{ww_0})\right\rangle.\end{aligned}$$

Now we consider the right hand side, which equals

$$\sum_{w\in W}(-1)^{l(w)}\prod_{\alpha>0}\frac{1-q^{-1}e^{-\alpha}(\tau_{\boldsymbol{z}})}{1-e^{-\alpha}(\tau_{\boldsymbol{z}})}\left\langle \mathcal{L}_{\lambda+\rho},\iota_w\right\rangle.$$

(To reduce clutter in the following calculations, we will write $e^\alpha(\tau_{\boldsymbol{z}})$ simply as e^α.) To isolate the w summand, we restrict to the fixed point F_w. Recall that

$$\iota_{w'}|_w=\begin{cases}\displaystyle\prod_{\alpha>0}1-e^{-w\alpha} & \text{if } w'=w\\ 0 & \text{otherwise.}\end{cases}$$

Thus, restricting to F_w gives

$$
\begin{aligned}
&(-1)^{l(w)} \prod_{\alpha>0} \frac{1-q^{-1}e^{-\alpha}}{1-e^{-\alpha}} \prod_{\alpha>0} 1-e^{-w\alpha} \\
&= (-1)^{l(ww_0)} \prod_{\substack{\alpha>0 \\ w^{-1}\alpha>0}} \frac{e^{\alpha}-q^{-1}}{1-e^{\alpha}} \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \frac{e^{\alpha}-q^{-1}}{1-e^{\alpha}} \prod_{\substack{\alpha>0 \\ w\alpha>0}} 1-e^{-w\alpha} \prod_{\substack{\alpha>0 \\ w\alpha<0}} 1-e^{-w\alpha}.
\end{aligned}
$$

Note that we can rewrite the fourth product so that it cancels with the denominators of the first:

$$
\prod_{\substack{\alpha>0 \\ w\alpha<0}} 1-e^{-w\alpha} = \prod_{\substack{\alpha<0 \\ w\alpha>0}} 1-e^{w\alpha} = \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} 1-e^{\alpha}
$$

and hence our full expression becomes

$$
\underbrace{(-1)^{l(ww_0)} \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} e^{\alpha}-q^{-1} \prod_{\substack{\alpha>0 \\ w\alpha<0}} 1-e^{w\alpha}}_{(w_0 \cdot b_w)|_w} \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \frac{e^{\alpha}-q^{-1}}{1-e^{\alpha}},
$$

and one can check that the bracketed piece is w_0 acting on b_w, localized at w. Hence the theorem follows.

Discussion

It is interesting to compare the results of Theorem 6 and Corollary 1 to our results in §5. Proposition 7 expresses the fixed point basis in terms of the geometric (Schubert) basis, and, symmetrically, Corollary 1 expresses the geometric (dual motivic Chern class) basis in terms of the fixed point basis. However, their proofs are not symmetric: Corollary 1 relied on the intermediary step of pairing each basis with the line bundle $\mathcal{L}_{\lambda+\rho}$, and hence was obtained from a fundamentally different type of lattice model.

Equating the left hand sides of (6.8) and (6.9) enables us to interpret the Iwahori and Tokuyama models pictorially as pairings–between $MC^{\vee}_{-q^{-1}}(C^{ww_0})$ and $\mathcal{L}_{\lambda+\rho}$ and between $MC^{\vee}_{-q^{-1}}(X^{id})$ and $\mathcal{L}_{\lambda+\rho}$, respectively, as depicted below in Figure 6.2.

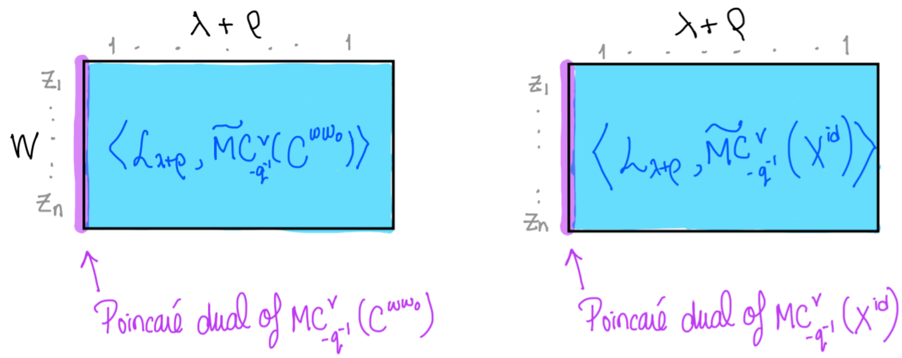


Figure 6.2: The Iwahori model (left) and Tokuyama model (right) as Poincaré pairings. (The Poincaré duals of $MC^\vee_{-q^{-1}}(C^{ww_0})$ and $MC^\vee_{-q^{-1}}(X^{id})$ can be calculated explicitly via [8], Theorem 6.2.)

Unfortunately, in contrast with the Frozen Pipes model, the terms involving the fixed point basis cannot be as straightforwardly identified with a piece of the lattice model. Figure 6.3 below depicts the Bethe ansatz-like process from §6.0.1 in the simple case where $\lambda = \varnothing$ and the lattice is 2×2. This process does not result in admissible states of the model; rather, it uses the commutation relations to break the lattice states into easily calculable chunks. Regardless, it seems noteworthy that the analysis of lattice models leading to equation (6.8) roughly follows the correspondence of Figure 4.2, in which the geometric basis is identified with the spin basis, and where the fixed point basis arises via the algebraic Bethe ansatz.

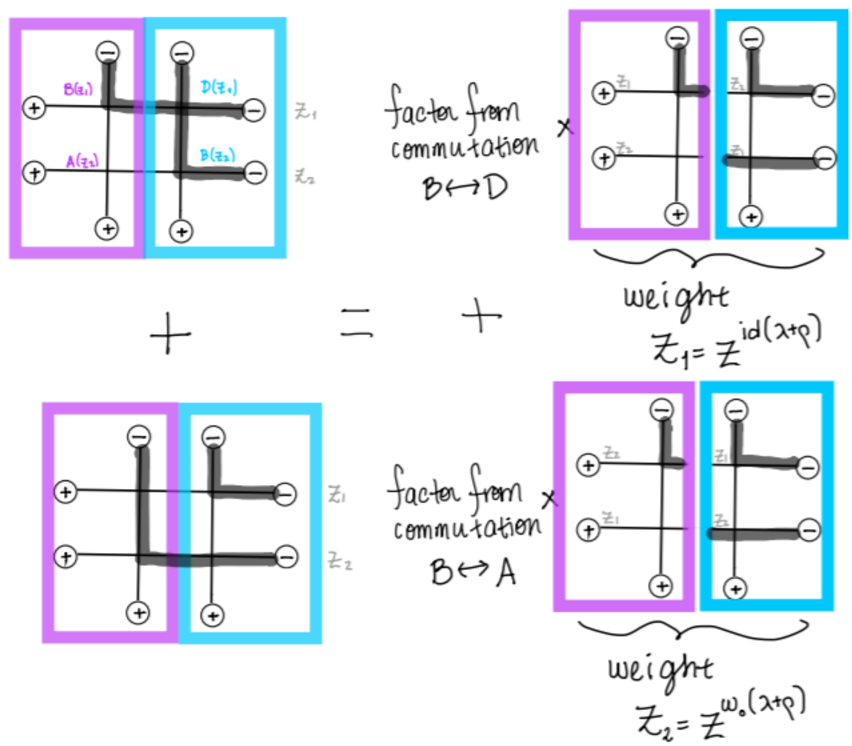


Figure 6.3: A diagrammatic description of the Bethe ansatz-like process in section 6.0.1, in the 2×2 case with $\lambda = \varnothing$. We take each state of the model (left), and use the commutation relations between the A, B, C, D operators to commute each column separately, until all A's and D's are below all B's (right). Note that the diagrams on the right are no longer admissible states.

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