LECTURE 24: CONTINUOUS FUNCTIONS (II)

Today: We'll prove some basic properties of continuous functions, such as $f+g$ is continuous or fg is continuous.

Recall Definition 1:

f is **continuous at** x_0 if, whenever x_n is a sequence that converges to x_0, then $f(x_n)$ converges to $f(x_0)$

Recall Definition 2:

f is **continuous** at x_0 if for all $\epsilon > 0$ there is $\delta > 0$ such that for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$

Note: Remember our convention that x_0, x, and x_n are assumed to be in the domain of f

Note: The book proves everything using sequences, but I will prove the results both using sequences and using $\epsilon - \delta$. This is not to torture you, but it is very important to be comfortable with $\epsilon - \delta$ proofs, as they will be crucial in Math 140B. So definitely thoroughly read the proofs below!

1. $f+g$ IS CONTINUOUS

Date: Friday, May 22, 2020.

Video: $f + g$ is continuous

Fact 1:

If f and g are continuous at x_0, then $f + g$ is continuous at x_0

Proof using Definition 1: Let x_n be a sequence converging to x_0. Then, since f is continuous at x_0, we get $f(x_n) \to f(x_0)$ and, since g is continuous at x_0, we have $g(x_n) \to g(x_0)$. But, by the sum law for limits of sequences (see section 9), we get:

$$(f+g)(x_n) = f(x_n) + g(x_n) \to f(x_0) + g(x_0) = (f+g)(x_0) \checkmark$$

Hence $f + g$ is continuous at x_0 $\square$

Note: Notice how the result about $f+g$ follows from the corresponding result for sequences! This will be pretty much true for all our proofs involving Definition 1.

Proof using Definition 2: (do not skip!)

Let $\epsilon > 0$ be given

Then, since f is continuous at x_0, there is $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2}$.

And, since g is continuous at x_0, there is $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$, then $|g(x) - g(x_0)| < \frac{\epsilon}{2}$.

But then, if $\delta = \min\{\delta_1, \delta_2\} > 0$, we get:

$$\begin{aligned}|(f+g)(x)-(f+g)(x_0)| &= |f(x)+g(x)-(f(x_0)+g(x_0))| \\ &= |f(x)-f(x_0)+g(x)-g(x_0)| \\ &\leq |f(x)-f(x_0)|+|g(x)-g(x_0)| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon \checkmark\end{aligned}$$

Hence $f+g$ is continuous at x_0 □

2. kf is continuous

As a tribute to kfC, let's prove that:

Fact 2:

If f is continuous at x_0, and k is a real number, then kf is continuous at x_0

Proof using Definition 1: If (x_n) is a sequence that converges to x_0, then, since f is continuous at x_0, $f(x_n) \to f(x_0)$, and therefore

$$(kf)(x_n) = k(f(x_n)) \to k(f(x_0)) = (kf)(x_0)\checkmark$$

And therefore kf is continuous at x_0 □

Proof using Definition 2: First of all, we may assume $k \neq 0$, because otherwise $kf = 0$, which is continuous.

Let $\epsilon > 0$, then, since f is continuous at x_0, there is $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \frac{\epsilon}{|k|}$ (we use absolute values because k might be negative)

Then, with the same δ, if $|x - x_0| < \delta$, we get:

$$|(kf)(x) - (kf)(x_0)| = |kf(x) - kf(x_0)| = |k| \, |f(x) - f(x_0)| < |k| \left(\frac{\epsilon}{|k|} \right) = \epsilon \checkmark$$

Therefore kf is continuous at x_0 □

Aside: If you've taken linear algebra, notice that Fact 1 says that continuous functions are closed under addition, and Fact 2 says that they are closed under scalar multiplication. Therefore, the set of continuous functions forms a vector space!

Note: Using Facts 1 and 2 and the fact that x^n is continuous for all $n \geq 0$, we get that polynomials like $4x^3 - 5x^2 + 4x + 1$ are continuous

Corollary:

If f and g are continuous at x_0, then $f - g$ is continuous at x_0

Proof: Since g is continuous at x_0, using Fact 2 above with $k = -1$, we get $-g = (-1)g$ is continuous at x_0.

Therefore, since f and $-g$ are continuous at x_0, by Fact 1, $f - g = f + (-g)$ is continuous at x_0 □

3. $|f|$ IS CONTINUOUS

In this small interlude, let's prove the following quick fact:

Fact 3:

If f is continuous at x_0, then $|f|$ is continuous at x_0

Proof using Definition 1: Suppose $x_n \to x_0$, then, since f is continuous at x_0, $f(x_n) \to f(x_0)$, and therefore $|f(x_n)| \to |f(x_0)|$ ✓

Hence $|f|$ is continuous at x_0 □

Proof using Definition 2: Let $\epsilon > 0$ be given. Then, since f is continuous at x_0, there is $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

With that same δ, if $|x - x_0| < \delta$, then by the reverse triangle inequality, which says $|a - b| \geq ||a| - |b||$, we have:

$$||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)| < \epsilon \checkmark$$

Therefore $|f|$ is continuous at x_0 □

4. fg IS CONTINUOUS

Video: fg is continuous

Now let's prove that the product of continuous functions is continuous:

Fact 4:

If f and g are continuous at x_0, then fg is continuous at x_0

Proof using Definition 1: Suppose $x_n \to x_0$. Then, since f is continuous at x_0, we have $f(x_n) \to f(x_0)$, and, since g is continuous at x_0, we have $g(x_n) \to g(x_0)$, and therefore, by the product law for limits (section 9), we have

$$(fg)(x_n) = (f(x_n))\,(g(x_n)) \to (f(x_0))\,(g(x_0)) = (fg)(x_0) \checkmark$$

Therefore fg is continuous at x_0 □

Proof using Definition 2:

STEP 1: Scratchwork

We need to estimate:

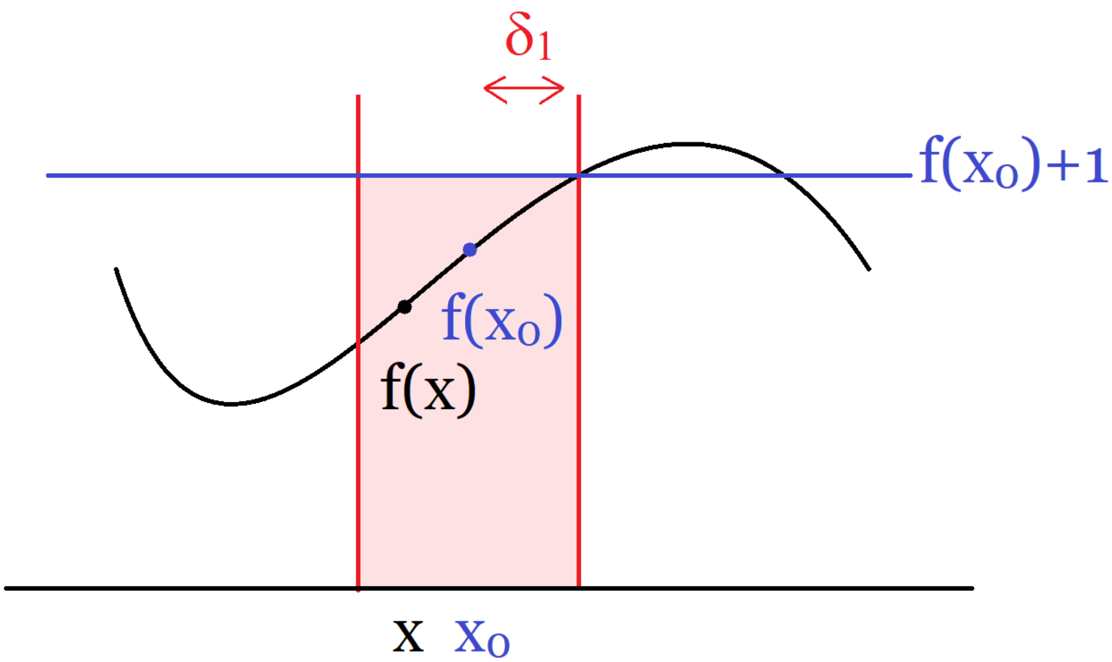
$$\begin{aligned}|f(x)g(x) - f(x_0)g(x_0)| &= |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)| \\ &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leq |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)|\end{aligned}$$

Now the $|f(x) - f(x_0)|$ and $|g(x) - g(x_0)|$ terms are good, since f and g are continuous at x_0. Moreover, the $|g(x_0)|$ term is good since it is constant.

The only problematic term is $|f(x)|$ since it depends on x. For this, use the fact that, since f is continuous, $f(x)$ is close to $f(x_0)$ (which is constant)

Since f is continuous with $\epsilon = 1$, we get that there is $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < 1$, but then

$$|f(x)| = |f(x) - f(x_0) + f(x_0)| \leq |f(x) - f(x_0)| + |f(x_0)| < 1 + |f(x_0)|$$



(In the picture above, notice that all the $f(x)$ in the red region are below the constant $f(x_0) + 1$)

Therefore, going back to our original inequality, we get:

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| \leq & |f(x)| \, |g(x) - g(x_0)| + |g(x_0)| \, |f(x) - f(x_0)| \\ \leq & (|f(x_0)| + 1) \, |g(x) - g(x_0)| + |g(x_0)| \, |f(x) - f(x_0)| \end{aligned}$$

We are finally ready for our actual proof:

STEP 2: Actual Proof:

Let $\epsilon > 0$ be given

Then, since f is continuous at x_0, there is $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < 1$, and therefore $|f(x)| \leq |f(x_0)| + 1$ (as before)

Now since g is continuous at x_0, there is $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$, then $|g(x) - g(x_0)| < \frac{\epsilon}{2(|f(x_0)|+1)}$ (the factor 2 is there because we have 2 terms)

Finally, since f is continuous at x_0, there is $\delta_3 > 0$ such that if $|x - x_0| < \delta_3$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2(|g(x_0)|+1)}$ (we have to use $|g(x_0)| + 1$ since $g(x_0)$ might be 0)

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$, then if $|x - x_0| < \delta$, then we get:

$$\begin{aligned}
|(fg)(x) - (fg)(x_0)| &= |f(x)g(x) - f(x_0)g(x_0)| \\
&\leq (|f(x_0)| + 1)\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)| \\
&< \cancel{(|f(x_0)| + 1)} \left(\frac{\epsilon}{2\cancel{(|f(x_0)| + 1)}}\right) + |g(x_0)| \left(\frac{\epsilon}{2\left(|g(x_0)| + 1\right)}\right) \\
&= \frac{\epsilon}{2} + \underbrace{\left(\frac{|g(x_0)|}{|g(x_0)| + 1}\right)}_{<1} \left(\frac{\epsilon}{2}\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon \checkmark
\end{aligned}$$

Therefore fg is continuous at x_0 $\square$

Note: This is why functions like $x\sin(x)$ or $e^x(x^2 + 1)$ are continuous, since they are products of continuous functions.

5. $\frac{f}{g}$ IS CONTINUOUS

Video: $\frac{f}{g}$ is continuous

In this section, we prove that quotients $\frac{f}{g}$ of continuous functions are continuous. For this, we need to first show that reciprocals $\frac{1}{f}$ of continuous functions are continuous.

Fact 5:

If $f \neq 0$ and f is continuous at x_0, then $\frac{1}{f}$ is continuous at x_0

Proof using Definition 1: If x_n is a sequence converging to x_0, then, since f is continuous at x_0, $f(x_n) \to f(x)$. By assumption $f(x_n) \neq 0$ for all n and $f(x) \neq 0$, so, by the results in section 9, $\frac{1}{f(x_n)} \to \frac{1}{f(x_0)}$ ✓

Therefore $\frac{1}{f}$ is continuous at x_0. □

Proof using Definition 2:

STEP 1: Scratchwork

This time we need to estimate

$$\left|\frac{1}{f(x)} - \frac{1}{f(x_0)}\right| = \left|\frac{f(x_0) - f(x)}{f(x)f(x_0)}\right| = \frac{|f(x) - f(x_0)|}{|f(x)|\,|f(x_0)|}$$

The $|f(x) - f(x_0)|$ term is good, and the $|f(x_0)|$ term is good as well (since it is constant)

The only term we need to control is the $|f(x)|$ term.

Note: Since we want $\frac{1}{|f(x)|}$ < something, we need $|f(x)| >$ something!

Before, for the product law, we used that $|f(x) - f(x_0)| < 1$. This doesn't *quite* work because we may have $|f(x_0)| - 1 < 0$ (see below). That's why we need a more subtle estimate. For this notice that $\frac{|f(x_0)|}{2} > 0$ (the choice for this will be clearer below)

Since f is continuous at x_0, with $\epsilon = \frac{|f(x_0)|}{2} > 0$, there is δ_1 such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$. But then, using again the triangle inequality (since we need $|f(x)| \geq$ something), we get

$$|f(x) - f(x_0)| \geq ||f(x)| - |f(x_0)|| \geq -(|f(x)| - |f(x_0)|) = |f(x_0)| - |f(x)|$$

(Here we used the fact that $|a| \geq -a$ for all a)

Therefore, we get

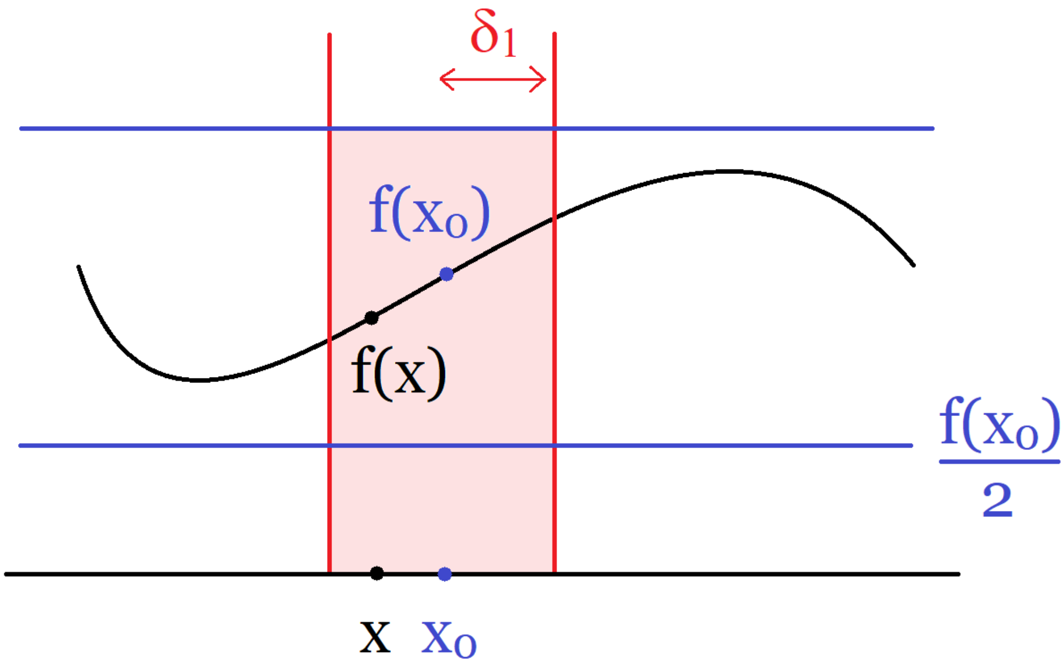
$$|f(x_0)| - |f(x)| \leq |f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$$

And therefore

$$|f(x)| > |f(x_0)| - \frac{|f(x_0)|}{2} = \frac{|f(x_0)|}{2} > 0$$

Hence

$$\frac{1}{|f(x)|} < \frac{2}{|f(x_0)|} \text{ (GOOD)}$$



(In the picture above, notice that in the red region, $f(x)$ is above $\frac{f(x_0)}{2}$)

Note: Had you chosen $|f(x) - f(x_0)| < 1$, you would have gotten $|f(x_0)| - 1$ in the above, which isn't necessarily positive! That's why we had to use $\frac{|f(x_0)|}{2}$ instead of 1

Hence, going back to our original identity, we get

$$\begin{aligned}\left|\frac{1}{f(x)} - \frac{1}{f(x_0)}\right| &= \frac{|f(x) - f(x_0)|}{|f(x)|\,|f(x_0)|} \\ &\leq \frac{|f(x) - f(x_0)|}{|f(x_0)|}\left(\frac{2}{|f(x_0)|}\right) \\ &= |f(x) - f(x_0)|\left(\frac{2}{|f(x_0)|^2}\right) \\ &\overset{?}{<} \epsilon\end{aligned}$$

Which gives $|f(x) - f(x_0)| < \frac{\epsilon}{2}|f(x_0)|^2$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given

Then, since f is continuous at x_0, there is $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$, which implies $|f(x)| > \frac{|f(x_0)|}{2}$, and therefore $\frac{1}{|f(x)|} < \frac{2}{|f(x_0)|}$

Moreover, since f is continuous at x_0, there is $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2}|f(x_0)|^2$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$, then, if $|x - x_0| < \delta$, then

$$\begin{aligned}\left|\frac{1}{f(x)}-\frac{1}{f(x_0)}\right| &= \frac{|f(x)-f(x_0)|}{|f(x)|\,|f(x_0)|}\\ &\leq \left(\frac{|f(x)-f(x_0)|}{|f(x_0)|}\right)\left(\frac{2}{|f(x_0)|}\right)\\ &= |f(x)-f(x_0)|\left(\frac{2}{|f(x_0)|^2}\right)\\ &< \left(\frac{\epsilon\cancel{|f(x_0)|^2}}{\cancel{2}}\right)\cancel{\left(\frac{2}{|f(x_0)|^2}\right)}\\ &= \epsilon\checkmark\end{aligned}$$

Hence $\frac{1}{f}$ is continuous at x_0 □

Corollary:

If f and g are continuous at x_0 with $g \neq 0$, then then $\frac{f}{g}$ is continuous at x_0

Proof: Since g is continuous at x_0 and $g \neq 0$, by the above, $\frac{1}{g}$ is continuous at x_0, and therefore, by the product law (Fact 4), $\frac{f}{g} = f\left(\frac{1}{g}\right)$ is continuous at x_0 □

Note: This is why rational functions like $\frac{x^3-1}{x^2+4}$ are continuous, and also why $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is continuous (whenever it's defined)

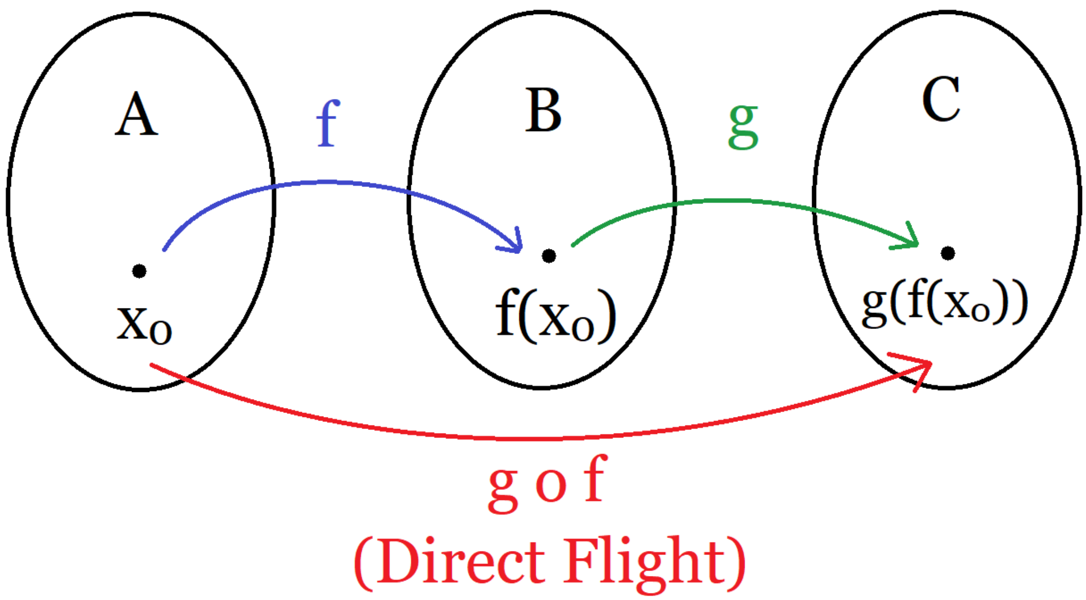
6. Composition is continuous

Video: $g \circ f$ is continuous

Definition:

If A, B, C are subsets of $\mathbb{R}$ and $f : A \to B$ and $g : B \to C$ are functions, then the **composition** $g \circ f : A \to C$ is defined by

$$(g \circ f)(x) = g(f(x))$$



Analogy: If you think of f as a layover from A to B and g as a layover from B to C, then $g \circ f$ is a direct flight from A to C

Let's show that the composition of two continuous functions is continuous!

Fact 6:

If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0

Proof using Definition 1: Suppose (x_n) is a sequence that converges to x_0. Then, since f is continuous at x_0, we have $f(x_n) \to f(x_0)$, but now, since g is continuous at $f(x_0)$, we have $g(x_n) \to g(x_0)$ and therefore:

$$(g \circ f)(x_n) = g(f(x_n)) \to g(f(x_0)) = (g \circ f)(x_0)\checkmark$$

And therefore $g \circ f$ is continuous at x_0 □

Note: It's because of this fact that, for instance $\cos(e^x)$ is continuous, being the composition of $\cos(x)$ and e^x. Similarly $\sin\left(\frac{1}{x}\right)$ is continuous except at $x = 0$ where it's undefined

Proof using Definition 2: Let $\epsilon > 0$ be given.

Since g is continuous at $f(x_0)$, there is $\delta' > 0$ such that

$$|x - f(x_0)| < \delta' \Rightarrow |g(x) - g(f(x_0))| < \epsilon$$

Note: Since the above is valid for all x, it is in particular valid with $f(x)$ instead of x (which is more specific)

Therefore there is $\delta' > 0$ such that

$$|f(x) - f(x_0)| < \delta' \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon$$

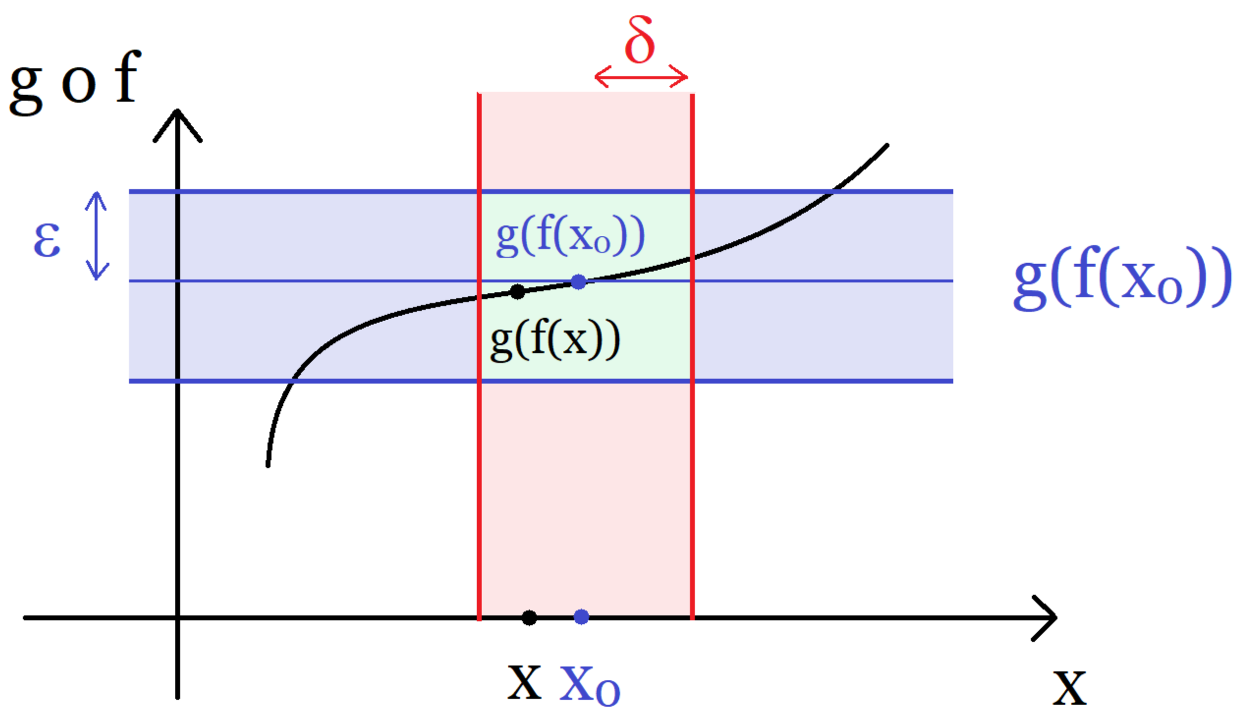
But now, since f is continuous at x_0, by the definition of continuity (but with δ' instead of ϵ), there is $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta'$.

Therefore, with δ as above, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta'$ and therefore

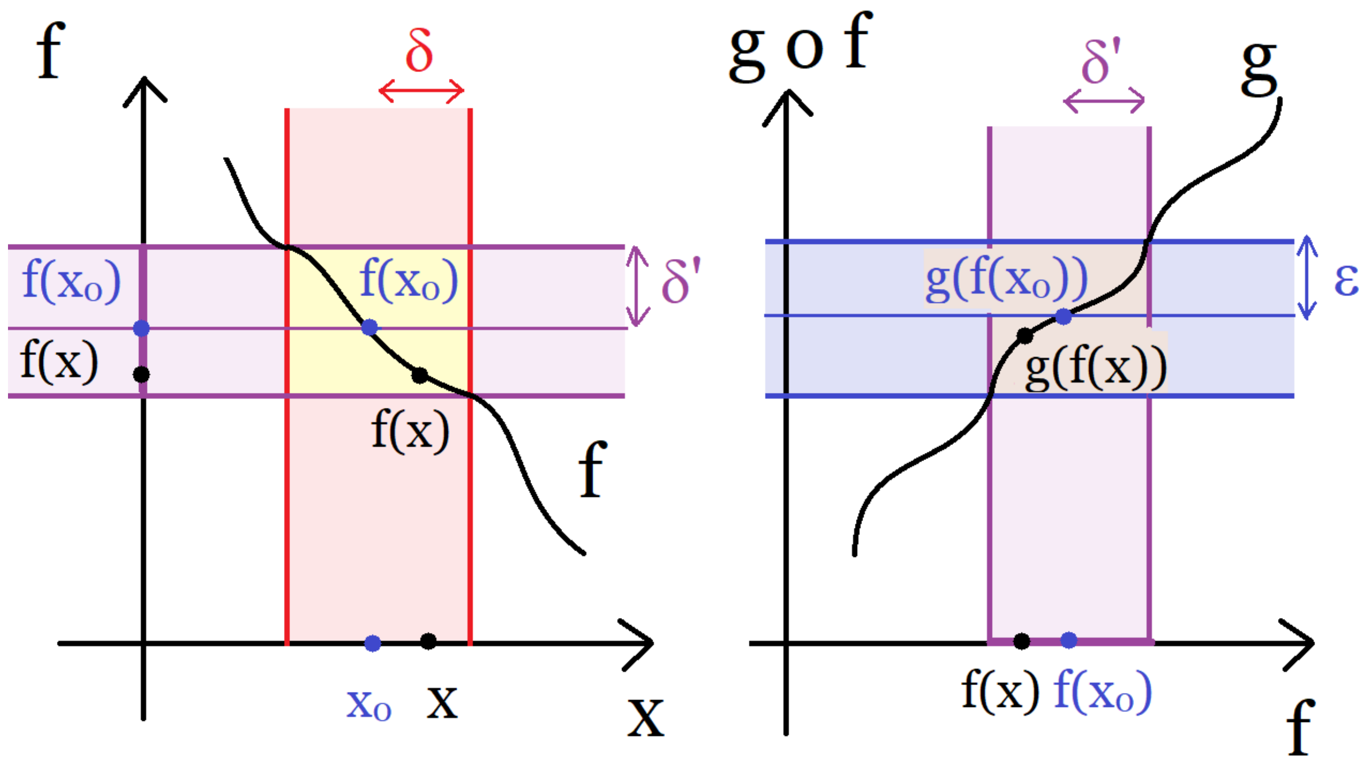
$$|(g \circ f)(x) - (g \circ f)(x_0)| = |g(f(x)) - g(f(x_0))| < \epsilon\checkmark$$

Therefore $g \circ f$ is continuous at x_0 □

Note: What is this proof saying intuitively? Our goal is to show that if x is close to x_0 then $g(f(x))$ is close to $g(f(x_0))$, as in the following picture. In other words, we need to show that if x is in the threshold region (in red), then $g(f(x))$ is in the good region (in blue/green), as in the following picture:



If x is in the threshold region (in red), then $f(x)$ is in the good region for f (in purple below).



Upshot: The good region for f (in purple, on the left) is the threshold region of g (also in purple, on the right)!

So if x is so close to x_0 that the purple region (on the left) is small, then the same purple region (but on the right) is so small that then we can guarantee that $g(f(x))$ is in the good region for $g \circ f$ (in blue), which is what we want.

7. $\max(f, g)$ IS CONTINUOUS

Video: $\max(f, g)$ is continuous

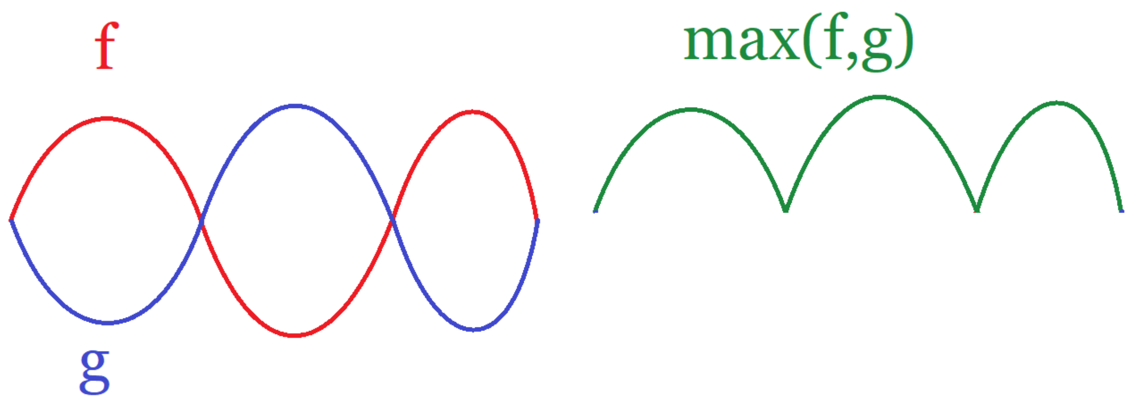
Note: This section is optional, but will be useful for the homework.

Finally, let's show that the maximum of f and g is continuous.

Definition:

$$\max(f,g)(x) = \begin{cases} f(x) \text{ if } f(x) \geq g(x) \\ g(x) \text{ if } g(x) \geq f(x) \end{cases}$$

In other words, at each x, $\max(f,g)$ is just the bigger one of $f(x)$ and $g(x)$



Fact 7:

If f and g are continuous at x_0, then $\max(f,g)$ is continuous at x_0

The proof of this relies on the following explicit formula for $\max(f,g)$

Claim:

$$\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$

Proof of Claim:

Case 1: $f(x) \geq g(x)$

Then $\max(f,g) = f(x)$, but also. since $f(x) - g(x) \geq 0$, we have $|f(x) - g(x)| = f(x) - g(x)$, and so

$$\begin{aligned}
\frac{1}{2}(f(x)+g(x)) + \frac{1}{2}|f(x)-g(x)| =& \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}(f(x)-g(x)) \\
=& \frac{1}{2}(f(x)+g(x)+f(x)-g(x)) \\
=& \frac{1}{2}(2f(x)) \\
=& f(x) \checkmark
\end{aligned}$$

Case 2: $g(x) \leq f(x)$

Similar, except you use $|f(x) - g(x)| = g(x) - f(x)$ since $f(x) - g(x) \leq 0$ here ✓ $\square$

Proof of Fact: Since f and g are continuous at x_0, $f+g$ is continuous at x_0, and therefore $\frac{1}{2}(f+g)$ is continuous at x_0.

But also $f-g$ is continuous at x_0, and therefore $|f-g|$ is continuous at x_0, and hence $\frac{1}{2}|f-g|$ is continuous at x_0, and therefore:

$$\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$

is continuous at x_0 (as the sum of two continuous functions) $\square$

Note: It then follows that the max of three functions f, g, h is continuous at x_0 because

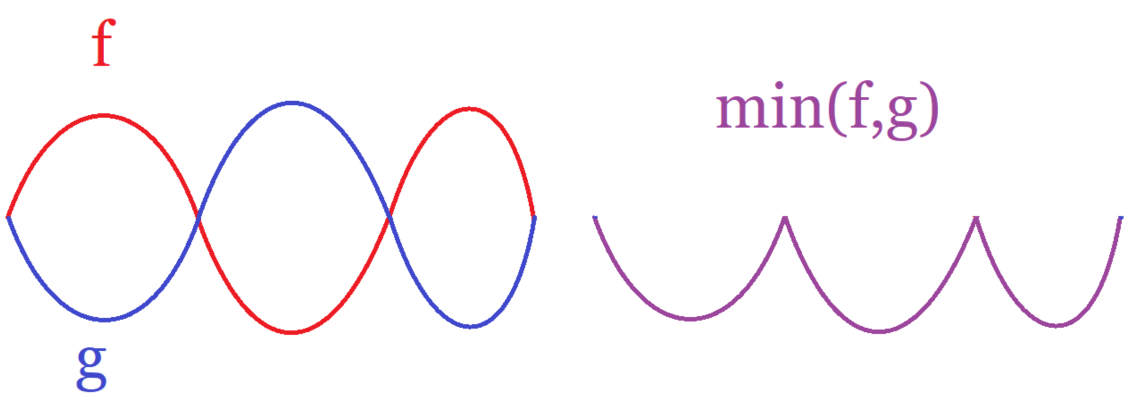
$$\max(f, g, h) = \max(\max(f, g), h)$$

And in fact, by induction, you can show that the max of finitely many continuous functions is continuous.

Remark: Similarly, you can define

Definition:

$$\min(f, g)(x) = \begin{cases} f(x) \text{ if } f(x) \leq g(x) \\ g(x) \text{ if } g(x) \leq f(x) \end{cases}$$



And similarly you can show

Fact:

If f and g are continuous at x_0, then $\min(f, g)$ is continuous at x_0

Proof: See Homework for details, but you either show (similar to above) that

$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$$

Or use that

$$\min(f, g) = -\max(-f, -g) \quad \square$$

(Compare this to $\inf(S) = -\sup(-S)$ from Chapter 1)