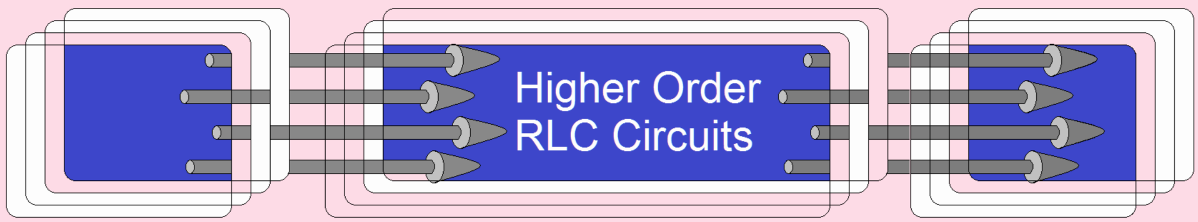
Part 3 - A.
1. Chapter 8 Schaums Outlines: Higher Order Circuits and Complex Frequency 6th Edition.
2. Relevant Chapters Engineering Circuit Analysis Hyat and Kemmerly 4th Edition.

Intermediate.

Circuiting PrerequisitesTo Laplace Transform Electric Circuits.



Part 3 - A
(Intermediate Level)

June 2020.

Higher Order Circuits and Complex Frequency.

Part 3A.

Engineering Circuit Analysis 4th Edition 1986, Hyat and Kemmerly:
(Today 2020 goes with the 9th Edition)

We are now about to begin the fourth major portion (textbook has seven parts, this is fourth) of our study of circuit analysis, a discussion of the concepts of complex frequency. This, we shall see, is a remarkable unifying concept which will enable us to tie together all our previously developed analytical techniques into one neat package. Resistive circuit analysis, steady state sinusoidal analysis, transient analysis, the forced response, the complete response, and the analysis of circuits excited by exponential forcing functions and exponential damped sinusoidal forcing functions will all become special cases of the general techniques of circuit analysis which are associated with complex frequency concept.

Comments:
We completed these topics indicated above shown again:
1). Resistive circuit analysis
2). Steady state sinusoidal analysis
3). Transient analysis
4). The forced response
5). The complete response
6). Analysis of circuits excited by exponential forcing functions
7). Exponential damped sinusoidal forcing functions

May not be as in depth as some like, but we did adequate example problems and plots to get a good understanding on the subject matter. Textbooks like Hyat and Kemmerly are few and far between - especially the 4th edition. Schaums Series/Outline is supplementary to main textbook. And in this case this Schaums textbook has been around since mid 1960's. Now in the 7th edition. Said that, I/We may have covered adequate the above requirements, to get started in this main topic, which we started in Part 1A.

Schaums Chapter 8 Contents:

Required skills from Hyat and Kemmerly included to relevant sections.

Methods to apply for this chapter:

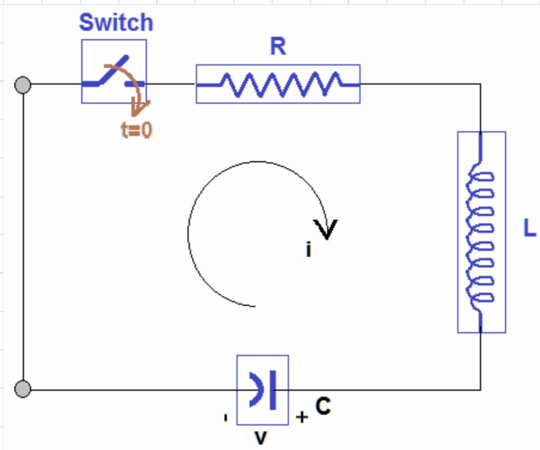
1. Circuit analysis using mesh, Thevenin, Norton,..........
2. Sketch the circuit
3. Apply methods worked in previous chapter 'Higher Order Circuits'.
4. Find if initial conditions apply.....
5. Apply the methods provided in Chapter 5 to solve circuit problems.......
6. Most circuits can be reduced to their equivalent resistance, capacitance, and inductance thru series or parallel calculations.
7. The final circuit connection-layout should be that it may be one of two; series RLC of parallel RLC circuit OR RL, RC, and LC.
8. Differential Equations? May look like we need a new one each time for a circuit but such is not the case. We have a few input sources and the response we seek is similar to the input voltage or current source. Similar to problems solved in chapter 5 'Higher Order Circuits'.
9. DE forms of solution of series and parallel RLC circuits should be the same they vary depending on voltage/current waveform equation form.
10. DE will NOT be an obstacle to solving the circuit problem.
    Use your math book and apply DE chapter.
    NOT need be an A student in math, average will do, re-read chapters.
11. Plot the answer's waveforms.
12. Do some analysis on the plots.
13. Make notes.
14. Be ready to use the method worked on the previous exercises.

Level: Intermediate.

Because this chapter has use for Differential Equations it is not at an advanced level.
We are merely using similar methods from previous chapter Higher Order Circuits.
With a particular DE form there is an expected particular solution for it.
We are not deriving or proving any of the DEs. These were already done in the Mathematics course. *DEs can be taught as a single topic in circuits course, its not done because it taught in the maths course. That can be a problem no EE examples.*

Highly Recommend:
Engineering Mathematics (For Electrical Engineering) 4th Edition by Croft, Davidson, Hargreaves, and Flint. Publisher: Pearson. Very Good Textbook for Electrical Engineering. Examples used in textbook many are based in electrical engineering.

8.2 Series RLC circuit:



There is NO voltage source in the circuit to the left.

That is obviously NOT normal.

However, we know capacitors discharge when the switch is turned off, and for a short duration discharge current into the circuit.

Though with no volt source, yet we can write applicable equations for anlysis, *whilst* the volt source can be inserted later.

Kirchoff conservation of voltage applied to the series RLC electric circuit:

Conservation? May not be the most appropriate choise of word, but we get tired of too many LAWS in engineering courses...excessive. Later we may say Norton's conservation of current at the electric circuit node. Leave it for the extra serious engineer to use the word law.

$v_R + v_L + v_c = 0$ voltage circuit; voltage loop equation.
Equal zero because there is no voltage source.

$Ri + L\cdot\left(\frac{di}{dt}\right) + \left(\frac{1}{C}\right)\int i\,dt = 0$ CORRECT.

$Ri + L\cdot\left(\frac{di}{dt}\right) + \left(\frac{1}{C}\right)\int i\,dt = 0$ differenting wrt dt

$R\cdot\frac{di}{dt} + L\cdot\left(\frac{d^2\ i}{dt^2}\right) + \left(\frac{1}{C}\right)\cdot i = 0$ just pull out the intergral symbol.

$L\cdot\left(\frac{d^2\ i}{dt^2}\right) + R\cdot\frac{di}{dt} + \left(\frac{1}{C}\right)\cdot i = 0$ rearranging for a 2nd order equation
2nd: di^2/dt^2, *1st:* di/dt, *constant:* i.

Above equation is good so why do we divide it by L?
Because we get R/L in the 2nd term and LC term in the 3rd term?
Or is it because the first term has unity (1) for the coefficient? Yes!
You ask your local engineer. It is for the 1 coefficient. L multiplied by C results in nothing significant, same for R divided by L.

$\left(\frac{d^2\ i}{dt^2}\right) + \left(\frac{R}{L}\right)\cdot\frac{di}{dt} + \left(\frac{1}{L\cdot C}\right)\cdot i = 0$ dividing by L

Differential Equation (DE) has a solution for the above form of expression.

$$\left(\frac{d^2 i}{dt^2}\right) + \left(\frac{R}{L}\right)\cdot\frac{di}{dt} + \left(\frac{1}{L\cdot C}\right)\cdot i = 0$$ <----Quadratic DE.

For quadratic equation results in 2 roots; i1 and i2.

$$\left[\frac{d^2\cdot(i1+i2)}{dt^2}\right] + \left(\frac{R}{L}\right)\cdot\frac{d(i1+i2)}{dt} + \left(\frac{1}{L\cdot C}\right)\cdot(i1+i2) = 0$$

$$\left(\frac{d^2\cdot i1}{dt^2}\right) + \left(\frac{R}{L}\right)\cdot\frac{di1}{dt} + \left(\frac{1}{L\cdot C}\right)\cdot i1 = 0 \qquad \left(\frac{d^2\cdot i2}{dt^2}\right) + \left(\frac{R}{L}\right)\cdot\frac{di2}{dt} + \left(\frac{1}{L\cdot C}\right)\cdot i2 = 0$$

$A1\cdot e^{s1\cdot t} + A2\cdot e^{s2\cdot t} = 0$ DE solution form. $i1(t) = A1\cdot e^{s1\cdot t}$ $i2(t) = A2\cdot e^{s2\cdot t}$

$A1\cdot s1e^{s1\cdot t} + A2\cdot s2e^{s2\cdot t} = 0$ <--- Its first derivative.

We next solve for s1 and s2.

A1 and A2 solved in initial conditions of circuit, and if there are new methods to solve.

We substitute s1 and s2 in solution: A1 s1 e^s1t + A2 s2 e^s2t

$s1^2 \;==>\; \left(\frac{di^2}{dt^2}\right)$ s1^2 represents the 2nd derivative of current i

$s1 \;==>\; \left(\frac{di}{dt}\right)$ s1 represents the 1st derivative of current i

Constant $==>$ i Constant represents current i.

$$s^2 + \left(\frac{R}{L}\right)\cdot s + \left(\frac{1}{L\cdot C}\right) = 0$$ CORRECT. We have a quadratice DE.

<----Quadratic equation form

Next the complete equation:

$$A1e^{s1t}\cdot\left(s_1{}^2 + \left(\frac{R}{L}\right)\cdot s_s + \left(\frac{1}{L\cdot C}\right)\right) + A2e^{s2t}\cdot\left(s_2{}^2 + \left(\frac{R}{L}\right)\cdot s_2 + \left(\frac{1}{L\cdot C}\right)\right) = 0$$ CORRECT.

DE is saying is:

$$s_1 = s_1{}^2 + \left(\frac{R}{L}\right)\cdot s_1 + \left(\frac{1}{L\cdot C}\right) \qquad s_2 = s_2{}^2 + \left(\frac{R}{L}\right)\cdot s_2 + \left(\frac{1}{L\cdot C}\right)$$

In other words s1 and s2 are the roots of: $s^2 + \left(\frac{R}{L}\right)\cdot s + \left(\frac{1}{L\cdot C}\right)$ Right on the answer.

Some explanation or steps may been left out compared to your textbook,
here I kept it short.

Remember we are working with complex frequency, s is the complex frequency.
s = sigma + jw. We solve for s much later after I get a better understanding of complex frequency.

Next we see the solutions to a quadratic equation

$$s_1 = -\left(\frac{R}{2\,L}\right) + \sqrt{\left(\frac{R}{2\,L}\right)^2 - \left(\frac{1}{LC}\right)} = -\alpha + \beta$$

$$s_2 = -\left(\frac{R}{2\,L}\right) - \sqrt{\left(\frac{R}{2\,L}\right)^2 - \left(\frac{1}{LC}\right)} = -\alpha - \beta$$

Where 2 in denominator came from? Quadratic equation divided by 2A. See equation below, A = 1 = Coeff of our 2nd order deriviative = 1. And (1/LC)? See below, so its a square term like B^2.

$\alpha = \left(\frac{R}{2\,L}\right)$ <--- Exponential damping coefficient. $\omega_0 = \left(\frac{1}{\sqrt{LC}}\right)$ <---Resonant frequency.

Where 2 in denominator came from? Quadratic equation divided by 2A. A = 1, Coeff of our 2nd order deriviative = 1. And (1/LC)? So its a square term like B^2.

$\beta = \sqrt{\alpha^2 - (\omega_0)^2}$ Alpha and Beta are parts of root s1 and s2.

We have some serious conditions that make the solutions unique to each condition. These are shown and applied in the examples later.
For example in our QUADRATIC EQUATIONS solution conditions:

$Ax^2 + Bx + C = 0$ $x1 = \frac{-B + \sqrt{B^2 - 4\,AC}}{2\,A}$ $x2 = \frac{-B - \sqrt{B^2 - 4\,AC}}{2\,A}$

$B^2 - 4\,AC$
- > 0 Roots are real and unequal.
- $= 0$ Roots are real and equal.
- < 0 Roots are imaginary.

Roots of equation can be imaginary (squar-root of -ve number) - Underdamped case.

$$\sqrt{\alpha^2 - (\omega_0)^2} = \sqrt{-1}\,\sqrt{\omega_0^2 - \alpha^2} = j\,\sqrt{\omega_0^2 - \alpha^2} = \omega_d$$ <---Natural resonant frequency

Sqrt(-1) ie j re-positions w0 and alpha above.

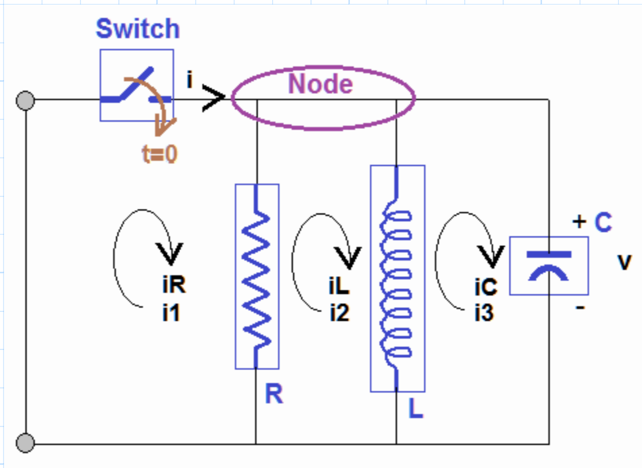
The conditions for series and parallel RLC circuits will be shown at end of parallel RLC circuit.
General idea expressed below so you got the mystery out. *Check with your textbook.*
In RLC circuit the three cases are called
1. over damped: series (alpha **>** omega0) and parallel (alpha^2 **>** omega0^2).
2. critically damped: series and parallel (alpha **=** omega0).
3. underdampled: series (alpha **<** omega) & for parallel (alpha^2 **<** omega0^2).

You have in depth explanation in your Electric Circuits textbook on this chapter.
Mostly math and the choice of DE for the circuit and solution!

End of the case of Series RLC electric circuit.
Please note the other engineering discipline like Mechanical Civil Process Chemical.....also use the same 2nd Order DE technique for solving their equations. EEs are not the only ones using this method in problem solving.

Next page the Parallel RLC circuit.
Maybe some steps need not be explained maybe similar.

8.3 Parallel RLC circuit:



Parallel RLC circuit, here we want to solve for voltage because the voltage is the same across the parallel branches. At the Node the voltage would be the same where all three passive elements R L and C are connected.

We use the Norton's node equation for current!

When the switch is closed, current flowing into the node generates a voltage, and at node identified in circuit, the voltage is the same. Sum of current of each branch of the three elements would sum to total circuit current.

$$v_{node} = v$$

$$i_R + i_L + i_C = i$$

$$\left(\frac{v}{R}\right) + \left(\frac{1}{L}\right)\cdot\int_0^t v\,dt + (C)\cdot\left(\frac{dv}{dt}\right) = 0$$ Equal zero because there is no voltage source.

$$\left(\frac{1}{R}\right)\cdot\frac{dv}{dt} + \left(\frac{1}{L}\right)\cdot v + (C)\cdot\left(\frac{dv^2}{dt^2}\right) = 0$$ differentiating

$$(C)\cdot\left(\frac{d^2\ v}{dt^2}\right) + \left(\frac{1}{R}\right)\cdot\frac{dv}{dt} + \left(\frac{1}{L}\right)\cdot v = 0$$ rearranging

$$\left(\frac{d^2\ v}{dt^2}\right) + \left(\frac{1}{RC}\right)\cdot\frac{dv}{dt} + \left(\frac{1}{LC}\right)\cdot v = 0$$ dividing by C to make the first term coefficient 1.

$s^2 ==> \left(\frac{d^2\ v}{dt^2}\right)$ s^2 represents the 2nd derivative of voltage v.

$s ==> \left(\frac{dv}{dt}\right)$ s represents the 1st derivative of voltage v.

Constant $==>$ v Constant represents voltage v.

$$s^2 + \left(\frac{1}{RC}\right)\cdot s + \left(\frac{1}{L\cdot C}\right) = 0$$ CORRECT. Different from the series RLC.

Next plug-in similar to series RLC

$$A1e^{s1t}\cdot\left(s_1^{\ 2} + \left(\frac{1}{RC}\right)\cdot s_s + \left(\frac{1}{L\cdot C}\right)\right) + A2e^{s2t}\cdot\left(s_2^{\ 2} + \left(\frac{1}{RC}\right)\cdot s_2 + \left(\frac{1}{L\cdot C}\right)\right) = 0$$ CORRECT.

What DE is saying is

$$s_1 = s_1^{\ 2} + \left(\frac{1}{RC}\right)\cdot s_1 + \left(\frac{1}{L\cdot C}\right)$$

$$s_2 = s_2^{\ 2} + \left(\frac{1}{RC}\right)\cdot s_2 + \left(\frac{1}{L\cdot C}\right)$$

In other words s1 and s2 are the roots of: $s^2 + \left(\frac{1}{RC}\right)\cdot s + \left(\frac{1}{L\cdot C}\right)$ Right on the answer.

Remember we are dealing with complex frequency,
NOT a typical or usual environment in beginner electric circuits. *We study s later section.*
Next we see something like a solution to a quadratic equation, but (1/LC) is incorrect, true, intentionally we made it 1/Sqrt(LC). So its square is (1/LC)

$$s_1 = -\left(\frac{1}{2\ RC}\right) + \sqrt{\left(\frac{1}{2\ RC}\right)^2 - \left(\frac{1}{LC}\right)} = -\alpha + \sqrt{\alpha^2 - \omega_0^{\ 2}}$$

$$s_2 = -\left(\frac{1}{2\ RC}\right) - \sqrt{\left(\frac{1}{2\ RC}\right)^2 - \left(\frac{1}{LC}\right)} = -\alpha - \sqrt{\alpha^2 - \omega_0^{\ 2}}$$

Where 2 in denominator above came from? Quadratic equation divided by 2A. A = 1, Coeff of our 2nd order deriviative = 1. And (1/LC)? So its a square term like B^2.

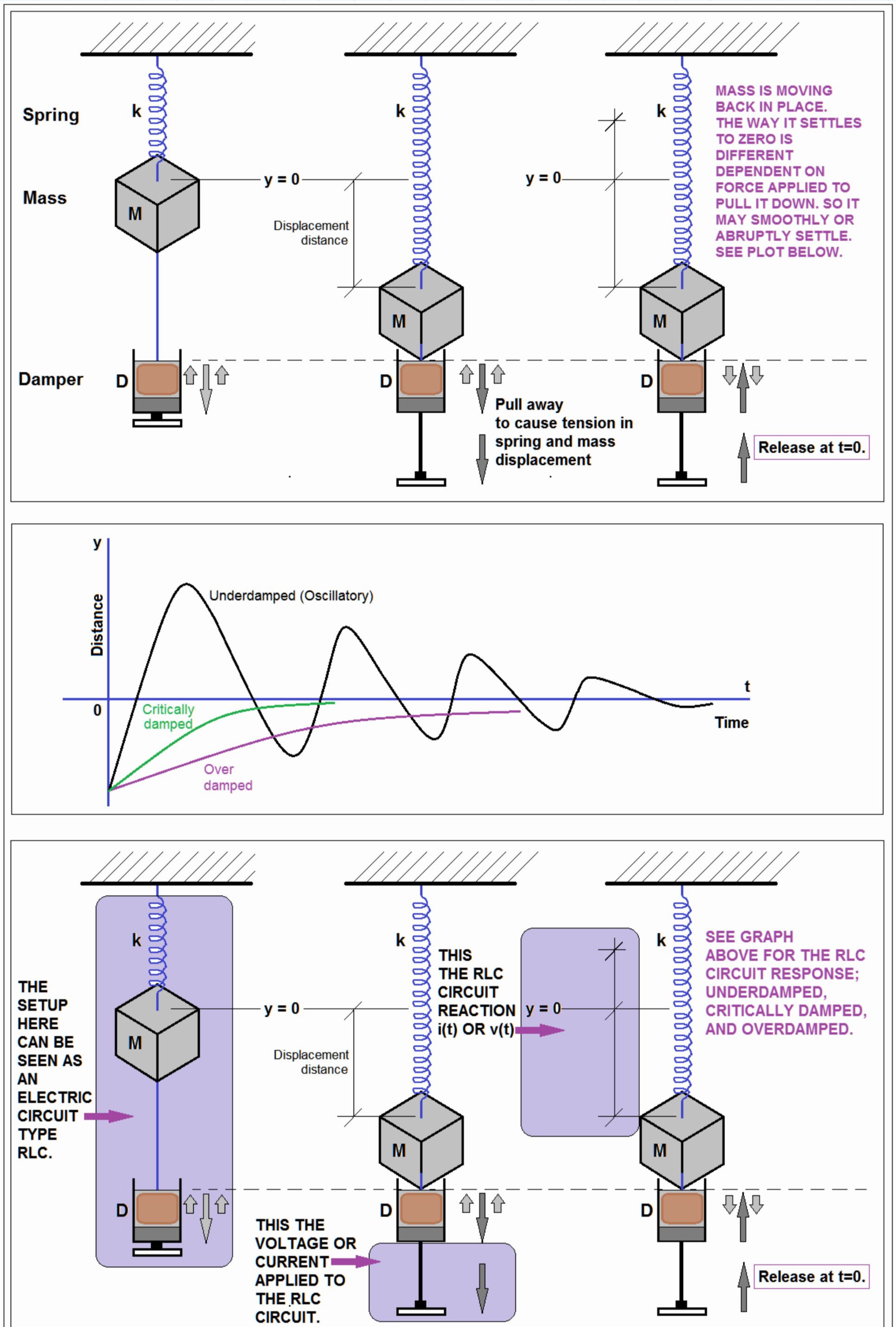
Where

$$\alpha = \left(\frac{1}{2\ RC}\right)$$ different from Series RLC

$$\omega_0 = \left(\frac{1}{\sqrt{LC}}\right)$$ same as Series RLC

Seen the series and parallel RLC. Other circuits with RL or RC or LC use similar approach.
Pull out Mathematics textbook work on DE solution forms for the circuit DE. Why the use of the word 'damped' see *my version* of the typical textbook figure next page. *Is there an expression 'dont be a damper'..meaning not to soften the impact or lower value? If not, you heard it here first.*
Reference from Schaum's Outline and other electric circuits textbooks. Look at the textbook in your hands.

The figure below attempts to show why the expression 'damped' used in circuits.
This is usually the way its presented using a mass damper combination for controls.



PLEASE CHECK WITH YOUR LOCAL ENGINEER-INSTRUCTOR. My attempt to make fit for electric circuit.

DAMPED - tight, held firmly. Critically damped, the response tight, over damped its less tighter than critically, and under damped is not tight its loose its oscillating.

We have the mass M, spring stiffness k, and damper D. Damper is some device or gadjet to cushion or damped the response. You can come up with your own ideas on how to present this.

The damper plays a role in the mass displacement that is why the response of the RLC circuit is similarly phrased - damped.

All disciplies civil,.....electrical use this mechanical analogy or setup.

My figure may be filled with errors you got the general idea.

Under damped (Oscillatory), Critically damped, and Over damped:

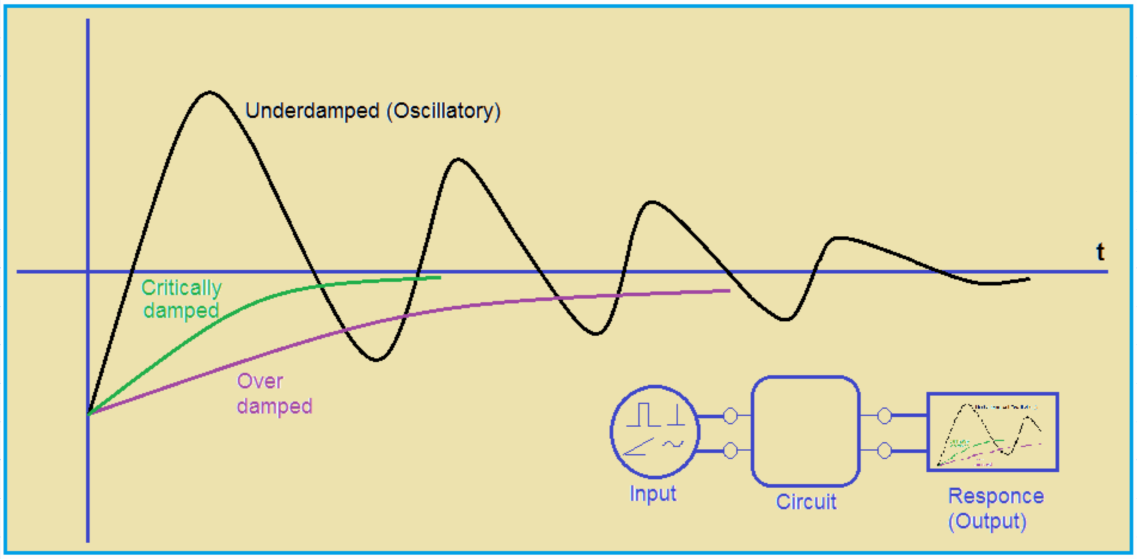


Figure above shows, dependent on input-source, dependent on circuit, the output response are three cases; under, critical, and over damped.

The CORRECT way to read it, "Depending on the type of input, a particular circuit may respond in one or several ways."

Case	Series RLC	Parallel RLC
Under damped: (Oscillatory)	$\alpha < \omega_0$	$\alpha^2 < \omega_0^2$
Critically damped:	$\alpha = \omega_0$	$\alpha = \omega_0$
Over damped:	$\alpha > \omega_0$	$\alpha^2 > \omega_0^2$
α :	$\left(\frac{R}{2\,L}\right)$	$\left(\frac{1}{2\,RC}\right)$
ω_0 :	$\left(\frac{1}{\sqrt{LC}}\right)$	$\left(\frac{1}{\sqrt{LC}}\right)$

Comments: It looks like in under damped the response is loose, not tense, up & down oscillating, maybe not reliable. Over damped is improving, stabler, no where near loose, but critically damped has a steeper linear region, tense, at the beginning before it settles to zero. Observations made purely based on the curves. Got it. Goes back to conditions above for alpha and omega_0.

Example 8.1: Case $\alpha > \omega_0$ Overdamped

A series RLC circuit. (Provided in solution).

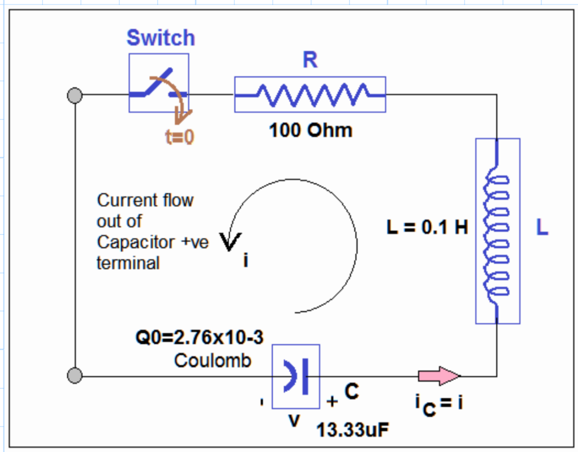
Capacitor C = 13.33 uF
Initial charge on the capacitor Qo = 2.67 x 10^-3 Coulomb.
Resistor R = 200 Ohm.
Inductor L = 0.10 H.
Switch is closed at t = 0. Allowing Capacitor to discharge.

Obtain the current transient?

Solution:

$R := 200$ $\quad L := 0.1$ $\quad C := 13.33 \cdot 10^{-6}$ $\quad Q_0 := 2.67 \cdot 10^{-3}$

Compute Alpha, Omega_o, and Beta:



$\alpha := \frac{R}{2 \cdot L} = 1000$ 1/s (per second)

$\omega_0 := \frac{1}{\sqrt{(L \cdot C)}} = 866.13$ 1/s

We have 'alpha > omega_o' condition for overdamped.

$\omega_0^{\ 2} \ = \ (\omega_0)^2 = 7.5 \cdot 10^5$ 1/s^2

$\beta := \sqrt{\left(\alpha^2 - (\omega_0)^2\right)} = 499.81$ 1/s $\qquad \beta = 500$ 1/s

We have 'alpha and beta' both real positive numbers.

Solution takes the form:

$i \ = \ e^{-\alpha \cdot t} \cdot \left(A_1 \cdot e^{\beta \cdot t} + A_2 \cdot e^{-\beta \cdot t}\right)$

$i \ = \ e^{-1000 \cdot t} \cdot \left(A_1 \cdot e^{500 \cdot t} + A_2 \cdot e^{-500 \cdot t}\right)$

Solve for A1 and A2.

Continued on next page.

e^-1000t? NOT that one in previous chapters on making the power of the exponent to the 1000th, rather its e^-alpha_t, alsp was calculated.
<---This you need a DE textbook in hand
OR Engineering Mathematics textbook.
Idea behind it in Chapter 5 was the suitable one is differentiable and remains close to the original function. Where exponents and sine/cosine comes to play. Lets NOT make it difficult for ourselves to proof and re-proof and re-proof again each time we face solutions from DE, seems now we only come across a few. Plot the results but please come to a STOP in making DE a life time experience.

Differential Equation is known for initial value conditions and boundary conditions.

The inductor L does not have a different current condition/value before t = 0, (-t), its condition is the same before t = 0 and after t =0 (t+). Switch is closed is at t =0.

Inductor stores energy by a magnetic field when a time varying current passes thru it. The energy stored is used by the inductor for its performance, and also returned to the source at other times. Stored in some cycles and returned to source in the others dependent on circuit conditions. Not storing when current is not present.
So, di/dt is critical for its functioning.

$i_L(0^{'+'}) = i_L(0^{'-'}) = 0$ i(t<-0) = i(t=0) = i(t>0+)...chapter 5.
CONTINUITY CONDITION CHAPTER 5.

Inductor condition at t =0:

$$i = e^{-1000\cdot t}\cdot\left(A_1\cdot e^{500\cdot t} + A_2\cdot e^{-500\cdot t}\right)$$

$0 = A_1 + A_2$ ...Eq 1
$-A_1 = A_2$
OR We come to this later on both the possible conditions.
$A_1 = -A_2$

For the capacitor C's voltage and charge we assumed it was left in OFF state for long period of time, completely discharged. Capacitor charge begins to discharge when external voltage is not present and circuit is closed.

Capacitor stores energy by an electric field when a time varying voltage is experienced across it. The energy stored is used by the capacitor for its performance, and also returned to the source at other times. Stored in one part of the cycle and returned to source in the rest/next. Storing when for a duration when voltage or circuit is off, this results in charge (current) released in the circuit. Known as a serioulis storage device/ element. So, dv/dt is critical for its functioning.

Capacitor voltage v = Q/C: $v_C(0^{'}) = v_C(0^{'-'})$

$\frac{Q_0}{C} = 200$ V Capacitor voltage: vC(-0)= vC(t=0) = vC(0+).

$v_C(0^{'-'}) = \frac{Q_0}{C} = 200.3$ V

$v_{c\_t\_0} := 200$ 200.3 round-off to whole number 200 V. Saying v_c_t_0 = vC(t=0).

When t =-0, current thru the inductor = v_c(0) / L <--Continuity condition, which works in solving simultaneous equations.

Inductor current:

$i_{L\_t\_minus\_0} := \frac{v_{c\_t\_0}}{L} = 2000$ Saying_iL_t_minus is iL(-0). *Will revert to this old chapter 5 writing style later after example 5 if suitable.*

Inductor dependent on di/dt so lets solve for it:

$$i = e^{-1000 \cdot t} \cdot \left(A_1 \cdot e^{500 \cdot t} + A_2 \cdot e^{-500 \cdot t}\right)$$

$$i = A_1 \cdot e^{-500 \cdot t} + A_2 \cdot e^{-1500 \cdot t}$$ multiplied thru e^-1000t

$$\frac{di}{dt} = -500\ A_1 \cdot e^{-500 \cdot t} - 1500\ A_2 \cdot e^{-1500 \cdot t}$$

Substitute $i_{L\_t\_minus\_0}$ for di/dt

$$2000 = -500\ A_1 \cdot e^{-500 \cdot t} - 1500\ A_2 \cdot e^{-1500 \cdot t}$$

$$2000 = -500\ A_1 - 1500\ A_2$$ when t = 0 next divide by 500

$$4 = -A_1 - 3\ A_2$$ ...Eq 2 <---Continuity condition led to solving simultaneous equations.

$$0 = A_1 + A_2$$ ...Eq 1 Shown here to solve for A1 and A2.

Add Eq 1 and 2

$$4 = 0 - 2\ A_2$$

$$A_2 = -2$$

Therefore A1 equal:

$$0 = A_1 - 2$$

$$A_1 = 2$$

Check:

$$0 = 2 - 2$$ CORRECT.

Now plug in A1 and A2 in the DE solution for i:

$$i = A_1 \cdot e^{-500 \cdot t} + A_2 \cdot e^{-1500 \cdot t}$$

$$i = 2 \cdot e^{-500 \cdot t} - 2 \cdot e^{-1500 \cdot t}$$

clear(t) <---*clear the variable i(t) for plot purpose.*

$i(t) := 2 \cdot e^{-500 \cdot t} - 2 \cdot e^{-1500 \cdot t}$ A. Answer.

Earlier we had a situation on two scenarios, presented here again.

$-A_1 = A_2$
OR <---We come to this later on both the possible conditions.
$A_1 = -A_2$

So the answer can also be as shown below, and in this case a flipped curve, maybe called a mirror image, see plot.

$i_{alternate}(t) := -2 \cdot e^{-500 \cdot t} + 2 \cdot e^{-1500 \cdot t}$ A. Alternate Answer.

The signs of A1 and A2 are fixed by the polarity of the initial voltage on the capacitor and its relationship to the assumed positive direction for the current. (Schaums Outline _ Nahvi and Edminister).

Comments:
A good introduction example, presented several points to consider in solution. Not necessarily easy or straight forward. A1 =+/-2 and A2=-/+2 <---Alternate Answers.

Plot i(t) on a graph on the next page. Hard to spot if its a overdamped curve, have to rely on equality conditions.

Review of Chapter 5:
There is no source in the circuit, no voltage source nor a current source.
NO SOURCE.

Why should we expect the current waveform to settle to zero, to approach zero?

Since there is no source, the circuit is SOURCE FREE, the current will have to settle to zero soon as the capacitor discharges completely into the circuit. This process will show a current rising to a peak then settling to zero.

So, our waveform is actually describing the current, ie charge per time, decaying in the circuit. *This is NOT a non-typical exercise, many engineering analysis involves source free circuit togther with the? INTIAL CONDITIONS OR BOUNDARY CONDITIONS.*

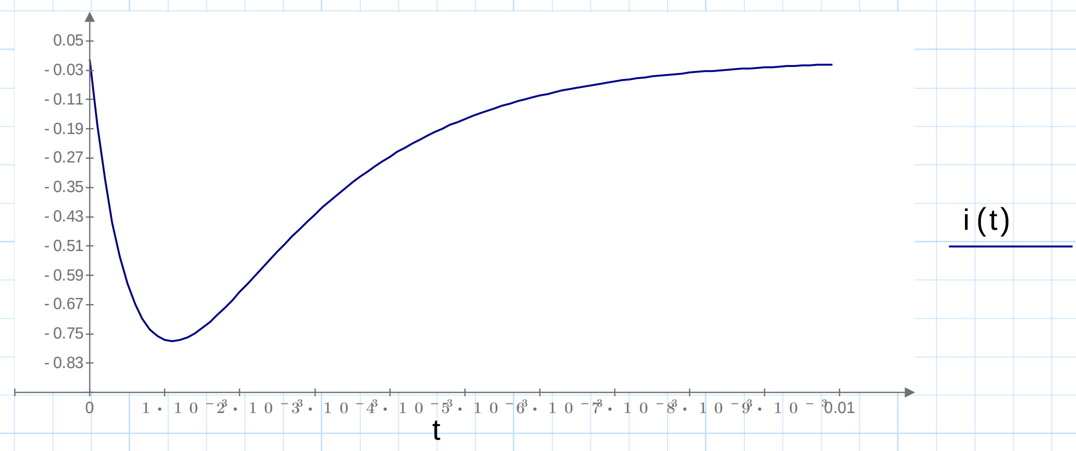
Set the t interval for plot to get the better of the plot.

cls(t) $t := 0, 0.0001 .. 0.0099$

$i(t) := 2 \cdot e^{-500 \cdot t} - 2 \cdot e^{-1500 \cdot t}$

Peak i(t) = 0.77 at time t = 1.11*10^-3 seconds.
Next the plot with the opposite signs; A1 = -2, and A2 = +2. *Similar result of course.*

$i(t) := -2 \cdot e^{-500 \cdot t} + 2 \cdot e^{-1500 \cdot t}$



Happy! Similar to Schaums Outline plot.
Maybe happy if you the reader go over the solution several times, so you get the approach to 'solving similar problems mastered'.....and correct any and all errors if any.

Comments: Looks like an over damped and critically damped curve, no oscillations, steepness of curve higher for critically compared to over damped. Look closely.
As the condition stated alpha>omega_0, it is over damped.

Example 8.2: Case $\alpha = \omega_0$ Critically damped.

A series RLC circuit, similar to example 1 with change in capacitor value.

Capacitor C = 10.0 uF
Initial charge on the capacitor Qo = 2.67 x 10^-3 Coulomb.
Resistor R = 200 Ohm.
Inductor L = 0.10 H.
Switch is closed at t = 0. Allowing Capacitor to discharge.

Obtain the current transient?

Solution:

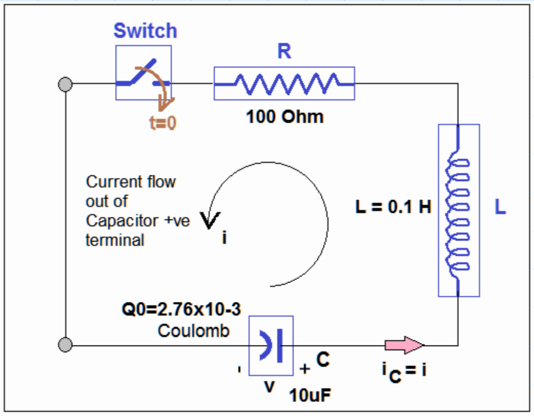
$R := 200 \quad L := 0.1 \quad C := 10.0 \cdot 10^{-6}$

$Q_0 := 2.67 \cdot 10^{-3}$

Compute Alpha, Omega_o, and Beta:

$\alpha := \frac{R}{2 \cdot L} = 1000$ 1/s (per second)

$\omega_0 := \frac{1}{\sqrt{(L \cdot C)}} = 1000$ (per second squared)



We have 'alpha = omega_o'. *Amazing with such low values you can actually get them to equal. Real World?*

$\omega_0^{\;2} = (\omega_0)^2 = 1 \cdot 10^6$ 1/s^2

$\beta := \sqrt{\left(\alpha^2 - (\omega_0)^2\right)} = 0$ 1/s $\quad \beta = 0$ 1/s

We have 'alpha and omega0' both real but beta is not a positive number instead zero.

Solution takes the form:

$i = e^{-\alpha \cdot t} \cdot (A_1 + A_2 \cdot t)$

$i = A1 \cdot e^{-1000 \cdot t} + A_2 \cdot e^{-1000 \cdot t} \cdot t$

<---Not the same solution form in example 8.1. Is it the conditon dictated it? Yes. We have multiplied t to the 2nd term. Example 8.1 term provided below.

Solve for A1 and A2.

Ex 8.1--> $i = e^{-1000 \cdot t} \cdot \left(A_1 \cdot e^{\beta \cdot t} + A_2 \cdot e^{-\beta \cdot t}\right)$

Because? Curve of each condition has to take on that shape, how else if the solution form (function) is not playing the game. So there is a fit there is a fix.

Continued on next page.

So there is a fit there is a fix....Karl Bogha. You may quote me in your textbook.

Similar initial value conditions to example 1.

$i_L(0'+') \quad = \quad i_L(0'-') \quad = \quad 0$

Inductor condition at t =0:

$$0 = A1 \cdot e^{-1000 \cdot 0} + A_2 \cdot e^{-1000 \cdot 0} \cdot 0$$

$$0 = A1 \qquad \text{...Eq 1}$$

Capacitor voltage v = Q/C: $\quad v_C(0') \quad = \quad v_C(0'-')$

$\frac{Q_0}{C} = 267 \quad V \qquad$ Capacitor voltage before vC(-0) = vC(0+).

$v_C(0'-') = \quad v_{c\_t\_minus\_0} := \frac{Q_0}{C} = 267 \qquad V \qquad$ <--- Equal v_c_(t=0).

$v_{c\_t\_0} := 200 \qquad$ 200.3 round-off to whole number 200 V.

When t =0, current thru the inductor = v_c(0-) / L

Inductor current:

$i_{L\_t\_minus\_0} := \frac{v_{c\_t\_0}}{L} = 2000 \qquad$ Same as example 1.

Inductor dependent on di/dt so lets solve for it:

$$i = e^{-\alpha \cdot t} \cdot (A_1 + A_2 \cdot t)$$

$$i = e^{-\alpha \cdot t} \cdot A_2 \cdot t \qquad \text{Since A1 = 0.}$$

$$\frac{di}{dt} = A_2 \cdot \left(-\alpha \cdot t \cdot e^{-\alpha \cdot t} + e^{-\alpha \cdot t}\right)$$

Substitute $i_{L\_t\_minus\_0}$ for di/dt and alpha = 1000.

$2000 \quad = \quad A_2 \cdot \left(-1000 \cdot t \cdot e^{-1000 \cdot t} + e^{-1000 \cdot t}\right)$
$2000 \quad = \quad -A_2 \cdot e^{-1000 \cdot t}$
$2000 \quad = \quad -A_2 \qquad$ when t = 0
$A_2 \quad = \quad -2000 \qquad$ when multiplied by - 1
OR
$-A_2 \quad = \quad 2000 \qquad$ So its a little vague, if A can be +/- then final value too is +/-.
We can say there are 2 possible values for A.

$A_1 = 0$

$A_2 = +/-\ 2000$

$i = e^{-\alpha \cdot t} \cdot A_2 \cdot t$ Substitute for A2

$i = e^{-1000 \cdot t} \cdot 2000 \cdot t$ Expression we were seeking A=2000.

$i = 2000 \cdot t \cdot e^{-1000 \cdot t}$ Answer.

$i_{alternate} = -2000 \cdot t \cdot e^{-1000 \cdot t}$ Answer. When A=-2000.

clear(t) <---*clear the variable i(t) for plot purpose.*

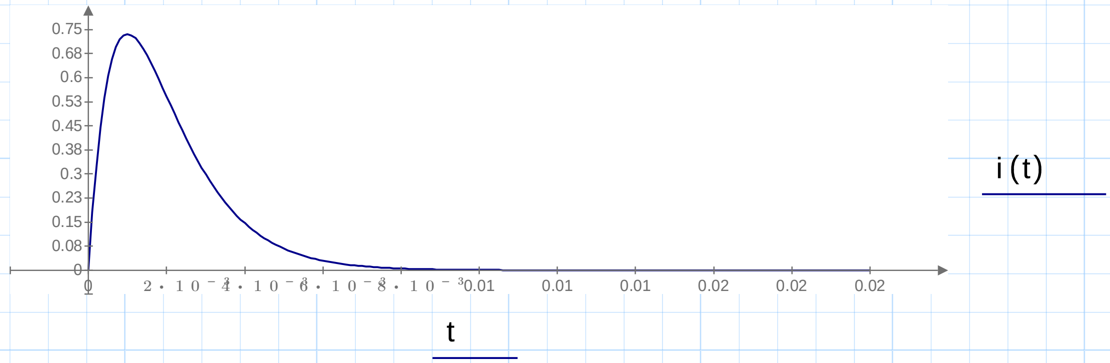
Again, the signs of A1 and A2 are fixed by the polarity of the initial voltage on the capacitor and its relationship to the assumed positive direction for the current. (Schaums Outline _ Nahvi and Edminister).

Plot i(t) on a graph next page.

clear (t) clear (i (t))

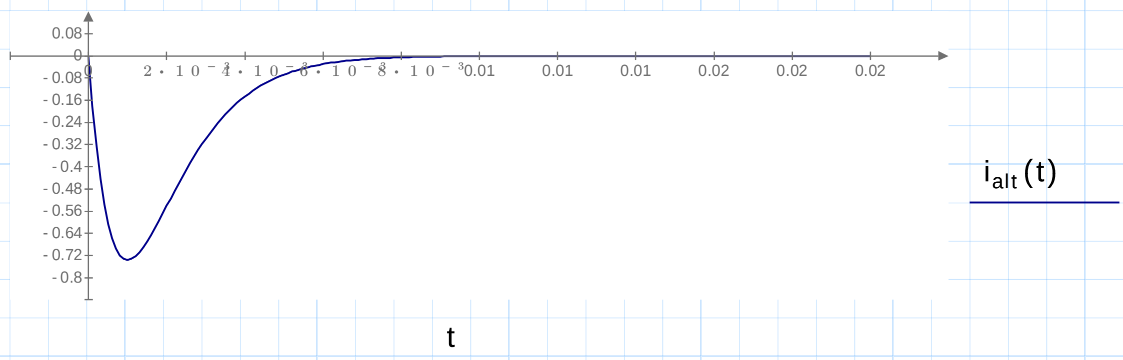
$t := 0, 0.0001 .. 0.02$

$i(t) := 2000 \cdot t \cdot e^{-1000 \cdot t}$



The other plot for A2 = - 2000.

$i_{alt}(t) := -2000 \cdot t \cdot e^{-1000 \cdot t}$



The curves are steeper here compared to previous over damped. So, now we see the curve difference, you can say the selected DE equation lend to the curve shape.

Nahvi & Edminister: The responses for the over damped and critially damped are quite similar. The engineer/student is encouraged to examine the results, selecting values for t, and comparing the currents. For example find the time when the current is 1.0 mA and 10 uA, and in each case the maximum current - (Page 182)

That can be done in Excel or any other Math based software.

Happy!
The last being the under damped (oscillatory) case, next.
This one with the *exciting* oscillating plot.

Example 8.3: Case $\alpha < \omega_0$ Under damped (Oscillation).

A series RLC circuit, similar to example 1 with change in capacitor value.

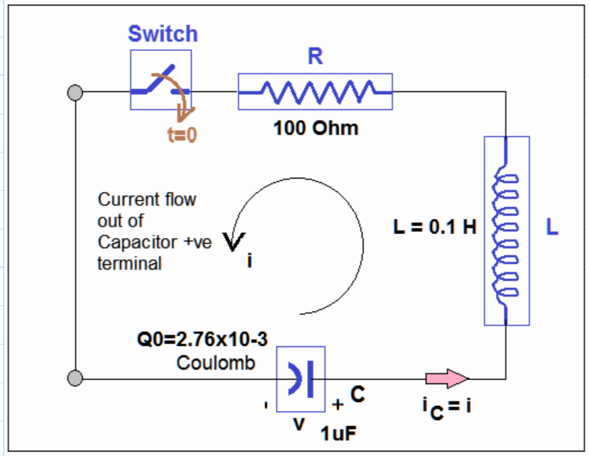
Capacitor C = 1.0 uF
Initial charge on the capacitor Qo = 2.67 x 10^-3 Coulomb.
Resistor R = 200 Ohm.
Inductor L = 0.10 H.
Switch is closed at t = 0. Allowing Capacitor to discharge.

Obtain the current transient?

Solution:

$R := 200$ $\quad L := 0.1$

$C := 1.0 \cdot 10^{-6}$ $\quad Q_0 := 2.67 \cdot 10^{-3}$



Compute Alpha, Omega_o, and Beta:

$$\alpha := \frac{R}{2 \cdot L} = 1 \cdot 10^{3} \text{ 1/s (per second)}$$

$$\omega_0 := \frac{1}{\sqrt{(L \cdot C)}} = 3.16 \cdot 10^{3}$$

$$\omega_0^{\,2} = (\omega_0)^2 = 1 \cdot 10^{7} \text{ 1/s^2}$$

Beta uses a different expression for alpha < omega:

$$\beta := \sqrt{\left((\omega_0)^2 - \alpha^2\right)} = 3 \cdot 10^{3} \text{ Rad/s} \qquad \beta = 3000 \text{ Rad/s}$$

Shown by calculations above alpha < omega.

Solution takes the form:

$$i = e^{-\alpha \cdot t} \cdot \left(A_1 \cdot e^{j \cdot \beta \cdot t} + A_2 \cdot e^{-j \cdot \beta \cdot t}\right)$$

<---Same solution form in example 8.1, but NOT same as 8.2 The 1000t is alpha_t. This form of exponential expression also exist in sinusoidal form. The sinusoidal form maybe the reason why we can see the oscillations!

OR sinusoidal form:

$$i = e^{-\alpha \cdot t} \cdot \left(A_1 \cdot \cos(\beta \cdot t) + A_2 \cdot \sin(\beta \cdot t)\right)$$

Ex 8.1--> $i = e^{-\alpha \cdot t} \cdot \left(A_1 \cdot e^{\beta \cdot t} + A_2 \cdot e^{-\beta \cdot t}\right)$

Ex 8.2--> $i = A1 \cdot e^{-\alpha \cdot t} + A_2 \cdot e^{-\alpha \cdot t} \cdot t$

Roots to the DEs above:

$s_1 = \alpha + j\,\beta \qquad s_1 = \alpha - j\,\beta$

Solve for A1 and A2.

Hello? We realise the sine/cosine terms are capable of generating oscillations with the exponential term generating varying amplitudes.

Similar initial value conditions to example 1.

$i_L(0'+') \quad = \quad i_L(0'-') \quad = \quad 0$

Inductor condition at t =0:

$$0 \quad = \quad e^{-\alpha\cdot t}\cdot\left(A_1\cdot\cos(\beta\cdot t) + A_2\cdot\sin(\beta\cdot t)\right)$$

$$0 \quad = \quad e^{-1000\cdot t}\cdot\left(A_1\cdot\cos(3000\cdot t) + A_2\cdot\sin(3000\cdot t)\right)$$

$$0 \quad = \quad e^{-1000\cdot 0}\cdot\left(A_1\cdot\cos(3000\cdot 0) + A_2\cdot\sin(3000\cdot 0)\right) \qquad \text{at } t = 0$$

$0 \quad = \quad A_1$ ...Eq 1

Capacitor voltage v = Q/C: $\qquad v_C(0') \quad = \quad v_C(0'-')$

$\frac{Q_0}{C} = 3\cdot 10^3\,V$ Capacitor voltage vC(-0) = vC(0) = vC(0+).

$v_C(0'-') = \qquad v_{c\_t\_minus\_0} := \frac{Q_0}{C} = 2.67\cdot 10^3 \quad V$ *Equal v_c_t=0.*

$v_{c\_t\_0} := 200$ 200.3 round-off to whole number 200 V.

When t =0, current thru the inductor = v_c(0-) / L = v_c(0+)/L...Chapter 5.

Inductor current:

$i_{L\_t\_minus\_0} := \frac{v_{c\_t\_0}}{L} = 2000$ *Same as example 1 and 2.*

Inductor dependent on di/dt so lets solve for it:

$$i \quad = \quad e^{-\alpha\cdot t}\cdot\left(A_1\cdot\cos(\beta\cdot t) + A_2\cdot\sin(\beta\cdot t)\right) \quad = \quad e^{-\alpha\cdot t}\cdot A_1\cdot\cos(\beta\cdot t) + e^{-\alpha\cdot t}\cdot A_2\cdot\sin(\beta\cdot t)$$

$$\frac{di}{dt} = \left(-\alpha\cdot e^{-\alpha\cdot t}\cdot A_1\cdot\cos(\beta\cdot t) - e^{-\alpha\cdot t}\cdot\beta\cdot A1\cdot\sin(\beta\cdot t)\right) + \left(-\alpha\cdot e^{-\alpha\cdot t}\cdot A_2\cdot\sin(\beta\cdot t) + e^{-\alpha\cdot t}\cdot\beta\cdot A_2\cdot\cos(\beta\cdot t)\right)$$

$$\frac{di}{dt} = \left(-1000\cdot e^{-1000\cdot 0}\cdot A_1\cdot\cos(3000\cdot 0) - e^{-1000\cdot 0}\cdot 3000\cdot A1\cdot\sin(3000\cdot 0)\right) + \left(-1000\cdot e^{-1000\cdot 0}\cdot A_2\cdot\sin(3000\cdot 0) + e^{-1000\cdot 0}\cdot 3000\ A_2\cdot\cos(3000\cdot 0)\right)$$

$\frac{di}{dt} = -1000\cdot A_1 + 3000\cdot A_2 \quad = \quad 0 + 3000\cdot A_2$ substitute di/dt = 2000

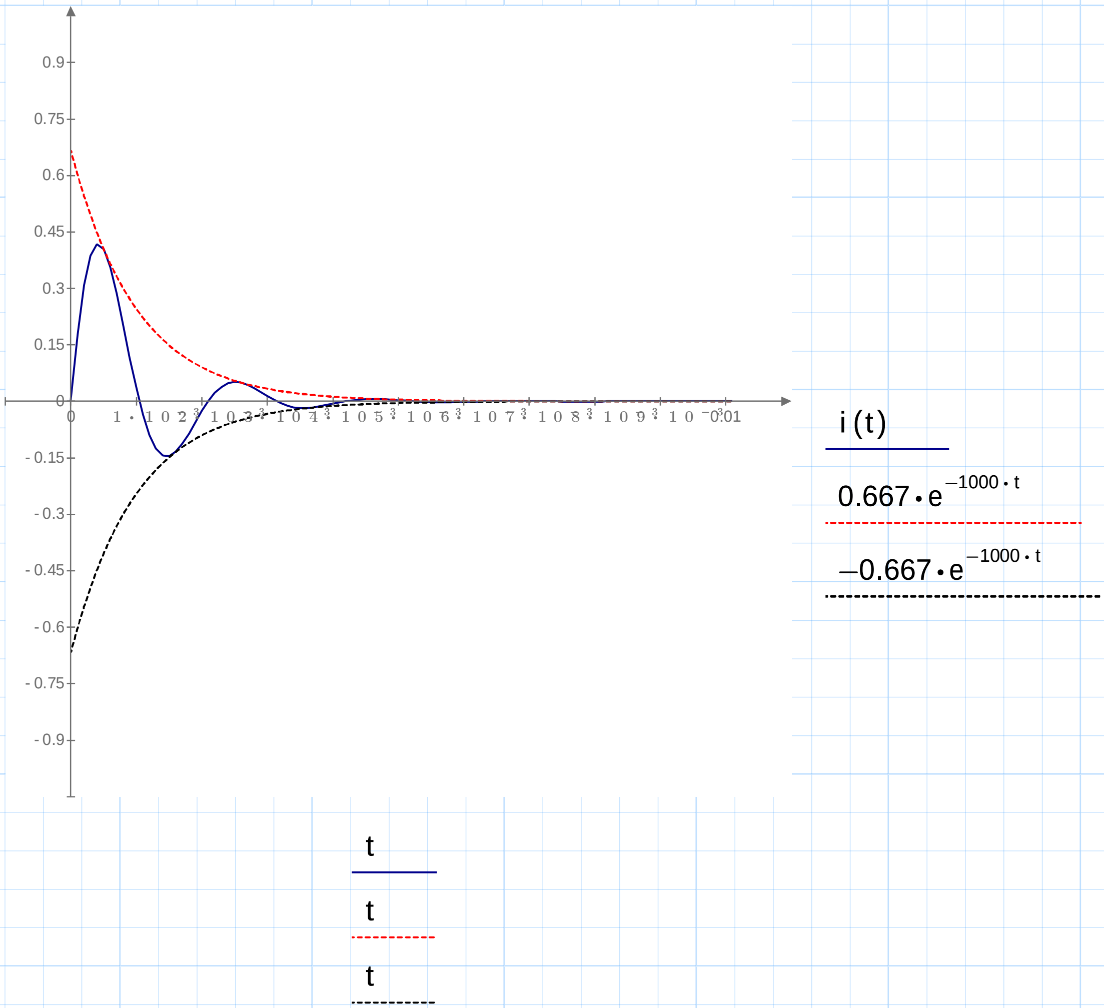
$2000 \quad = \quad 3000\cdot A_2 \qquad A_2 := \frac{2000}{3000} = 0.667$

Substitute for A1 and A2 in i(t):

$i(t) := 0.667 \cdot e^{-1000 \cdot t} \cdot \sin(3000 \cdot t)$ A. Answer.

clear(t) <---*clear the variable i(t) for plot purpose.*

Plot i(t) on a graph on the next page.



The oscillating wave is enveloped by the wave function +/- 0.667 (e ^ - 1000t). This shown in dash.

Oscillatory current has a radian frequency of Beta rad/s and is damped by the exponential term e^-alpha t.

$i(t) := e^{-1000 \cdot t} \cdot 0.667 \cdot \sin(3000 \cdot t)$

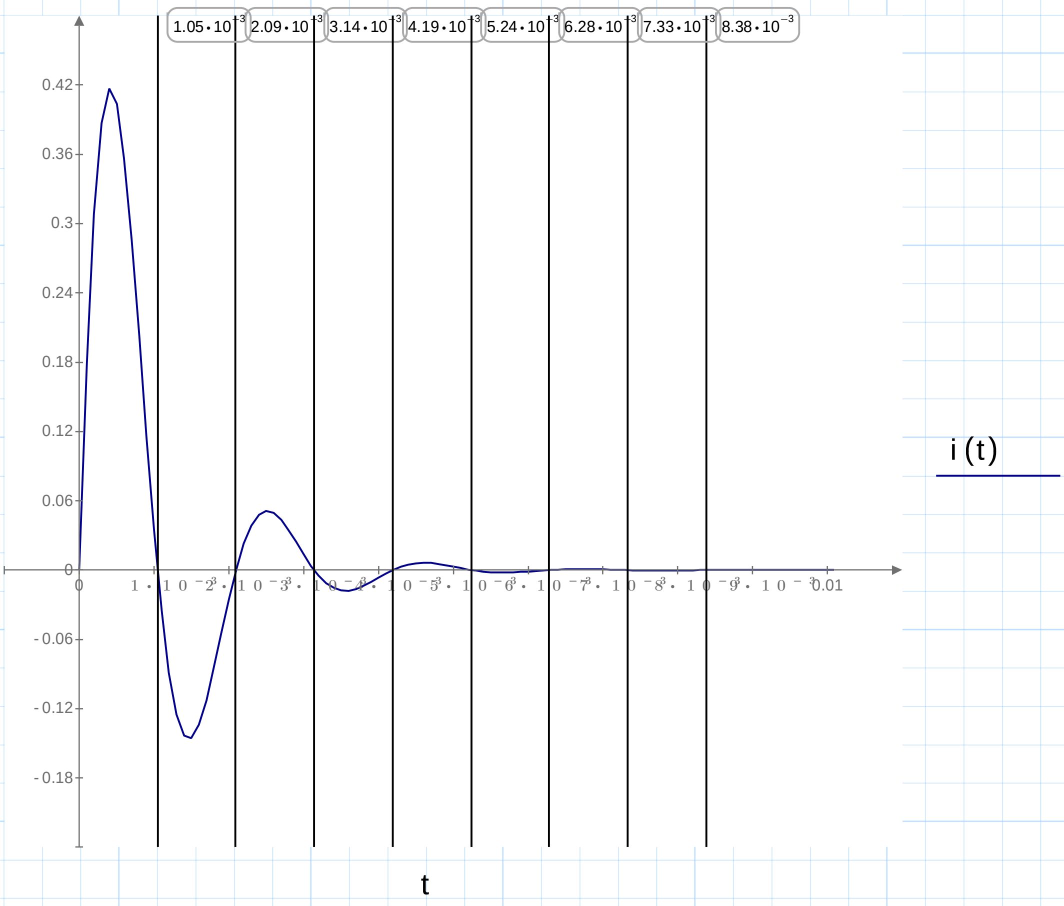
$\beta := 3000$ $\omega := 3000$ $f := \frac{\omega}{2 \cdot \pi} = 477.46$ $T := \frac{1}{f} = 2.09 \cdot 10^{-3}$

2 PI would be where 'i' intersects the x-axis - 1st cycle (up then down) at T.

This is 2PI/2 (180 deg) ---> $T_{half} := \frac{T}{2} = 1.05 \cdot 10^{-3}$ PI ? $T = 2.09 \cdot 10^{-3}$

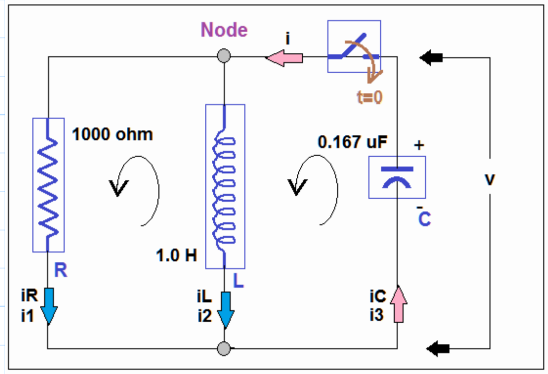
3PI/2? $\frac{3 \cdot T}{2} = 3.14 \cdot 10^{-3}$ 2PI? $2 \cdot T = 4.19 \cdot 10^{-3}$ 5 PI/2? $\frac{5 \cdot T}{2} = 5.24 \cdot 10^{-3}$

3PI? $3 \cdot T = 6.28 \cdot 10^{-3}$ 7 PI/2? $\frac{7 \cdot T}{2} = 7.33 \cdot 10^{-3}$ 4PI? $T \cdot 4 = 8.38 \cdot 10^{-3}$



In this oscillation case, we see the waveform completely comes to zero at the 4th cycle. Maybe AMAZING where the waveform crosses x-axis each time at T/2 and T (PI and 2 PI). Its time wise (t) a cycle but shape wise its not, the waveform is deminishing to zero, not the same shape each cycle, starts at t = 0 s and ends at t = 8.38*10^-3 s.

Example 8.4: Case $\alpha^2 > \omega_0^2$ Overdamped



A PARALLEL RLC circuit.

Capacitor C = 0.167 uF
Initial voltage on Capacitor = 50.0 V
Resistor R = 1000 Ohm.
Inductor L = 1.0 H.

Switch is closed at t = 0.
Allowing Capacitor to discharge.

Obtain the voltage when switch is closed at t = 0?

Solution:

$R := 1000$ $L := 1.0$ $C := 0.167 \cdot 10^{-6}$ $V_0 := 50.0$ clear (t)

Compute Alpha^2, and Omega_o^2,:

$\alpha := \frac{1}{2 \cdot R \cdot C} = 2994$ 1/s (per second) $\alpha^2 = 8.96 \cdot 10^6$ 1/s^-2

$\omega_{0\_squarred} := \frac{1}{(L \cdot C)} = 5.99 \cdot 10^6$ Over damped its squared for parallel circuit.

We have 'alpha^2 > omega_o'^2; over damped condition met.

DE for the circuit voltage: $v = A_1 \cdot e^{s1 \cdot t} + A_2 \cdot e^{s2 \cdot t}$

We seen this equation before. Ok but note in series RLC it was same over damped condition.

Solutions for roots s1 and s2 use the Parallel RLC notes:

$$s_1 := -\alpha + \sqrt{(\alpha^2) - (\omega_0^2)} = -2994 + \sqrt{8.96 \cdot 10^6 - 5.99 \cdot 10^6} = -1271$$

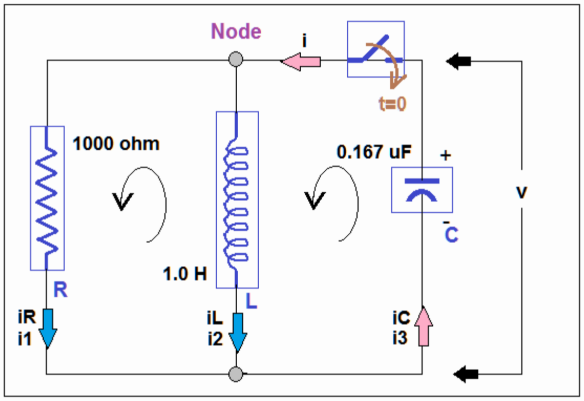
$$s_2 := -\alpha - \sqrt{(\alpha^2) - (\omega_0^2)} = -2994 - \sqrt{8.96 \cdot 10^6 - 5.99 \cdot 10^6} = -4717$$

At time t =0, the voltage of the circuit is 50 V, initial condition.
v(t=0) = v(t+) = 50 V. Looks good? Maybe. Good for now.
At the instant the switch closes its t =0, there is voltage present. Current has to build up from 0 to some value. Since its a nodal equation in parallel RLC we are concerned with the? Voltage. Series circuit case current flow from the capacitor. Here parallel circuit.

Continued next page.

$$\frac{dv}{dt} = s_1 \cdot A_1 \cdot e^{s1 \cdot t} + s_2 \cdot A_2 \cdot e^{s2 \cdot t}$$

$$\frac{dv}{dt} = s_1 \cdot A_1 + s_2 \cdot A_2 \quad \text{at time } t = 0.$$



RLC parallel circuit with node. You may sketch a better looking circuit. Take into consideration its a source free circuit and capacitor is the source of voltage and current.
Remember: DC voltage makes capacitor open circuit. Here that is not the case when Capacitor is discharing, voltage and current varies.

$$V_0 = A_1 + A_2$$

At t = 0, capacitor is discharging near maximum voltage. So the coefficients A1 and A2 combined will have the maximum voltage in the circuit coming from the capacitor branch. So s1 and s2 can be neglected in equation which is exponential...Vo = A1e^s1t + A2e^s2t...at t=0. The exponential character at t=0. CORRECT.

$$V_0 = A_1 + A_2 \qquad V_0 := 50 \text{ V.}$$

Discussion:
In the series RLC circuit for the inductor L, where current played the role. Here we do not know what the initial current is, so we may not solve for vL = L (di/dt).
Inductor needs a varying current (di/dt) then multiply it to L gives voltage across inductor L. We procced using another method to obtain that voltage. Continuity?

So first try resistor and capacitor experiencing same voltage across it, and the nodal (current) equation at node will be:

$$\frac{V_0}{R} + C\left(\frac{dv}{dt}\right) = 0$$

At time t = 0, no initial current in inductor L.
Continuity condition, iL(-0) = iL(0) = iL(0+) = 0. Correct.

$$\frac{dv}{dt} = -\frac{V_0}{RC}$$

Rearranging and next line dv/dt at time t = 0

$$\frac{dv}{dt} = s_1 \cdot A_1 + s_2 \cdot A_2$$

Engineer plug in dv/dt in next.

$$-\frac{V_0}{RC} = s_1 \cdot A_1 + s_2 \cdot A_2$$

Lets question what can A2 possibly equal?

$$A_2 = V_0 - A_1 \qquad \text{Correct, A1 + A2 = Vo.}$$

$$-\frac{V_0}{RC} = s_1 \cdot A_1 + s_2 \cdot (V_0 - A_1)$$

$$-\frac{V_0}{RC} = s_1 \cdot A_1 + s_2 \cdot V_0 - s_2 \cdot A_1$$

$$A_1 \cdot (s_2 - s_1) = V_0\left(s_2 + \left(\frac{1}{RC}\right)\right)$$

$$A_1 = \frac{V_0\left(s_2 + \left(\frac{1}{RC}\right)\right)}{s_2 - s_1}$$

Solutions for roots s1 and s2 solved on previous page(s):

$s_1 := -1271$

$s_2 := -4717$

$$A_1 := \frac{V_0 \cdot \left(s_2 + \left(\frac{1}{R \cdot C}\right)\right)}{(s_2 - s_1)} = -18.44$$

<---6th Ed Schaum's Outline page 184 has an error, A1 = 155.3. They used s2 = -s2 in (s2 + (1/RC)). This got them 155.3 you may check. It then goes to solve for A2 = 50 - 155.3 = - 105.3. Sometimes author-engineer intentionally place the error so you catch it. Check with your local enigneer.

$$A_2 := V_0 - A_1 = 68.44$$

Now we can plug in A1, A2, s1, and s2 into the expression for v:

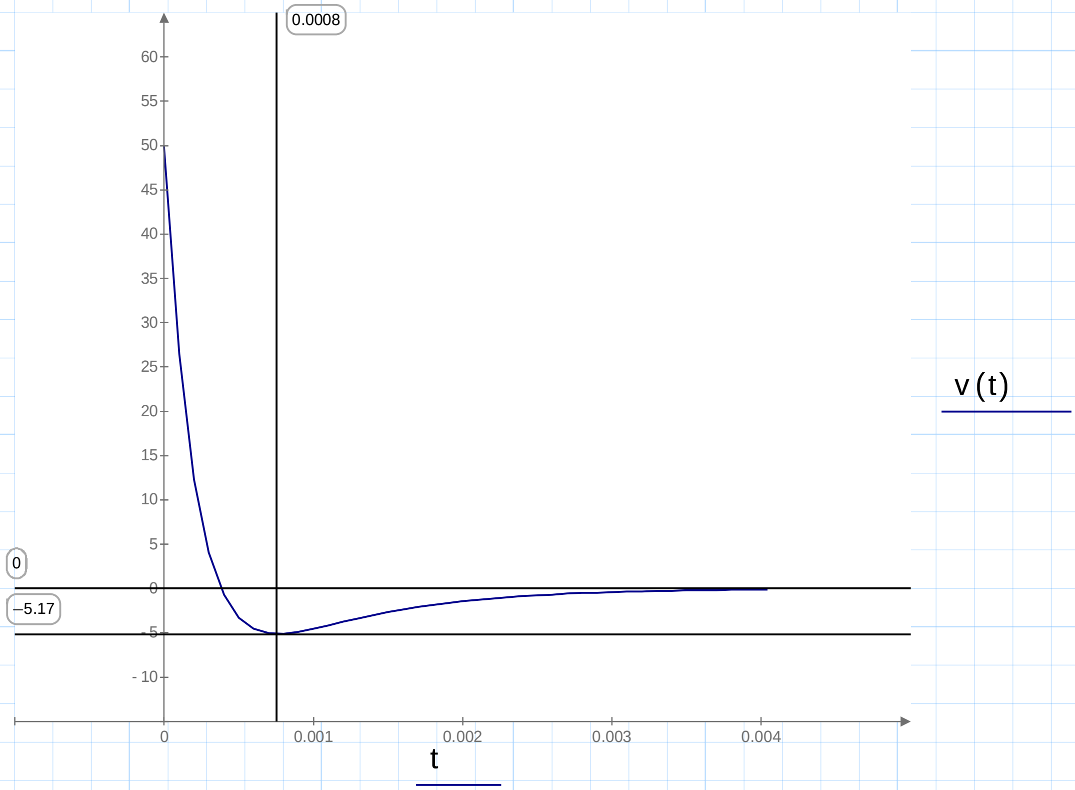
$$v = A_1 \cdot e^{s1 \cdot t} + A_2 \cdot e^{s2 \cdot t}$$

$v(t) := -18.44 \cdot e^{-1271 \cdot t} + 68.44 \cdot e^{-4717 \cdot t}$ V. Answer.

$v(t) := 155.3 \cdot e^{-1271 \cdot t} - 105.3 \cdot e^{-4717 \cdot t}$ V. *Schaums Error Answer.*

Rest of the answers or values are the same.

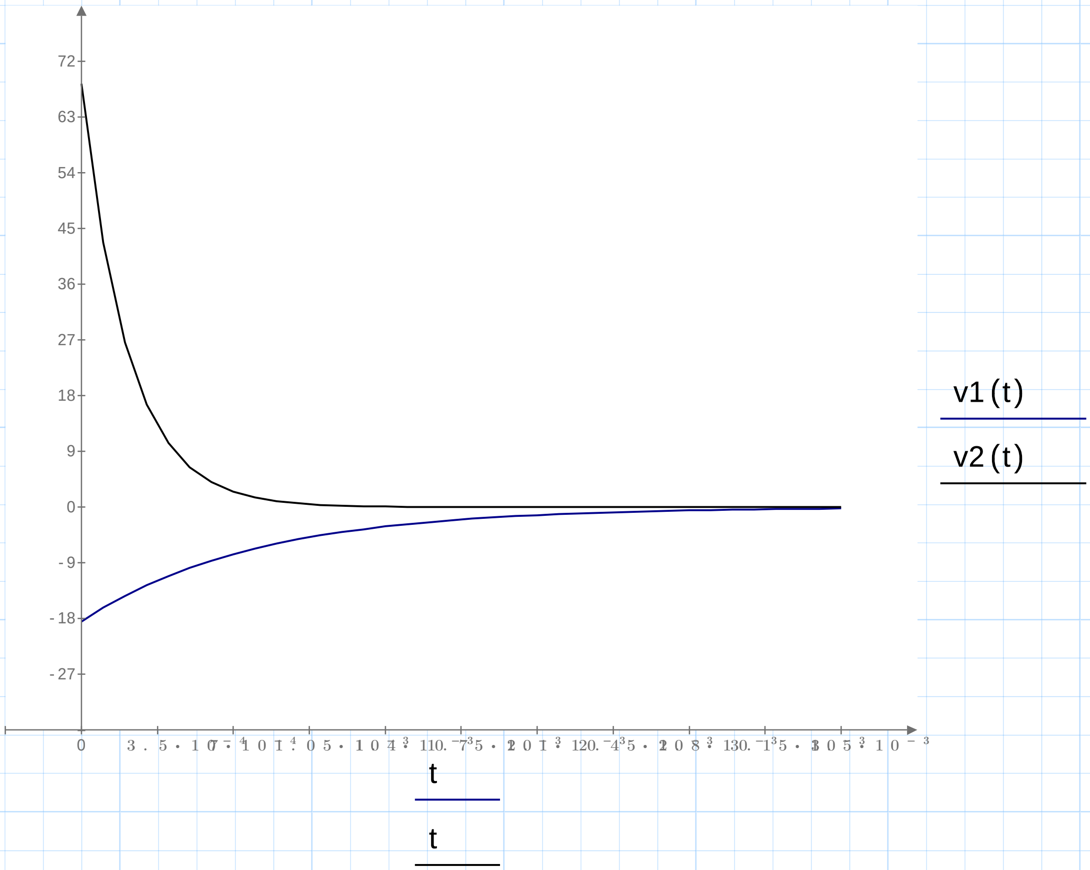
Plot of: $v(t) := -18.44 \cdot e^{-1271 \cdot t} + 68.44 \cdot e^{-4717 \cdot t}$



The curve does not take on the over damped until t= 0.0008 seconds or close to 1 ms. At this time the voltage is -5.17V. Then the curve rises with a low slope to settle at 0V. The shape of under damped is there, but it starts after a sharp drop from from t=0 to t=0.8 ms.
Peak voltage is 50V as seen on the y-axis at t=0.
Plot settles to zero as expected.
So this should be correct.
Any errors you may be able to catch/spot/identify/.........

Next page plot by parts of the expression to get better curve visualisations, and why the plot went negative voltage.

$v1(t) := -18.44 \cdot \left(e^{-1271 \cdot t}\right)$ $\qquad$ $v2(t) := 68.44 \cdot e^{-4717 \cdot t}$

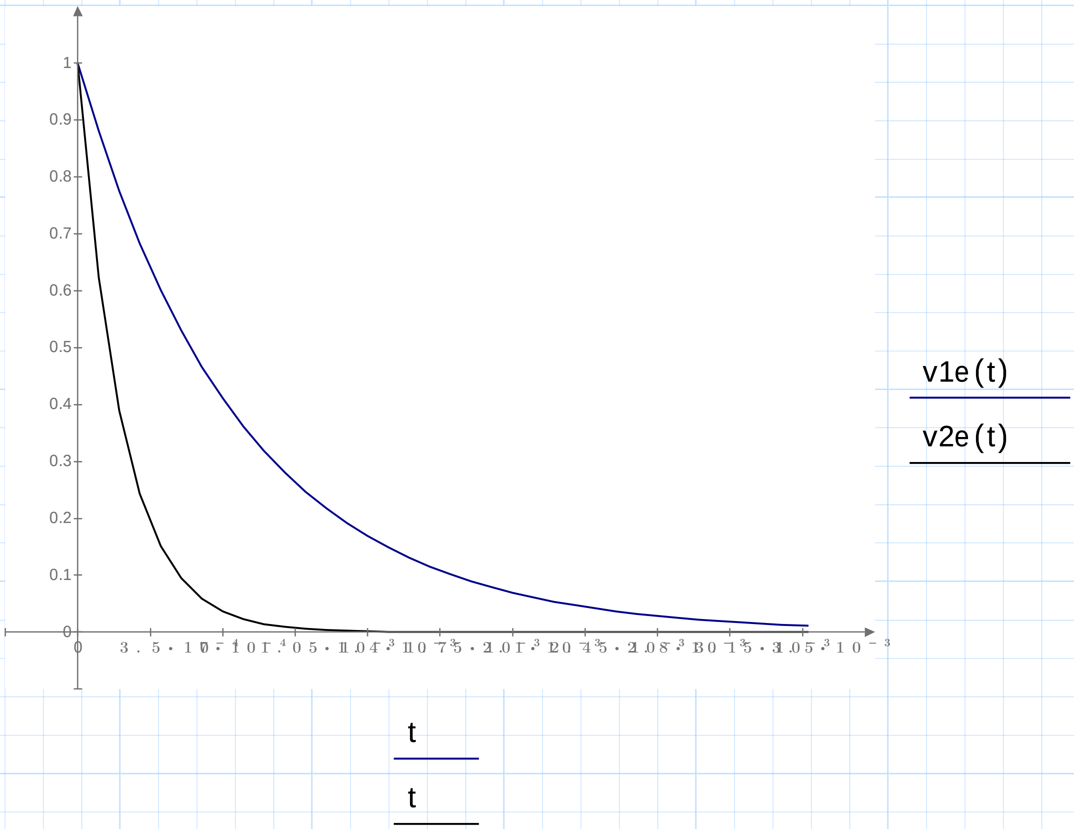


Each part of the function starts first on the y-axis at its coefficent A1 and A2. So we understand now why the curve goes negative part v2(t) goes negative.

2 plot of exponential terms here plotted both positive:
The coefficients A1 and A2 provided the +ve and -ve sign.
Just to see the exponential term curve both are positive side settling to 0.

$$v(t) := -18.44 \cdot e^{-1271 \cdot t} + 68.44 \cdot e^{-4717 \cdot t}$$

$$v1e(t) := e^{-1271 \cdot t} \qquad v2e(t) := e^{-4717 \cdot t}$$



The -4771 exponential power term is clearly steeper from -1271.

Here the exponential terms do not form an envelope for the function v(t).
Correct because its NOT an oscillating output, its over damped NOT underdamped.
This was checked you may plot to check. *Next page we see the author-engineer solution plotted, thorugh wrong, for some reason you may be curious!*

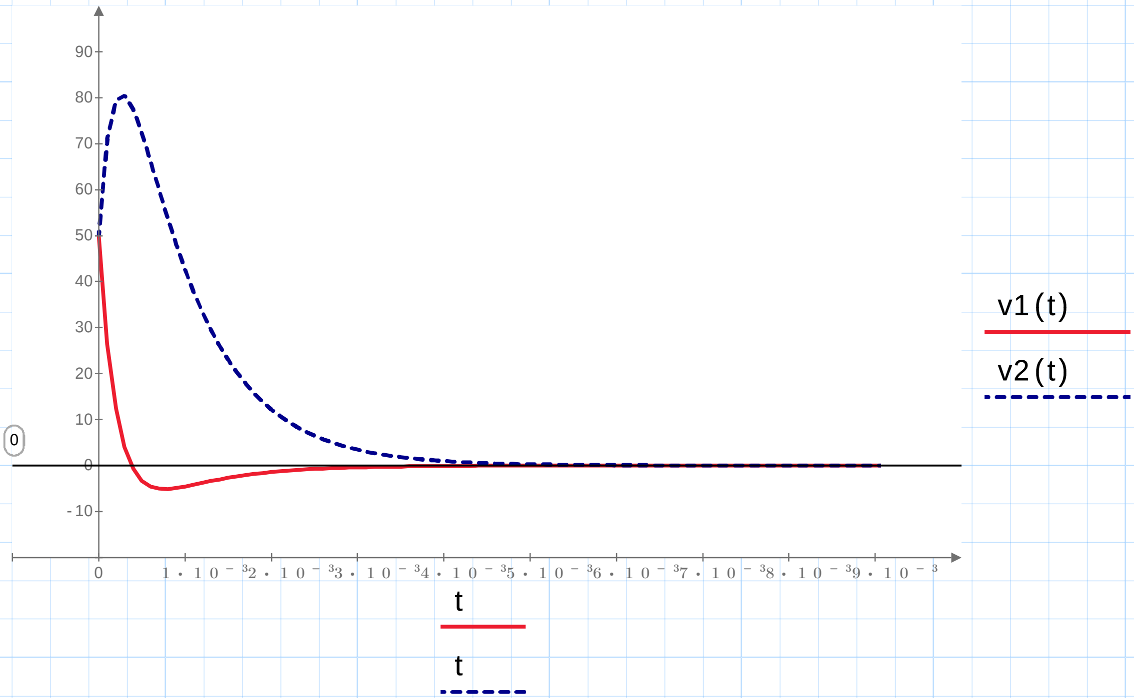
Next the underdamped (oscillatory) circuit. Followed by the PARALLEL critically damped, why PARALLEL critically is last here, because there is NO need for it said the author-engineers. Reasons provided later check your textbook.

Plot of both solutions to compare. v1(t) is the solution worked here and v2(t) is the Schaums solution.

cls(t)

$$v1(t) := -18.44 \cdot e^{-1271 \cdot t} + 68.44 \cdot e^{-4717 \cdot t}$$

$$v2(t) := 155.3 \cdot e^{-1271 \cdot t} - 105.3 \cdot e^{-4717 \cdot t}$$



The plots do raise questions.

v1(t) has the shape we are looking for and there is nothing wrong with it going negative first.

v2(t) rises to a peak then drops but on the way down it does slope upside down to what we expect, but that is ok just so it settles toward 0 - shown dashed.

That was why I provided both the plots. The Schaums function is not wrong by plot appearance. You take your pick or check with your local enigneer/ lecturer. My pick is v1(t).

Example 5: Case $\omega_0^2 > \alpha^2$ Underdamped OR Oscillatory.

A PARALLEL RLC circuit.

Capacitor C = 3.57 uF
Initial voltage on Capacitor = 50.0 V
Resistor R = 200 Ohm.
Inductor L = 0.28 H.
Switch is closed at t = 0. Allowing Capacitor to discharge.

Obtain the voltage function when switch is closed at t = 0?

Solution:

$R := 200$ $L := 0.28$ $C := 3.57 \cdot 10^{-6}$ $V_0 := 50.0$ cls(t)

Compute Alpha^2, Omega_o^2,:

$\alpha := \frac{1}{2 \cdot R \cdot C} = 700$ 1/s (per second) $\alpha^2 = 4.9 \cdot 10^5$ 1/s^-2

$\omega_{0\_squarred} := \frac{1}{(L \cdot C)} = 1 \cdot 10^6$ for parallel circuit its squared for over damped.

We have 'omega_o'^2 greater than 'alpha'^2 ; under damped condition.

w_d (omega-d) is radian frequency = SQRT(w_0^2 + alpha^2)

$\omega_d := \sqrt{(\omega_{0\_squarred}) - (\alpha^2)}$ $\omega_d = 714$

DE for the circuit voltage: $v = e^{-\alpha \cdot t} \cdot (A_1 \cdot \cos(\omega_d \cdot t) + A_2 \cdot \sin(\omega_d \cdot t))$

At time t =0, the voltage of the circuit is 50 V, initial condition.
v(t=0) = v(t+) = 50 V. Substitute v = 50 at t = 0 for the expression v,
with w_d, and alpha:

$50 = e^{-700 \cdot 0} \cdot (A_1 \cdot \cos(714 \cdot 0) + A_2 \cdot \sin(714 \cdot 0))$

$50 = e^0 \cdot (A_1 \cdot \cos(0))$

$50 = A_1$

$A_1 := 50$ Solved for A1 next solve for A2.

Node equation, sum of currents at node equal 0.

$$i_R + i_L + i_C = 0$$

$$\frac{V_0}{R} + \left(\frac{1}{L}\right)\int_0^t v\,dt + C\left(\frac{dv}{dt}\right) = 0$$

$$\frac{V_0}{R} + C\left(\frac{dv}{dt}\right) = 0$$

Here at t=0, C supplies current to inductor, inductor sees current gradually building up from 0. Capacitor voltage starts at some value then decays, likewise the capacitor current. So iL(-0)=iL(0+)=0.

$$\frac{dv}{dt} = -\frac{V_0}{RC} \quad \text{rearranging}$$

$$v = e^{-\alpha\cdot t}\cdot\left(A_1\cdot\cos\left(\omega_d\cdot t\right) + A_2\cdot\sin\left(\omega_d\cdot t\right)\right)$$

$$\frac{dv}{dt} = \left(-\alpha\cdot e^{-\alpha\cdot t}\cdot A_1\cdot\cos\left(\omega_d\cdot t\right) - e^{-\alpha\cdot t}\cdot\omega_d\cdot A1\cdot\sin\left(\omega_d\cdot t\right)\right) + \left(-\alpha\cdot e^{-\alpha\cdot t}\cdot A_2\cdot\sin\left(\omega_d\cdot t\right) + e^{-\alpha\cdot t}\cdot\omega_d\cdot A_2\cdot\cos\left(\omega_d\cdot t\right)\right)$$

$$\frac{dv}{dt} = \left(-\alpha\cdot A_1\cdot\cos\left(\omega_d\cdot t\right)\right) + \left(\omega_d\cdot A_2\cdot\cos\left(\omega_d\cdot t\right)\right) \quad \text{at } t = 0$$

$$\frac{dv}{dt} = \left(-\alpha\cdot A_1\right) + \left(\omega_d\cdot A_2\right) \quad \text{at } t = 0,\ \cos(0) = 1.$$

$$\frac{dv}{dt} = \left(\omega_d\cdot A_2\right) - \left(\alpha\cdot A_1\right) \quad \text{rearranging}$$

Substitute $\frac{dv}{dt} = -\frac{V_0}{RC}$

$$\left(\omega_d\cdot A_2\right) - \left(\alpha\cdot A_1\right) = -\frac{V_0}{RC}$$

$$\left(\omega_d\cdot A_2\right) = \left(\alpha\cdot A_1\right) - \frac{V_0}{RC}$$

$$A_2 = \left(\frac{1}{\omega_d}\right)\cdot\left(\left(\alpha\cdot A_1\right) - \frac{V_0}{RC}\right)$$

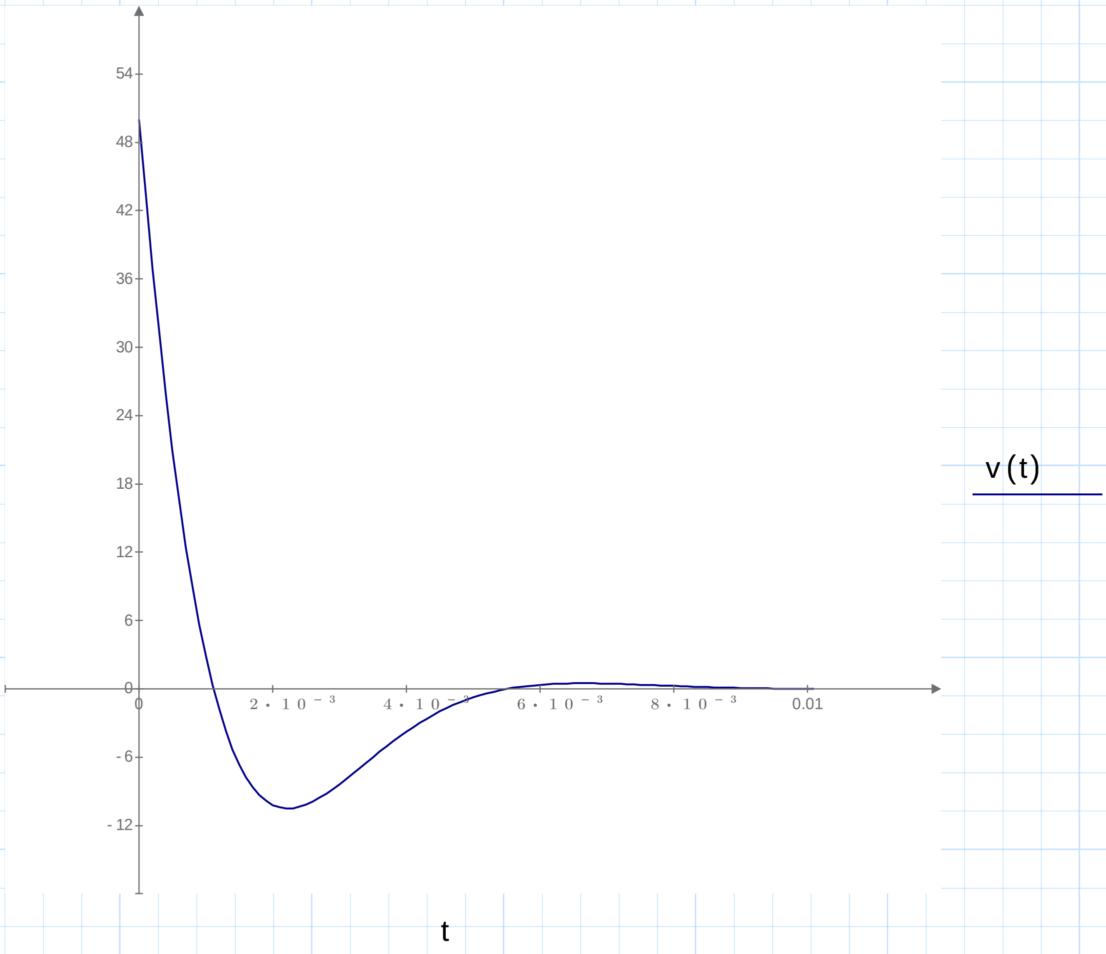
Next substitute values to solve for A2.

$$A_2 = \left(\frac{1}{\omega_d}\right)\cdot\left(\left(\alpha\cdot A_1\right) - \frac{V_0}{R\cdot C}\right) = -49$$

$v = e^{-700\cdot t}\cdot\left(50\cdot\cos(714\cdot t) - 49\cdot\sin(714\cdot t)\right)$ V. Answer.

Next we plot the oscillatory graph for v(t).

$$v(t) := e^{-700\cdot t}\cdot\left(50\cdot\cos(714\cdot t) - 49\cdot\sin(714\cdot t)\right)$$



Slight oscillation, good enough, and the plot settles to zero at 0.01 seconds.

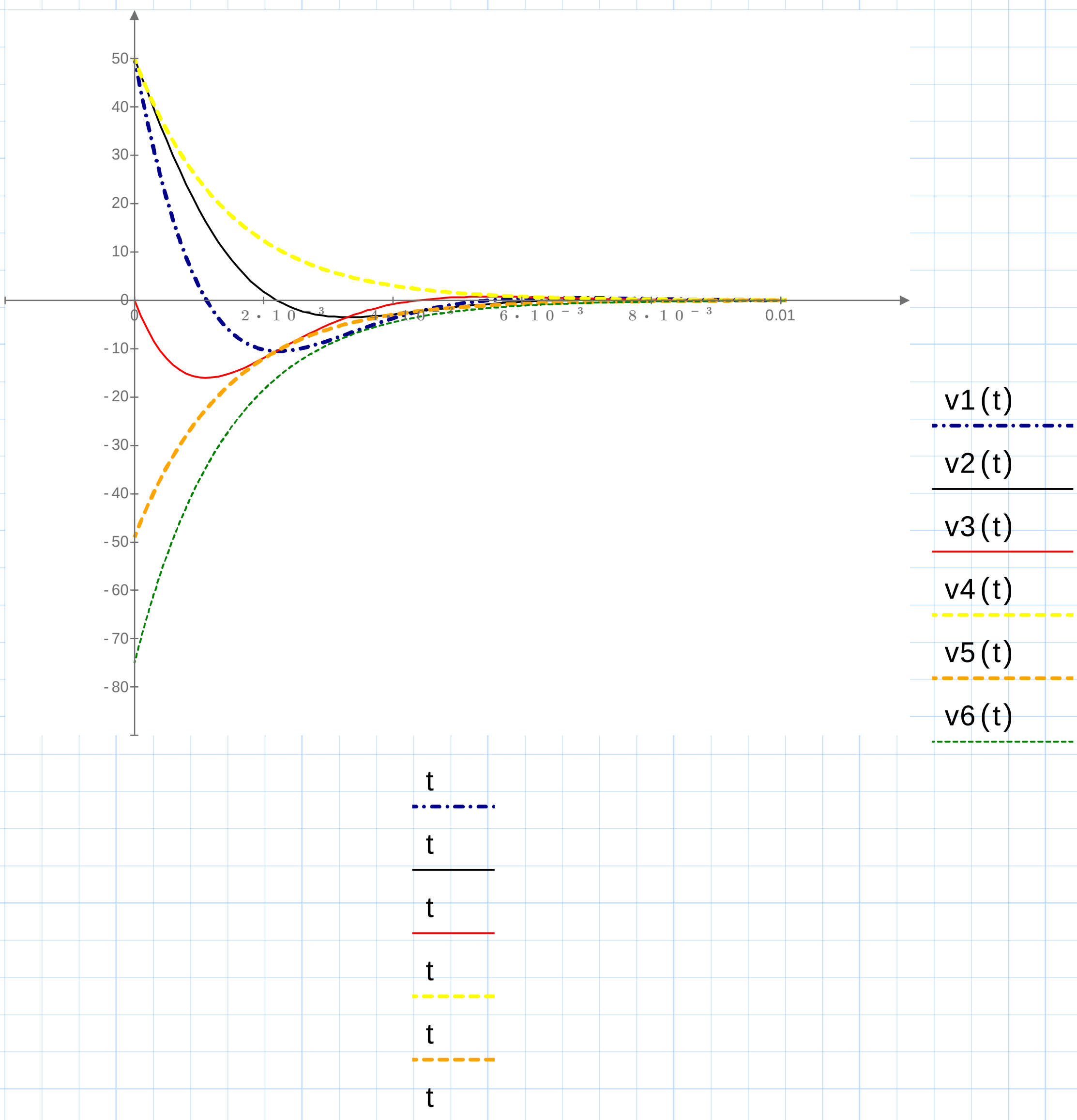
Next we plot each term separately, try to get the envelope on the oscillation if possible, though the oscillation is low with 2 crossings on the x-axis.

clear(t)

$v1(t) := e^{-700 \cdot t} \cdot (50 \cdot \cos(714 \cdot t) - 49 \cdot \sin(714 \cdot t))$

$v2(t) := e^{-700 \cdot t} \cdot 50 \cdot \cos(714 \cdot t)$ $\quad v3(t) := -e^{-700 \cdot t} \cdot 49 \cdot \sin(714 \cdot t)$

$v4(t) := 50 \cdot e^{-700 \cdot t}$ $\quad v5(t) := -49 \cdot e^{-700 \cdot t}$ $\quad v6(t) := -75 \cdot e^{-700 \cdot t}$



The functions v4(t) = 50e^-700t, and v5(t) = -49e^-700t maybe were to provide the envelope above and below for v1(t) which is the solution v(t). The bottom envelope did not capture the curve, so adjusting it manaully resulted with v6(t). Not that we were able to get that envelope from the solution, rather for our learning outcome. The sine and cosine terms, functions v2(t) and v3(t) both show a very low oscillation. Sine term starts at 0 since sin(0) = 0. Cosine term starts at 50. *Its Math!*

Parallel RLC Circuit Critically Damped.

The critically damped case will not be examied for the parallel RLC circuit, since it has little or no real value in the circuit design.

In fact, it is merely a curiosity, since it is a set of circuit constants whose response, while damped, is on the verge of oscillation (under damped).
(page 185 Schaums Outline 6th Ed Nahvi & Edminister).

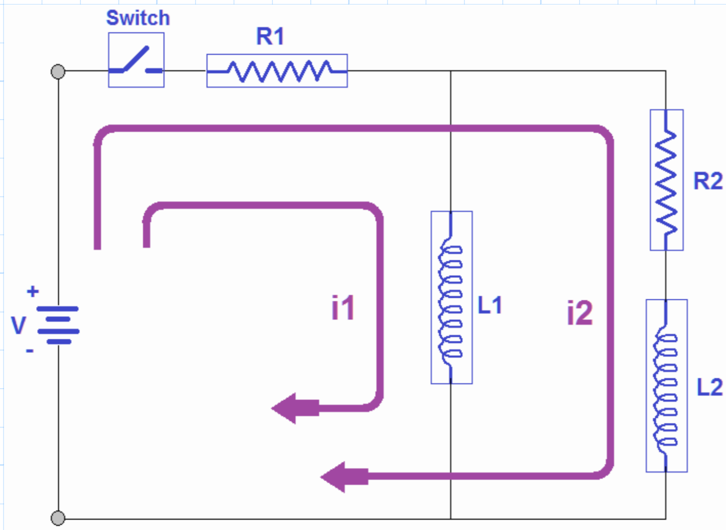
This may be sufficient for the theorectical understanding, thru the several cases examined over, critical, and under damped for series and parallel RLC circuit.

Next section 8.4 continues with the same chapter in Schaums Outline Chapter 8 Higher-Order Circuits and Comple Frequency.

At this time the aim is to get much of the 's = sigma + j omega' material covered, so the 'complex frequency s' subject matter becomes comprehensible, then apply the Lapalce methods to solve electric circuits.

We come to s in later part of this PDF under the heading complex frequency thru the Hyat Kemerly textbook.

8.4 Two-Mesh circuits (2nd order differential equation created - d/dt^2).



Form 2 loop equations to solve the circuit. We set two currents i1 and i2 then apply Kirchoff Voltage.

One loop runs the outer perimeter of the circuit, the other the left side loop. Many ways to form loops, just so they intersect/overlap or pick up currents in opposite and or same directions.

$$R1 \cdot i1 + L1 \cdot \left(\frac{di1}{dt}\right) + R1 \cdot i2 = V \qquad \text{Eq 1.....i1 loop.}$$

$$R1 \cdot i1 + (R1 + R2) \cdot i2 + (L2) \cdot \left(\frac{di2}{dt}\right) = V \qquad \text{Eq 2....i2 loop.}$$

With 2 loops and an inductor in each loop, the equation(s) to solve would result in a 2nd order equation.

Differentiate Eq 1:

$$R1\left(\frac{di1}{dt}\right) + L1 \cdot \left(\frac{d^2\ i1}{dt^2}\right) + R1 \cdot \left(\frac{di2}{dt}\right) = 0 \quad \text{next arrange in order}$$

$$L1 \cdot \left(\frac{d^2\ i1}{dt^2}\right) + R1\left(\frac{di1}{dt}\right) + R1 \cdot \left(\frac{di2}{dt}\right) = 0 \quad \text{Eq 3}$$

From Eq 1, solve for i2:

$$i2 = \frac{\left(V - R1 \cdot i1 - L1 \cdot \left(\frac{di1}{dt}\right)\right)}{R1}$$

From Eq 2, solve for di2/dt:

$$(L2) \cdot \left(\frac{di2}{dt}\right) = V - R1 \cdot i1 - (R1 + R2) \cdot i2$$

$$\left(\frac{di2}{dt}\right) = \frac{(V - R1 \cdot i1 - (R1 + R2) \cdot i2)}{L2} \qquad \text{next substitute i2}$$

$$\left(\frac{di2}{dt}\right) = \left(\frac{1}{L2}\right)\left[V - R1 \cdot i1 - (R1 + R2) \cdot \frac{\left(V - R1 \cdot i1 - L1 \cdot \left(\frac{di1}{dt}\right)\right)}{R1}\right] \quad \text{Eq 4}$$

Substitute di2/dt in Eq 3

$$L1\cdot\left(\frac{d^2\ i1}{dt^2}\right)+R1\left(\frac{di1}{dt}\right)+R1\cdot\left(\frac{di2}{dt}\right) = 0$$ Eq 3...provided here again

$$L1\cdot\left(\frac{d^2\ i1}{dt^2}\right)+R1\left(\frac{di1}{dt}\right)+R1\cdot\left(\frac{1}{L2}\right)\left[V-R1\cdot i1-(R1+R2)\cdot\frac{\left(V-R1\cdot i1-L1\cdot\left(\frac{di1}{dt}\right)\right)}{R1}\right] = 0$$

Expand last term (3rd term) on the LHS:

$$\frac{R1\cdot V}{L2}-\frac{R1\cdot R1\cdot i1}{L2}-\left(\left(\left(\frac{R1\cdot V}{L2}\right)+\left(\frac{R2\cdot V}{L2}\right)+\left(\frac{R1\cdot(R1+R2)}{L2}\right)\cdot i1+\left(\left(\left(\frac{R1\cdot L1}{L2}\right)+\left(\frac{R2\cdot L1}{L2}\right)\right)\right)\cdot\left(\frac{di1}{dt}\right)\right)\right)$$

Arranging like terms in order:

$$L1\cdot\left(\frac{d^2\ i1}{dt^2}\right)$$

$$\left(R1-\left(\left(\frac{R1\cdot L1}{L2}\right)+\left(\frac{R2\cdot L1}{L2}\right)\right)\right)\cdot\left(\frac{di1}{dt}\right)$$

$$\left(-\left(\frac{R1\cdot R1}{L2}\right)-\left(\frac{R1\cdot(R1+R2)}{L2}\right)\right)\cdot i = \frac{-R1R1+R1R1+R1R2}{L2} = \left(\frac{R1R2}{L2}\right)\cdot i$$

$$\left(\frac{R1\cdot V}{L2}\right)-\left(\frac{R1\cdot V}{L2}\right)+\left(\frac{R2\cdot V}{L2}\right) = \left(\frac{R2\cdot V}{L2}\right)$$ Constant (can become RHS term)

Divide by L1 so first term (di2/dt) coefficient equal 1, rows below left to right:

$$\left(\frac{d^2\ i1}{dt^2}\right) \qquad \left(\left(\frac{R1}{L1}\right)+\left(\frac{R1}{L2}\right)+\left(\frac{R2}{L2}\right)\right)\cdot\left(\frac{di1}{dt}\right)$$

$$\left(\frac{R1R2}{L1\cdot L2}\right)\cdot i1 \qquad \left(\frac{R2\cdot V}{L1\cdot L2}\right)$$ Constant (RHS term)

The differential equation becomes:

$$\left(\frac{d^2\ i1}{dt^2}\right)+\left(\left(\frac{R1}{L1}\right)+\left(\frac{R1}{L2}\right)+\left(\frac{R2}{L2}\right)\right)\cdot\left(\frac{di1}{dt}\right)+\left(\frac{R1R2}{L2}\right)\cdot i1 = \left(\frac{R2\cdot V}{L1\cdot L2}\right)$$

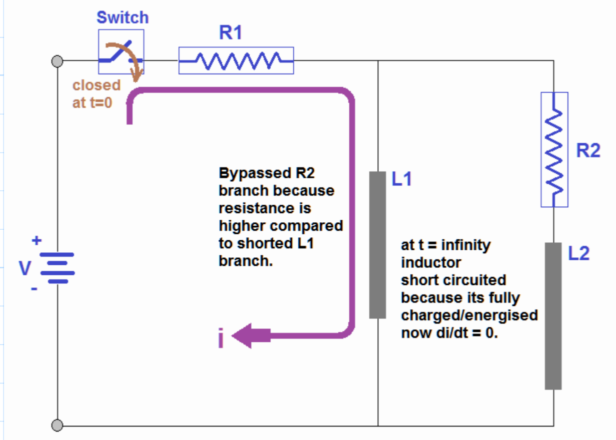
$$\left(\frac{di1^2}{dt^2}\right)+\left(\frac{R1\cdot L1+R1\cdot L2+R2\cdot L1}{L1\cdot L2}\right)\cdot\left(\frac{di1}{dt}\right)+\left(\frac{R1R2}{L1\cdot L2}\right)\cdot i1 = \left(\frac{R2\cdot V}{L1\cdot L2}\right)$$ CORRECT.

We have a differential equation of 2nd order.

$$\left(\frac{d^2 i1}{dt^2}\right)+\left(\frac{R1\cdot L1+R1\cdot L2+R2\cdot L1}{L1\cdot L2}\right)\cdot\left(\frac{di1}{dt}\right)+\left(\frac{R1\cdot R2}{L1\cdot L2}\right)\cdot i1 = \left(\frac{R2\cdot V}{L1\cdot L2}\right)$$

Equation above LHS equal the constant at RHS side (R2V/L1L2).

Discussion: Inductor OPEN and CLOSED CIRCUIT: A fully discharged inductor (no current from it) initially acts as an open circuit (time t<0, voltage dropped to 0 no supply of current) when faced with the sudden application of voltage (t=0) then being charged/ energised fully to it's acceptable final level of current, it acts as a short circuit (t>0 or infinity, current pases with no voltage drop across it, a conductor with near 0 resistance).



When switch is closed, inductor L1 and L2 get fully charged-energised, it becomes a short circuit. Thus current passes thru L1 by-passing L2 because the resistor in the loop i2 has higher resistance, whilst L1 branch offers no resistance. See figure to left. See later for di1/dt=0.

Steady state solution:

Current at i(t=infinity) when the inductors are fully charged, makes a short circuit at L1, then the current i(t) in the circuit is:

$$i(t) = \frac{V}{R1}$$

Using circuit analysis see figure above.
i(t) = i1(t), they are the same path/branch.

Alternate: Just so happens from our 2nd order differential equation above, at time t = infinity, (t>>0), the current from the source V supplied to the inductors would become constant when inductors L1 and L2 are fully energised/charged. Here di/dt and d^2i/dt^2 = 0.
We can solve for i in the expression.

$$(0)+\left(\frac{R1\cdot L1+R1\cdot L2+R2\cdot L1}{L1\cdot L2}\right)\cdot(0)-\left(\frac{R1\cdot R2}{L1\cdot L2}\right)\cdot i1 = \left(\frac{R2\cdot V}{L1\cdot L2}\right)$$

$$\left(\frac{R1\cdot R2}{L1\cdot L2}\right)\cdot i1 = \left(\frac{R2\cdot V}{L1\cdot L2}\right) \qquad i1 = \frac{V}{R1}$$ Correct.

Transient solution:

Here RHS equal zero since voltage source was removed from the circuit, and analysis includes inductor initial conditions iL(-0) = iL(0+) = 0 Amps. Continuity condition.

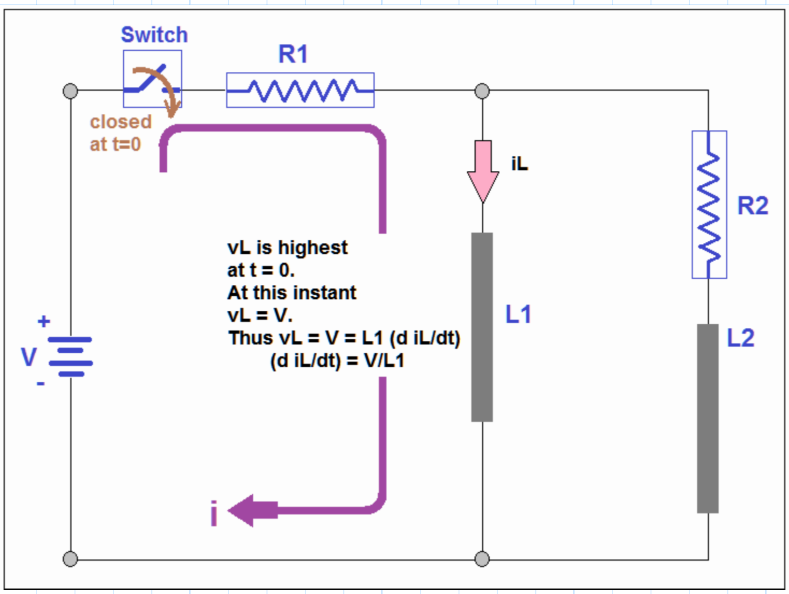
$$\left(\frac{d^2\ i1}{dt^2}\right)+\left(\frac{R1\cdot L1+R1\cdot L2+R2\cdot L1}{L1\cdot L2}\right)\cdot\left(\frac{di1}{dt}\right)-\left(\frac{R1R2}{L1\cdot L2}\right)\cdot i1 \quad = \quad 0$$

Differential equation above now can be seen in the 's' differential order form; where d^2i1/dt^2 : s^2, di1/dt : s, and constant term i. We done this in previous pages.

$$s^2+\left(\frac{R1\cdot L1+R1\cdot L2+R2\cdot L1}{L1\cdot L2}\right)\cdot s-\left(\frac{R1R2}{L1\cdot L2}\right) = 0$$

The 3rd term became a constant, instead of an i term.

Transient solution of equation above will be solved by the roots, s1 and s2, together with the initial conditions (continuity):



Refer to the figure to the left for the expression for diL/dt which here is di1/dt.
This applies to his circuit connection and components.

We are able to isolate di1/dt at time t=0. Inductor is NOT in short circuit condition.

Derivative of the current would be a higher order term than the current, which mathematically logic tells us this will be the transient current condition.

Current i1(0+) = 0 from continuity condition.
Voltage vL is the branch parallel to voltage source V.
vL (0+) = V. This is the maximum the circuit can provide.
We got vL at t=0+, we know vL = L(di1/dt).
Can we solve for di1/dt at time t=0+?
Cleaver them engineers.

$$i_1(0+) \quad = \quad 0$$

$$vL1(0+) \quad = \quad L1\left(\frac{di1}{dt}\right)$$

Here vL1 = V ---> $\frac{di1}{dt} = \frac{vL1}{L1} = \frac{V}{L1}$

Now for our circuit the initial condion at t=0+: $\frac{di1}{dt} = \frac{V}{L1}$

Once i1 is solved i2 can be solved in Eq 1.

There will be a damping factor that ensures the transient will ultimately die-out. Depending on the values R1, R2, L1, and L2, the transient can be over or under damped (oscillatory).

In general the current expression will be: i1 = (transient) + V/R1

The transient part will have a value of V/L1 at t =0+
and a value of 0 as t approaches infinity.
Zero at t>>0, the inductor has short circuited.

Obviously as time approaches infinity like a few seconds the current will die-out in the transient case since all the energy in the inductor been consumed in the circuit.
Fully discharged inductor is an open circuit.

Differential equation generated through mesh analysis, then solved thru continuity condition for t=0+ and derivative of i with respect to t at t=0.
Reference material Schaums Outline electric Circuits, Nahvi & Edminister.

Comment:
Schaums does not have a RLC circuit analysis example and solved problem directly related to this learning, but it has one supplementary problems (unsolved) which is an RLC circuit.

Hyat & Kemmerly has a subject matter example on an RLC circuit.
My hope is to go thru this theory and example to gain problem solving skills for RLC circuits. It is a long one! Reading the excerpt below is tough, would expect likewise for it's problem solving.

Hyat & Kemmerly page 213: We must now consider those RLC circuits in which dc sources are switched into the network and produce forced responses that do not vanish as time becomes infinite. The general solution is obtained by the same procedure that was followed in RL and RC circuits: the forced response is determined completely the natural response is obtained as a suitable functional form containing the appropriate number of arbitrary constants; the complete response is written as the sum of the forced and the natural responses; and the initial conditions are then determined and applied to the complete response to find the values of the constants. It is this last step which is quite frequently the most troublesome to students. Consequently, although the determination of the initial conditions is basically no different for a circuit containing dc sources than it is for the source-free circuits which we have already covered in some detail, this topic will receive emphasis in the example below. Most of the confusion in determining and applying the initial conditions arises for the simple reason that we do not have laid down for us a rigrous set of rules to follow.

Okay some idea I have from our RL and RC circuits, we feel with RLC we need to apply some formulas to help in the solution which like the alpha omega0 and beta. NOT true. We got initial conditions which we are a little confident, Yes. But now we have differential equations with terms di/dt, d^2i/dt and initial conditions apply to them as well. The form of current or voltage equation and then the form of solution that best suits them. Some idea we have. But its not solid yet. We got skills to build from the explanation following (here voltage is the response) and then the example.

Notes made from Engineering Circuit Analysis 4th Edition, page 214:

Complete voltage response of a 2nd order system consists of a forced reponse, which is a constant for a dc excitation system,

$$v_f(t) = V_f$$

and a natural response

$$v_n(t) = Ae^{s1t} + Be^{s2t}$$

Thus,

$$v(t) = V_f + Ae^{s1t} + Be^{s2t}$$

From our previous pages we seen how to reach to s1, s2, and Vf. So now we assume these have been determined. Remaining is how to find A and B.

$$v(t) = V_f + Ae^{s1t} + Be^{s2t}$$

The equation above shows the functional interdependence of A, B, v, and t, and substitution of the known value of v at t = 0+, this provide us with the single equation relating A and B. This is the easy part.

Another relationship between A and B is necessary, unfortunately, and this is normally obtained by taking the derivative of the response,

$$\frac{d(v(t))}{dt} = s1Ae^{s1t} + s2Be^{s2t} \quad \text{at } v(0+') = 0$$

and inserting in it the known value of dv/dt at t = 0+. This is the value that goes in the LHS of the equation above.

Now, there is no reason why this process cannot be continued.
We used d(v(t)/dt above, we can use d^2(v(t)/dt^2 at t=0+ for a third relationship between A and B.

$$\frac{d^2(v(t))}{dt^2} = (s1)^2 Ae^{s1t} + (s2)^2 Be^{s2t} \quad \text{at } v(0+') = 0$$

From the 1st and 2nd derivative we attain 2 equations which allows us to solve simultaneously for A and B. Provided we get the initial values correctly for both derivatives.

So how do we get the value of v and dv/dt at t = 0+ ?

Since its a RLC circuit, and we are seeking voltage response, we say lets focus on the capacitor it is defined as iC = C (dvC/dt). So dv/dt is in the expression it maybe how we work it to get vC.

$$i_C(t) \quad = \quad C\cdot\left(\frac{dvC}{dt}\right)$$

When a component gets energised or charged, its value is going to rise gradually, maybe exponential, sinusoidal, linear,.... whatever it is, it will have a value we can be satisfied with at time t=-0, t=0, and t=0+, which usually we say its the same at the start leading to t(-0) = t(0+) = 0. Correct.

$$i_C(t) \quad = \quad C\cdot\left(\frac{dvC}{dt}\right) \qquad \text{at } t(0+')$$

From the expression above for the capacitor C voltage we can say there is an initial value for (dv/dt), and for the capacitor C current an initial value for iC(t), at time t(0+). We need FIRST establish a value for capacitor C current, then we shall automatically establish the initial value of (dv/dt). From the expression below.

$$\frac{dvC}{dt} \quad = \quad \left(\frac{1}{C}\right)\cdot i_C(t) \qquad \text{at } t(0+')$$

Finding the initial value for (dv/dt) at t=0+ may have the student-graduate engineer fail the first time or more times..... compared to the initial condition for v at t=0+.

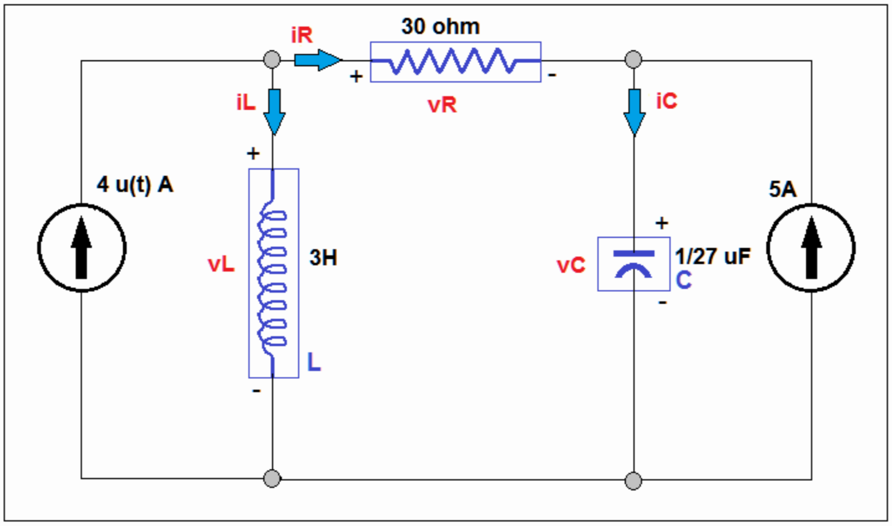
If intead of Capacitor we had Inductor, then similarly applying on the expression below:

$$v_L(t) \quad = \quad L\cdot\left(\frac{diL}{dt}\right) \qquad \text{at } t(0+)$$

For inductor L, inductor current is the response, initial value of (di/dt) must be closely related to the initial value of the inductor voltage, at time t=0+.
We taken care for Capacitor and Inductor.

Should there be variables, other than capacitor voltage and inductor current, these can be determined by expressing their initial values and the initial values of their derivatives in terms of the corresponding values of vC and iL.<--- Long winded in other words play it the same way.

Example: RLC Circuit Complete Response.
Hyat & Kemmerly 4th ed page 215.

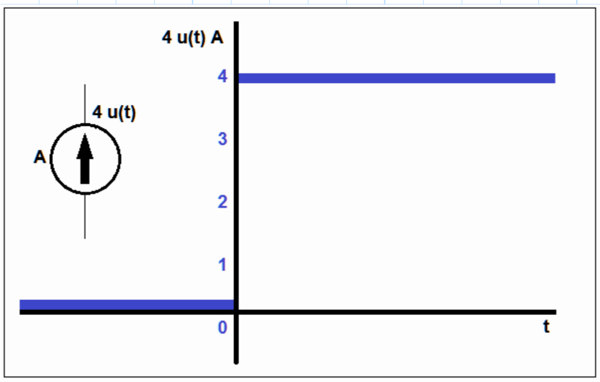


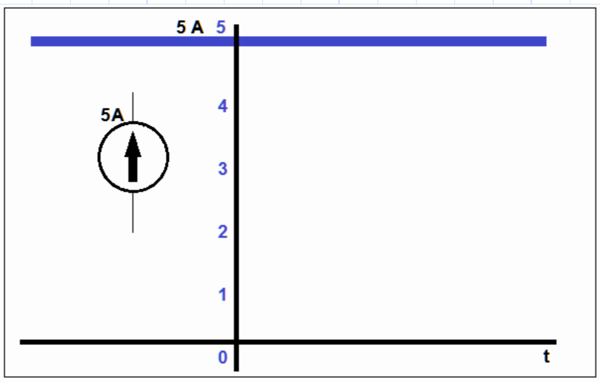
Illustrate the procedure and find a*lll* these values by the careful analysis of the circuit in the figure above. To simplify the analysis numerically an unrealistically large capacitor is used.

Objective:

To find the value of each current and voltage at both t=0- and t=0+;
with these quanitities known, the required derivatives may be easily calculated.

Solution:





At time t=0-, the 4u(t) source was not supplying current. But the current source on the RHS supplied current of 5A for time t=0-. At t=0 and t=0+ both the LHS and RHS are supplying current i.e. 4u(t) A and 5 A.

t = 0- :
LHS current source supplied 5A and this supply was constant.
So with constant current, the components L and C would reach their steady state conditions, to the maximum allowable current and voltage, and similary for R dependent on value of current supplied. Eventually all the components R L and C would reach a steady state value for the time t=0-.

In this case, when we form the circuit loop equations, the expression on the right side of the equation will be some voltage. This is the forcing function the term on the RHS of equation. *Most likely since, we have current 5A, its likely the equation we form are mesh loops. If we had voltage then we formed the node equations most likely - If you got this get that, that thinking. Surely there is a name for it.* We we have the forcing function in the time t=0-.

We have the forced function, we need to find the forced response which will take the form of the forcing function's integral or derivative. As we seen in the previous examples, the forcing function is the RHS term its integral or derivative.

The integral of the forcing function, a linearly increasing function of time, is not present in this circuit for it can occur only when a constant current is forced through a capacitor OR a constant voltage is maintained across an inductor.
This situation should not normally be present because the capacitor voltage or inductor current would assume an unrealistic infinite value at t = 0-. - Hyat & Kemerly page 215.

Discussion: *Lets deviate for a **short discussion** on the paragraph above - **You may skip this discussion.***

First the integral and derivative terms for L and C.

Capacitor:

$$i_C = C\left(\frac{dv_C}{dt}\right)$$

$$\frac{dv_C}{dt} = \frac{i_C}{C}$$

$$dv_C = \left(\frac{i_C}{C}\right) dt$$

$$\int 1\, dv_C = \int_{t1}^{t2} \left(\frac{i_C}{C}\right) dt$$

$$v_C = \int_{t1}^{t2} \left(\frac{i_C}{C}\right) dt$$

Inductor:

$$v_L = L\left(\frac{di_L}{dt}\right)$$

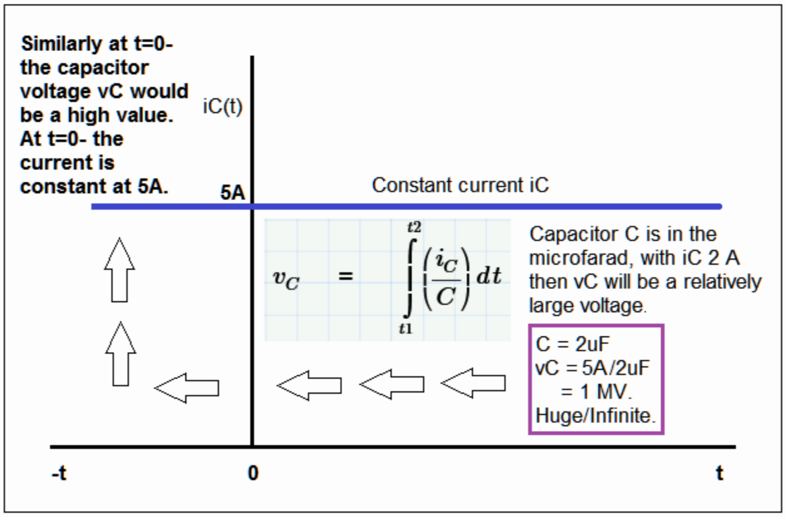
$$\frac{di_L}{dt} = \frac{v_L}{L}$$

$$di_L = \left(\frac{v_L}{L}\right) dt$$

$$\int 1\, di_L = \int_{t1}^{t2} \left(\frac{v_L}{L}\right) dt$$

$$i_L = \int_{t1}^{t2} \left(\frac{v_L}{L}\right) dt$$

We work capacitor case above, we create an example with new value of capacitor.



Capacitor voltage an unrealistic high value or infinite value at t=0-

$$v_C = \int_{t1}^{t2} \left(\frac{i_C}{C}\right) dt$$

Repeated here: The integral of the forcing function, a linearly increasing function of time, is **not present in this circuit** for it can occur **only when** a constant current is forced through a capacitor OR a constant voltage is maintained across an inductor.
What does the constant current has to do with the forcing function?
The forcing function, found on the RHS of the node/loop equation, could be a numerical value or an expression. Whatever it is, its integral has to be linearly increasing function of time. Our circuit has a constant current, 5A and 4u(t) ie 4A.

$$i_C = C\left(\frac{dv_C}{dt}\right)$$

<---vC(t) has to be some function thats differentiable, so that it can be multiplied by C to get a constant current.

The forcing function has to be some expression that when its integral is taken it results in a linear increasing function. Since we represented vC in an integral form above, vC = integral of (iC/C)dt. Obviously iC has to be some function of t or constant, here its constant and it can be integrated. Also derivative of capacitor voltage can be a concern. So, the function vC should be one when its derivative is taken it results in a constant, if its a term in t then its increasing with time.

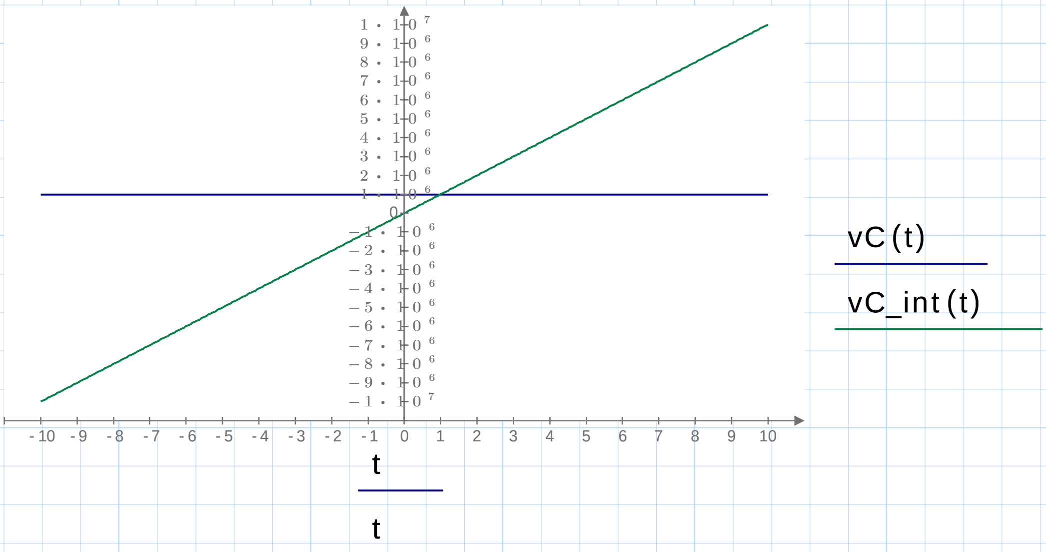
Try work it with values: iC = 5A, and vC = 1 MV.
*Now to get 1MV the forcing function vC(t) OR v(t) across the capacitor must be equal to (1x10^6)**t** when it is differentiated we get (1x10^6) or 1MV.*
*When the integral is taken of the voltage function vC(t) = (1x10^6) it is equal to (1x10^6)**t**, the linear increasing function.*

COMMENT: I said would try. Plot next page.
You check with your local engineer/lecturer.

clear (t) $\quad vC(t) := 1 \cdot 10^6$ Capacitor voltage function vC(t) linearly increasing.
Next its integral vC_int(t).

Integral of f(t): $= \int 1 \cdot 10^6 \, dt = 1 \cdot 10^6 \cdot t \qquad vC\_int(t) := 1 \cdot 10^6 \cdot t$

So here at t(0-) the voltage is high negative value, as in the plot below.



The green line is the linear relationship expression sought to show.
Maybe correct my attempt.
Its derivative is the horizontal straight line at 1x10^6.

May not be best discussion. *May been better if it was writen in German than my explanation required translation with my excuse in the translation process.* Sorry.
End of discussion.

Continued with solution on next page.

A constant voltage across the capacitor requires zero current through it - open circuit.

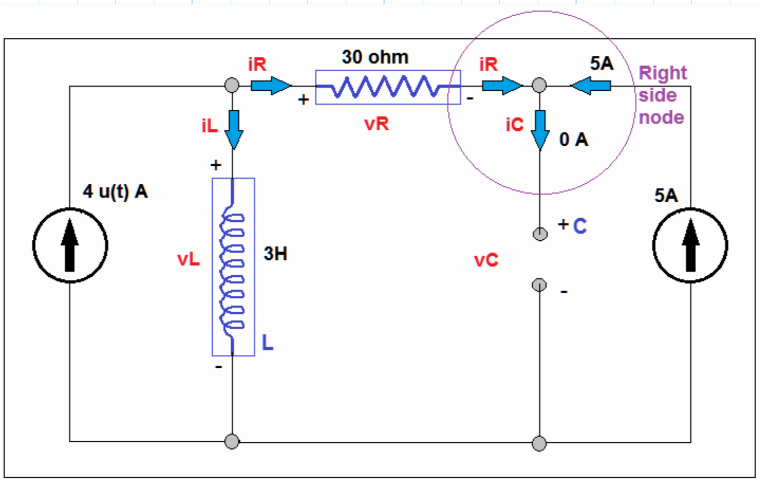
A constant current thru the inductor requires a zero voltage across it - closed circuit.

Now for the inductor L, a constant current through the inductor requires a zero voltage at time t=0-,

$$v_L(0-') = 0$$

and a constant voltage across the capacitor requires zero current through it,

$$i_L(0-') = 0$$



Apply Kirchoff current at right side node.

For time t = 0-

iR(0-)+ 5A = 0
iR(0-) = -5A.

$R := 30$ Ohms $\quad L := 3$ H $\quad C := \left(\frac{2}{27}\right)$ F

$i_R(0-') = -5$ A

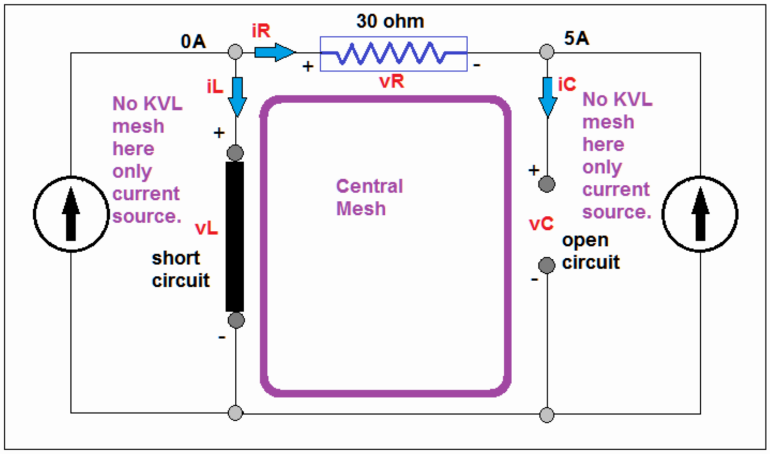
$i_{R\_0\_minus} := -5$

$v_{R\_0\_minus} := i_{R\_0\_minus} \cdot R = -150$

$v_R(0-') = 150$ Ohms

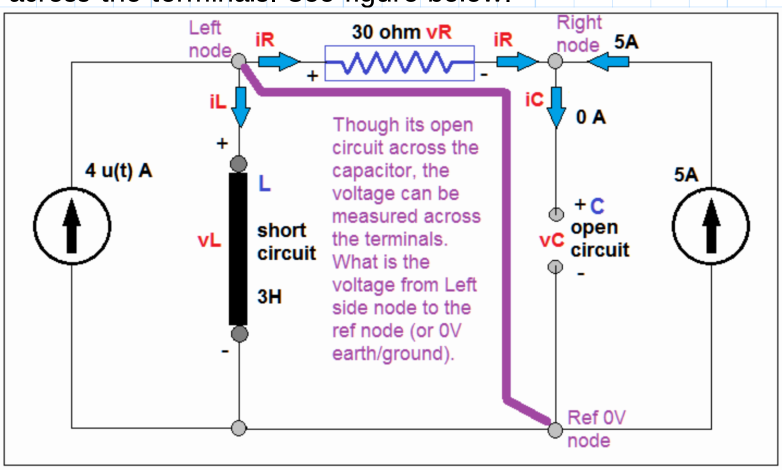
I have to do iR_0_minus because the variable setting does not allow iR(0-). It cannot take the -ve sign in the parenthesis. Same everywhere else here.

Next we move to the central mesh. Figures on next page.



From observation there is only one mesh, as opposed to three. The left side and right side mesh do not have voltage because they are only current source branches.

The central mesh need to be analysed on how to get the voltage, mesh analysis. Since iC = 0A open circuit across capacitor. However voltage can be measured across the terminals. See figure below.



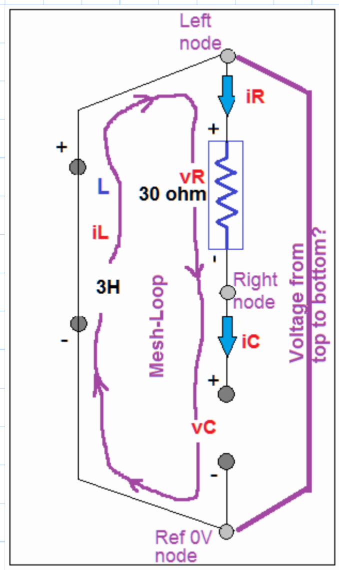


Figure to the right is a modification of the figure above. We made the mesh loop into a straight line So we can get the voltage in the loop seen straight with the top left node having the highest voltage. We re-positioned the resistor the circuit is the same for the mesh.

KCL central mesh:

$$v_L + v_R(0-') + v_C(0-') = 0$$

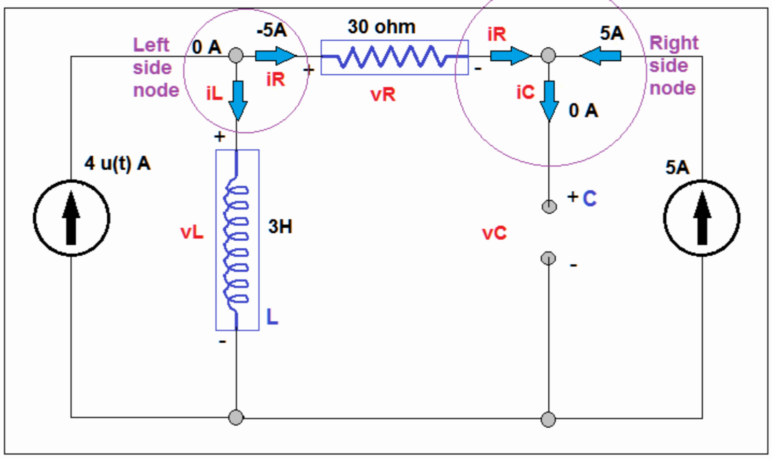
$$0 - 150 + v_C(0-') = 0$$

$$v_C(0-') = 150 \quad V$$

$$v_L = 0$$

There are many circuit analysis methods building the skills requires solving circuit problems. Its not always possible to get it first time, for example in this case vC(0-). Maybe how we see the circuit connections and apply the connected method !
You teach me!

Next we try for the inductor current at t = 0-.



Left side node, we have 0 current entering the node from 4u(t). Shown on figure.

Form a current node equation at left side node, for time t=0-.

Current node equation at left side node at time t = 0- :

$0 - i_R - i_L \quad = \quad 0$ Here we did a sum of currents at node equal 0.

$0 - (-5) - i_L \quad = \quad 0$

$5 - i_L \quad = \quad 0$

$i_L(0-') \quad = \quad 5 \quad A.$ Seemed easy which similar to the way we did in the right side node.

According to the textbook at this stage the engineer-authors wrote all the variables for time t=0- have been determined. We recap what we have found.

$v_R(0-') = -150$ V $\qquad i_R(0-') = -5$ A

$v_L(0-') = 0$ V $\qquad i_L(0-') = 5$ A

$v_C(0-') = 150$ V $\qquad i_C(0-') = 0$ A

Looks like we got all of them for t=0-.

We are informed, the derivatives of the variables for t=0- are all zero, in this problem we are told they are of little interest to us, NOT all problems but in this problem. Visibly so since none of the values have a function in time v(t) all were constants.

t = 0 + :

Next time increases by an 'incremental amount'. From t = 0- to t = 0+.
During this interval meaning at t = 0, the left hand side source 4u(t)A becomes active. As such most of the voltages and current at t=0- will change abruptly.
We are told most because some will not change? Yes.
We are told to focus on those quantities which cannot change, being the inductor current (diL/dt=0), and capacitor voltage (dvC/dt=0). Both of these must remain constant during the switching interval. DC source in both left and right sides.

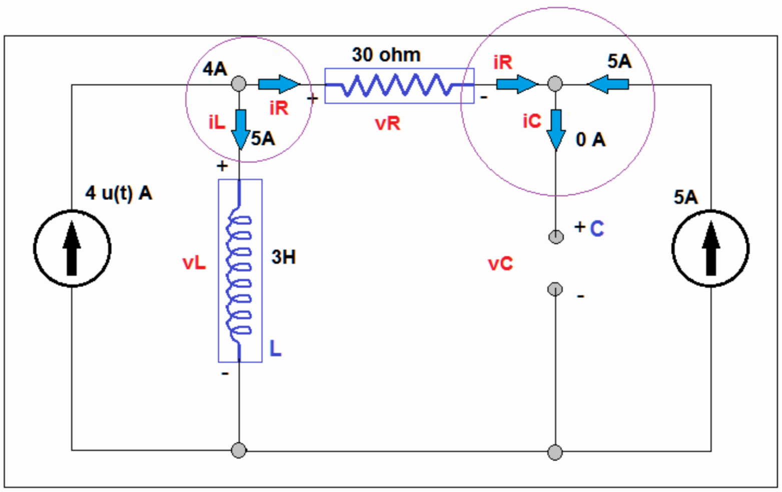
For that matter it makes our calculations a little simpler here in this circuit.

We now have iL(0+) = iL(0-) and vC(0+) = vC(0-) :

$i_L(0+') = 5 \quad A$

$v_C(0+') = 150 \quad V$

Reader Discuss: Current iC changed but vC did not change. Suggestion capacitor fully charged could not increase further its voltage with change of increased current. iC(0+) calculated in next page.



The updated figure to the left.

4u(t) A supplied 4A at t=0.

We solve for iR(0) at left node.

Then move to the right node.

Left node at t = 0+ :

Sum of current at node: $4 - i_R - i_L = 0$

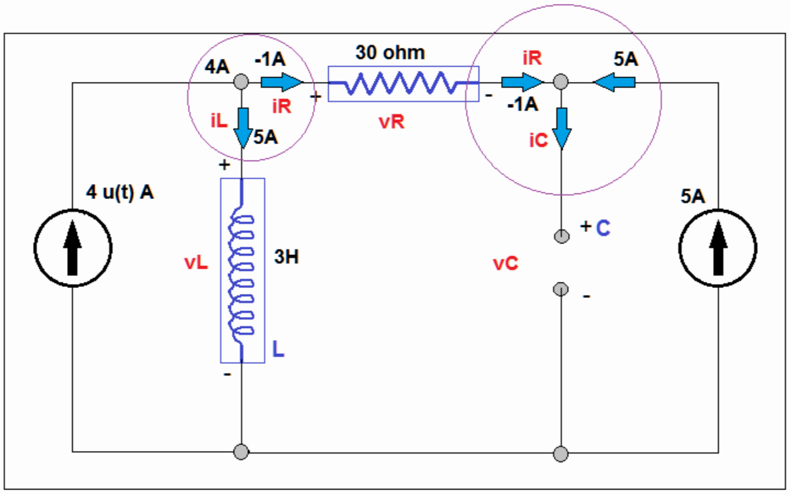
$4 - i_R - 5 = 0$

$i_R(0+') = -1 \quad A$

$v_R(0+') = -1 \cdot 30 = -30 \quad V$

Now we got the resistor voltage at t = 0+. A new resistor voltage because of change in resistor current.

Moving to the right side node for t = 0+.



Discussion: What remains is the capacitor current and voltage. We had iC(0-) = 0. But now there is an abrupt change in current, since left side source 4A has come ON. It will cause voltages to change. We cannot expect a vC change because of current change. We see dvC/dt exist for t=0+, when 4u(t)A came on at t=0.

Calculate iC at t = 0+ using current node at t = 0+ :

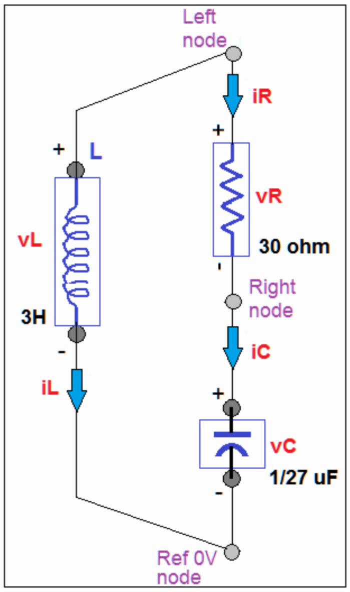
$$i_R(0+') + 5 - i_C(0+') \ = \ 0 \quad A$$

$$i_C(0+') \ = \ -1 + 5$$

$$i_C(0+') \ = \ 4 \quad A$$

We do a central mesh to solve for the inductor voltage, now since the inductor has voltage across it the mesh changes from t = 0- conditions.
Note: We do NOT use vL = L (d_iL/dt) to solve for vL, instead apply circuit analysis.

Figure to the right assists in solving for the inductor voltage vL(0+). According to our previous similar figure, now we have vL parallel to the branch where vR and vC are in series.



$$v_L(0+') \ = \ v_R(0+') + v_C(0+')$$

$$v_L(0+') \ = \ -30 + 150 \ \ = \ 120$$

$$v_L(0+') \ = \ 120 \quad V$$

We recap the 6 circuit values for t = 0 + on the next page.

Values of 6 variables at t = 0 + :

$v_R(0+') = -30$ V $\quad i_R(0+') = -1$ A

$v_L(0+') = 120$ V $\quad i_L(0+') = 5$ A

$v_C(0+') = 150$ V $\quad i_C(0+') = 4$ A

Next finding the derivatives dvR/dt, dvL/dt, dvC/dt - voltage
and
diR/dt, diL/dt, diC/dt - current

Apply the L and C defining equations, example vL = L (di/dt).
Derivatives for t = 0- and t = 0+. We have a good reason to conclude for t = 0- they are zero. Any way we work t = 0- as an exercise for t = 0+.

1st Derivatives t = 0- (You know from 0- values derivatives will be 0) :

Values we calculated prior for t=0- are shown below:

$v_R(0-') = -150$ V $\quad i_R(0-') = -5$ A

$v_L(0-') = 0$ V $\quad i_L(0-') = 5$ A

$v_C(0-') = 150$ V $\quad i_C(0-') = 0$ A

diL(0-)/dt :

$$v_L(0-') = L\cdot\left(\frac{di_L(0-')}{dt}\right)$$

$$\left(\frac{di_L(0-')}{dt}\right) = \frac{v_L(0-')}{L} = \frac{0}{3} = 0 \text{ A}$$

As expected, we got a 0 value in expression.

dvC(0-)/dt :

$$i_C(0-') = C\cdot\left(\frac{dv_C(0-')}{dt}\right)$$

$$\frac{dv_C(0-')}{dt} = \frac{i_C(0-')}{C} = \frac{0}{\left(\frac{1}{27}\right)} = 0 \text{ A}$$

As expected, we got a 0 value in expression.

diR(0-)/dt:

Lets write a current node equation at the left side node,
the source 4u(t=0') = 0A.

$$i_{4ut} - i_L - i_R = 0$$

$$0 - i_L - i_R = 0 \qquad \text{Equation t = 0-}$$

Differentiate the above equation:

$$0 - \left(\frac{di_L}{dt}\right) - \left(\frac{di_R}{dt}\right) = 0$$

$$\left(\frac{di_R(0-')}{dt}\right) = -\left(\frac{di_L(0-')}{dt}\right)$$

$$\frac{di_R(0-')}{dt} = -\left(\frac{d(5)}{dt}\right) = 0 \qquad \text{As expected}$$

diC(0-)/dt:

Lets write a current node equation at the right side node,
the current source supplies 5A.

$$i_{5A} - i_C + i_R = 0 \qquad \text{Going into node -ve, and iR is known -ve.}$$

$$5 - i_C - 5 = 0$$

Differentiate the above equation:

$$0 - \left(\frac{di_C(0-')}{dt}\right) - \left(\frac{di_R(5)}{dt}\right) = 0$$

$$\left(\frac{di_C(0-')}{dt}\right) = 0 \qquad \text{As expected}$$

dvR(0-)/dt:

$$v_R(0-') = i_R(0-')\cdot R$$

We have right side values next take the 1st derivative.
R a constant term its derivative is 0.

$$\frac{dv_R(0-')}{dt} = R\cdot\left(\frac{di_R(-5)}{dt}\right) = 30\cdot 0 = 0 \text{ V. As expected.}$$

dvL(0-)/dt:

$$v_L(0-') = L\left(\frac{di_L(0-')}{dt}\right)$$ This is the expression for inductor voltage.

$$v_L(0-') = L\cdot\left(\frac{d(5)}{dt}\right)$$ Should we take the derivative of this expression we get Ldi/dt but the RHS of expression is alredy zero for the 1st derivative result. Evaluate and next derivative.

$$v_L(0-') = L\cdot 0 = 0$$

$$\frac{dv_L(0-')}{dt} = \frac{Ld^2\ 0}{dt^2} = 0$$

Correct since the inductor at t=0- is the same as t<<0 here the inductor is a short circuit, voltage is near zero. As expected since our vL(0-) = 0V.

Alternate method we can form a loop equation for the central mesh.

$$v_R(0-') + v_C(0-') + v_L(0-') = 0$$ Next the derivative of the equation.

$$\frac{dv_R(0-')}{dt} + \frac{dv_C(0-')}{dt} + \frac{dv_L(0-')}{dt} = 0$$

$$\frac{d(-150)}{dt} + \frac{d(150)}{dt} + \frac{dv_L(0-')}{dt} = 0$$

$$0 + 0 + \frac{dv_L(0-')}{dt} = 0\ \ V$$

$$\frac{dv_L(0-')}{dt} = 0\ \ V$$

Recap derivatives values for t = 0- :

$$\frac{dv_R(0-')}{dt} = 0\ \ V \qquad \frac{di_R(0-')}{dt} = 0\ \ A$$

$$\frac{dv_L(0-')}{dt} = 0\ \ V \qquad \frac{di_L(0-')}{dt} = 0\ \ A$$

$$\frac{dv_C(0-')}{dt} = 0\ \ V \qquad \frac{di_C(0-')}{dt} = 0\ \ A$$

For the next set of calculations we see some numerical values instead of all zero(s).

1st Derivatives t = 0+ :

Values we calculated prior for t=0- shown below:

$v_R(0+') = -30 \quad V \qquad i_R(0+') = -1 \quad A$

$v_L(0+') = 120 \quad V \qquad i_L(0+') = 5 \quad A$

$v_C(0+') = 150 \quad V \qquad i_C(0+') = 4 \quad A$

diL(0+)/dt :

$$v_L(0+') = L \cdot \left(\frac{di_L(0+')}{dt}\right)$$

$$\left(\frac{di_L(0+')}{dt}\right) = \frac{v_L(0+')}{L} = \frac{120}{3} = 40 \text{ A/s}$$

Units: Ampere per second.

dvC(0-)/dt :

$$i_C(0+') = C \cdot \left(\frac{dv_C(0+')}{dt}\right)$$

$$\frac{dv_C(0+')}{dt} = \frac{i_C(0+')}{C} = \frac{4}{\left(\frac{1}{27}\right)} = 108 \quad V/s$$

diR(0+)/dt:

Lets write a current node equation at the left side node,
the source 4u(t=0') = 0A.

$$i_{4ut} + (-i_L) + (-i_R) = 0$$
$$4 - i_L - i_R = 0$$

Convention into node +ve, out of node -ve.

Differentiate the above equation:

$$0 - \left(\frac{di_L}{dt}\right) - \left(\frac{di_R}{dt}\right) = 0$$

$$\left(\frac{di_R(0+')}{dt}\right) = -\left(\frac{di_L(0+')}{dt}\right)$$

$$\frac{di_R(0+')}{dt} = -40 \quad A/s.$$

diC(0-)/dt:

Lets write a current node equation at the right side node,
the current source supplies 5A. iR and 5A into node, iC out of node.

$$i_{5A} - i_C + i_R = 0$$

Differentiate the above equation:

$$0 - \left(\frac{di_C(0+')}{dt}\right) + \left(\frac{di_R(0+')}{dt}\right) = 0$$

$$\left(\frac{di_C(0+')}{dt}\right) = \left(\frac{di_R(0+')}{dt}\right)$$

$$\left(\frac{di_C(0+')}{dt}\right) = -40 \quad A/s$$

dvR(0-)/dt :

$v_R(0+') = i_R(0+') \cdot R$ Next take the 1st derivative.
R a constant term.

$$\frac{dv_R(0+')}{dt} = R \cdot \left(\frac{di_R(0+')}{dt}\right) = 30 \cdot (-40) = -1200 \quad V/s.$$

dvL(0+)/dt:

We form a loop equation for the central mesh.

$v_R(0+') + v_C(0+') + v_L(0+') = 0$ Next the derivative of the equation.

$$\frac{dv_R(0+')}{dt} + \frac{dv_C(0+')}{dt} + \frac{dv_L(0+')}{dt} = 0$$

$$-1200 + 108 + \frac{dv_L(0+')}{dt} = 0$$

$$\frac{dv_L(0+')}{dt} = 1200 - 108 = -1092 \quad V/s$$

We solved the problem to the textbook answers in this theory section for t=0+. Textbook did not do the calculations for t=0- because all result were zero.

Recap derivatives values for t = 0+ :

$$\frac{dv_R(0+')}{dt} = 1200 \quad V/s \qquad \frac{di_R(0+')}{dt} = -40 \quad A/s$$

$$\frac{dv_L(0+')}{dt} = -1092 \quad V/s \qquad \frac{di_L(0+')}{dt} = 40 \quad A/s$$

$$\frac{dv_C(0+')}{dt} = 108 \quad V/s \qquad \frac{di_C(0+')}{dt} = -40 \quad A/s$$

That solved it for this problem in regards to the solutions. Surprising we could calculate derivative values for voltages and currents. Not the usual resistive calculation.

One key thing in the solution completed was 'each capacitor voltage and each inductor current must remain constant during the switching interval'.

This being from t = 0- to t = 0+.

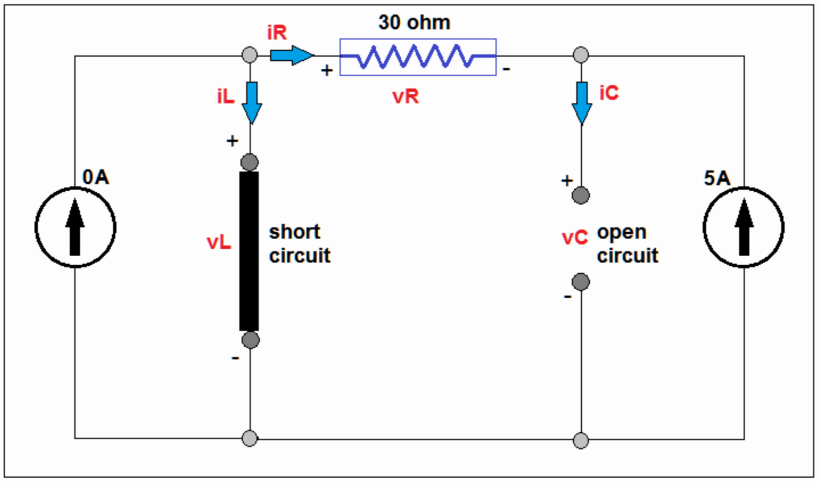
This was our focus in the early part of the problem.

There is another method to solve this problem.

It must be 'studied-worked' because it has applications in circuit problem solving.

Steps for alternate solution:

Below is the circuit for when t = 0-. Here the inductor is short circuited, and the capacitor is open circuited.



The three voltages and currents can be found by resistive circuit analysis.

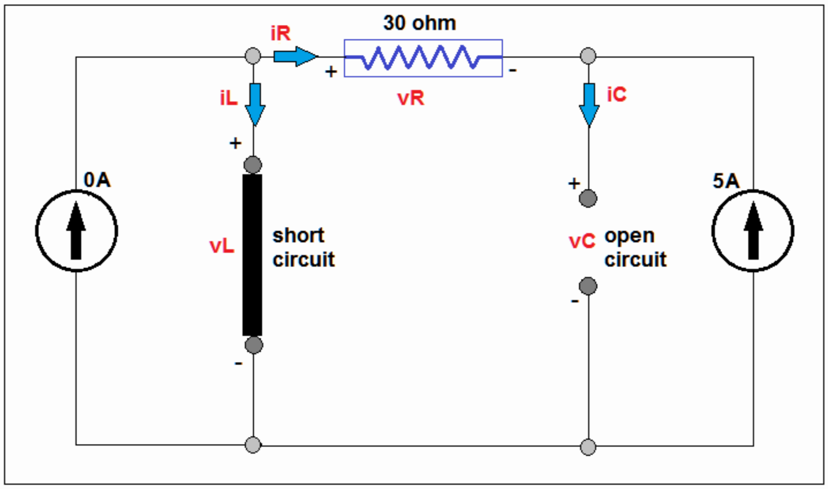
Resistive because we do not have capacitor and inductor by their defining expression rather short and open circuit conditions.

Next we can upgrade our circuit with the following, we are still in dc conditions with 4u(t) OFF. In the t = 0-.

1). Inductor is shorted, this means there is low voltage across it, if not near zero, and it only allows current to flow thru. From being an inductor it now may be subsituted for a current source.

2). The capacitor is open circuit, its not allowing current to pass. If its not allowing current to pass we may say its developed a high resistance. The high resistance can be modelled into a voltage source, because if its open circuited it may not have appreciable current flow but some small micro level or mili level amps maybe passing thru. Call it imperfections of the capacitor.
So we say V=IR, R is so huge and I small, it results in a voltage, so we say its a voltage source.

Upgraded circuit provided below.

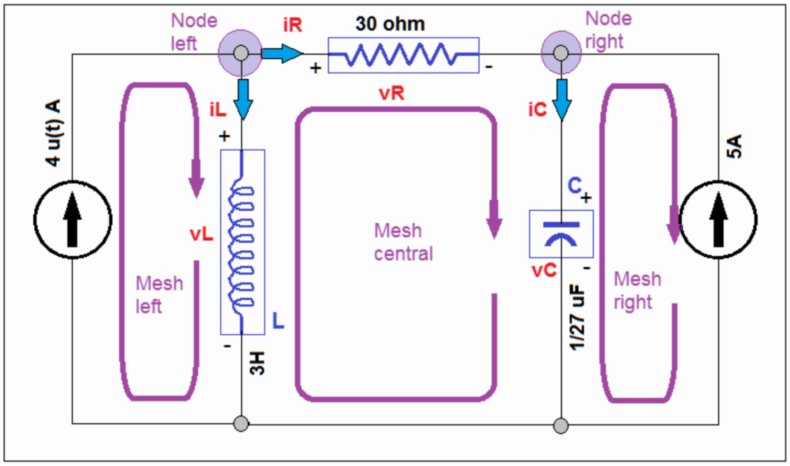


We show current source supplying 5A, this came from the iL(0-) calculation, and voltage source is 150V, this came from the vC(0-) calculation.

Emphasised again, we fixed L and C to the main solving step, each capacitor voltage and each inductor current must remain constant during the switching interval.

Use the circuit analysis methods for the circuit above for t=0-, and apply 4u(t) turning ON at t=0+.

Before we leave, the method we would had first attempted, but were deviated away is shown in the figure below - mesh analysis. That would been good for a complicated circuit, instead we used some new simple methods which were enlightening. Next page has the typical approach which we bypassed.



Three loops and two nodal equations.

Using the methods for t = 0 - first then progressing into t = 0 +.

We apply the initial condition for L and C in t = 0-. The results maybe produce some new/additional values along the way of the solution during the mesh analysis, but the final values will be the same at t= 0 +.

Next we continue with our solution for the response vC(t) for the original circuit. With both sources dead 4u(t) and 5A. The circuit is a series RLC circuit. Where we have only the central mesh. Yes.

$$v_C(t) \quad = \quad V_0 + A1 \cdot e^{s1t} + A2 \cdot e^{s2t}$$

$$s_1 = -\left(\frac{R}{2L}\right) + \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)} = -\alpha + \beta$$

$$s_2 = -\left(\frac{R}{2L}\right) - \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)} = -\alpha - \beta$$

Where

$$\alpha = \left(\frac{R}{2L}\right)$$

$$\beta = \sqrt{\alpha^2 - (\omega_0)^2}$$

$$\omega_0 = \left(\frac{1}{\sqrt{LC}}\right)$$

Series RLC circuit expression for calculating s1 and s2.

$R := 30 \qquad L := 3 \qquad C := \left(\frac{1}{27}\right)$

$$\alpha := \left(\frac{R}{2 \cdot L}\right) = 5$$

$$\omega_0 := \left(\frac{1}{\sqrt{L \cdot C}}\right) = 3$$

$s1 := -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s1 = -1$

$s2 := -\alpha - \sqrt{\alpha^2 - \omega_0^2} \qquad s2 = -9$ *Both as the textbook has for s1 and s2.*

We calculated vC(0-) = 150V, therefore the initial forced response at t=0- is 150V, this is when the 5A supplies current to the circuit. Substitute s1 and s2 into the expression for vC(t):

RLC circuit capacitor --> is the voltage source $$v_C(t) = 150 + A1 \cdot e^{-t} + A2 \cdot e^{-9t}$$

$$v_C(0-') = v_C(0+')$$ Continuity condition.

For the longest time t<<0 the vC(0-) = vC(0+) = 150V.

Therefore at t=0 :

$$v_{C\_0\_plus}(t) = 150 + A1 + A2$$

$$150 = 150 + A1 + A2$$

$$0 = A1 + A2$$ Equation 1

We take the derivative of vC(t) for creating the next equation, remember we already solved for the derivative values of capacitor voltage previously.

$$\frac{dv_C(t)}{dt} = -A1 \cdot e^{-t} - 9 \cdot A2 \cdot e^{-9t}$$

$$\frac{dv_C(t)}{dt} = 108 \quad V/s$$ <--- derivative value of capacitor voltage

$$108 = -A1 \cdot e^{-t} - 9 \cdot A2 \cdot e^{-9t}$$

At t=0+

$$108 = -A1 - 9\ A2$$ Equation 2

So you heard it from your innner mind plug in for t=0!

Simultaneously solving for equation 1 and 2:
OR using matrix.

$$Coefs := \begin{bmatrix} 1 & 1 \\ -1 & -9 \end{bmatrix} \quad InvCoefs := Coefs^{-1} = \begin{bmatrix} 1.13 & 0.13 \\ -0.13 & -0.13 \end{bmatrix} \quad RHS := \begin{bmatrix} 0 \\ 108 \end{bmatrix}$$

$$A1\_A2 := InvCoefs \cdot RHS = \begin{bmatrix} 13.5 \\ -13.5 \end{bmatrix} \quad A1 := 13.5 \quad A2 := -13.5$$

Complete response of vC(t):

$$v_C(t) := 150 + 13.5 \cdot e^{-t} - 13.5\ e^{-9t}$$

$$v_C(t) := 150 + 13.5\left(e^{-t} - e^{-9t}\right)$$ Answer.

In the RLC circuit capacitor is the voltage source. With the solution above our circuit's voltage source under condition source free was achieved.

The theory example had a final answer, provided above.
Next on pages 218 and 219 is a summary of some of the steps taken.

Summary:

1). Decide if its a series or parallel RLC circuit.
2). This leads to selecting the correct alpha.
3). Next calculate omega0.
4). We have alpha and omega0.
Next meet one of the 3 conditions:
5). alpha > omega0 — circuit is overdamped
solve for s1 and s2
natural response fn(t) = A1e^(s1t) + A2e^(s2t)
6). alpha = omega0 — circuit is critically damped
solve for s1 and s2
natural response fn(t) = e^(-alpha)t (A1t + A2)
7). alpha < omega0 — circuit is underdamped
solve for s1 and s2
natural response is:
fn(t) = e^(-alpha)t **(** A1(cos (w*d*)t + A2(sin (w*d*)t **)**
where (w*d*) = sqrt(w0^2 - alpha^2)
8). Last decision: If there are no independent sources acting in circuit after switching or discontinuity is completed, then the circuit is source-free and the natural response comprises the complete response. IF independent sources are still present then the circuit driven by a forced response must be determined. The complete response is then f(t) = f_f(t) +f_n(t).

In short when no source present during the analysis time t, source free and the circuit is in its own natural voltage supply from vC, this results in the natural response, OTHERWISE with a source present then the circuit has the additional forced response to be found and the complete response is a sum of forced and natural responses.

Next introduction to complex frequency. This will bring this part to end. Next part will pick up with continuing topics in complex frequency and frequency response in some what detail. From Hyat and Kemerly 4th Ed. After which we pick back-up with Schaums.

Complex Frequency.
Chapter 13 Hyat Kemmerly 4th Ed.
Critical Topic *(For most Electrical Engineers this is simple, not here especially not for me).*
STUDY NOTES.

Comments:
Here I/We keep the wording and explanation to the minimum as possible.
I seen so many expressions I should be able to make sense of it, or leat on the 2nd try.
Having to read excessive lietrature in this topic can cause many to drop the matter at hand, its tiresome, demanding, rewards are far of in the future,... By now some past experience in a math or engineering course may remind us of the steps and explanation.
This is my aim but I may fail and end up needing to have some explanation so coming back to it in the future I do not have to search for answers or grapple with the content.

13.1 Introduction:

1.---> $v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta)$ <---Exponentially damped sinusoidal function, previous pages/chapter we know why we have exponent term.

σ <--- in expression above is negative usually.
if its positive the amplitude may increase, we know how exponentials can be, times difficult to visualise, so better to plot the functions.

Let sigma and omega be zero then we get a constant voltage:

$$v(t) = V_m \cdot \cos(\theta)$$

2.---> $v(t) = V_0$ and when theta =0 deg, voltage is Vo for t=0 initial voltage.

But if we let sigma only equal 0 we get the general sinusoidal voltage, the one we usually see:

3.---> $v(t) = V_m \cdot \cos(\omega t + \theta)$ $\cos(0\ \mathbf{deg}) = 1$

And if we let omega only equal 0 we get the exponential voltage:

$$v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\theta)$$

4.---> $v(t) = V_0 \cdot e^{\sigma t}$ ...when theta =0 deg, cos(0 deg) = 1, Vm cos (theta) = Vo..., Why Vo? Vm for sinusoidal maximum, so Vo for exponential's constant term.

We have number 1 the damped sinusoid and in it includes the special cases number 2 dc (constant), number 3 sinusoidal (general expression) and number 4 exponential function.

Between 3 and 4 above we can see a relationship
when the phase angle (theta) equal zero.

$$v(t) = V_m \cdot \cos(\omega t + \theta) \quad \text{----->} \quad v(t) = V_m \cdot \cos(\omega t)$$

5. $v(t) = V_m \cdot \cos(\omega t) = V_0 \cdot e^{j \omega t}$

Compare 4 and 5:

4. $v(t) = V_0 \cdot e^{\sigma t}$

5. $v(t) = V_0 \cdot e^{j \omega t}$

σt and $j \omega t$ <---we see this difference and some similarity.

Exponent in 4 is real while in 5 its imaginary.

Similarity gets improved when we make sigma be frequency.
Omega = 2 Pi f.
Now we have sigma = frequency.

We can write like this maybe you see it better : $e^{\sigma t} \cdot e^{jwt} = e^{(\sigma + j \omega) t}$

Yes we can write it like this, deviating from the expression above to show the mathematical side of the exponent.

This allows us to assign one real part and the other imaginary.

σ Real fequency. And f the typical frequency is also real. So we name sigma 'neper frequency'.

ω Imaginary frequency

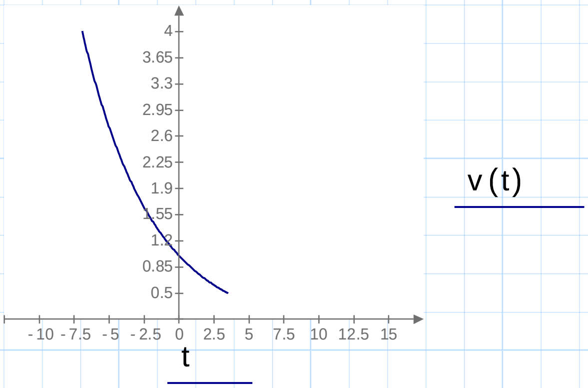
Neper is the dimensionless unit of the exponent of e.
Example e^7t, is '7 neper frequency'. t is in seconds but we've made it dimensionless. Its frequency we have the per second as well. e^7t is 7 nepers per second. Omega is 2 Pi f where f is frequeny per second. Now say dimensions are the same so sigma the real and jw the imaginary parts.

$s = \sigma + j \omega$ ---> It is typical to call the real part σ

and the imaginary part ω
not jw just w.

How is the plot for sigma<0, and we set Vo=1.

$V_0 := 1$ $\sigma := -0.2$ $v(t) := V_0 \cdot e^{\sigma \cdot t}$



We have a plot here with sigma = -0.2.

When sigma<0, the amplitude of the forcing function of source (v or i) can have very large values in the distant past (t = 0-). We see this in the plot in the -t direction much higher than Vo=1 value.

"In figure above, v(t) in time -t or - infinity the forcing function can go so high dependent on expression, but initial conditions can place a restriction on this. With initial conditions specified the application of the forcing function v(t) at a specified instance of time produces thereafter a response identical to a forced response without any transient response - *examples of this appear later*. Its said its near impossible in the lab to generate damped sinusoidal or exponentail forcing functions accurate for all time. In the lab we may produce approximations for circuits whose transient response do not last very long." - *I maybe should had kept away from this but this is explained further in coming section.*

13.2 Complex frequency:

A function is written in the form: $f(t) = K \cdot e^{st}$

K and s are complex constants, independent of time, their behaviour defined by the complex frequency s. Basically the complex frequency is just the factor that multiplies t, in this complex exponential form.

This form above is necessary until we are able to determine the complex frequency of a function by inspection. In other words write it in this form first.

Lets work on this form on a familiar forcing function $v(t) = V_0$

Rewrite as: $v(t) = V_0 \cdot e^{(0)\,t}$ Why zero in exponent? It would cause the power of exponent 0 when s=0, to st =0 makes for time t=0.

Complex frequency of a DC current or voltage is thus zero since s=0.

Next form here ---> $v(t) = V_0 \cdot e^{st}$

To here ---> $v(t) = V_0 e^{\sigma t}$ <---Only got sigma no jw.

We know the Re + Im parts are: $\sigma + j\ \omega$

So make jw = 0, hence s and sigma take the same place in the expression.

We have the Re part only: $\sigma + j0 = \sigma$

Now both of same standing in terms of expression: $V_0 e^{\sigma t}$ and $V_0 e^{st}$

Now we apply v(t) in the sinusoidal form: $v(t) = V_m \cdot \cos(\omega t + \theta)$

We want to work on the sinusoidal into the complex exponential.

Euler expression for cosine:

$$V_m \cos(\omega t + \theta) = \left(\frac{1}{2}\right) \cdot \left(e^{j\theta} + e^{-j\theta}\right) \qquad j := \sqrt{-1}$$

Applying it on sinusoidal expression:

$$v(t) = V_m \cdot \cos(\omega t + \theta) = V_m \left(\frac{1}{2}\right) \cdot \left(e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}\right)$$

$$= \left(\frac{V_m}{2} e^{j\theta}\right) \cdot e^{j\omega t} + \left(\frac{V_m}{2} e^{-j\theta}\right) \cdot e^{-j(\omega t)}$$

$$K_1 = \left(\frac{V_m}{2} e^{j\theta}\right) \quad K_2 = \left(\frac{V_m}{2} e^{-j\theta}\right) \quad s_1 = e^{j\omega t} = e^{j(2\pi f)1t} \quad s_2 = e^{-j\omega t} = e^{-j(2\pi f)2t}$$

$$v(t) = K_1 e^{s1t} + K_2 e^{s2t}$$

Observations on v(t) exponential expression above:

1). Sum of 2 complex exponentials which give 2 complex frequencies.
2). Complex frequency of 1st term s = s1 = jw = j(2 Pi f1) and the
2nd term s = s2 = -jw = -j(2 Pi f2).
3). Values of s are conjugates; s2 = s1*.
4). Two values of K are also conjugates, K1 = (Vm/2)(e^j(theta), and so
K2 = K1* = (Vm/2)(e^-j(theta).
5). Entire 1st and 2nd terms are conjugates, their sum should be a real quantity.

Previous pages we had four forms of voltage expression. We want to look at number 1 the sinusoidal expression, provided again below.

1. $v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta)$ <---Exponentially damped sinusoidal function.

$$v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta) = \left(\frac{V_m}{2}\right) \cdot e^{\sigma t} \cdot \left(e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}\right)$$

$$v(t) = \left(\frac{V_m}{2} e^{j\theta}\right) \cdot e^{(\sigma + j\omega) t} + \left(\frac{V_m}{2} e^{-j\theta}\right) \cdot e^{(\sigma - j\omega) t}$$

Now observations on v(t) exponential expression above:

1). Sum of 2 complex exponentials which give 2 complex frequencies.
2). Complex frequency of 1st term s = s1 = sigma + jw, and the
2nd term s = s2 = sigma - jw. (*w = 2 Pi f in s1 and s2*)
3). Values of s are conjugates; s2 = s1*.
4). Two values of K are also conjugates, K1 = (Vm/2)(e^j(theta), and so
K2 = K1* = (Vm/2)(e^-j(theta).
5). Entire 1st and 2nd terms are conjugates, their sum should be a real quantity.

We therefore find once again that a conjugate complex pair of frequencies is required to describe the exponentially damped sinusoid,
s1 = sigma + jw, and s2 = s2* = sigma - jw.

Returning to page 1 and 2 we said of the exponentially damped sinusoidal function:

1.---> $v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta)$ *<---Exponentially damped sinusoidal function*

Said this: We have number 1 the damped sinusoid and in it includes the special cases number 2 dc (constant), number 3 sinusoidal (general expression) and number 4 exponential function.

NOW, In general, neither sigma nor omega is zero. We see that the exponentially varying sinusoidal waveform is the general case, the constant sinusoidal waveform the special case, and exponential waveform the special case.

$v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta)$ <---Exponentially sinusoidal - GENERAL CASE.

$v(t) = V_0$ <--- Constant - SPECIAL CASE.

$v(t) = V_0 \cdot e^{\sigma t}$ <--- Exponential - SPECIAL CASE.

Comment: *So much of electrical engineering is math, its math governing or moulding the phnomenon rather than engineering exploiting the phenomenon - Joe Stein.*

Exercise:

Identify the complex frequencies associated with the functions below.

1. $v(t) = 0$

Function above comes across as a constant. It would not have a complex frequency.
Complex frequency is s1 and s2.

$s = 0$ Answer.

2. $v(t) = 5\,e^{-2t}$

Function above is exponential.

4.---> $v(t) = V_0 \cdot e^{\sigma t}$

$s = \sigma + j\,\omega$

$= -2 + 0$

$s = -2 + j0$ Answer.

3. $v(t) = 2\,\sin 500t$

Function above is sinusoidal.

3.---> $v(t) = V_m \cdot \sin(\omega t + \theta)$ <---Sin term is imaginary, and cos term is real.

$\omega = 500$ Because its sin its j? Both jw. *Not a problem.*

$j\,\omega = j500$

$s1 = 0 + j500$

$s1 = j500$ Answer.

$s2 = s1^{conj} = 0 + \_negate\_j \cdot 500 = 0 - j \cdot 500$

$s2 = -j \cdot 500$ Answer. Just change sign on jw - conjugate.

4. $v(t) = 4\,e^{-3t}\,\sin(6\,t + 10\,°)$

Function above is exponential sinusoidal.

1.---> $v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta)$ <---Exponentially sinusoidal function.

Here we got two sources of s. Each term provides a source.
Thru inspection we know the sinusoidal term is providing omega.
Hence, the other exponential term provide sigma. And s = sigma + jw. *Helps.*

Sinusoidal:

$\sin(6\,t + 10\,°)$ 10 deg is the phase angle.

$\omega = 6$ $j\,\omega = j6$

$s1 = 0 + j6$

$s1 = j6$

$s2 = s1^{conj} = -(j6) = 0 - j6$

$s2 = -j6$

Exponential:

$4\,e^{-3t}$

$s = \sigma + j\,\omega$

$s = -3 + 0$

Combine both for s1 = sigma + jw, and s2 = sigma - jw

$s1 = -3 + j6$ Answer.

$s2 = -3 + (-j6) = -3 - j6$ Answer.

Discussion:
Go over that exercise again and refer when needed.
The engineers belief we should get it the first sight. WRONG.
Some can not all. *I cant most cannot.*
If the sinusoidal was a cosine term? That makes the sinusoidal term of the last exercise what? Real. And would the exponential term not provide a real part too? Obviously the waveform would not work with two real terms, they may be added?
WRONG. *Its not a math exercise* its recognising the expression for s1 and s2.
If its cosine the form it takes is jw if its sine it takes jw. *Not a problem.*

We have one example below from Schaums for cosine.

$v(t) \quad = \quad 2 \cdot e^{-5t} \cdot \cos(2t - 120°)$

Sinusoidal:
$\cos(2t - 120°)$ 120 deg is the phase angle.
$\omega = 2 \qquad \omega = j2$
$s1 = 0 + j2$
$s1 = j2$
$s2 = s1^{conj} = 0 - ('+j2) = 0 - j2$
$s2 = -j2$

Exponential:
$2e^{-5t}$
$s = \sigma + j\omega$
$s = -5 + 0$

Combine both for s1 = sigma + jw, and s2 = sigma - jw
$s1 = -5 + j2$ Answer.
$s2 = -5 - (j2) = -5 - (-j2) = -5 + j2$ Same as s1? WRONG.
$s2 = -5 - j2$ Answer. It must be conugate to s1, -5+j2, drop the math on it work to the identity of conjugate; -jw-->jw, jw--->-jw.

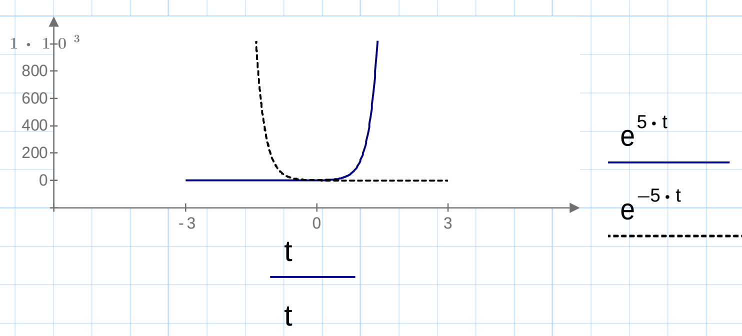
Next the hard part going in REVERSE.
Construct the function given s1 and s2.
We start by resorting to the form: $K \cdot e^{st} = K \cdot e^{(\sigma + jw)t}$

Example1: $s = 5 + j0$
$s = \sigma + j\omega$

Real part: 5
Im part w: 0

$f(t) = K \cdot e^{(5 + j0)t}$

$f(t) = K \cdot e^{5t}$



This is all the information we have. K? Well if K is real its a real physical system. In electrical it need not be real, it can be imaginary. With exponent +5 the curve will rise upward (blue) and -5 it drops to 0 (black).
Exp +ve rising and Exp -ve decreasing.

We start by resorting to the form: $K \cdot e^{st} = K \cdot e^{(\sigma + jw)\,t}$

Example2: $s = j10$

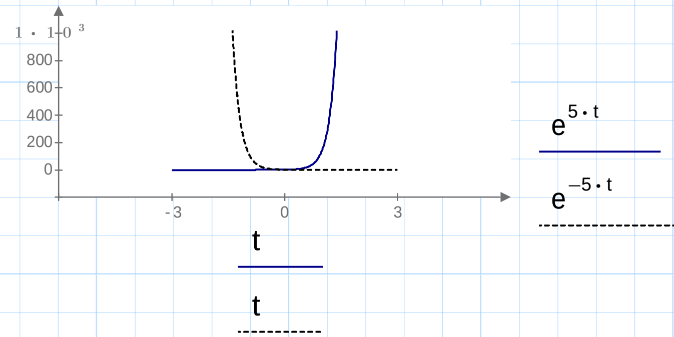
$s = \sigma + j\ \omega$

Real part: 0

IM part w: 10

$f(t) = K \cdot e^{(0 + j10)\,t}$

$f(t) = K \cdot e^{j10\,t}$



Here we have only an imaginary value j10.
Continued after discussion.

Discussion:

How do you plot an exponential term with the power of j?
Can this be plotted e^(sqrt(-1)*10t) ? No but the y-axis can be made Im. So by plotting e^(10t) may imply the - j axis. Each value is multiplied to j (sqrt (-1)).
Is the value of e^j10t same as e^10t .....then just label the y-axis Im?
Because we have a Re and Im component which make up the vector.

R = r (cos(theta) + j sin(theta)).

$R = r \cdot \cos(\theta) + j \cdot r \cdot \sin(\theta)$

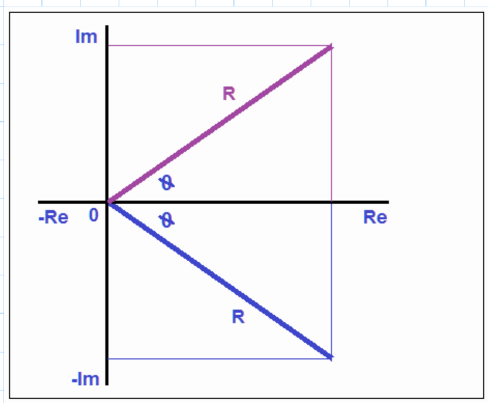
$x = r \cdot \cos(\theta)$

$y = j \cdot r \cdot \sin(\theta)$ <--- see 'j' here as a necessary unit.

$R = \sqrt{x^2 + y^2}$

$y = j \cdot r \cdot \sin(\theta)$:

y can be +ve and -ve located on +Im and -Im axis;
r sin(theta) = +ve 1st quadrant
-r sin(theta) = -ve 4th quadrant
and we do have to apply j^2 as we see
coming... causes y-axis value to change.



$R = \sqrt{\left(r^2 \cdot \cos(\theta)^2\right) + j^2 \cdot \left(r^2 \cdot \sin(\theta)^2\right)}$

Above j^2 results in -ve ---> $\left(\sqrt{-1}\right) \cdot \left(\sqrt{-1}\right) = -1$

$R = \sqrt{\left(r^2 \cdot \cos(\theta)^2\right) - \left(r^2 \cdot \sin(\theta)^2\right)}$

$R^2 = r^2 \left(\cos(\theta)^2 - \sin(\theta)^2\right)$

Problem or lucky its negative or if it were +ve then the expresion is R=r!!!!

Correct me, if we had no -ve sign or say no j^2 which caused
the -sign then we would get:

$$R = \sqrt{\left(r^2 \cdot \cos(\theta)^2\right) + \left(r^2 \cdot \sin(\theta)^2\right)}$$

$$R^2 = r^2 \left(\cos(\theta)^2 + \sin(\theta)^2\right)$$

$R^2 = r^2 \ (1) \ = r^2$ This isnt correct in the sense we have angle theta.
Angle theta is the same angle for the cos and sin terms.

So now we have dependent on where the angle theta is for the sine term.

$r \cdot \sin(\theta) = -y$ <--- this case
$r \cdot \cos(\theta) = x$ <--- this case

$$R = \sqrt{\left(r^2 \cdot \cos(\theta)^2\right) + j^2 \cdot \left(r^2 \cdot \sin(\theta)^2\right)}$$

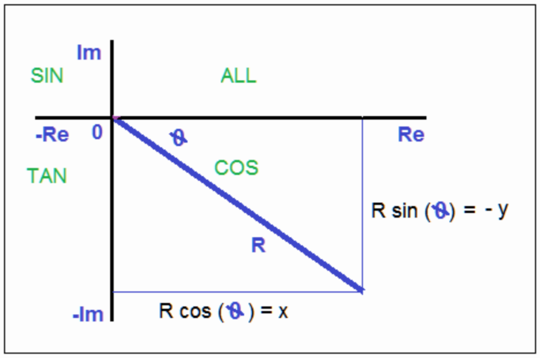
$$R = \sqrt{x^2 - \left(j^2\right)(-y)^2}$$

$$R = \sqrt{x_{value} - (-1)\ (y)^2}$$

$$R = \sqrt{x_{value} - (-1)\ y_{value}}$$

$$R = \sqrt{x_{value} + y_{value}}$$

Lets confine to sine term, here the sine term can be +ve and -ve dependent on which quadrant it is in. Sine is +ve in 1st and 2nd, and negative in 3rd and 4th quadrant.



Now we see either case +ve or -ve for y, we shown turns out postive we worked on -y above. Were calculating the vector distance which would be? Positive in any quadrant. So the angle gives the location respective to quadrant. *I was WRONG. Is not the 1st time.* Hope that took some hidden conspiracy or suspicion out. There was a reaosn why I deviated here, coming next from the author engineers.---

Continuing with textbook. *This part is implicit tricky want to take the mystery out of K1 and K2.*

A purely imaginary value of s for example j 10, our example, can never be associated with a real quantity; the function form Ke^(j10)t, can also be writen as K(cos 10t + j sin 10t), and obviously this possesses both real and imaginary parts.

$$Ke^{j10t} = K(\cos(10\ t) + j \cdot \sin(10\ t))$$

To form a real function we need to have in this case conjugate values of s, s1 = 10j and s2 = -10j. And both terms have j, yes. Usually thats the case. x1 = 2+3j, conjugate x2 = 2-3j. Correct. And same for K, it has conjugate pair values K1 and K2.

We can identify either s1=j10 or s2=-j10 with a sinusoidal voltage at the radian frequency or 10 rad/s (*w* = 10 rad/s). Remember its j*w* so w is the 10. CORRECT.

Can we guess what K could be given only *w*? NO.
I say no, they will surprise me. Reason I said no was, seems like *I am searching for more than one variable and only got something like 1 equation*. *Rare Moment* - ***Joke.***
s1 given and K1 was given its lets make up K1, has to come from somewhere.

Lets make $K1 = 6 - j \cdot 8$ Lets work the magnitude of K1 next.

$K1 = \sqrt{6^2 - j^2 \cdot 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$ $K1 = 10$

(Negative sign in the square root expression). Where is the j? $j := \sqrt{-1}$ $j^2 = -1$

$K1 = \sqrt{6^2 - j^2 \cdot 8^2}$ $-j^2 = 1$

$= -j\sqrt{6^2 + 8^2}$

$= -j\sqrt{6^2 + 8^2}$

$= -j\sqrt{36 + 64}$

$K1 = -j10$

The -j out side the expression is not multiplying both terms 6 and 8, because these now, 6 and 8, are both real. ***Hello? Wrong!*** *Why not after all j is imaginary - it was a make fit to begin in math. So if we got a K1 real we can show a K2 is likewise real and of the same value.*

You may **disagree** there be no -ve sign there, could be your mathematics? But is it a problem? Because its 2 constant magnitudes K1 and K2, with j10 and -j10.

$-j\sqrt{6^2 + 8^2}$ ---> Put j back in but both terms $\sqrt{-j^2 \cdot 6^2 - j^2 \cdot 8^2} = \sqrt{36 + 64}$ $R = 10$
Ok maybe its wrong, the magnitude of K1 = 10 calculated usual way, with j removed.
Got sorted mystery not in K1 or K2, their magnitude is? the same, solved it. Its 10.

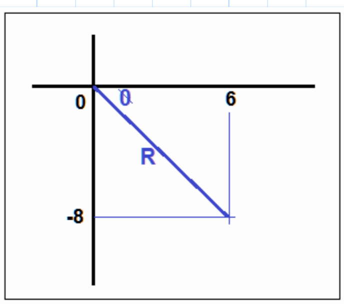
$R = \sqrt{6^2 + (-8)^2} = 10$

Discussion wise maybe half a case there, and why not they end up with the same magnitude, *so what if multiplied 6 by -j^2*
- I wouldnt call that an engineering joke. BAD JOKE. ...but?....

What about the phase angle, we know from exercises/examples from Part 2 we can get the phase angle.
Sketch to the right says where we find the angle.

$\tan(\theta) = \frac{-8}{6}$ $\theta = \operatorname{atan}\left(\frac{-4}{3}\right)$

$\theta = \operatorname{atan}\left(\frac{-4}{3}\right) = -53.13$ **deg**



<---Vector angle theta, and R is used to represent magnitude K1 or K2 or K. We know K1=K2=K=10.

Negative sign for direction of angle theta in 4th quadrant.

$$v(t) = K_1 e^{s1t} + K_2 e^{s2t}$$

$$s_2 = s_1^{conj}$$

$$K_2 = K_1^{conj}$$

The expression for the voltage v(t) is:

$$v(t) = 20\cos(10t - 53.1°)$$

I said where did the 20 come from? K = 10---> $v(t) = 10\cos(10t - 53.1°)$
The -53.1 was sorted.
10t was jw where w=10, that was sorted.
Except the 20?

Go back 6 or 7 pages, the expression's note number 5.

$$v(t) = \left(\frac{V_m}{2} e^{j\theta}\right) \cdot e^{(\sigma + j\omega)t} + \left(\frac{V_m}{2} e^{-j\theta}\right) \cdot e^{(\sigma - j\omega)t}$$

Now observations on v(t) exponential expression above:

5). Entire 1st and 2nd terms are conjugates, their sum should be a real quantity.

$$v(t) = 2\left(\frac{V_m}{2} e^{j\theta}\right) \cdot \left(e^{(\sigma + j\omega)t} + e^{(\sigma - j\omega)t}\right)$$

So we left it half the amplitude on each side, which we now show multiplied by 2.
So our -j10 or j10 is 10 for the half amplitude and times 2 is 20.
Or add 10 + 10 = 20.

$$\frac{V_m}{2} = 10 \qquad 2 \cdot \left(\frac{V_m}{2}\right) = 2 \cdot (10) = 20$$

So now our voltage expression is something what the engineer have.

$$v(t) = 20\cos(10t - 53.1°)$$ <--- Real sinusoid.

'In a similar manner, a general value for s such as 3 - j5, can be associated with a real quantity only if it is accompanied by its conjugate 3 + j5.'

$K_1 = 6 - j8$
$K_2 = K_1^{conj}$
$K_2 = 6 + j8$
$K = 10$

You agree we started with 6-j8, we could had made it 3-4j, this results in magnitude 5, then v(t) = 10 cos(5t - 53.1). We know how to relate or show 6 - j8 has a conjugate but the K1 and K2 magnitude are the same, angle in opposite directions, that makes the other coordinate 6 + j8, but their magnitude is? the same.

We seen 2 forms of the solution below:

$$f(t) = K \cdot e^{j10\,t}$$

$$v(t) = 20\cos(10\,t - 53.1\,°)$$

We have one more form we can reveal. This what the engineers said:

Speaking loosely again, we may think of either of these sinusoidal two conjugate frequencies as describing an exponentially increasing sinusoidal function:

$$v(t) = e^{st}\cos(wt) \qquad s = \sigma + j\,\omega$$

$$v(t) = e^{6\,t}\cos(8\,t) \qquad s_1 = 6 + j8 \qquad s_2 = 6 - j8$$

v(t) for both s1 and s2 conjugate terms.

A general value for s such as 6 - j8, can be associated with a real quantity only if it is accompanied by its conjugate 6 + j8.' Shown below how.

$$s^2 = s_1 \cdot s_2$$

$$= (6 + j8)\cdot(6 - j8)$$

$$= 36 - j48 + j48 - j^2\,64$$

$$= 36 - j^2\,64$$

$$s^2 = 6^2 - j^2\,8^2$$

$$s = 6 - j\,8$$

<--- s for s1 and s2, plus we know likewise magnitude of s is real. Hello!

$$v(t) = e^{3\,t}\cos(5\,t) \qquad s_1 = 3 + j5 \qquad s_2 = 3 - j5$$

We did a little exercise before completing sentence above, so now we complete it.

Speaking loosely again, we may think of either of these sinusoidal two conjugate frequencies as describing an exponentially increasing sinusoidal function e^(6t) cos(8t), the specific amplitude and phase angle will again depend on the specific values of the conjugate complex K's.

General Points:

$v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta)$ <---Exponentially sinusoidal

1. The sigma the real part of s is associated to the exponential variation.

2. Sigma is negative the function decays as t increases.

3. Sigma is positive the function increases as t increases.

4. Sigma is zero the sinusoidal amplitude is constant.

5. Larger the magnitude of sigma, greater is the rate of increase or decay.

6. Omega *w* describes the sinusoidal variation; cos(*w* t + theta).

7. Omega *w* is radian frequency.

8. Large magnitude of w (omega) indicates a more rapidly changing function of time.

9. Larger magnitude of sigma, omega, or the magnitude of s (sigma + j omega) indicate a more rapidly changing function.

Complex frequency---> $s = \sigma + j\omega$

Specifically without confusion:

s is called complex frequency - complex nepers per second or
- complex radians per second

σ is neper frequency - nepers per second

ω is radian frequency - radians per second

f is cyclic frequency - cycles per second

Some of the points above we knew from the mathmatical behaviour of the exponent, here we are applying to electrical. That may have helped reduce difficulties in identifying their units.

Exercise below is more mathematical then electrical but when we apply the points from the previous page it should help build in mind the waveform.

Exercise:
Specify all the complex frequencies present in the time function:

a). $\left(2-3\,e^{-7\,t}\right)\cos(5\,t)$

b). $2-3\,e^{-7\,t}\cos(5\,t)$

c). $2\cos(5\,t)-5\sin(10\,t)\cos(2\,t)$

d). $3+2\cos(7\,t-30\,°)$

Solution:
We need to have some v(t) expression in mind so we can at least use that for starters. We have the 4 we went thru provided below.

1.---> $v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta)$ exponentially damped sinusoidal function
2.---> $v(t) = V_0$ constant voltage
3.---> $v(t) = V_m \cdot \cos(\omega t + \theta)$ general sinusoidal voltage
4.---> $v(t) = V_0 \cdot e^{\sigma t}$ exponential voltage:

a): $\left(2-3\,e^{-7\,t}\right)\cos(5\,t) = 2\cdot\cos(5\,t) - e^{-7\,t}\cos(5\,t)$

$2\cdot\cos(5\,t)$: $j\,\omega\,t = j5t$ $\quad j\,\omega = j5$ $\quad j\,\omega^{conj} = -j5$

$e^{-7\,t}\cos(5\,t)$: $\sigma = -7$ $\quad j\,\omega = j5$ $\quad j\,\omega^{conj} = -j5$

$s_1 = -7 + j5 \qquad s_2 = -7 - j5$

Complex frequencies: $j5 \quad -j5 \quad -7+j5 \quad -7-j5$ 1/s (per second)
Answer.

b). $2-3\,e^{-7\,t}\cos(5\,t)$

2 : 0 constant.

$3\,e^{-7\,t}\cos(5\,t)$: $\sigma = -7$ $\quad j\,\omega = j5$ $\quad j\,\omega^{conj} = -j5$
$s_1 = -7 + j5 \qquad s_2 = -7 - j5$
Complex frequencies: $0 \quad -7+j5 \quad -7-j5$ 1/s (per second) Answer.

c). $2\cos(5t) - 5\sin(10t)\cos(2t)$

$2\cos(5t)$: $j\omega = j5$ $\quad j\omega^{conj} = -j5$

$5\sin(10t)\cos(2t)$:
Use function-product trig identity:

$$\sin(\theta)\cos(\beta) = \left(\frac{1}{2}\right)\cdot\sin(\theta+\beta) + \left(\frac{1}{2}\right)\cdot\sin(\theta-\beta)$$

Lets look at it for sine we done it for cosine:
Euler expression for sine:

$$V_m\sin(\omega t+\theta) = \left(\frac{1}{2j}\right)\cdot\left(e^{j\theta} - e^{-j\theta}\right) \qquad j := \sqrt{-1}$$

Applying it on sinusoidal expression:

$$v(t) = V_m\cdot\sin(\omega t+\theta) = V_m\left(\frac{1}{2j}\right)\cdot\left(e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right)$$

$$= \left(\frac{V_m}{2j}e^{j\theta}\right)\cdot e^{j\omega t} - \left(\frac{V_m}{2j}e^{-j\theta}\right)\cdot e^{-j(\omega t)}$$

$$K_1 = \left(\frac{V_m}{2j}e^{j\theta}\right) \quad K_2 = \left(\frac{V_m}{2j}e^{j\theta}\right) \quad s_1 = e^{j\omega t} = e^{j(2\pi f)1t} \quad s_2 = e^{-j\omega t} = e^{-j(2\pi f)2t}$$

$$v(t) = K_1e^{s1t} - K_2e^{s2t}$$

Returning to our solution.

$$5\sin(10t)\cos(2t): \left(\frac{1}{2}\right)\cdot\sin(10t+2t) + \left(\frac{1}{2}\right)\cdot\sin(10t-2t)$$

$$\left(\frac{1}{2}\right)\cdot\sin(12t) + \left(\frac{1}{2}\right)\cdot\sin(8t)$$

$j\omega = j12$ $\quad j\omega^{conj} = -j12$

$j\omega = j8$ $\quad j\omega^{conj} = -j8$

Complex frequencies: $j5 \quad -j5 \quad j8 \quad -j8 \quad j12 \quad -j12$ 1/s (per second)
Answer.

d). $3+2\cos(7t-30°)$

3 : 0 constant.

$2\cos(7t-30°)$: $j\omega = j7 \quad j\omega^{conj} = -j7$

Complex frequencies: $0 \quad j7 \quad -j7$ 1/s (per second) Answer.

Next the lets say opposite side to the problem we got the complex frequencies and need to form the function expression, which can be v(t), i(t),.....

Exercise.

Write the general form of a real voltage having components at the complex frequencies:

a). -7, 5 (1/second).

We are given only two complex frequencies, which in our mind first thought, well truthfully in my mind was s = sigma + jw. Which maybe close.
It should been s1 and s2 were given, because these are complex frequencies.
s1 = sigma1 + jw1
s2 = sigma2 - jw2
So these two frequencies are the only two.
What could their v(t) look like?

$$v(t) = K_1e^{s1t} + K_2e^{s2t}$$

The values of K1 and K2 were not given which we can leave them as A and B or A1 and A2. *But 'MY' thinking on K1 and K2 were the same - maybe in sinusoidal expression.*

$s1 = -7$
$s2 = 5$
$v(t) = e^{-7t} + e^{5t}$ No magnitude infront of v(t) so we place A and B.

$v(t) = Ae^{-7t} + Be^{5t}$

b). 0, -4, -2 + j7 (1/second).

We are given -2 + j7 this works or fits directly in the exponentially varying sinusoidal equation. Leaving 0 and -4 to fit in another part of the expression; but its all one expression. *I had to look at the solution provided in the textbook.* Maybe not you go ahead. I apologise. My question was do we need a conjugate for -2 + j7.

We dont because, s is made up of s1 and s2, in the exponential sinusoidal form, here that s = -2 + j7. The other real values 0 and -4 could only fit in the same expression we had in part a, here they are s1 and s2.

$$v(t) = K_1 e^{s1t} + K_2 e^{s2t}$$

$$v(t) = Ae^{0\,t} + Be^{-4\,t}$$

$$v(t) = A + Be^{-4\,t}$$

In the prior exercise part b we had:

$$3\,e^{-7\,t}\cos(5\,t) \quad : \quad \sigma = -7 \qquad j\,\omega = j5 \qquad j\,\omega^{conj} = -j5$$

$$s_1 = -7 + j5 \qquad s_2 = -7 - j5$$

Here we have -2 + j7 to fit in an expression and no coefficients are given so we place some constant C.

$$v(t) = C\,e^{-2\,t}\cos(7\,t)$$

So now the complete expression: $v(t) = A + Be^{-4\,t} + C\,e^{-2\,t}\cos(7\,t)$

Do we insert the phase angle?
We can, it can be 0 deg or some theta degree(s).
Makes the answer more electrical.

$$v(t) = A + Be^{-4\,t} + C\,e^{-2\,t}\cos(7\,t + \theta)$$ Answer.

c). -3, -3 + j3, 3 - j3 (1/second).

-3 ? We see the K1e^st.
-3 + j3 ? We see the exponentially varying sinusoidal expression.
3 - 3j ? This looks like a conjugate to -3 + j3? Not.
Conjugate would be -3 - j3.
REMEMBER here this is not s1 and s2 making up s, rather s for one expression and another s for another expression.
We have 2 exponentially varying sinusoidal expressions.

$-3 \quad : \quad Ae^{-3\,t}$

$-3 + 3j \quad : \quad Be^{-3\,t}\cos(3\,t + \theta)$

$3 - 3j \quad : \quad Ce^{3\,t}\cos(3\,t + \phi)$

Discussion on 3 - 3j: NOT cos(-3t) or -ve sign infront of C? No, remember its w we seek not j, in jw. The negative sign can be on j, (-j)w. No coefficients were given. WRONG. Can be -3t. Answer provided as shown below. Math wise cos(-3t + phi) is good. If we plotted from -t to t + the y-axis values may be swapped -ve/+ve, may show same curve symmetrical on t=0. So Leave it cos(3t+phi).

$$v(t) = Ae^{-3\,t} + Be^{-3\,t}\cos(3\,t + \theta) + Ce^{3\,t}\cos(3\,t + \phi)$$ Answer.

Phasors, before we start with section 13.3.

Fortunately, we got some more theory. I call it theory because in my time that was how the student world associated it, theory and solving problems. Thoery may be a high level case for some and not fit some textbook explanations, but if you examine the KVL loops and the Nodal KCL equations they are more theory, if they were laws they obviously were common sense to begin with. What goes in comes out, sum of what goes in a node is the sum of what comes out. Law! Agreed. You knew you'll turn around.
We especially I want to get a little complacent on this, then have to grapple later with how to solve problems. We have a few simple example problems here in the theory.

PHASORS?.....because of the exponential term.

We are NOT using phasor here, but it may look like it. Review on phasors several equations presented Phasors were used more in 3 phase circuit analysis. Usually shown bold italic. Power Systems uses phasors. Just the equations and no explanation. We use cosine term for Re and sin for Im, keep it cosine term because we try to stay in the real.

We are looking at UNDAMPED sinusoid, not worked on by external elements, free.

Trig values of sin and cos:

$\sin(30\ \mathbf{deg}) = 0.5 \qquad \cos(90\ \mathbf{deg} - 30\ \mathbf{deg}) = 0.5 \qquad \cos(60\ \mathbf{deg}) = 0.5$

$\sin(\theta) = \cos(90 - \theta)$ <---*Gets sine term to cosine (real).*

$V_m = V_m e^{j0°} \qquad I_m = I_m e^{j\phi°}$

$v(t) = V_m \cos(\omega t + 0°)$ ----> $V_m \angle 0°$ <---Phasor.

$V_m e^{j(\omega t + 0°)}$ ----> $V_m \angle 0°$ ----> $V_m e^{j0°}$ <---Phasor form on degree only.

$v(t) = V_m \cos(\omega t + \phi°)$ ----> $V_m e^{j(\omega t)}$ <---Phasor (Always Cosine - Re).

$I_m \cos(\omega t + \theta)$ Sinusoidal form of current

$I_m e^{j(\omega t + \phi)}$ Complex form of current

$i(t) = I_m \cos(\omega t + \phi°)$ ----> $I_m \angle \phi°$ <---Phasor.

$i(t) = \mathrm{Re}\left(I_m e^{j(\omega t + \phi°)}\right)$ ----> $I_m \angle \phi°$ <---Phasor.

$I = I_m e^{j(\phi)}$ <---Phasor complex form.

$I = I_m e^{j(\phi)}$ ----> $I_m \angle \phi°$ <---Phasor.

We go from i(t) to I and I to i(t). Time domain to Frequency domain *(bold italic)*, and Frequency domain to Time domain.

Convert i(t)/v(t) to I/V (time to frequency domain):

Given i(t) write sinusoidal func i(t) in t-domain, write i(t) as a cosine wave with ph angle. sin(wt) written as cos(wt - 90), express cosine wave as real part of a complex quantity using Euler's identity, drop Re and suppress e^jwt. *Phasors are shown in capital letter bold and italic.*

Example v(t) to ***V*** :

The phase angle of the cosine angle is the angle on the phasor. ***<---COSINE.***

$v(t) = 100\cdot\cos(400\,t - 30\,°)$ Cosine term-->Re Part.

$v(t) = 100\cdot e^{j\,(400\,t - 30\,°)}$

$V = 100\angle -30\ \text{deg}$ <--- Answer

Example i(t) to ***I*** :

$i(t) = 1000\cdot\sin(400\,t + 150\,°)$

$\sin(400\,t + 150) = \cos(400\,t + 150 - 90) = \cos(400\,t + 60)$

$i(t) = 1000\cdot e^{j\,(400\,t + 60\,°)}$

$I = 1000\angle 60\ \text{deg}$ <--- Answer

Example i(t) to ***I*** :

$i(t) = 8\cdot\sin(\omega\,t - 20\,°)$

$\sin(\omega\,t - 20) = \cos(\omega\,t - 20 - 90) = \cos(\omega\,t - 110)$

$i(t) = 8\cdot e^{j\,(\omega\,t - 110\,°)}$

$V = 8\angle -110\ \text{deg}$ <--- Answer

Example i(t) to ***I*** :

$i(t) = 6\cdot\sin(\omega\,t) - 2\cos(\omega\,t)$

$\sin(\omega\,t - 0) = \cos(\omega\,t - 0 - 90) = \cos(\omega\,t - 90)$

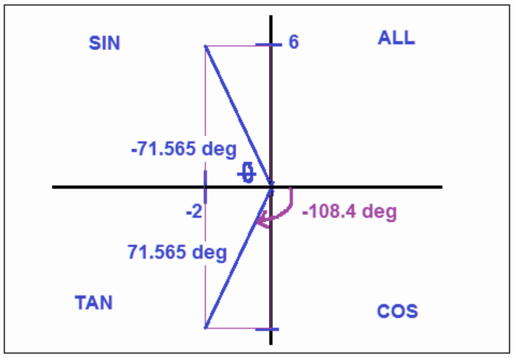
$= 6\cdot e^{j\,(\omega\,t - 90\,°)} - 2\cdot e^{j\,(\omega\,t - 0\,°)}$

$mag := \sqrt{6^2 + 2^2} = 6.325$ $phAng := atan\left(\frac{6}{-2}\right) = -71.565\ \mathbf{deg}$

-71.565 deg in the 3th quadrant, and into the clockwise direction.
180 deg - 71.565 deg.

$phAng := (-180 + 71.565)\ \mathbf{deg} = -108.435\ \mathbf{deg}$

See figure for angle measurement--->



$I = 6.325\angle -108.4\ \text{deg}$

Comments: A little troubling since we really just can't make is work the simple math way, plus we got to meet the convertion procedure.

Convert I/V to i(t)/v(t) (frequency to time domain):

Given phasor I in polar form write complex expression in exponential form, reinsert/multiply by e^jwt, replace real part operator Re. Obtain time domain expression by applying Euler's identity. Resulting cosine expression maybe changed to sine wave by increasing the argument by 90 degs.

Example ***V*** to v(t) :

$V = 115\angle -45$

$v(t) = 115\cos(\omega t - 45)$ <---Answer.

$= 115\sin(\omega t - 45 + 90)$

$v(t) = 115\sin(\omega t + 45)$ <---Answer in sine expression.

Example ***V*** to v(t) :

$V = 8\angle -110\ \text{deg}$

$v(t) = 8\cos(\omega t - 110)$ <---Answer.

$= 8\sin(\omega t - 110 + 90)$

$= 8\sin(\omega t - 20)$ <---Answer in sine expression.

Example ***I*** to i(t) :

$I = 6.325\angle -108.4\ \text{deg}$

We did this from i(t) to I it was a 2 term expression, now we need to reverse it.
But it does not have to reverse back as a 2 term expression.

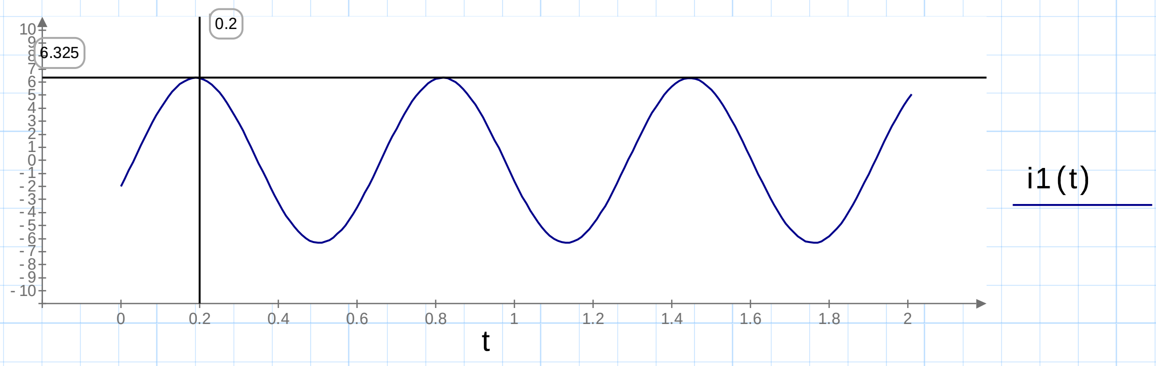
$i(t) = 6.325\cos(\omega t - 108.4)$ <---Answer.

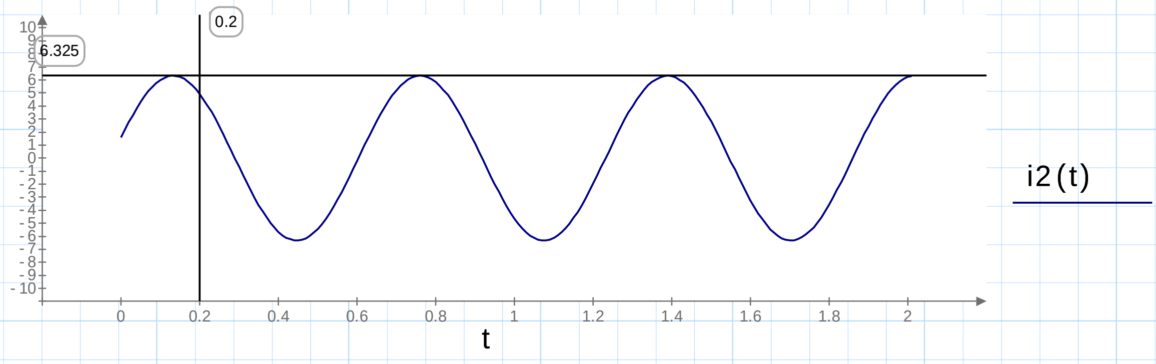
$= 6.325\sin(\omega t - 108.4 + 90)$

$= 6.325\sin(\omega t - 18.6)$ <---Answer in sine expression.

clear(t) $t := 0, 0.01 .. 10$

$\omega := 10$ $i1(t) := (6\cdot\sin(\omega\cdot t) - 2\cos(\omega\cdot t))$ $i2(t) := 6.325\sin(\omega\cdot t - 18.6)$

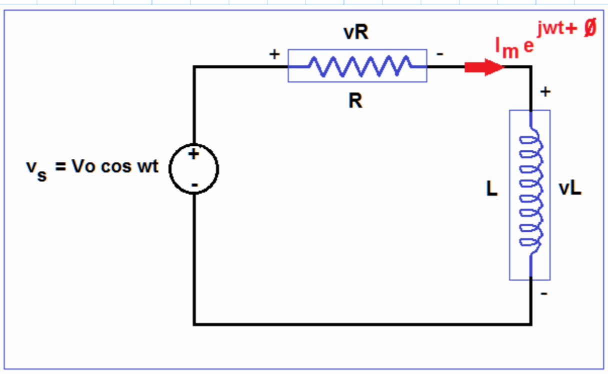




Both the wave forms i1(t) and i2(t) are identical.
So we need not fuss over the exact expression as the original i(t).
Our conversion expression gives the same results.

Again math magic prevailed. No need to fuss again in the future.
Math dictates. I dont want to do further exercises in the textbook, this took me days and night, without reading the book I was doing it like simple trig-right angle thinking, WRONG. Next continuing with the electric circuits subject matter.

Exponential terms in RL circuit loop equation:



Series RL circuit.

$$\cos(\omega t) = \text{Re}\ e^{j\omega t}$$

$$V_m \cos(\omega t) = V_m e^{j\omega t}$$

Complex response i: $I_m e^{j(wt+\phi)}$

$$\frac{di}{dt} = jwI_m e^{j(wt+\phi)}$$

KVL loop: $$Ri + L\left(\frac{di}{dt}\right) = V_s$$

$$R\left(I_m e^{j(wt+\phi)}\right) + L\left(jwI_m e^{j(wt+\phi)}\right) = V_m e^{j\omega t}$$

$$R\left(I_m e^{j(wt+\phi)}\right) + L\left(jwI_m e^{j(wt+\phi)}\right) = V_m e^{j\omega t}$$ Disappear jwt OR as Engineer-Authors wrote suppress jwt.

$$R\left(I_m e^{j\phi}\right) + L\left(jwI_m e^{j\phi}\right) = V_m$$

$$I_m e^{j\phi} = \frac{V_m}{R + jwL}$$

We seen something like this before in RLC circuits. Using the right angle Pythogra.

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + j^2 \cdot w^2 L^2}}$$ <--- Squared then square-root same thing.

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + j^2 \cdot w^2 L^2}} \cdot e^{j\left(-\tan^{-1}\left(\frac{wL}{R}\right)\right)}$$ <--- Complex response.

$$I_m = \frac{V_m}{\sqrt{R^2 - w^2 L^2}}$$ $j^2 = -1$

$$\phi = -\tan^{-1}\left(\frac{wL}{R}\right)$$

Resistance real x axis, and jwL Im y axis, so -wL/R = tan (angle). Thats why we see it that way always, Im - y-axis.

For the complex response of i(t): $i(t) = I_m \cos(\omega t + \phi)$

$$i(t) = \frac{V_m}{\sqrt{R^2 + w^2 L^2}} \cdot \cos\left(\omega t - \tan^{-1}\left(\frac{wL}{R}\right)\right)$$

Lets go back a few steps and apply phasors:

$R\left(I_m e^{j\phi}\right) + L\left(jwI_m e^{j\phi}\right) = V_m$ <--- From the KVL loop equation.

$V = V_m \angle 0° = V_m e^{j0°}$ <---Phasor.

$I = I_m e^{j(\phi)}$ <---Phasor.

Substitute V and I:

$R(I) + L(jwI) = V$ <--- From the KVL loop equation.

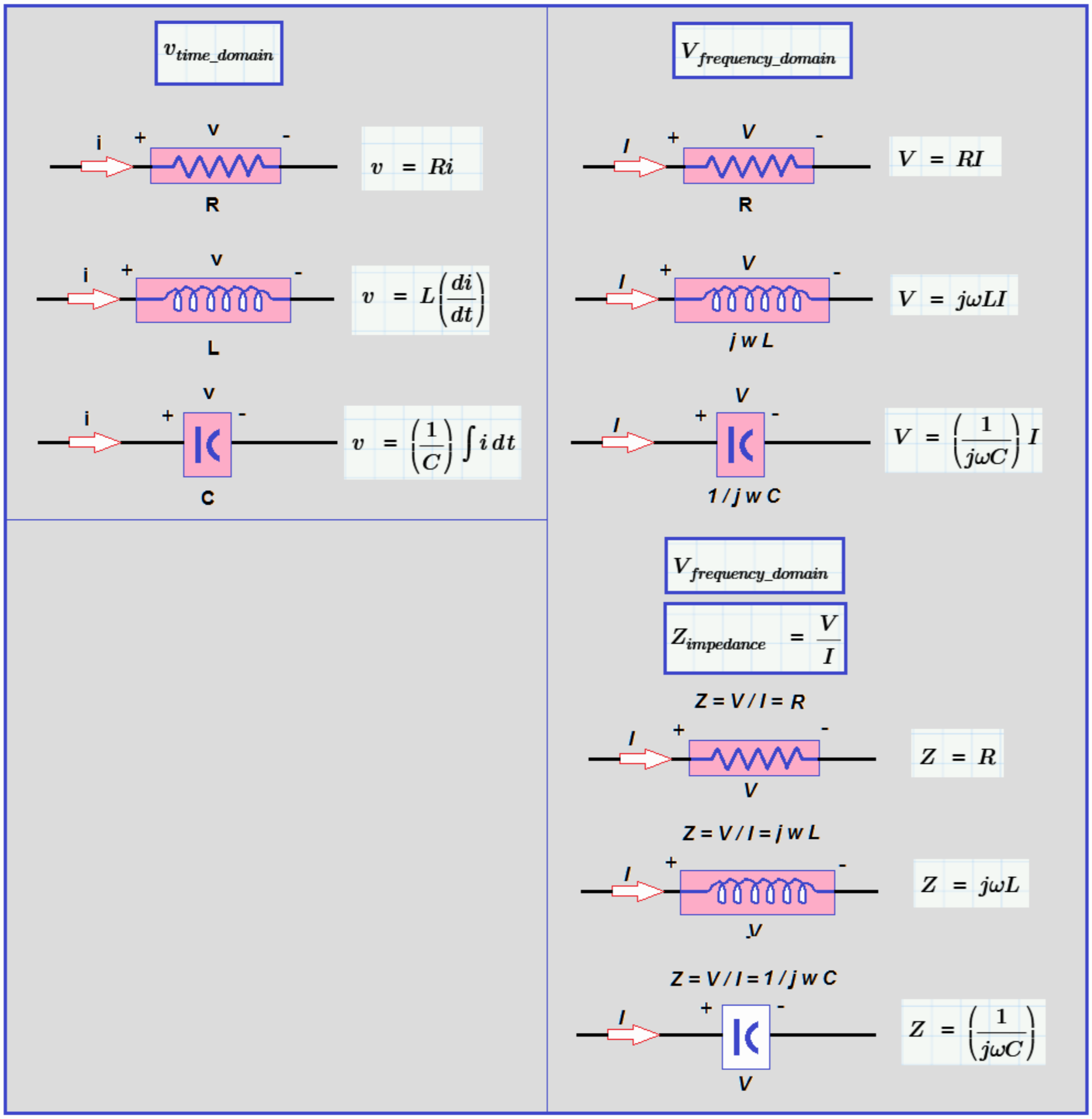
$R(I) + j\omega L(I) = V$

$I(R + j\omega L) = V$ **I and V can be in polar or exponential form.**

Table: Comparison of time and frequecy domain for R L and C.

v_{time_domain}	$V_{frequency\_domain}$	$Z_{impedance\_freq\_domain} = \frac{V}{I}$
$v = Ri$	$V = RI$	$Z_R = R$
$v = L\left(\frac{di}{dt}\right)$	$V = j\omega L I$	$Z_L = j\omega L$
$v = \left(\frac{1}{C}\right)\int i\,dt$	$V = \left(\frac{1}{j\omega C}\right) I$	$Z_C = \left(\frac{1}{j\omega C}\right)$

These kind of expressions for RLC we studied in sinusoidal steady state, phasors, etc. Equation provided below.



Next section 13.3.
Any errors or slight in explanation please check with your textbook and local engineer.

13.3 Damped Sinusoidal Forcing Function:

Luckily we are coming to an explained exmple in this section. We needed an example. We have 2 examples, after Hyat & Kemerly, from Schaums.

We have: $v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)$

$v(t) = V_m e^{\sigma t} e^{j(\omega t + \theta)}$ <---positive

$s = \sigma + j\omega$

$v(t) = V_m e^{j \cdot \theta} e^{(\sigma + j\omega) t}$ This we studied and this is identified to the damped sinusoid.

Conjugate---> $v(t) = V_m e^{\sigma t} e^{j(-\omega t - \theta)}$ <--- negative, so we have a *-jwt*.

$= V_m e^{-j \cdot \theta} e^{(\sigma - j\omega) t}$

Recently in the previous pages we did a RL circuit, we did a voltage loop equation. Shown again below.

KVL loop: $Ri + L\left(\frac{di}{dt}\right) = V_s$

$R\left(I_m e^{j(wt+\phi)}\right) + L\left(jwI_m e^{j(wt+\phi)}\right) = V_m e^{j\omega t}$

$R\left(I_m e^{j(wt+\phi)}\right) + L\left(jwI_m e^{j(wt+\phi)}\right) = V_m e^{j\omega t}$

----> $R\left(I_m e^{j\phi}\right) + L\left(jwI_m e^{j\phi}\right) = V_m$ <----

In the last equation stays the exponential term: $e^{j\phi}$

The angle Phi can be replaced for Theta: $e^{j\theta}$

$R\left(I_m e^{j(wt+\theta)}\right)$ <--- This form can be used to express v(t) and expanded as below.

$v(t) = V_m e^{j\theta} e^{j(\omega t)}$ <--- This is the undamped exponentially varying sinusoidal expression.

$v(t) = V_m e^{j \cdot \theta} e^{(\sigma + j\omega) t}$ <--- We place concern to this expression. Damped sinusoid.

See the difference between $v(t) = V_m e^{j\theta} e^{j(\omega t)}$
and
$v(t) = V_m e^{j\cdot\theta} e^{(\sigma + j\omega)t}$

Sigma is the difference, the positive part of the complex frequency s = sigma + jw.

$s = \sigma + j\omega$ Damped sinusoid.
$\sigma = 0$
$s = 0 + j\omega$ Undamped sinusoid.

Now we have in expression s = sigma + jw both the undamped and damped condition.

Previous chapter we were *educated* or traaained in the situation something like this:
We have a forcing function and our response will take on a similar function.
We also benefit by the form of the forcing function that is easily intergrated and differentiated such that to retain some form of the original forcing function.
something like this, you may correct it. Its been some time.

Now in the complex frequency form, the real part of the complex frequency forcing function produces the real part of the response. Likewise the imaginary part of the forced function produces the imaginary part of the response.

What kind of forcing function can be given?
A complex forcing function. Of course we spent some pages on it already.
But with a little difference. Our source generators are real not imaginary they produce real values. So our forcing function need to be the real part of the complex function. This results with the response, the real part of this complex function is the time domain forced response.

So we put the equations in a row $v(t) = \text{Re}\left(V_m e^{j\theta} e^{st}\right)$ Real forcing function

Apply ---> $V_m e^{j\theta} e^{st}$

Resultant forced response ---> $I_m e^{j\theta} e^{st}$

of course the *I* and *V* value of ph angle is different, can be same for resistive load.

Real part of which is the desired time domain forced response ----> $i(t) = \text{Re}\left(I_m e^{j\theta} e^{st}\right)$

What are we looking for in the current i(t) expression above: I_m Amplitude
θ Phase angle

Next a plot.

This plot exercise came about on the difficulty in visualing the real and imaginary part in the complex exponentially varying sinusoidal function.

clear(t)

Given the expression: $v_e(t) := 60 \cdot e^{-2 \cdot t} \cdot e^{j \cdot (4 \cdot t + 10\,\mathbf{deg})}$

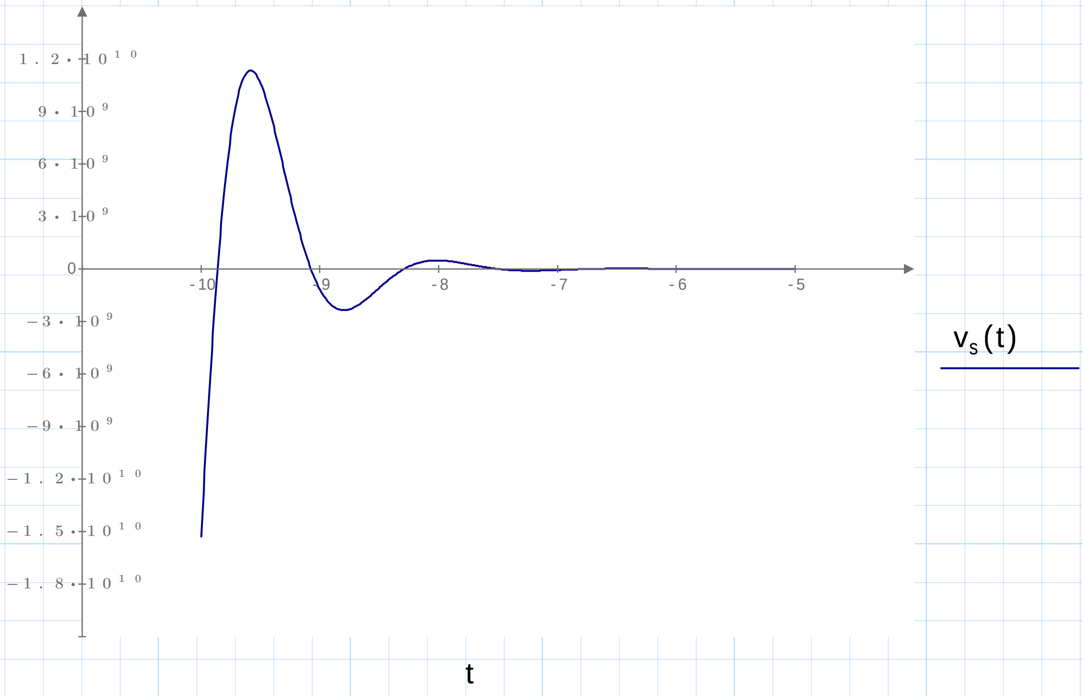
$v_e(t) := 60 \cdot e^{j \cdot 10\,\mathbf{deg}} \cdot e^{(-2 + j \cdot 4) \cdot t}$ re-written

$s = -2 + j4$

You can give it a try on how to plot v_e(t).

If you find it troubling or a nuisance for now maybe next time you can separate the plot into two plots; one real plot one imaginary plot. However, maybe we can turn it around into a sinusoidal expression with exponential term. So the j, imaginary part, is pulled out of the expression provided we know the cosine is for real part and sine for imaginary. We concern with real part. *You discuss this plot if its good.*

Exponentially varying sinusoidal expression: $v_s(t) := 60 \cdot e^{-2 \cdot t} \cdot \cos(4 \cdot t + 10\,\mathbf{deg})$ <--- Plot.



Next a circuit example on the material covered for complex frequency! Fina*l*l y !

Short Complex Number Exercise *(Refresher)*:

$j := \sqrt{-1}$ $\qquad j^2 = -1$

Exercise 1. Find the conjugate of s and s1*s2:

$s = -2 + j4$

$s1 = -2 + j4$

$s2 = s2^{conj} = -2 - j4$ <---For the conjugate ONLY change the sign of the imaginary part of the number.

$s1 \cdot s2 = (-2 + j4)(-2 - j4)$

$= 4 + j8 - j8 - j^2\ 16$

$= 4 - (-1)\ 16$

$= 4 + 16$

$s1 \cdot s2 = 20$

Exercise 2. Rationalise the denominator of: $\dfrac{20}{(-4 + j5)}$

$s1 = -4 + j5$

$s2 = s1^{conj} = -4 - j5$

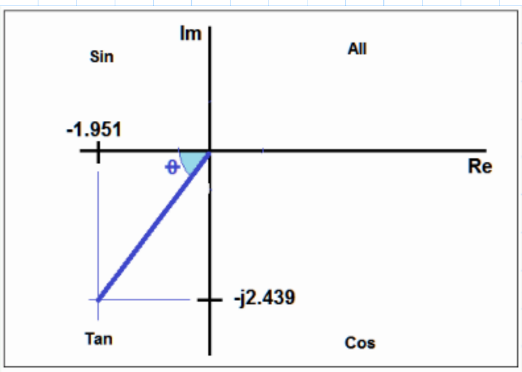
Multiply by **(-4-j5) / (-4-j5)**, cojugate term, to 'rationalise the denominator'.

$$\frac{20\ (-4 - j5)}{(-4 + j5)\ (-4 - j5)} = \frac{-80 - j100}{16 + j20 - j20 - j^2\ 25}$$

$$= \frac{-80 - j100}{16 - (-1)\ 25}$$

$$= \frac{-80 - j100}{16 + 25} = \frac{-80 - j100}{41}$$

$= -1.951 - j2.439$ <---See it as horizontal and vertical axis, the j is not relevant to the square root calculation.



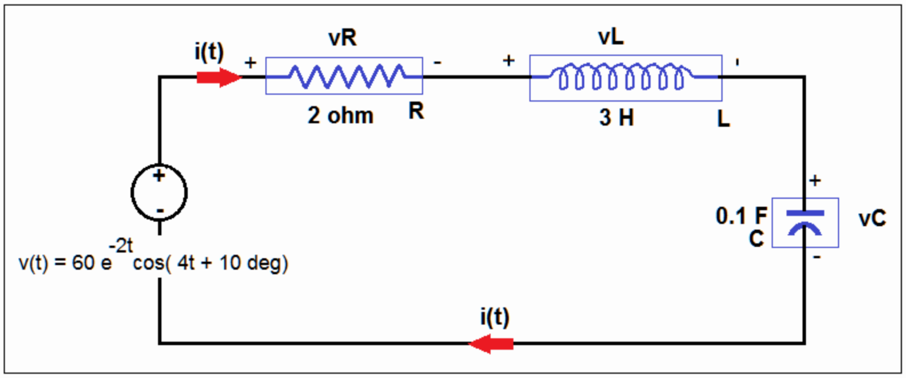
$\text{Magnitude} := \sqrt{(-1.951)^2 + (-2.439)^2} = 3.123$

$\text{Phase\_Angle} := \text{atan}\left(\dfrac{-2.429}{-1.951}\right) = 51.228$ **deg**

Phase Angle is in the 3rd quadrant where tangent is positive.

Example From Section 13.3:

Solve series RLC circuit for the forced response.



Solution:

$R := 2 \qquad L := 3\ H \qquad C := 0.1\ F$

$v(t) = 60 \cdot e^{-2 \cdot t} \cdot \cos(4 \cdot t + 10°)$ <---Forcing function.

Forced response takes the similar form, so here its surprising we can make that as provided below, easy:

$i(t) = I_m \cdot e^{-2 \cdot t} \cdot \cos(4 \cdot t + \phi)$ <--- Forced response.

We need to solve for Im and phase angle Phi.

We create a Real forcing function: $v(t) = 60 \cdot e^{-2 \cdot t} \cdot \cos(4 \cdot t + 10°)$

$$v(t) = \mathrm{Re}\left(60 \cdot e^{-2 \cdot t} \cdot e^{j \cdot (4 \cdot t + 10\,deg)}\right)$$

$$v(t) = \mathrm{Re}\left(60 \cdot e^{j \cdot 10\,deg} \cdot e^{(-2 + j \cdot 4) \cdot t}\right)$$

$$v(t) = \mathrm{Re}\left(60 \cdot e^{j\,\theta} \cdot e^{s \cdot t}\right)$$ <--- Form.

Droping Re steps:

Forcing function form: $v(t) = V \cdot e^{j\,\theta} \cdot e^{st}$

$$V \cdot e^{j\,\theta} = 60 \cdot e^{j \cdot 10\,deg}$$

$$V = 60\angle 10\ deg$$ <--- Phasor form.

$$s = -2 + j4$$

$$v(t) = V \cdot e^{st} = v(t) = 60\angle 10 \cdot e^{st}$$ <--- *Dropped Re left with complex forcing function; phasor and exponential form.*

We have Ve^st, now our forced response will take a similar form:

$$v(t) = V \cdot e^{st} = 60\angle 10 \cdot e^{st}$$

$i(t) = I \cdot e^{st}$ <---Forced response form.

Where: $I = I_m \angle \theta$ Same as we did for voltage.

Next we do the voltage conservation, *voltage loop*, sum of voltages equal 0 or equal the forcing function. This will have the integral and differentiation terms, called the integrodifferential equation. Exactly as we done in previous chapter.

$$v(t) = Ri + L\left(\frac{di}{dt}\right) + \left(\frac{1}{C}\right)\int i\,dt$$

$$v(t) = 2i + 3\left(\frac{di}{dt}\right) + \left(\frac{1}{0.1}\right)\int i\,dt$$

$$v(t) = 2i + 3\left(\frac{di}{dt}\right) + 10\int i\,dt$$

As we done before, dont worry about forgetting thats natural and expected.
Not really natural because our memory is exposed to external and internal imposing factors which requires we refresh thru opening past notes/textbook, except for super humans.

$v(t) = 60\angle 10 \cdot e^{st}$ substitute it for v(t)

$i(t) = I \cdot e^{st}$ intergtate it, and differentiate it, then substitute in.

$$\frac{di}{dt} = sIe^{st} \qquad \int i\,dt = \left(\frac{1}{s}\right) Ie^{st}$$

$60\angle 10 \cdot e^{st} = 2\ I \cdot e^{st} + 3\ sIe^{st} + \left(\frac{1}{s(0.1)}\right) Ie^{st}$ Next cancel out e^st.

$60\angle 10 = 2\ I + 3\ sI + \left(\frac{10}{s}\right) I$ Next factor RHS.

$60\angle 10 = I\left(2 + 3\ s + \left(\frac{10}{s}\right)\right)$ Next solve for I.

$$I = \frac{60\angle 10}{\left(2 + 3\ s + \left(\frac{10}{s}\right)\right)}$$

Next some explanation before evaluating for I.
First thought was how do we solve for I with s in the expression.

LHS we have current.
RHS we have voltage 60 at 10 degs divided by (2 + 3s + (1/10s)), this latter expression must be resistance because LHS is current I, and RHS has to be V/R.
However the form of the resistance is in an 's' expression.

s = sigma + jw.

In electric circuits the phasor studies introduce us to impedance Z and admittance Y.

Component	Phasor	Impedance	Admittance
	$\frac{V}{I}$	$Z = \frac{V}{I}$	$Y = \frac{1}{Z}$
R	R	R	$\frac{1}{R}$
L	$j\ \omega\ L$	sL	$\frac{1}{sL}$
C	$\frac{C}{j\ \omega}$	$\frac{1}{sC}$	sC

Look at the phasor and impedance columns.
Notice at L and C rows, where *jw* is in phasor thats where *s* is in impedance. L lines up for phasor and impedance, but for C its numerator at phasor and denominator at impedance.
So can we fix C's case?

$C = 0.1 \quad \frac{1}{sC} = \frac{1}{0.1\ s}$ Fits, works, match the impedance format.

OR $C = \frac{1}{10} \quad \frac{1}{sC} = \frac{1}{s\left(\frac{1}{10}\right)} = \frac{10}{s}$ Engineers used this format for this solution, whole number benefit, we know now both work, which is good to know because capacitor can be 2 F or 0.2 F.

$$I = \frac{60\angle 10}{\left(2 + 3\ s + \left(\frac{10}{s}\right)\right)}$$ <--- This is the equation we were looking to match.

$2 + 3\ s + \left(\frac{10}{s}\right)$ <--- $R = 2 \quad L = 3 \quad C = \frac{10}{s}$

Next we plug in s = -2 + j4 in the denominator to solve for I.

$$2 + 3\,s + \left(\frac{10}{s}\right) = 2 + 3\,(-2 + j4) + \frac{10}{(-2 + j4)}$$

$$= 2 - 6 + j12 + \frac{10}{(-2 + j4)}$$

Multiply by (-2-j4)/(-2-j4), cojugate term, to 'rationalise the denominator' of just the right most term.

$$\frac{10\,(-2 - j4)}{(-2 + j4)\,(-2 - j4)} = \frac{-20 - j40}{4 + j8 - j8 - j^2\,16}$$

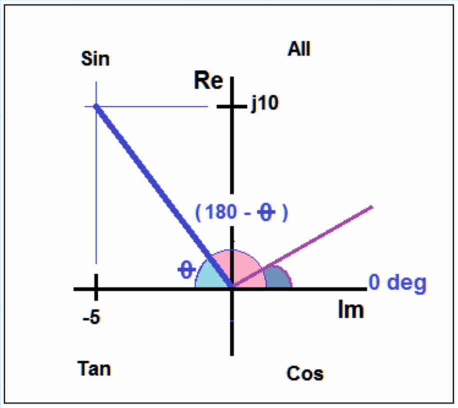
$$= \frac{-20 - j40}{4 + 16}$$

$$= \frac{-20 - j40}{20}$$

$$= -1.0 - j2.0$$

$$= 2 - 6 + j12 - 1 - j2$$

$$= -5 + j10$$



$$Mag_1 := \sqrt{(-5.0)^2 + (10.0)^2} = 11.18$$

$$\theta_1 := \operatorname{atan}\left(\frac{10}{-5}\right) = -63.435\ \mathbf{deg}$$

Next we see to the appropriate angle, see figure to right, 180 deg - Theta1.

$$(180\ \mathbf{deg} - 63.435\ \mathbf{deg}) = 116.565\ \mathbf{deg}$$

This now is the angle with the reference to zero degrees, anti-clockwise positive.

The phasor of denominator: $\left(2 + 3\,s + \left(\frac{10}{s}\right)\right) = 11.18\angle 116.565\ \text{deg}$

$$I = \frac{60\angle 10\ \text{deg}}{11.18\angle 116.565\ \text{deg}}$$

$$I_m := \frac{60}{11.18} = 5.367 \quad \phi := 10\ \mathbf{deg} - 116.565\ \mathbf{deg} = -106.565\ \mathbf{deg}$$

$$I = 5.37\angle -106.6\ \text{deg}$$

Required forced response in time domain: $i(t) = I_m \cdot e^{-2 \cdot t} \cdot \cos(4 \cdot t + \phi)$

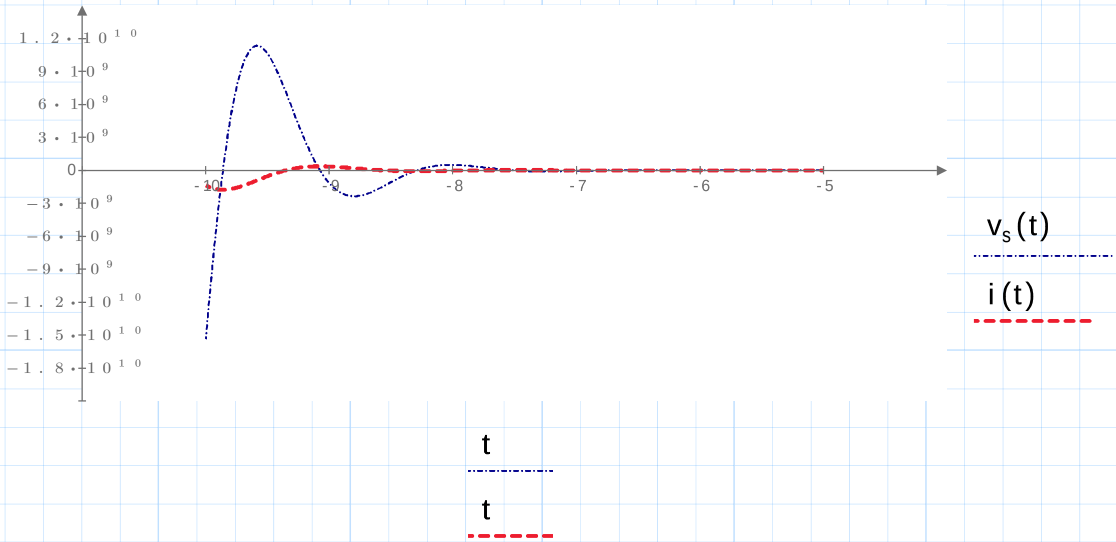
Answer: $i(t) = 5.37\ e^{-2 \cdot t} \cdot \cos(4 \cdot t - 106.6\ °)$

Plot of forcing function and forced response next page.

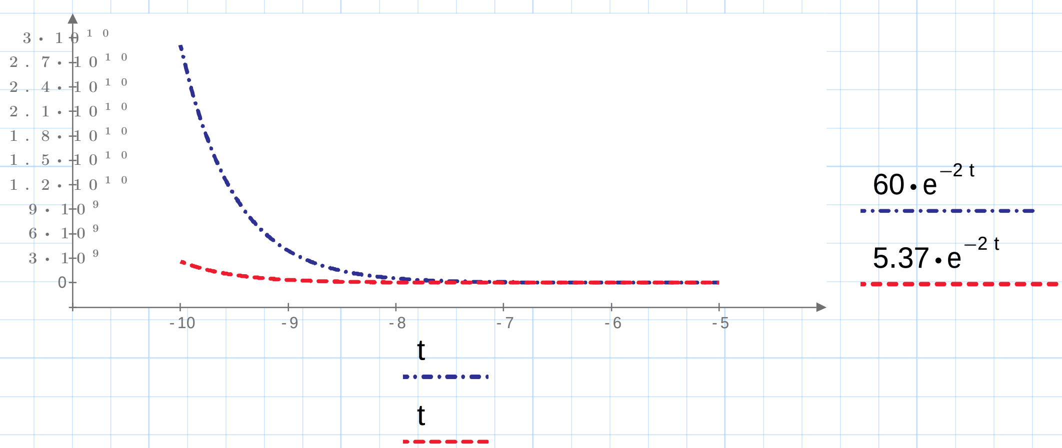
clear (t)

$v_s(t) := 60 \cdot e^{-2 \cdot t} \cdot \cos(4 \cdot t + 10\ \mathbf{deg})$ <--- Forcing function

$i(t) := 5.37 \cdot e^{-2 \cdot t} \cdot \cos(4 \cdot t - 106.6\ \mathbf{deg})$ <--- Forced response



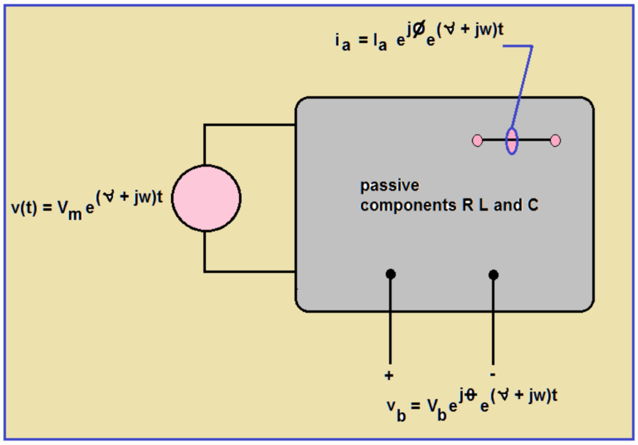
Comments:
The current (response) is much smaller compared to the voltage source. Both decay eventually. This probably due to the circuit component values? Very high starting voltage and the current is very low. This is in the time t<0. Numerical answer evaluated is correct to the textbook. See below its the exponential term.



It is troubling to see the circuit's math solution has such vast contrast between v(t) and i(t). Question is was that a real world circuit example? Maybe no. Voltate to 10^10! end and then 10^9 for i(t), close maybe but so huge. Math application just working the math. Its possible with these circuit component values but the vs(t) has to provide 60e^-2t, that is a very high voltage. Exponent term cause of high value.

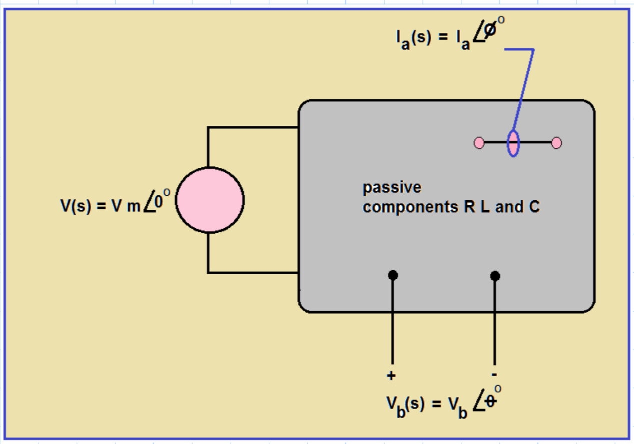
13.4 Z(s) and Y(s) and
8.6 Generalised Impedance (R L C) in s-domain:

We show the time domain circuit and the equivalent s-domain circuit.



Time domain:

v(t) = voltage source - forcing function
ia = forced response
vb = voltage across circuit component or branch voltage



s-domain (also frequency domain):

$s = \sigma + j\omega$

Component	Phasor	Impedance	Admittance
	$\frac{V}{I}$	$Z = \frac{V}{I}$	$Y = \frac{1}{Z}$
R	R	R	$\frac{1}{R}$
L	$j\omega L$	sL	$\frac{1}{sL}$
C	$\frac{C}{j\omega}$	$\frac{1}{sC}$	sC

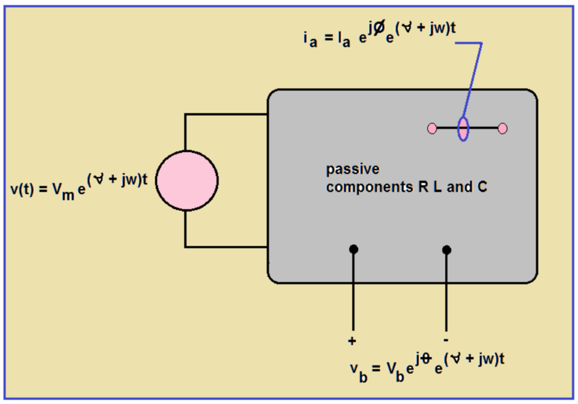
This section we get to example problem solving, whats missing theory wise catch it in textbook. Should be ok going thru most you seen.

Example 6.8: Series RL circuit.

A series RL circuit with R = 10 ohm and L = 2H, has an applied voltage
v = 10 e^-2t cos(10t + 30 deg).

Obtain the current i by an s-domain analysis.

Solution:



$$v(t) = 10\,e^{-2t}\cos(10\,t + 30\ \text{deg})$$

$$s = -2 + j10$$

Polar form V: $10\angle 30\,°$

Exponential: $e^{(-2+j10)\,t} = e^{st}$

$$v = 10\angle 30\,°e^{st}$$

$i = Ie^{st}$ The form of response based on the forcing function v.

$$\frac{di}{dt} = sIe^{st}$$

Voltage conservation or voltage *loop* equation:

$$v = Ri + L\left(\frac{di}{dt}\right)$$

$$10\angle 30\,°e^{st} = R\left(Ie^{-st}\right) + L\left(sIe^{-st}\right) = 10 \cdot Ie^{st} + 2 \cdot sIe^{st} = Ie^{st}\,(10 + 2 \cdot s)$$

$$Ie^{st} = \frac{10\angle 30\,°e^{st}}{(10 + 2 \cdot s)}$$ Cancelling e^st next by dividing by e^st

$I = \dfrac{10\angle 30\,°}{(10+2\cdot s)}$ Next substitute value of s and solve for I.

$I = \dfrac{10\angle 30\,°}{10+2\cdot(-2+j10)} = \dfrac{10\angle 30\,°}{10-4+j20} = \dfrac{10\angle 30\,°}{6+j20}$

$6+j20$ ----> $Mag := \sqrt{(6)^2+(20)^2} = 20.881$ $PhAng := atan\left(\dfrac{20}{6}\right) = 73.301\ \mathbf{deg}$

$I = \dfrac{10\angle 30\,°}{20.881\angle 73.301\,°}$ $I = \dfrac{10\angle 30\,°}{20.881\angle 73.301\,°}$

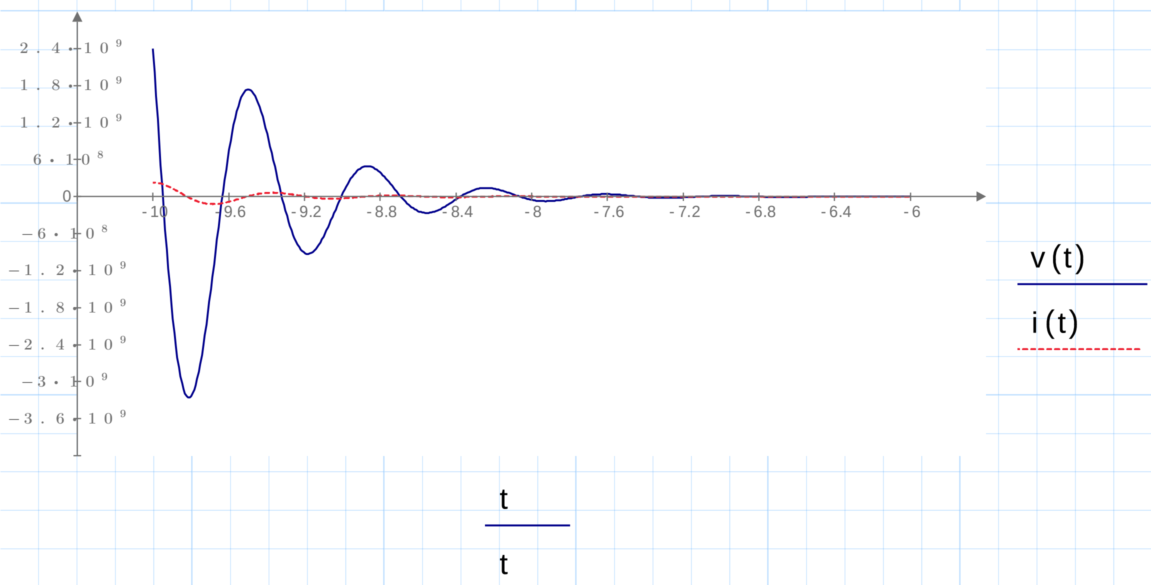
$\dfrac{10}{20.881} = 0.479$ $30\ \mathbf{deg} - 73.301\ \mathbf{deg} = -43.301\ \mathbf{deg}$

$I = 0.479\angle -43.3\ deg$

$v(t) := 10\ \mathbf{e}^{-2t}\cos(10\ t + 30\ \mathbf{deg})$ <--- this was the form of voltage in time domain.

$i(t) := 0.48\ \mathbf{e}^{-2t}\cos(10\ t - 43.3\ \mathbf{deg})$ Answer. In time domain.

Comment: Method of solution was in s-domain and it easily got the answer at end in time domain.
Plot of v(t) and i(t) below. **clear** (t)



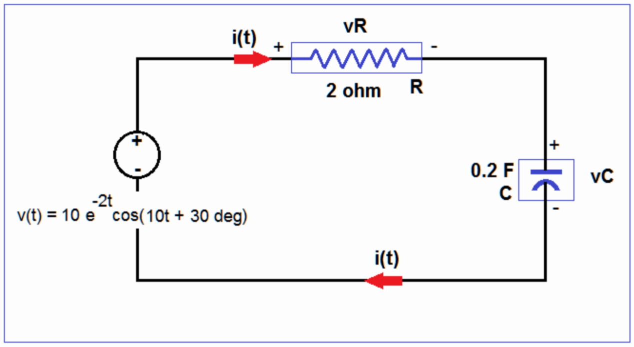
Comments: Plot observations 1). forcing function v(t) was oscillatory ie under damped 2). i(t) does oscillate, hard to say if it is underdamped, critically damped or over damped. You discuss.

Example 6.9: Series RC circuit.

A series RC circuit with R = 10 ohm and C = 0.2H, has an applied voltage
v = 10 e^-2t cos(10t + 30 deg).

Obtain the current i by an s-domain analysis.

Solution:



With the exception to the capacitor similar circuit to previous example. So we may try to reduce some already done steps.

$$v(t) = 10\, e^{-2t} \cos(10\,t + 30\ \text{deg})$$

$$s = -2 + j10$$

Polar form V: $10\angle 30\,°$

Exponential: $e^{(-2+j10)\,t} = e^{st}$

$$v = 10\angle 30\,° e^{st}$$

$i = Ie^{st}$ The form of response based on the forcing function v.

$$\int i\,dt = \left(\frac{1}{s}\right) Ie^{st}$$

Voltage conservation or voltage loop equation:

$$v = Ri + \left(\frac{1}{C}\right)\int i\,dt$$

$$10\angle 30\,° e^{st} = R\left(Ie^{-st}\right) + \left(\frac{1}{Cs}\right) Ie^{-st} = 10 \cdot Ie^{st} + \left(\frac{5}{s}\right)\cdot Ie^{st} = Ie^{st}\left(10 + \frac{5}{s}\right)$$

$$Ie^{st} = \frac{10\angle 30\,° e^{st}}{\left(10 + \frac{5}{s}\right)}$$ Cancelling e^st next by dividing by e^st

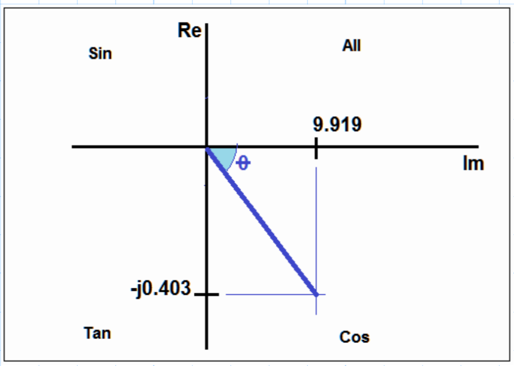
$$I = \frac{10\angle 30^\circ}{\left(10 + \frac{5}{s}\right)}$$ Next substitute value of s and solve for I.

$$I = \frac{10\angle 30^\circ}{10 + \frac{5}{(-2 + j10)}}$$ <--- Little complexity in denominator needs rationalising.

$$\frac{5}{(-2 + j10)} \cdot \frac{(-2 - j10)}{(-2 - j10)} = \frac{-10 - j50}{4 + j20 - j20 - j^2\ 100} = \frac{-10 - j50}{124}$$

$$= -0.081 - j0.403$$

$$I = \frac{10\angle 30^\circ}{10 - 0.081 - j0.403} = \frac{10\angle 30^\circ}{9.919 - j0.403}$$



<---Figure not to scale or proportion

Solve ----> $9.919 - j0.403$

$$Mag := \sqrt{(9.919)^2 + (0.403)^2} = 9.927$$

$$PhAng := atan\left(\frac{-0.403}{9.919}\right) = -2.327\ \mathbf{deg}$$

$$I = \frac{10\angle 30^\circ}{9.919\angle -2.327^\circ}$$

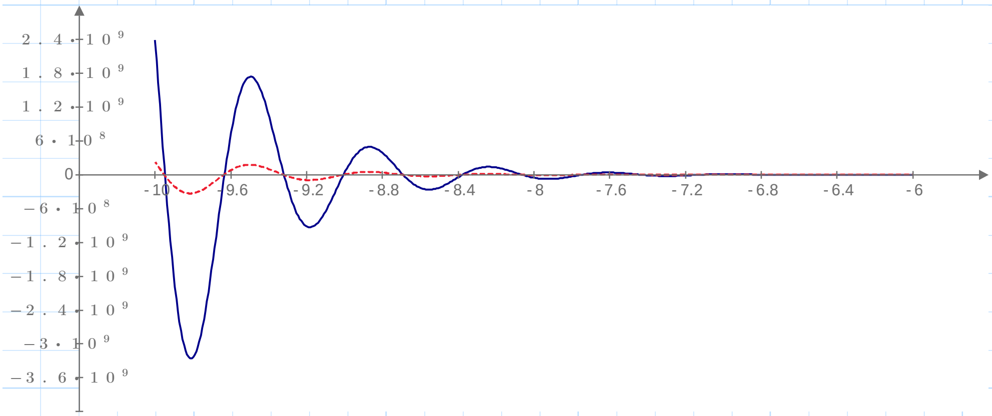
$$I = 1.008\angle 32.327^\circ$$ Angle: $(30 - (-2.327)) = 32.327$

$i_{RC}(t) := 1.01\ \mathbf{e}^{-2t} \cos(10\ t + 32.3\ \mathbf{deg})$ Answer. In time domain.

$v_{RC}(t) := 10\ \mathbf{e}^{-2t} \cos(10\ t + 30\ \mathbf{deg})$ <--- voltage in time domain.

Comment: Method of solution was in s-domain and it easily got the answer at end in time domain.
Plot of v(t) and i(t) below.

clear (t)



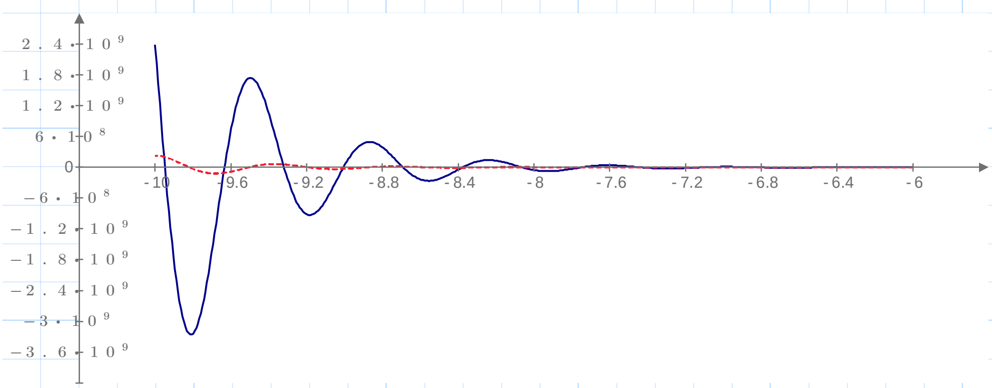
$v_{RC}(t)$

$i_{RC}(t)$

t

t

Comments: Plot observations 1). forcing function v(t) was oscillatory ie under damped 2). i(t) does oscillate, hard to say if it is underdamped, critically damped or over damped. Same as previous example plot so you discuss. We had inductor and now capacitor. The plot shape of course is dependent on the equation. We have a change in phase angle in the current i(t) from L to C. Previous plot provided below for comparison, near same.



v(t)

i(t)

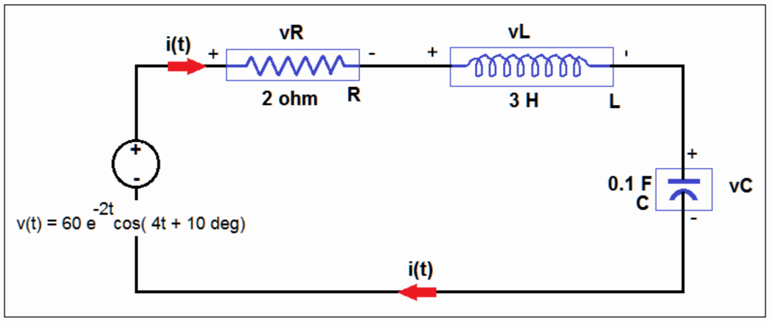
t

t

Example Series RLC circuit in Frequency Domain:

New method to solve 'Example From Section 13.3' *<--Did this past 2 examples ago.*

Circuit in time domain below convert to frequency domain.



$$v(t) = 60\, e^{-2t} \cos(4t + 10 \text{ deg})$$

$$s = -2 + j4$$

$$\sigma = -2$$

$$\omega = 4$$

$$V = 60\angle 10 \text{ deg}$$

$$Z_L = sL = 3\,(-2 + j4) = -6 + j12$$

<--- My everytime mistake I see impdeance and not frequency, j12 is jw <---w = 2 Pi f = 12 <--- radian frequency.

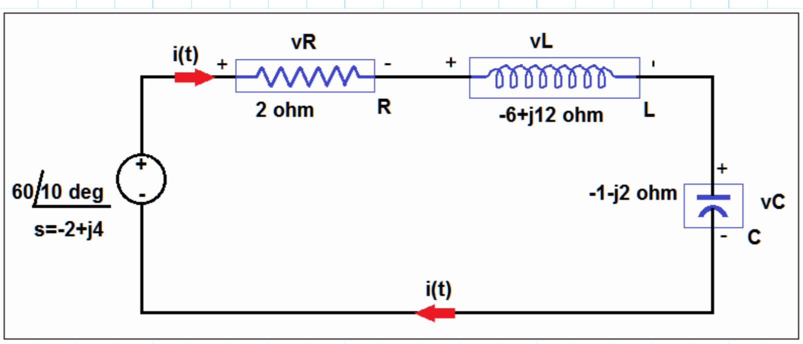
$$Z_C = \frac{1}{sC} = \frac{1}{(-2+j4)\cdot\left(\frac{1}{10}\right)} = \frac{10}{-2+j4}$$

<--- Rationalise denominator

$$= \frac{10\,(-2-4j)}{(-2+j4)\,(-2-j4)}$$

$$= \frac{-20-40j}{4+j8-j8-j^2\,16} = \frac{-20-40j}{20}$$

$$Z_C = -1-2j$$



Frequency domain equivalent of resistive circuit.

Next we solve for current I. Using the usual resistive circuit analysis.

$$I = \frac{V}{Z} = \frac{60\angle 10}{2 + (-6 + j12) + (-1 - 2j)} = \frac{60\angle 10}{-5 + j10}$$

$-5 + j10$ ---> $Mag := \sqrt{(-5)^2 + (10)^2} = 11.18$

$PhAng := atan\left(\frac{10}{-5}\right) = -63.435$ **deg**

Without sketching the angle figure, angle is in the 2nd quadrant.
Angle is to the x-axis, so now we need the angle measured from 0deg.

$PhAng := (180 - 63.435)$ **deg** $= 116.565$ **deg**

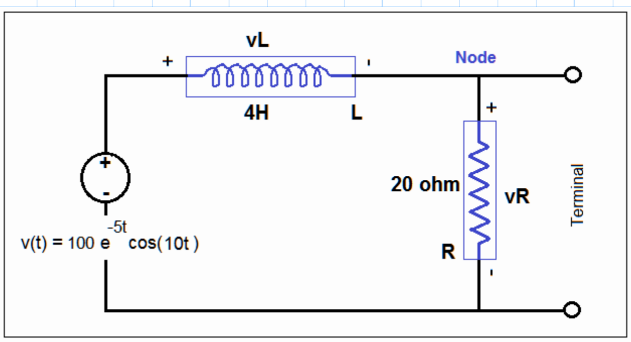
$$I = \frac{V}{Z} = \frac{60\angle 10}{11.18\angle 116.565} \quad \frac{60}{11.18} = 5.367 \quad (116.565 - 10)\ \textbf{deg} = 106.565\ \textbf{deg}$$

$I = 5.37\angle 106.6\ °$ Answer.

Note:
All the circuit analysis techniques we studied and applied in our circuits course such as
- mesh
- nodal
- superposition
- thevenin
- norton

all valid here.

Example: Convert to Thevenin equivalent.



$v(t) = 100\ e^{-5\ t}\ \cos(10\ t)$
$s = -5 + j10$
$\omega = 10$
$V = 100\angle 0$
$Z_L = sL = 4\ (-5 + j10) = -20 + j40$

Circuit connection at node indicates its parallel, to the right is another connection, we identify as terminal.

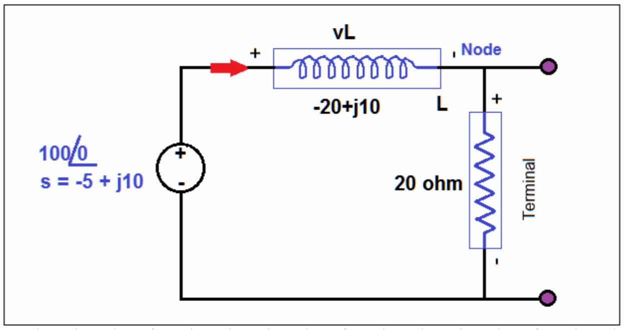
Thevenin equivalent in this case is the equivalent resistance seen across terminal.

$$Z_{th} = \frac{20 \cdot (-20 + j40)}{20 + (-20 + j40)} = \frac{-400 + j800}{j40}$$

Note: $\frac{1}{j} = -j$

Here we can divide directly $Z_{th} = j10 + 20 = 20 + j10$ Answer.

Next solve for open circuit voltage, ie across the terminals.



Shown in pink is the circuit equivalent impedance, the voltage we seek is across the 20 ohm resistor.

Equivalent circuit in frequency domain.

Apply voltage division:

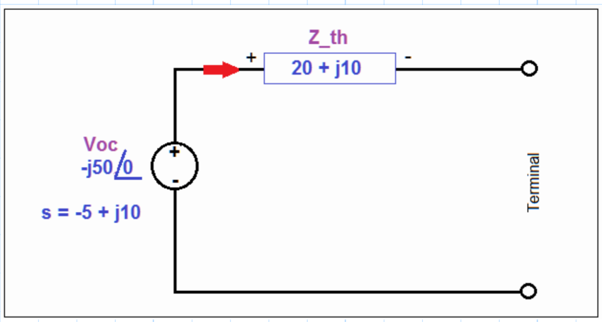
We apply the total circuit resistance: $Z_{Total} = 20 + (-20 + j40) = j40$

$$V_{open\_circuit} = 100\angle 0 \cdot \left(\frac{20}{j40}\right)$$

$$\frac{20 \cdot (-j40)}{j40 \cdot (-j40)} = \frac{-j800}{-j^2\ 1600} = \frac{-j800}{1600} = -j0.5$$

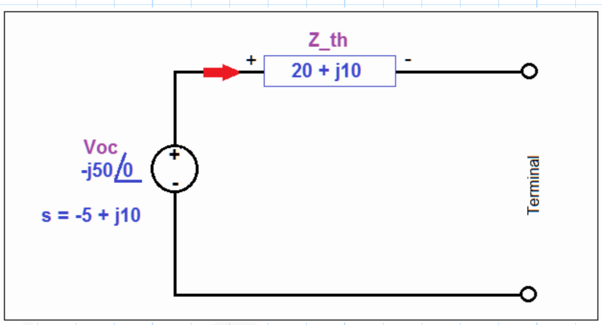
$$V_{open\_circuit} = 100\angle 0 \cdot (-j0.5)$$

$V_{open\_circuit} = -j50\angle 0$ Answer.



<--- Frequency domain Thevenin equivalent.

Now lets say we added a component to the circuit across the open terminals.



<---Insert L = 4H.

$Z_{L\_add} = sL = 4\ (-5 + j10) = -20 + j40$

Now find frequency domain current?

We update our resistance of the circuit:

$$Z_{th} = 20 + j10$$

$$Z_{th\_add\_L} = (20 + j10) + (-20 + j40) = j50$$

$$I = \frac{V_{open\_circuit}}{Z_{th\_add\_L}} = \frac{-j50\angle 0}{j50} = -1\angle 0$$ Frequency domain current

Corresponding time domain current?

$$s = -5 + j10$$

$$i(t) = (-1)\, e^{-5t} \cos(10\, t + 0\, deg)$$

$$i(t) = -e^{-5t} \cos(10\, t)$$ Answer.

Concluded a simple example from section 13.4 of Hyat Kemerly 4th Ed.
Essentially demonstrating the ease of using a new circuit analysis technique in the frequency domain.

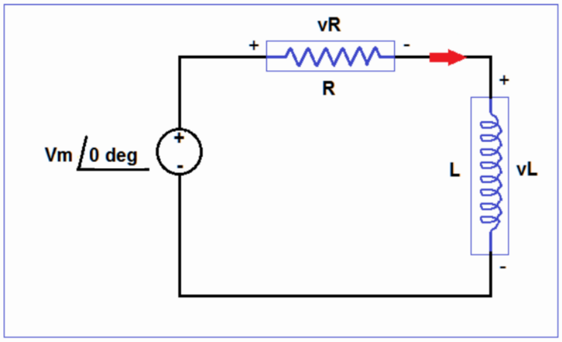
Next we need a little extra time to look at response as a:
1). function of omega and
2). function of sigma.

This will facilitate understanding of the pole and zero in complex frequency plane.

For the function of omega we use the sinusoidal steady state response from chapter 10 The Sinusoidal Steady State Response (Hyat & Kemerly).

13.5 Frequency Response as a function of sigma:

We have a series RL circuit with frequency domain voltage source Vm at 0 degs.



Current I is obtained as a function of s, by dividing voltage by impedance.

$$I = \frac{V_m \angle 0}{R + sL}$$

If we make w (omega) equal zero, 2 Pi f = 0 because f=0,
then s = sigma + j0.

$\omega = 0$
$s = \sigma + j0$
$s = \sigma$ <--- Yes.

Definition wise, when f =0, it makes w = 0, we can no longer be in the frequency domain. We are in the time domain.

$$v_s = V_m e^{(\sigma + j0)\,t}$$

$$v_s = V_m e^{\sigma t}$$ <--- Yes.

We had in frequency domain I equal... $I = \frac{V_m \angle 0}{R + sL}$

Now with the s = sigma: $I = \frac{V_m}{R + \sigma L}$ <--- No variable t here.

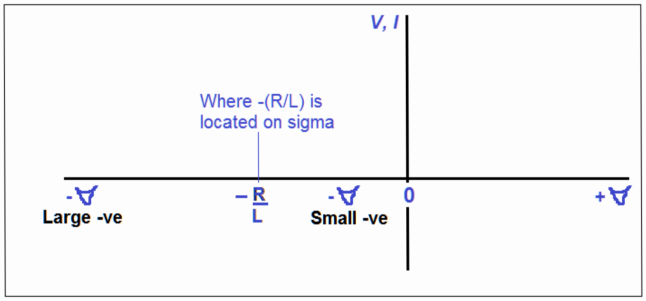
Before we move on to the next sentence, if we multiply the RHS by (1/L) to numerator and denominator:

$$I = \frac{V_m}{L}\left(\frac{1}{\frac{R}{L} + \sigma}\right) = \frac{V_m}{L}\left(\frac{1}{\sigma + \frac{R}{L}}\right)$$

<---We look at this form it has sigma isolated to itself.

And if instead we use v_s = Vm e^(sigma)t for Vm,
we get the time domain:

$$i(t) = \frac{V_m e^{\sigma t}}{R + \sigma L}$$ <---Making it time domain.



<--- sigma shown on x-axis.

I. Case when sigma is large negative number:

$e^{-(large)\ t}$ ---> exponentially decreasing rapidly

$V_m e^{-(large)\ t}$ ---> which makes the voltage decrease rapidly

$$I = \frac{V_m}{L}\left(\frac{1}{(-large) + \frac{R}{L}}\right) = \frac{V_m}{L}('-small\_value)$$

We see now the benefit of using this form, isolating sigma.

$I = small$ ---> our current is small in amplitude when sigma is a -ve large number.

II. Case when sigma is smaller negative closer to zero number:

$e^{-(small)\ t}$ ---> exponentially decreasing slowly in comparison to case I.

$V_m e^{-(small)\ t}$ ---> which makes the voltage decrease slower in comparison.

$$I = \frac{V_m}{L}\left(\frac{1}{(-small) + \frac{R}{L}}\right) = \frac{V_m}{L}('-ve\_Or\_+ve\_larger\_in\_comparison\_to-caseI)$$

$I = larger\_than\_case\_I$ ---> current is larger in amplitude when sigma is a -ve small number.

III. Case when sigma close to -(R/L):

$e^{-\left(\frac{R}{L}\right) t}$ ---> dependent on values of R/L but its negative, decreasing exponentially.

$V_m e^{-\left(\frac{R}{L}\right) t}$ ---> voltage decrease.

$$I = \frac{V_m}{L}\left(\frac{1}{-\left(\frac{R}{L}\right) + \frac{R}{L}}\right) = \frac{V_m}{L}\left(\frac{1}{small}\right) = \frac{V_m}{L}('large)$$

<---(1/small) equal infinity we say arbitrarily large, in comparison to case I and II.

$I = Large$ ---> our current is largest in amplitude when sigma is -R/L.

Introduction to POLES and ZEROS.

Response as a function of Omega (2 Pi f):
- In The Sinusoidal Steady State Response.

Pages 273-276 of Hyat and Kemerly 4th ed.

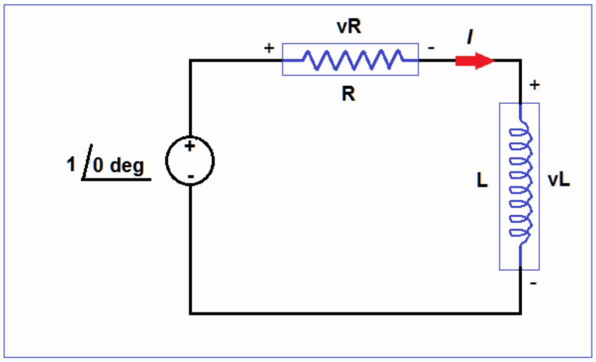
In the power industry the frequency is constant for 3 phase transmission. 50 Hz or 60 Hz, depending on region power company. Other wise frequency f and especially radian frequency Omega plays an important role in most areas of electrical, and mechanical engineering. Our study here under a sinusoidal source condition.

Less likely the radian frequency plays a similar role in an exponentially varying source the curve is not oscillating for one instance.

To keep it short as possible equations will be presented with short notes, you and I be able to follow thru.

$V_s = V_s \angle \theta$ <---Phasor form also polar.

$V_s = V_s \cos(\omega t + \theta)$



Series RL circuit:

$$I = \frac{V_s}{R + j\,\omega L}$$

Impedance: $$Z = \frac{V_s}{I}$$

Admittance: $$Y = \frac{I}{V_s} \qquad Y = \frac{\dfrac{V_s}{R + j\,\omega L}}{V_s}$$ <---Series RL circuit.

$$Y = \frac{1}{R + j\,\omega L}$$ <---This admittance can be interpretated as current produced by a voltage source of magnitude 1 at 0 deg.

Magnitude of the response: *Refer past notes.*

$$|Y| = \frac{1}{\sqrt{R^2 + j^2\,\omega^2\,L^2}}$$

We seen this derivation in previous notes. j^2 = -1

$$|Y| = \frac{1}{\sqrt{R^2 - \omega^2\,L^2}} \quad ---> \quad |Y| = \frac{1}{\sqrt{R^2 + \omega^2\,L^2}}$$

Right Angle - Pythogaras rule, Hypotenuse, place -wL if in -ve direction of graph. **<--1**

Angle of the response:

$$angY = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

-wL/R results in 4th quadrant; -ve. **<---2**

Equation 1 and 2 are mag and ph angle of response, both presented as a function of omega. Omega is the *format* we need to plot.

From previous notes, time constants :
Current response for example: $i = Ae^{\frac{-t}{\tau\_RL}} = Ae^{\frac{-t}{\frac{L}{R}}} = Ae^{-\left(\frac{R}{L}\right)t}$

RL circuit time constant tau = L/R $\tau_{RL} = \frac{L}{R} \qquad \frac{1}{\tau_{RL}} = \frac{R}{L}$ <--- t - axis

To attain a plot lets give values to our components:

$R := 2 \quad L := 1 \quad V := 1\angle 0$ **deg**

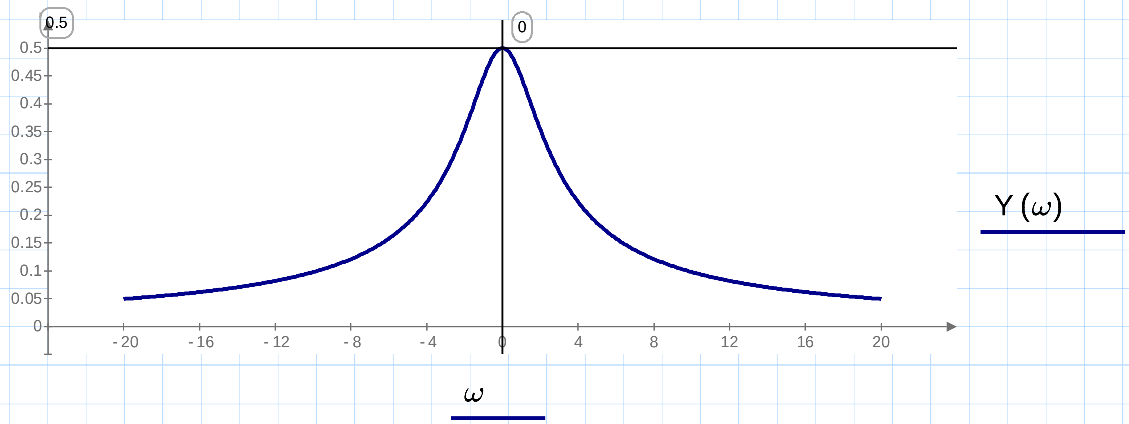
$Y(\omega) := \frac{1}{\sqrt{R^2 + \omega^2 \cdot L^2}}$

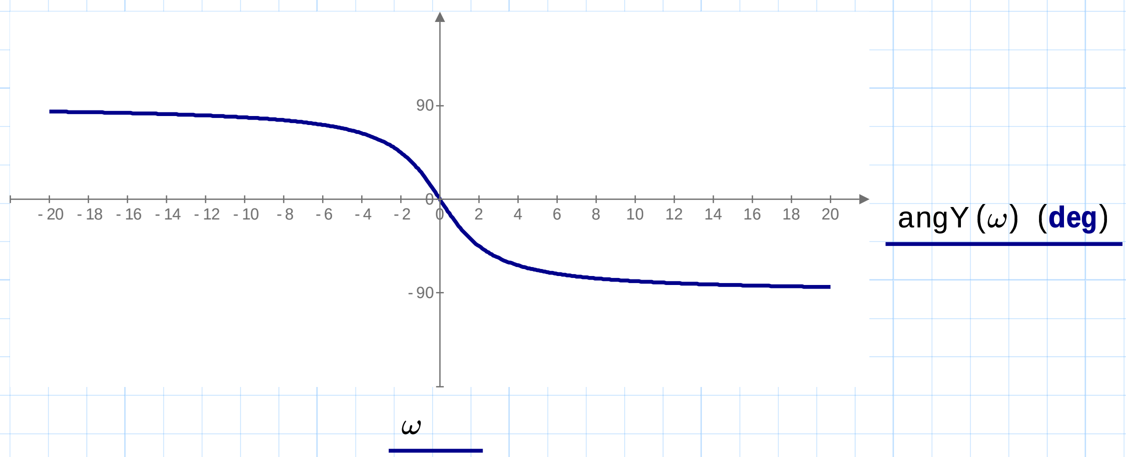
At w = 0 $Y(0) = \frac{1}{\sqrt{2^2 + 0^2 \cdot 1^2}} = 0.5$

At w = 0 $Y(0) = \frac{1}{R} = 0.5$

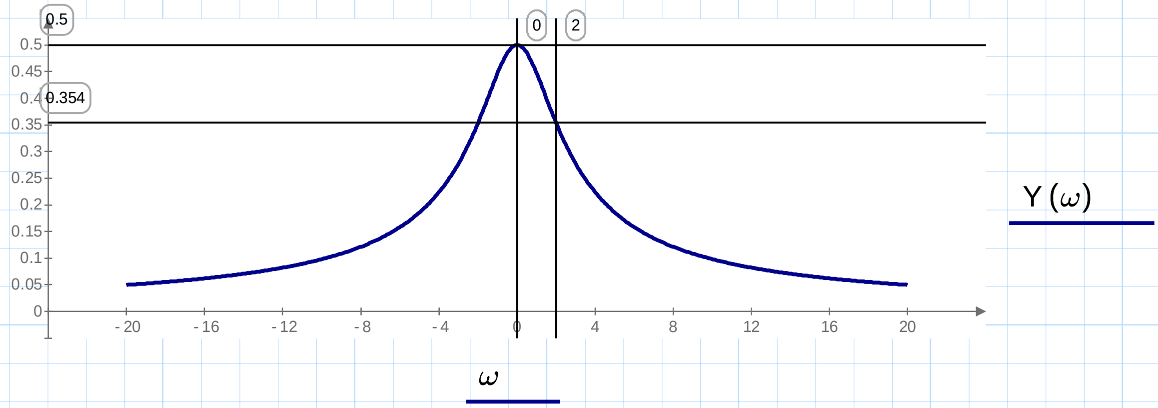
$angY(\omega) := -atan\left(\frac{\omega \cdot L}{R}\right)$

w: $\frac{R}{L} = 2 \quad \frac{2\ R}{L} = 4 \quad \frac{4\ R}{L} = 8 \quad \frac{6\ R}{L} = 12 \quad \frac{8\ R}{L} = 16 \quad \frac{10\ R}{L} = 20$





Exactly same in textbook plots above. We got the general idea. May seen this before in circuits course, if not its not out of ordinary hundreds of circuit plots have magnitude and angle plotted separately. *No where near a genius yet. A century away. Sorry.*



Magnitude vs radain frequency in the plot above is symmetrical.
w can be -ve.

v(t) = 50 cos(100t + 30 deg) ....w = 100
v(t) = 50 cos(-100t + 30 deg)....w = -100

At ***w =0*** the magnitude = 1/R, here 0.5 = 1/2 for plot above.
So you can conclude any sinusoidal response can be treated as discussed above for an RL circuit.

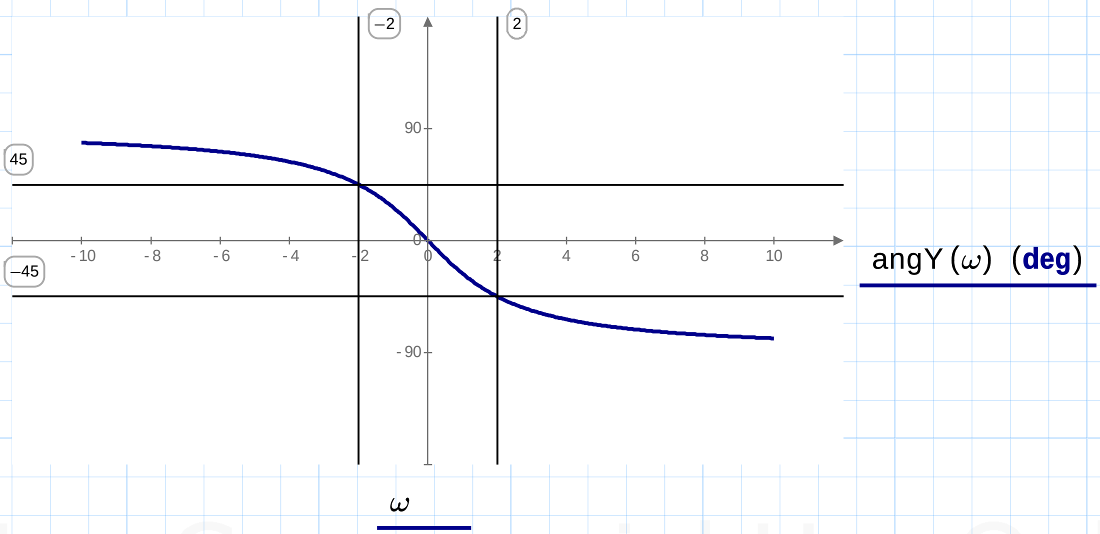
w1: $\frac{R}{L} = 2$ ***w2:*** $\frac{2\ R}{L} = 4$ ***w3:*** $\frac{4\ R}{L} = 8$ ***w4:*** $\frac{6\ R}{L} = 12$ ***w5:*** $\frac{8\ R}{L} = 16$

The magnitude of Y for w0 = 0: $\omega_0 = 0$ $Y_{\omega 0} := \frac{1}{R} = 0.5$

Multiply by 0.707: $\omega_1 = 2$ $Y_{\omega 1} = 0.707 \cdot (Y_{\omega 0}) = 0.354$

The phase angle for w1 at +/-2 is 45 degrees.
See plot below.

We saw this in plot. Unfortunatley it did not work for the other w2, w3,....



We seen some equations and plots but it did not have a written statement that went with it. Below from Hyat-Kemerly:

> 'The points w1 = R/L = 2 and -w1 = -R/L = -2 are marked on the plot. At these radian frequencies (2 and -2) the magnitude (Y) is 0.707 times the maximum magnitude at zero frequency, (w0 = 0; 2 Pi 0 = 0), and the phase angle has a magnitude of 45 degrees.'

From that paragraph above we progress to the following:

> 'At the frequency at which the admittance magnitude (Y) is 0.707 times its maximum value (Y = 0.5 at w0), the current magnitude is 0.707 times its maximum value, and the average power supplied by the source is 0.707^2 or 0.5 times its maximum value. *Comment: The maximum value will be on the power plot at w=0.*
>
> It is NOT very strange that w = R/L (here w = 2) is identified as a half power frequency.'

So, what they the engineers are saying is if we have a current versus radian frequency plot, at w=2 we have a current value which equals 0.707 times current value at w=0.

I = V/Z, Y = 1/Z, **I = Y V.**
So we see its possible for Y to provide a relationship to I because I = Y V.
Cleaver engineers!

Remember NOT for all radian frequencies on the plot ONLY for w1, and in our example w1 = R/L.

Some numbers you seen before:

$\sqrt{2} = 1.414 \qquad \frac{\sqrt{2}}{2} = 0.707 \qquad 0.707^2 = 0.5$ *Thats where 0.5 came about for the half power frequency.*

Power = VI.
The forcing function is v(t).
The forced response is i(t).
In the p(t) plot, where we have power as the y-axis, and w as the x-axis, at the half power frequency we have the average power supplied. Average power, not maximum, minimum,....but average. Got it! So we find what the half power frequency we got the average power supplied on the curve. Maximum power will be at w=0.

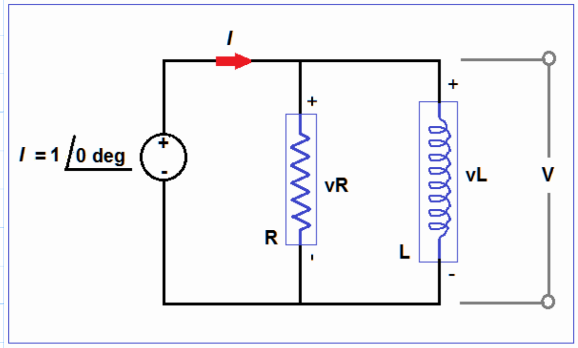
Next another example this time an LC circuit. New things to apply. LC circuit is a little harder more involved.

Parallel LC circuit:

Forcing function is current, we seek the voltage as forced response.

$I_s = I_s \angle 0$ deg <---Phasor form also polar.

Current can be leading or lagging, here leave it in phasor form at simple 0 deg. If leading voltage then v(t) need add a phase angle. Keep simple need not a sinusoidal form of I.



Paralle LC circuit:

$$V = I \cdot Z$$

$$Z_L = j\,\omega L$$

$$Z_C = \frac{1}{j\,\omega C}$$

$$Z_{total} = \frac{Z_L \cdot Z_C}{Z_L + Z_C} = \frac{(j\,\omega L)\left(\frac{1}{j\,\omega C}\right)}{(j\,\omega L) + \left(\frac{1}{j\,\omega C}\right)} = \frac{(j\,\omega L)\left(\frac{1}{j\,\omega C}\right)}{(j\,\omega L) - j \cdot \left(\frac{1}{\omega C}\right)} = \frac{\frac{j}{j} \cdot \left(\frac{\omega L}{\omega C}\right)}{j \cdot \left(\omega L - \frac{1}{\omega C}\right)}$$

$$Z_{total} = \frac{\left(\frac{L}{C}\right)}{j \cdot \left(\omega L - \frac{1}{\omega C}\right)}$$

Next we try to factor for w.

$$Z_{total} = \frac{\left(\omega \cdot \frac{C}{L}\right) \cdot \left(\frac{L}{C}\right)}{j \cdot \left(\omega \cdot \frac{C}{L}\right) \cdot \left(\omega L - \frac{1}{\omega C}\right)} = \frac{(\omega)}{j \cdot \left(\omega^2\ C - \frac{1}{L}\right)} = \frac{(\omega)}{j \cdot C \cdot \left(\omega^2 - \frac{1}{LC}\right)}$$

Need to get t (1/LC) in denominator.

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \omega_0^2 = \frac{1}{LC}$$

In series and parallel circuit we have w0 = 1 / (SQRT LC)). We need to get it in that form.

$$Z_{total} = -j \cdot \frac{(\omega)}{C \cdot \left(\omega^2 - \omega_0^2\right)}$$

Did a substitution for w0^2, next we factor the omega parenthesis. And -j = (1/j).

$$(\omega - \omega_0)\ (\omega + \omega_0) = \omega^2 + \omega\,\omega_0 - \omega\,\omega_0 - w_0^2$$
$$= \omega^2 - \omega_0^2$$

$$Z_{total} = \frac{-j}{C} \cdot \frac{(\omega)}{(\omega - \omega_0)\ (\omega + \omega_0)}$$

How does the j get suppressed or disappear?

$$\frac{(\omega)}{(\omega - \omega_0)\ (\omega + \omega_0)}$$

<---The left most term is all radian frequency, and we have -j, and that makes it -jw.

Engineering college year 2 course of 4 year program OR year 1 of 3 year program. Re-fresher OR Self Study. Graduate Study Review.
May be used in New Zealand, US, Malaysia, India, Pakistan, UK, and other Common Wealth Country engineering colleges.
Any errors and omissions apologies in advance.

$\frac{-1}{C} \cdot j \cdot \frac{(\omega)}{(\omega - \omega_0)\,(\omega + \omega_0)}$ in a form like this ---> $-\frac{1}{C} \cdot j\ \omega$

We seen -w +w on the x-axis, the j now says its an imaginary term, it can be on the +ve and -ve side of the axis.
'-jw 0 jw', and s = sigma + jw.
s can have roots s1 = sigma + jw
s2 = sigma - jw.
So the -jw in our expression is not a problem.
My first reaction each time I see -jw its how do I manipulate or work that, not a problem s = sigma +/- jw from our studies.

So next we take the absolute value of Z, this we seen in most our math and engineering course work.

$$|Z| = \left| \frac{1}{C} \cdot \frac{-j\,(\omega)}{(\omega - \omega_0)\,(\omega + \omega_0)} \right|$$

<---- What happens here?
On the plot axis -jw 0 jw
same as -w 0 w
Absolute value of -j takes it out of the expression.

$j = \sqrt{-1}$
$-j = -\sqrt{-1}$

$$(-j)^2 = \left(-\sqrt{-1}\right) \cdot \left(-\sqrt{-1}\right) = '+(-1)$$

$$(-j)^2 = -1$$

$$\left|(-j)^2\right| = 1$$

Sometimes you just take the -ve sign out but since its j, I did the square term first. You know that wasn't necessary.

$$|Z| = \frac{1}{C} \cdot \frac{(\omega)}{(\omega - \omega_0)\,(\omega + \omega_0)}$$

<---- By letting w0 = 1/SQRT(1/LC) and factoring the expression for the input impedance, the magnitude of the impedance may be writen in a form which enables those frequencies to be identified at which the response is zero or infinite - Hyat Kemerly

Key note: Frequencies for which the response is zero or infinite.

Such frequencies are termed critical frequencies.

Explanation on this coming.

Next, we proceed to create plots of |Z| and ang Z versus frequencies.

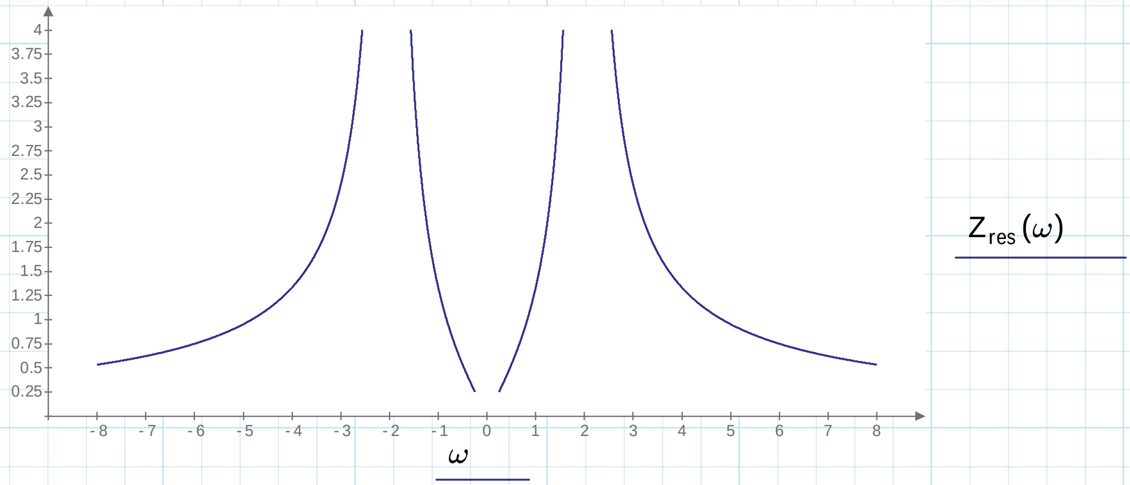
$R := 2 \quad L := 1 \quad C := 0.25 \quad \omega_0 := \frac{1}{\sqrt{L \cdot C}} = 2$ <---w0 not same to previous RL circuit.

w0: $\omega_0 = 2 \quad 2 \cdot \omega_0 = 4 \quad 3 \cdot \omega_0 = 6 \quad 4 \cdot \omega_0 = 8 \quad 5 \cdot \omega_0 = 10 \quad 6 \cdot \omega_0 = 12$

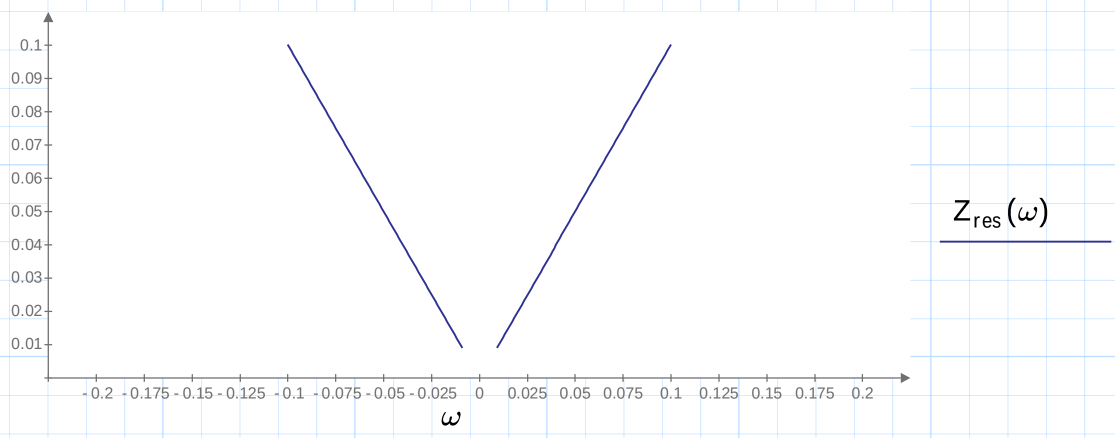
$\omega := -10.0, -9.999 .. 10.0$ <--- Initialisation for w

$Z(\omega) := \frac{1}{C} \cdot \frac{(\omega)}{(\omega - \omega_0) \, (\omega + \omega_0)}$ <--- Calculating Z(w)

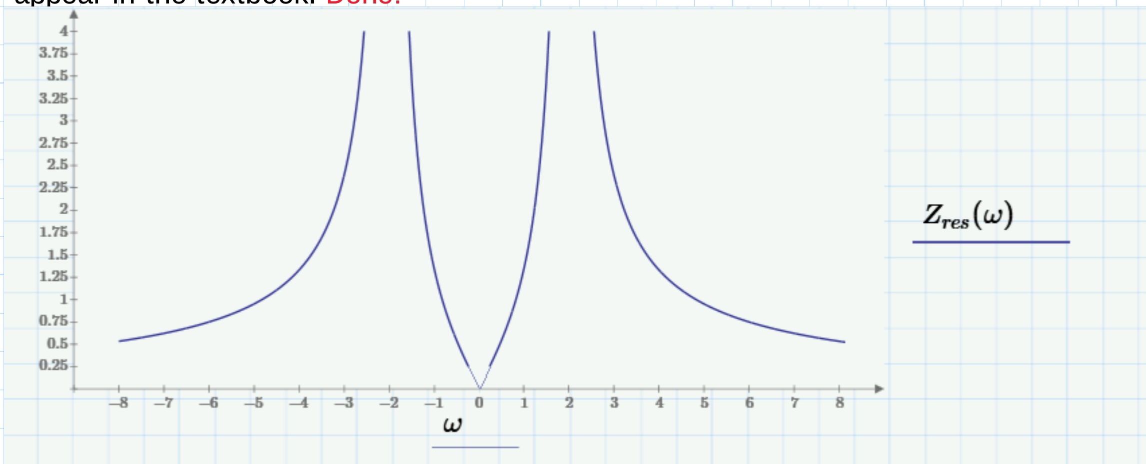
$Z_{res}(\omega) := |Z(\omega)|$ <--- Taking the magnitude of Z(w), otherwise we only get half the plot. The subscript res for response. Z response.



The middle part of plot at w=0 the 2 inner curves do not meet at 0 because of a division by zero when w=0 in Z(w). Shown below for clarity, when coming closer to w=0 by using a smaller interval. 0.01 closer to 0, compared to 0.25 to 0.



So in the plot below the part near to ***w=0*** has been sketched in just as it appear in the textbook. Done!



Next the angular plot for Z(w):

$$Z_{total} = \frac{\left(\frac{L}{C}\right)}{j \cdot \left(\omega L - \frac{1}{\omega C}\right)}$$ Lets try making the numerator 1, multiply by C/L.

Lets use Z now to represent Z total, we know its the circuit total impedance anyway.

$$Z = \frac{\left(\frac{C}{L}\right)\cdot\left(\frac{L}{C}\right)}{j \cdot \frac{C}{L} \cdot \omega L - \frac{C}{L} \cdot \frac{j}{\omega C}} = \frac{1}{j \cdot C\omega - \frac{j}{L\omega}}$$

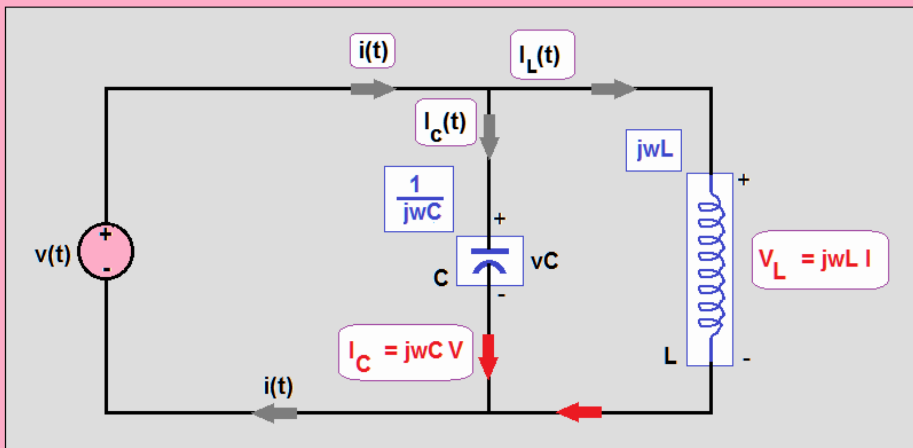
$$Z = \left(\frac{1}{j}\right) \frac{1}{\left(C\omega - \frac{1}{L\omega}\right)} = -\frac{j}{\left(C\omega - \frac{1}{L\omega}\right)}$$ <----Phase angle plot this?

$$-\frac{j}{\left(C\omega - \frac{1}{L\omega}\right)}$$ <--- How do I get the phase angle thru the inverse tangent expression?

$$angZ = -atan\left(\frac{\frac{1}{L\omega}}{C\omega}\right)$$ OR $$angZ = -atan\left(\frac{C\omega}{\frac{1}{L\omega}}\right)$$ Either of these any correct? No.

I attempted several combinations all failed. According to the engineers this has to be done thru inspection. *Tan^-1 (y/x) = ? deg*. Does not exist you sketch it. You cant get tan^-1 to result in 90 degs. *Reason I emphasised on this was because the results are at 90 deg in the graph of the textbook.*

First I read-up on the lead or lag for the inductor and capacitor. Figure contents below may need correcting check with your textbook and course notes.



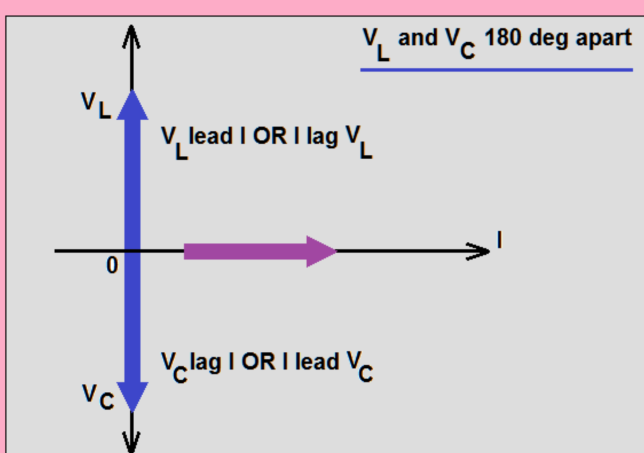
Rough Notes:

Inductor

1. Angle of the factor jwL is exactly +90 deg, and
   I must therefore lag V by 90 deg in an inductor.
   - Voltage V applied across the inductor terminals, it has to come ON first, V is leading, and eventually 90 degs later current I is appearing, potential V is needed to create the electric field which brings the current. 90 degs is a quarter cycle ahead.

Capacitor

2. Angle of the factor jwC is -90 deg, and
   I leads V by 90 degs in a capacitor. Here the capacitor has to be charged-up, and charge relates to current. When discharging, current response is maximum due to increasing voltage that occured 90 degs earlier. So when current is travelling voltage was 90 degs behind. Tank filling up, current flowing in, current first, as level rises voltage rises.
   A tank filled up, full voltage, open the tap, current flow first. I here is 90 degs ahead of V.



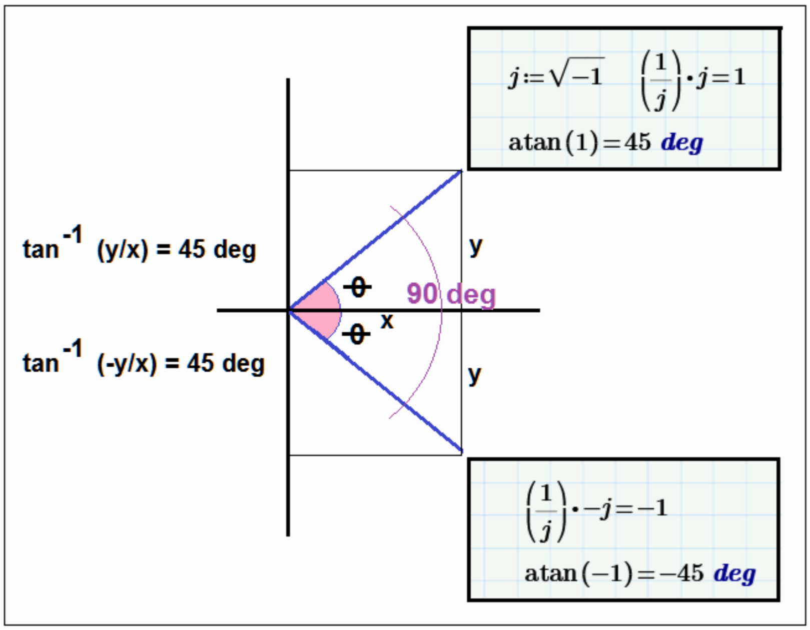
Thru inspection? I give it a try. Make 'fit-force' to match the answer.
You got a better solution go by it.

Takes me for-ever...thats okay I'm not interested in rebuilding Rome.
'Rome wasn't built in a day'. Not interested.

Math on j from college days, maybe this may help, only remember so much......You give it a better try. So I want to inspect it not evaluate.

$j := \sqrt{-1}$ $\quad \left(\frac{1}{j}\right)\cdot j = 1$ $\quad \left(\frac{1}{j}\right)\cdot -j = -1$

$\operatorname{atan}(1) = 45$ **deg** $\quad \operatorname{atan}(-1) = -45$ **deg**



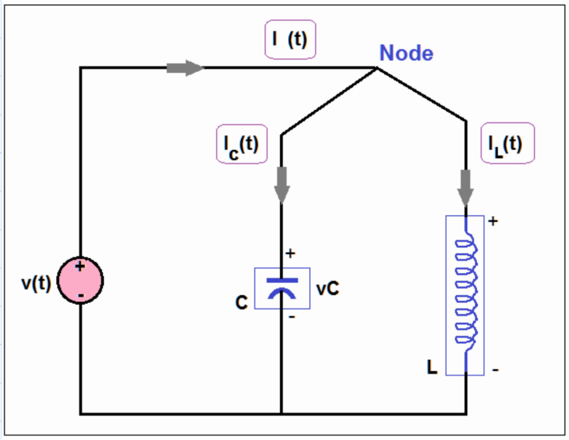
We can get the tangent of 45 degrees, and in this case can show a 90 degree between the two. Just in case if its needed in the inspection.

Not all lecturers will teach you that, some may not know depending on their experience, my UG ones were a little intuitive. Which may be bad because its a little harder to pass their test.

$$-\frac{j}{\left(C\,\omega - \frac{1}{L\,\omega}\right)}$$

<--- How do I get the phase angle thru the inverse tangent expression?

Continued next page.



Correct this discussion for any errors.

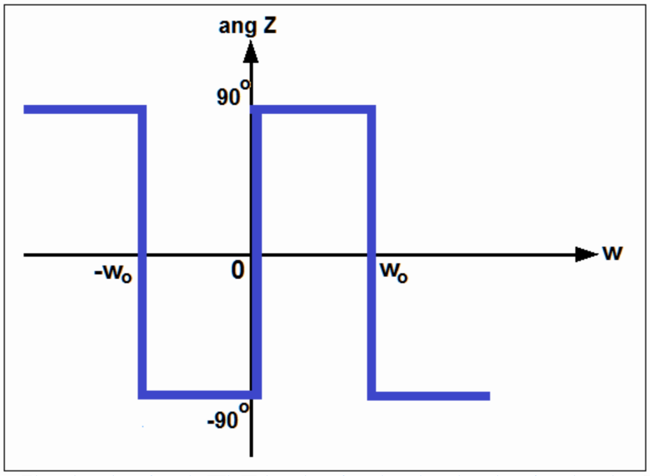
Lets assume the switch just got turned on, not shown here, so at time t<0 everything 0.

We are looking at t=0 the switch is ON.

Current starts to flow and comes to the node. Splits both directions, one to C and the other to L. We know how the capacitor and inductor work. Here the behaviour of the capacitor provide storage of charge to release as current at the right event. The inductor its a little more mysterious its v = L di/dt its providing voltage from the changing current. Both L and C have the same voltage across them in a parallel circuit. So I am saying the current I is the player here in this discussion.

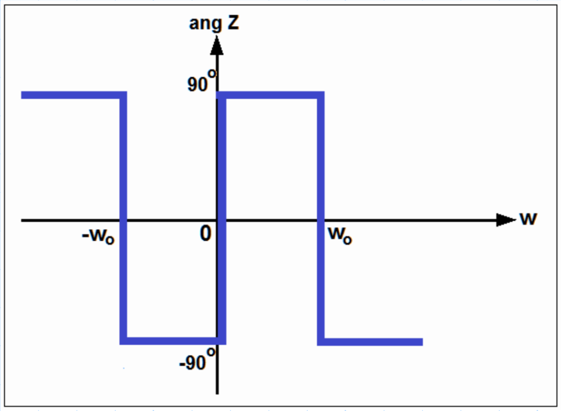
Capacitor starts getting charged current is increasing and potential across its terminal rises. What is the condition here?
1. Ic leads v(t) which we now say v(t) is V (phasor form).
2. -90 degrees lead. -ve 90 meaning I was there 1/4 cycle first, to the right of t=0.

Inductor starts getting current its increasing and potential across its terminal rises. What is the condition here?
1. Ic lags v(t) which we now say v(t) is V (phasor form).
2. 90 degrees lag. +ve 90 meaning I was there 1/4 cycle late; to the left of t=0.



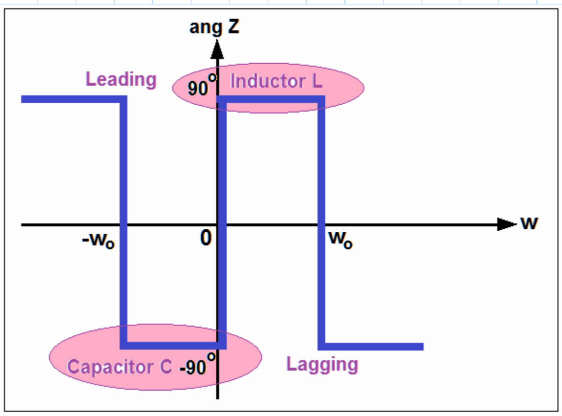
<----This is the answer the plot provided in textbook.

Lets explain the vertical line first.



L and C get their current at the same time, when one is at 90 deg +ve because its lagging (L), the other is at -90 deg -ve because its leading (C).

All this at that very same w0 and w0 is 1/sqrt(LC).
So we cycle thru each multipe of w0 plot wise.



So maybe why the engineers, Hyat and Kemerly, highlighted only one time period before and after w=0. We have one leading C and one lagging L. We had in a previous figure V_L and V_C 180 degs apart, straight line. Vertical line. And lets say j for inductor +ve vertical half of vertical line and -j of capacitor -ve vertical half of vertical line.
Maybe you agree.

Next the horizontal line.

Between -w0 and 0 we have no change in angle, there is a group of frequencies, and the reaction/response is no change remains at -90 degs. This is whats expected of a capacitor stay at -90 deg. But why over an interval of frequencies?

$$Z_{total} = \frac{\left(\omega \cdot \frac{C}{L}\right) \cdot \left(\frac{L}{C}\right)}{j \cdot \left(\omega \cdot \frac{C}{L}\right) \cdot \left(\omega\, L - \frac{1}{\omega\, C}\right)} = \frac{(\omega)}{j \cdot \left(\omega^2\ C - \frac{1}{L}\right)} = \frac{(\omega)}{j \cdot C \cdot \left(\omega^2 - \frac{1}{LC}\right)}$$

We had this expression in our earlier solution, term to the right may provide this answer.

$$\frac{(\omega)}{j \cdot C \cdot \left(\omega^2 - \frac{1}{LC}\right)} \text{ ---> } \frac{(\omega)}{j \cdot C \cdot \left(\omega^2 - {\omega_0}^2\right)} \text{ ---> }$$

$\frac{1}{LC}$ ---> ${\omega_0}^2$

$\omega^2 - {\omega_0}^2$ <--- This is that interval of frequency for L and C (w-w0)(w+w)<---interval.

So to keep it tight on the proposed solution the loose ends you can tie up if any. Check with your local engineer.

Zero and Pole:

We done the series RL and then did the parallel LC.

$$|Z| = \frac{1}{C} \cdot \frac{(\omega)}{(\omega - \omega_0)\ (\omega + \omega_0)}$$

<---- Going back a few pages we derivied this equation for the magnitude of Z.

By letting w0 = 1/SQRT(1/LC), and factoring the expression for the input impedance, the magnitude of the impedance may be writen in a form which enables those frequencies to be identified at which the response is zero or infinite - Hyat Kemerly

My/Our concern is with these frequencies where the response is zero or infinite.

Some frequencies give a zero response some frequencies give an infinite response.

Such frequencies are termed critical frequencies, and their early identification simplifies the construction of the response curves.

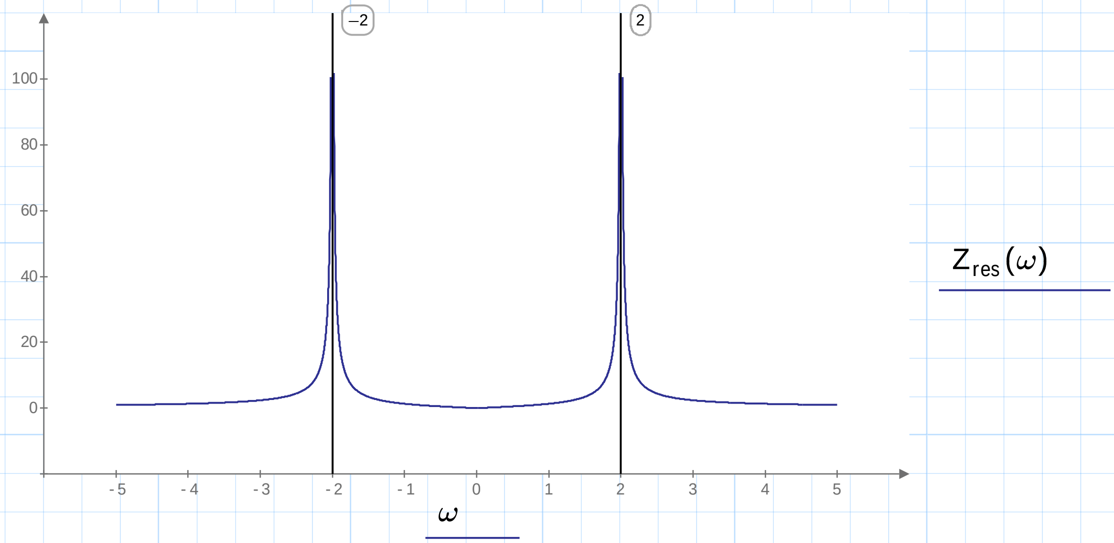
We note first that the response has zero amplitude at w = 0; when this happens, we say that the response has a zero at w = 0, and we describe the frequency at which it occurs as a zero.

<---- Zero

Discussion: *The opposite of zero response is maximum response or yet higher infinite response. We get infinite when we have something of value and divide it by something so small near 0, so let say its become zero, so 100/0 = infinite, which really was 100/0.0000001 but to get it so small its just the same as 0. But when we have 0/100 it equal 0 because we got nothing to begin with and if you have nothing, you divide nothing by 100 you got nothing. So we appreciate infinite more than? nothing.*

Continued next page with an adjustment on a previous plot.

We plotted this graph before now plotted with higher amplitude values.



We inserted vertical lines at w=-2 and w = 2.
Here the response is suddenly much higher than the other radian frequencies w.

We see the infinite amplitude in the plot at -w0 = -2 and w0 = 2,
we got w0 = 1/sqrt(LC). For frequencies <-w0 and >w0 the amplitude approaches zero.

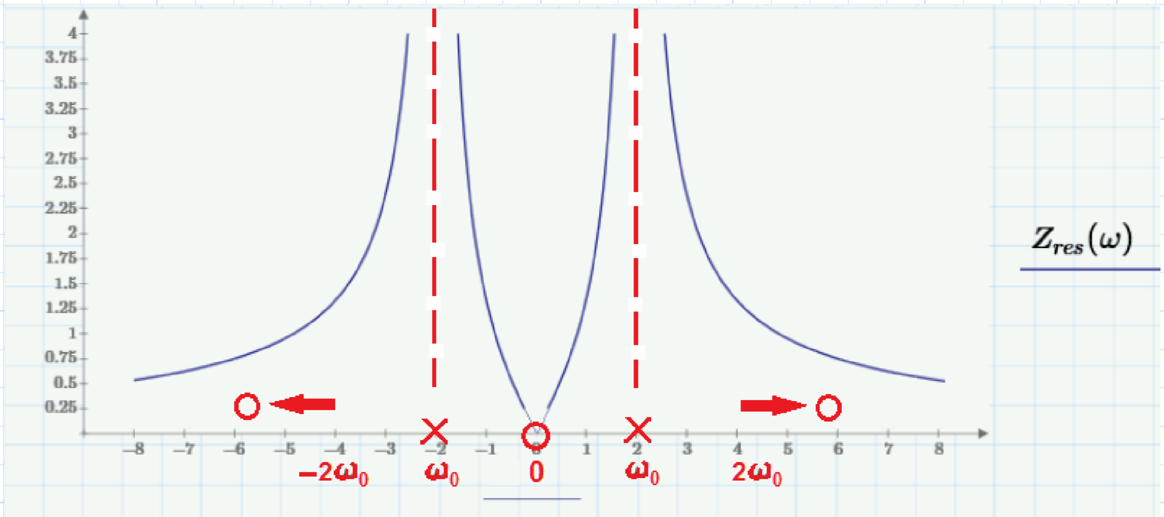
Response of infinite amplitude is noted at w = w0 and w = -w0; these frequencies are called poles, and the response is said to have a pole at each of these frequencies. **<---- Pole**

Finally, we note that the response approaches zero as w --> infinity and thus w = +/- infinity is also zero. **<---- Zero**

Note: It is customary to consider plus infinity and minus infinity as being the same point. The phase angle of the response at very large positive and negative values of w need not be the same however.

Read a 2nd time, correct any errors, and if its satisfactory, then lets say we got zeros and poles identified.

In some textbooks zeros and poles are identified in a mathematical evaluation kind of explanation, if I am right thats mostly in controls course they assume you got this from a circuits course. So here we got some understanding on zeros and poles related to radian frequencies. Next how to identify them on a graph using markers/symbols.



Location for critical frequencies are marked on the w-axis using small circles for zeros and crosses for poles. Poles or zeros at infinity frequencies should be indicated by an arrow near the axis, as shown figure above.

The actual drawings of the graph is made easier by adding broken vertical lines as asymptotes at each pole location. The completed graph of magnitude versus w (radian frequency) shown above where the slope at the origin is not zero.

Pages 273-276 of Hyat and Kemerly 4th ed.

I/We come to end of Part 3A.
Next Part 3B I/We pick up here in complex frequency.
After which continue with Schaums chapter 8.

Tentative Plan:
It looks like there is parts A, B, C, D, and E.
Parts C and D will be the end of Schaums chapter 8 solved and unsolved problems.
Part E on some special topics needed for Laplace for electric circuits.
Hopefully that covers the pre-requisites for Laplace from the electric circuits side.
Which I can attest now is not a 100% coverage, you may agree.
One of which is the in depth math side of Laplace that we cannot cover here that has to reside in the maths course a self study refresher is encouraged thru my/your/our textbook(s).

Apologies in advance for any errors and omissions.