Introduction to Non-Gaussian Random Fields: a Journey Beyond Gaussianity

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Outline

Why Gaussian ?

Some good reasons for using Gaussian Random Fields (RF)

- Fully characterized with two moments
- Likelihood accessible
- Conditional expectation is linear
- Stability under linear combinations, marginalization and conditioning

But data are rarely Gaussian

Environmental / climatic data are often

- positive: grade, composition, ...
- in an interval: humidity, ...
- skewed: pollution, temperature, ...
- long tailed: rain, grade, ...

Need to go beyond the Gaussian world

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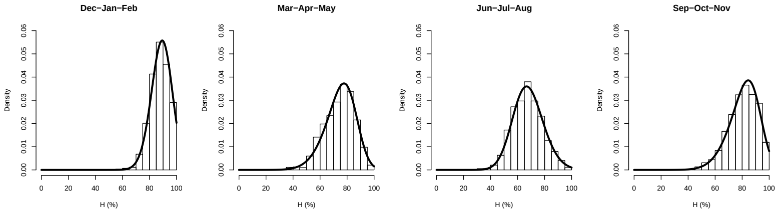
Need to go beyond the Gaussian world

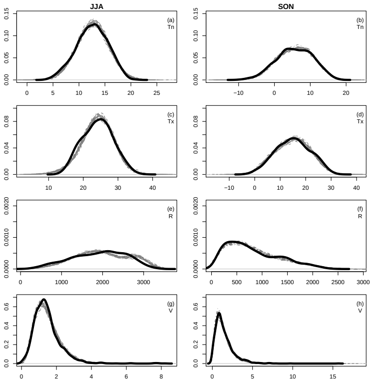
Humidity

Humidity per season (as a %), in Toulouse (France)



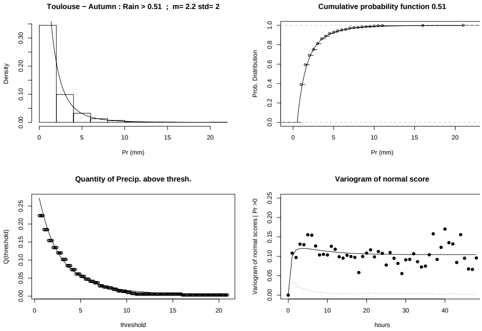
4 climatic variables

Tn, Tx, R and W in Toulouse, summer and autumn



Rain in Toulouse (autumn)

Histogram, cpf, and quantity above threshold



Leaving the Gaussian world, but not to far...

There is a need for

- Non Gaussian RF, but which model ?
- With good mathematical properties, i.e. easy to handle

$\Rightarrow$ playing with Gaussian RFs

- Tranforming : transformed multi- and bi- Gaussian RFs
- Thresholding : Excursion sets
- Truncating : Truncated Gaussian and transformed Gaussian RFs
- Conditioning : Skew-normal RFs

General Outline

- ✓ Some reminders on Gaussian RFs
- ✓ Rfs with Gaussian marginals that are not Gaussian RFs
- ✓ Transformed multi- and bi- Gaussian RFs
- ✓ Quite specific tranformation: thresholding
  $\rightarrow$ Random Sets
- ✓ Truncated (transformed) Gaussian RFs
- ✓ Skew-normal RFs

Illustrated with applications !

Some reminders

Gaussian RF

A RF is (multi-) Gaussian if all its finite-dimensional distributions are multivariate Gaussian.

Characterization

A stationary Gaussian RF is characterized by its expectation and its covariance function, $C(h)$

Bochner's theorem

The covariance function is semi positive definite function; it is the Fourier Transform of a positive bounded measure.

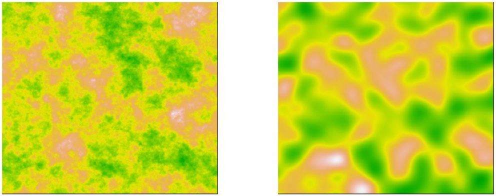
$$C(h) = \int e^{2\pi \imath \langle u,h \rangle} F(du), \quad \text{with} \quad \int F(du) < \infty$$

Some reminders

Regularity of a stationary RF

- ✓ A RF is mean-squared continuous iif its covariance function is continuous at $h = 0$
- ✓ A RF is mean-squared differentiable everywhere iif its covariance function has a second derivative at $h = 0$

$C(h) = e^{-||h||/a}$ $\qquad$ $C(h) = e^{-||h||^2/a^2}$

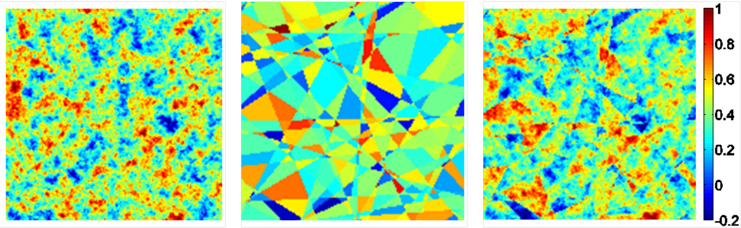


Outline

Non Gaussian Fields with Gaussian marginals

Same $\mathcal{N}(0,1)$ pdf; same exponential covariance

[Garrigues, Allard and Baret (2007)]



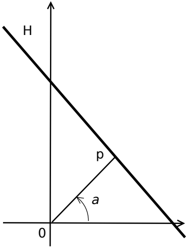
Gaussian RF — Poisson Line RF — Mixture

Poisson tesselation

Recall

A hyper-plance in $\mathbb{R}^d$ is specified by a direction $\alpha \in S_d^+$ and a location $p \in \mathbb{R}$

$$H(\alpha, p) = \{x \in \mathbb{R}^d \,|< x, \alpha >= p\}.$$



Poisson tesselation

Definition 1

A network of Poisson hyperplanes is parametrized by a Poisson process in $S_d^+ \times \mathbb{R}$. They define Poisson cells.

Definition 2

Consider a Poisson hyperplane process on $\mathbb{R}^d$. To each Poisson cell, associate an independent random variable. This defines a Poisson hyperplane RF on $\mathbb{R}^d$.

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Covariance of Poisson cell models

Proposition

The covariance function of a Poisson hyperplane RF is

$$C(h) = \sigma^2 e^{-a||h||} = \sigma^2 \rho(h), \quad h \in \mathbb{R}^d$$

Sketch of the proof: The intersection of the Poisson hyperplanes with any line defines a 1D Poisson point process with intensity, say a.

Example

Poisson lines in $\mathbb{R}^2$ parametrized by a Poisson process in $[0, \pi[\times\mathbb{R}$ and i.i.d. $\mathcal{N}(\mu, \sigma^2)$ Gaussian random variables define a marginal Gaussian Poisson cell model.

Illustration

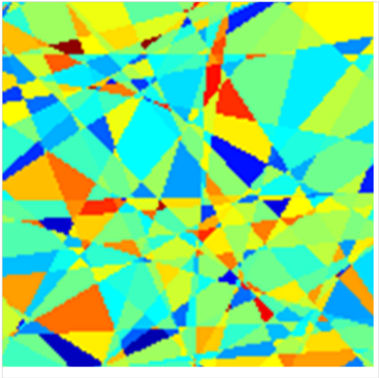


Illustration (ctd)

Use variogram of order 1 (madogram)

$$\gamma_1(h) = 0.5E[|Y(x+h) - Y(x)|]$$

Gaussian RF: Let $G \sim \mathcal{N}(0, \tau^2)$. We know $E[|G|] = \sqrt{2\tau^2/\pi}$. Then,

$$\gamma_1(h) \propto \sqrt{\gamma(h)}$$

since $Y(x+h) - Y(x) \sim \mathcal{N}(0, 2\gamma(h))$.

Poisson RF: Consider $A = \{w, w+h \in \text{same cell}\}$: $P(\bar{A}) = 1 - \rho(h) = \gamma(h)$.

$$E[|Y(x+h) - Y(x)| \mid A] = 0 \ \text{ and } \ E[|Y(x+h) - Y(x)| \mid \bar{A}] \propto \sigma^2$$

Thus $\gamma_1(h) \propto \gamma(h)$.

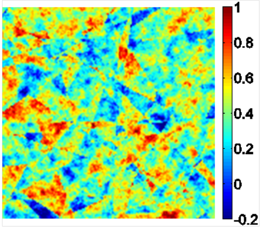
Mixture RF:

Define

$$Y_m(x) = \sigma \left(w Y_G(x) + \sqrt{1 - w^2} Z_P(x) \right) + \mu$$

where $Z_G(\cdot)$ and $Z_P(\cdot)$ are $(0, 1)$ Gaussian and Poisson RF with same exponential covariance.

If $\gamma_G(h) = \gamma_P(h) = \gamma(h)$, then $\gamma(h)$ is the variogram of $Y_m(\cdot)$ for all w.



Mixture RF:

Proposition

Garrigues, Allard and Baret (2007) obtain

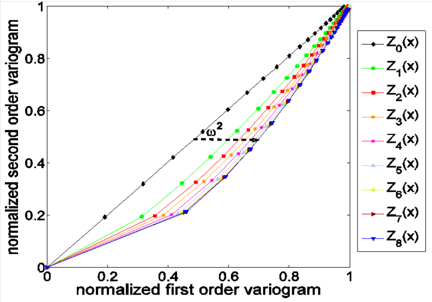
$$\gamma_1(h) = \frac{\sigma}{\pi}\left[w(1-\gamma_{2,P}(h))\sqrt{\gamma_{2,G}(h))} + \gamma_{2,P}(h)\sqrt{w^2\gamma_{2,G}(h) + (1-w^2)}\right]$$

Proof: Condition on A; use independence of Z_P in different cells

In summary

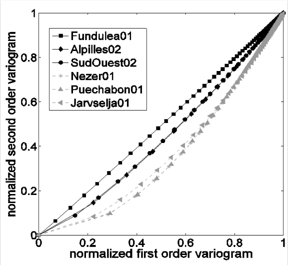
Relationship between first and second order variograms:

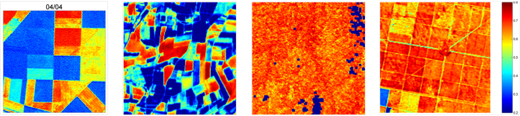
- Gaussian RF: quadratic
- Poisson RF: linear
- Mixture RF: intermediate



Application

Modeling remote sensing images (NDVI)
Fit simultaneously first and second order variograms
w is a degree of tesselation of landscape (agriculture)





Outline

General framework

Assume $Y(x)$ is a $(0, 1, \rho(h))$ stationary Gaussian RF on a domain $\mathcal{D}$. Let $\phi(\cdot)$ be a one-to-one mapping. Then consider,

$$Z(x) = \phi(Y(x)), \quad x \in \mathcal{D}.$$

- Transform the data: $Y_i = \phi^{-1}(Z(x_i))$
- Use all nice Gaussian properties
- Back-transform predictions/simulations with ϕ
- Pay attention to non linearities in case of prediction!

Two theoretical frameworks:

1. Transformed Multi-Gaussian Random Field
2. Transformed Bi-Gaussian Random Field

Lognormal Random Fields

Definition

Use an exponential function for $\phi(y)$

$$Z(x) = e^{\mu + \sigma Y(x)}, \quad x \in \mathcal{D}$$

is said to be a lognormal RF.

Using the general result $E[e^{aY}] = e^{a^2/2}$ for $Y \sim \mathcal{N}(0,1)$, leads to:

$$\begin{aligned} E[Z(x)] &= m = & e^{\mu} e^{\sigma^2/2} \\ \mathrm{Cov}(Z(x), Z(x+h)) &= C(h) = & m^2 \left(e^{\sigma^2 \rho(h)} - 1\right) \\ \mathrm{Var}[Z(x)] &= C(0) = & m^2 \left(e^{\sigma^2} - 1\right) \end{aligned}$$

Lognormal Random Fields

Denoting $\gamma(h) = 1 - \rho(h)$ the variogram of $Y(\cdot)$ and $\Gamma(h)$ the variogram of $Z(\cdot)$,

$$\Gamma(h) = m^2 e^{\sigma^2}(1 - e^{-\sigma^2\gamma(h)})$$

What if $Y(\cdot)$ is not 2nd order stationary ? Matheron (1974)

- μ and σ^2 no longer exist
- need to condition on a domain $V \supset \mathcal{D}$
- there exists m_V and A_V such that, for $x, y \in V$.

$$\begin{aligned} E[Y(x)] &= m_V \\ \text{Cov}(Y(x), Y(y)) &= A_V - \gamma(x - y), \end{aligned}$$

Lognormal Random Fields

Localy stationary log-normal RF (Matheron, 1974)

Let $Y(x) \sim \mathrm{IRF}(\gamma(h))$, conditioned on V as above. Then $Z(x)$ is a locally (i.e. on V) stationary lognormal RF with

$$\begin{aligned} E[Z(x)] &= M_V = & e^{m_v} e^{A_V/2} \\ \Gamma(h) &= & M_V^2 e^{A_V} \left(1 - e^{-\gamma(h)}\right) \end{aligned}$$

[See also Schoenberg's theorem]

- Exponential flavour of $\Gamma(h)$
- Finite range on V !!

Using lognormal Random Fields

Data $Z_i = Z(x_i) > 0$, $i \in I = \{1, \ldots, n\}$
Goal : predicting Z_0 at an unsampled location

1. Compute log-data $Y_i = \ln Z_i$, $i \in I$
2. Estimate the varigram $\gamma(\cdot)$ of $Y(\cdot)$
3. Predict $Y^* = E[Y_0 \mid (Y_i)_{i \in I}] = \sum_{i \in I} w_i Y_i$ (Gaussianity !)
4. Back transform $Z^* = e^{Y^* + \sum_{i \in I} \sum_{j \in I} w_i w_j \gamma_{ij}/2}$

Note: used as driving intensity for non homogeneous point processes
[Møller, Syversveen, Waagepetersen, 1998]

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Box-Cox transformation

For positive values $Z(x)$

Box-Cox transformation

$$\phi_\lambda^{-1}(z) = \frac{z^\lambda - 1}{\lambda} \quad \text{if} \quad \lambda \neq 0; \qquad \phi_0^{-1}(z) = \ln z,$$

- Similar derivations; need to use

$$E[(\mu + \sigma Y)^p] = \sigma^p \{-\imath\sqrt{2}\,\text{sgn}(\mu)\}^p U\left(-\frac{1}{2}p, \frac{1}{2}, -\frac{1}{2}(\mu/\sigma)^2\right)$$

  where U is a Kummer's confluent hypergeometric function
- Beware of bias correction !

Introduction

- ✓ *n*-multivariate gaussianity for any *n* is a strong assumption, which can not be checked in practice — not speaking of testing
- ✓ Bi-variate gaussianity is a weaker condition, that can be checked to a certain extent

Is there a less demanding theory ?

Decomposition with Hermite polynomials

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Decomposition with Hermite polynomials

Hermite polynomials

- Denote $g(y)$ and $G(y)$ the $\mathcal{N}(0,1)$ pdf and cpf.
- Consider the space Hilbert space $L^2(G)$ of functions $\phi(\cdot)$ such that $\int \phi^2(y)g(y)dy < \infty$
- Consider the Hermite polynomials H_n

$$H_n(y)g(y) = \frac{d^n}{dy^n g(y)} = yH_{n-1}(y) - (n-1)H_{n-2}(y),$$

with $H_0(y) = 1$ and $H_1(y) = -y$.

- In addition, for $k \geq 1$, $E[H_k(Y)] = 0$, $\text{Var}[H_k(Y)] = k!$

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Hermite polynomials

- The normalized Hermite polynomials $\chi_n(y) = H_n(y)/\sqrt{n!}$ form an othonormal basis of $L^2(G)$ w.r.t. gaussian density, i.e.

$$\int_{-\infty}^{\infty} \chi_n(y)\chi_m(y)g(y)dy = \delta_{nm} \Leftrightarrow E[\chi_n(Y)\chi_m(Y)] = \delta_{nm}$$

- Let $\phi \in L^2(G)$. Then,

$$Z = \phi(Y) = \sum_{k=0}^{\infty} \varphi_k \chi_k(Y) \quad \text{with} \quad \varphi_k = E[\phi(Y)\chi_k(Y)]$$

- Thus $E[\phi(Y)] = \varphi_0; \quad \text{Var}[\phi(Y)] = \sum_{k=1}^{\infty} \varphi_k^2$

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Hermite polynomials (ctd)

- For a Bi-Gaussian pair U, V

$$g_\rho(u,v) = \sum_{k=0}^{\infty} \rho^k \chi_k(u)\chi_k(v)g(u)g(v)$$

- For a Bi-Gaussian vector $(Y(x), Y(x+h))$ with correlation $\rho(h)$.

$$\mathrm{Cov}[\phi(Y(x)), \phi(Y(x+h))] = \sum_{k=1}^{\infty} \varphi_k^2 \rho^k(h)$$

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Example 1

$\phi(Y) = e^{\mu+\sigma Y}$. Then,

$$\varphi_k = (-1)^k e^{\mu+\sigma^2/2} \frac{\sigma^k}{\sqrt{k!}}, \quad k \geq 0$$

i.e.

$$E[Z(x)] = \varphi_0 = e^{\mu+\sigma^2/2} = m \text{ and } \mathrm{Cov}_Z(h) = m^2(e^{\sigma^2 \rho(h)} - 1)$$

identically to multi-gaussian RF.

Example 2

$Z_y(x) = \phi(Y(x)) = \mathbf{1}_{Y(x) \geq y}$. Then,

$$\varphi_k = -g(y)\frac{\chi_{k-1}(y)}{\sqrt{k}}, \quad k \geq 1$$

and $\varphi_0 = 1 - G(y)$. Hence

$$\mathrm{Cov}_{Z_y}(h) = g(y)^2 \sum_{k=1}^{\infty} \frac{\chi_{k-1}^2}{k} \rho(h)^k.$$

Checking for bi-gaussianity

- ✓ Transform data $Z(x_i)$ into Gaussian scores $Y(x_i)$
- ✓ Crossplots $Y(x)$, $Y(x+h)$ should be elliptical
- ✓ $\gamma_{Y,1}(h)$ should be proportional to $\sqrt{\gamma_{Y,2}(h)}$
- ✓ Denote $\gamma_{Y,2}(h)$ a variogram fitted on $Y(\cdot)$. Then,

$$\gamma_{Z,2}(h) = \sum_k \varphi_k^2 \{1 - \gamma_{Y,2}(h)\}^2$$

should fit on $Z(\cdot)$

Disjunctive Kriging of $Z(x_0)$

- ✓ Estimate φ_k from empirical cpf
- ✓ For each k, do (simple) Kriging of $\chi_k(Y(x_0))^*$
- ✓ $\phi(Y(x_0))^* = \sum_k \varphi_k \chi_k(Y(x_0))^*$
- ✓ $\sigma^2_{DK} = \sum_k \varphi_k \sigma^2_k$ with $\sigma^2_k = \mathrm{Var}[\chi_k(Y(x_0))^* - \chi_k(Y(x_0))]$

Outline

Definitions

Indicator function

Consider a $(0,1)$ Gaussian stationary RF $Y(x)$ on $\mathbb{R}^d$ with covariance function$\rho(h)$. Set a threshold $y \in \mathbb{R}$. Chose $\phi(Y) = \mathbf{1}_{Y \geq y}$, i.e.

$$Z_y(x) = 1 \text{ if } Y(x) \geq y; \;\; X(x) = 0 \text{ otherwise.}$$

Excursion sets

$$X_y = \{x \in \mathbb{R}^d : Y(x) \geq y\}$$

is the y-level excursion set of Y.

The X_ys are Random Sets verifying

$$u \leq v \Longrightarrow X_u \supset X_v.$$

Proposition (Lantuéjoul, 2002)

The variogram γ_{Z_y} of $Z_y(x)$ is

$$\gamma_{Z_y}(h) = \frac{1}{2\pi}\int_{\rho(h)}^{1} \frac{1}{\sqrt{1-r^2}} e^{-y^2/(1+r)} dr.$$

Proof

$$\gamma_{Z_y}(h) = \int_{-\infty}^{y}\int_{y}^{\infty} g_\rho(u,v)dudv \text{ with } g_\rho(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-u^2+v^2-2\rho uv/(1-\rho^2)}.$$

Direct computation yields

$$\frac{\partial g_\rho}{\partial \rho} = \frac{\partial^2 g_\rho}{\partial u \partial v},$$

from which

$$\frac{\partial \gamma_{Z_y}}{\partial \rho}(h) = \int_{-\infty}^{y}\int_{y}^{\infty} \frac{\partial^2 g_\rho}{\partial u \partial v} dudv = -g_\rho(y,y).$$

Hence

$$\gamma_{Z_y}(h) = \int_{\rho}^{1} g_r(y,y,)dr = \frac{1}{2\pi}\int_{\rho(h)}^{1} \frac{1}{\sqrt{1-r^2}} e^{-y^2/(1+r)} dr.$$

Regularity of excursion sets

Denote $\gamma(h) = 1 - \rho(h)$.
Perform the change of variable $r = \cos(2t)$. Then,

$$\gamma_y(h) = \frac{1}{\pi} \int_0^{\arcsin \sqrt{\gamma(h)/2}} \exp\left(-\frac{y^2}{2}(1 + \tan^2 t)\right) dt$$

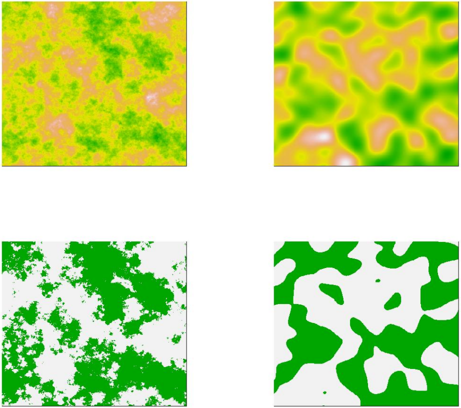
For $\gamma(h) \approx 0$

$$\gamma_y(h) \approx \frac{1}{\pi\sqrt{2}} \sqrt{\gamma(h)} e^{-y^2/2}, \quad h \approx 0$$

- If $\gamma(h) \propto |h|^2$ near $h = 0$, X_y has finite specific perimeter
- If $\gamma(h) \propto |h|^\alpha$, $\alpha < 2$ near $h = 0$, X_y, has infinite specific perimeter

Illustration

$C(h) = e^{-||h||/a}$ $\quad$ $C(h) = e^{-||h||^2/a^2}$



Covariance function of excursion set

We have seen

$$\gamma_{Z_y}(h) = \int_\rho^1 g_r(y, y,)dr = \frac{1}{2\pi} \int_{\rho(h)}^1 \frac{1}{\sqrt{1-r^2}} e^{-y^2/(1+r)} dr.$$

Open problem

- ✓ According to the above equation, the mapping $\gamma \to \gamma_{Z_y}$ is one-to-one
- ✓ But, not all variograms can correspond to an excursion set (later...)
- ✓ What is the general form of variograms, or covariance functions of excursion sets ?
- ✓ Lantuéjoul (2002) shows that $1 - e^{-||h||/a}$ is the variogram of an excursion set

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Some applications

- ✓ Geometry of petroleum reservoirs
  [Heresim Group, 1992-1993] (in Fontainebleau) for setting up the method
  [Allard, 1994] for conditional simulations with connectivity constraints
  [Emery, 2007] for extension to pluri-Gaussian framework
- ✓ Latent variable of non-homogeneous point processes
  [Myllymäki and Penttinen 2009]

Outline

Introduction

We already have seen

- Excursion sets
- Poisson hyperplanes tesselation

It is time to present some theory on Random Sets

- Let X be a random set in $\mathbb{R}^d$, and $Z(x) = \mathbf{1}_X(x)$ be its indicator function:

$$\mathbf{1}_X(x) = 1 \Leftrightarrow x \in X; \quad \mathbf{1}_X(x) = 0 \Leftrightarrow x \notin X, \qquad x \in \mathbb{R}^d$$

- X can be a set of points, segments, lines, objects (balls), + finite or infinite unions and intersections of those
- Cannot be characterised by the family of finite distributions of the type $P(x_1 \in X, \ldots, x_n \in X, y_1 \notin X, \ldots, y_m \notin X)$

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Theory of random closed sets (Matheron, 1975)

A Random Closed Set X on $\mathbb{R}^d$ is fully characterized by its avoiding functional

$$Q(K) = P\{X \cap K = \emptyset\}, \quad K \text{ compact set} \subset \mathbb{R}^d$$

Its complement is the hitting functional

$$T(K) = P\{X \cap K \neq \emptyset\} = 1 - Q(K).$$

Works even if X is a countable set of points.

First and second moments

Consider X to be regular, (i.e. no infinitely thin components or any isolated points)

- Considering $K = \{x\}$ yields the local proportion:

$$p(x) = 1 - Q(\{x\}) = P(x \in X)$$

X stationnary $\Longleftrightarrow p(x) = p = 1 - q$ for all x

- Considering $K = \{x, x+h\}$ yields the non centered covariance.

$$q(x, x+h) = Q(\{x, x+h\}) = P(x \neq X, x+h \neq X).$$

X stationnary $\Leftrightarrow q(x, x+h) = q(h)$

- Associated variogram:

$$\begin{aligned} \gamma(h) &= 0.5E[(Z(x) - Z(x+h))^2] = 0.5P\{Z(x) \neq Z(x+h)\} \\ &= 0.5\{P(x \in X, x+h \notin X) + P(x \notin X, x+h \in X)\} \\ &= P(x \notin X) - P(x \notin X, x+h \notin X) \\ &= q - q(h) \end{aligned}$$

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✓ Excursion sets: $X = \{x : Y(x) \geq y\}$. Exponential variogram OK

✓ Poisson hyperplanes tesselation: each cell is in X, independently with probability p: Exponential variogram

✓ Boolean model:

$$X = \bigcup_{\xi \in PP} A_i(\xi),$$

where

- PP is a Poisson point process
- $A_i(\xi)$ is a random objet $\sim A$ translated at ξ

$$q = e^{-\theta E[|A|]}; \quad \gamma(h) = 1 - 2q + q^2(e^{\theta E[|A \cap A_h|]} - 1)$$

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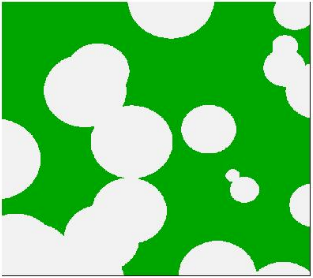
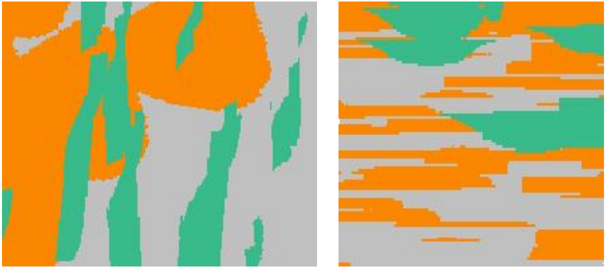
Example of a Boolean model



More realistic examples of a Boolean model

Petroleum reservoir: channels and lenses



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- ✓ Other object models: random token model, dead leaves, boolean random functions...

Indicator variogram: behaviour at the origin

Recall

$$\gamma(h) = 0.5E[\{\mathbf{1}_X(x) - \mathbf{1}_X(x+h)\}^2]$$

which is equivalent to

$$P(x \in X, x+h \notin X) = \gamma(h) = P(x \notin X, x+h \in X)$$

As $h \to 0$, $\gamma(h)$ conveys information about the boundary of X

Specific surface (Matheron, 1975)

Let $\sigma^{(d)}$ denote the specific $(d-1)$-volume of X. Assume X is isotropic

$$\sigma^{(d)} = \frac{d\omega_d}{\omega_{d-1}}\gamma'(0)$$

with ω_d is the d-volume of the unit ball in$\mathbb{R}^d$.

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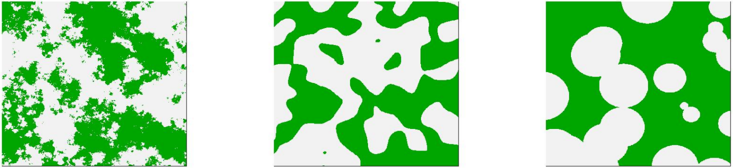
- If $\gamma(h)$ has linear behavior at 0, $\sigma^{(d)}$ is finite
- If $\gamma(h)$ is parabolic at the origin, $\sigma^{(d)} = 0$!! Degenerate case
- If $\gamma'(0) = \sigma^{(d)}$ is infinite: fractal RS



Indicator variogram: triangular inequality

Since

$$\{\mathbf{1}_X(x) - \mathbf{1}_X(x+h)\}^2 = |\mathbf{1}_X(x) - \mathbf{1}_X(x+h)|$$

and using

$$|\mathbf{1}_X(x) - \mathbf{1}_X(x+h+h')| \leq |\mathbf{1}_X(x) - \mathbf{1}_X(x+h)| + |\mathbf{1}_X(x+h) - \mathbf{1}_X(x+h+h')|$$

the variogram must satisfy

$$\gamma(h+h') \leq \gamma(h) + \gamma(h')$$

Consider $\gamma(h) \approx h^\alpha$, when $h \approx 0$. Then, chosing $h = h'$ yields

$$(2h)^\alpha \leq 2h^\alpha \Leftrightarrow \alpha \leq 1.$$

Excludes all regular variograms such as Gaussian or Matern with $\kappa > 1/2$.

Indicator variogram: open problem

Not all variograms can be the variogram of a random set. Must

- be bounded
- verify triangular inequality
- not be too regular

Is there a general characterization ?

Indicator variogram: open problem

Is there a general characterization ?

Conjecture, Matheron (1975)

Let $(x_i)_{i=1,n}$ be a finite sequence of points, and $(\varepsilon_i)_{i=1,n}$ a sequence of values in $\{-1, 0, 1\}$ such that $\sum_{i=1,n} \varepsilon_i = 1$. An indicator variogram is a bounded, conditionally definite negative function fulfill the condition

$$\sum_{i=1,n} \sum_{j=1,n} \varepsilon_i \varepsilon_j \gamma(x_i - x_j) \leq 0.$$

It is a necessary condition (containing the triangular inequality); is it sufficient ?

Outline

A la maison

Outline

The Closed Skew-Normal (CSN) distribution

General idea

$$\text{new density} = \text{constant} \times \text{density} \times \text{cpf}$$

La densité $CSN_{n,m}(\mu, \Sigma, D, \nu, \Delta)$

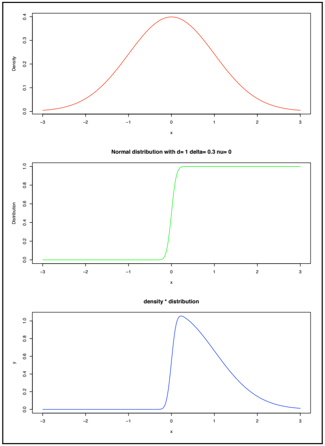
$$f(y) = \frac{1}{\Phi_m(0; \nu, \Delta + D^t \Sigma D)} \phi_n(y; \mu, \Sigma) \Phi_m(D^t(y - \mu); \nu, \Delta)$$

- If $D = 0$: $N_n(\mu, \Sigma)$
- If $m = 1$: skew-normal distribution (Azzalini, 1985; Azzalini, 1986)

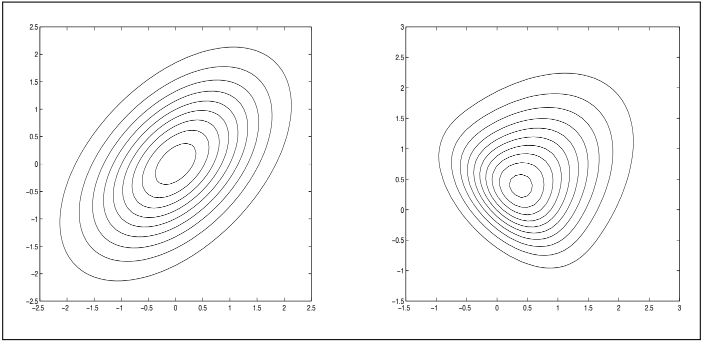
Dominguez-Molina, 2002, Azzalini et al. 1996

Example

$m = n = 1; \mu = 0, \sigma^2 = 1,\ d = 1,\ \nu = 0.3,\ \Delta = 0.3$



Gaussian and CSN bivariate density



Some properties of CSN distributions

Linearity

$$A \times CSN_{n,m}(\mu, \Sigma, D, \nu, \Delta) \sim CSN_{r,m}(A\mu, \Sigma_A, D_A, \nu, \Delta_A)$$

where

$$\Sigma_A = A\Sigma A^T, \quad D_A = D\Sigma A^T \Sigma_A^{-1}, \quad \Delta_A = \Delta + D\Sigma D^T - D_A \Sigma_A D_A^T$$

Sum (particular case)

$$N(\mu, \Sigma) + CSN_{n,m}(\psi, \Omega, D, \nu, \Delta) \sim CSN_{n,m}(\psi + \mu, \Omega + \Sigma, D, \nu, \Delta)$$

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Some properties of CSN distributions

Conditioning

Consider $Y = (Y_1, Y_2) \sim CSN_{n,m}(\mu, \Sigma, D, \nu, \Delta)$.
Then, $Y_2 | Y_1 = y_1$ is

$$CSN(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}, D_2, \nu - D_1 y_1, \Delta)$$

Moment generating function

$$M(t) = \frac{\Phi_m(D^t\Sigma t; \nu, \Delta + D\Sigma D^T)}{\Phi_m(0; \nu, \Delta + D\Sigma D^T)} \exp\{\mu^T t + \frac{1}{2}(t^T\Sigma t)\}$$

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Some properties of CSN distributions

First moment

$$EY = \frac{\partial}{\partial t} M_Y(t)\Big|_{t=0} = \mu + \Sigma D' \psi,$$

where

$$\psi = \frac{\Phi_q^*(0; \nu, \Delta + D\Sigma D')}{\Phi_q(0; \nu; \Delta + D\Sigma D')},$$

and, for any positive definite matrix Ω

$$\Phi_q^*(s; \nu, \Omega) = [\nabla_s \Phi_q(s; \nu, \Omega)]',$$

where $\nabla_s = \left(\frac{\partial}{\partial s_1}, ..., \frac{\partial}{\partial s_q}\right)'$ is the gradient operator.

Some properties of CSN distributions

Second moment

$$\begin{aligned} \mathrm{E}\left(\boldsymbol{y}\boldsymbol{y}'\right) &= \left.\frac{\partial^2}{\partial \boldsymbol{t}\partial \boldsymbol{t}'} M_{\boldsymbol{y}}\left(\boldsymbol{t}\right)\right|_{\boldsymbol{t}=\mathbf{0}} \\ &= \Sigma + \boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\psi}' D\Sigma + \Sigma D\boldsymbol{\psi}\boldsymbol{\mu}' + \Sigma D'\Lambda D\Sigma \end{aligned}$$

$$\Lambda = \frac{\Phi_q^{**}\left(\mathbf{0};\boldsymbol{\nu},\Delta + D\Sigma D'\right)}{\Phi_q\left(\mathbf{0};\boldsymbol{\nu};\Delta + D\Sigma D'\right)},$$

$$\Phi_q^{**}\left(\boldsymbol{t};\boldsymbol{\nu},\Omega\right) = \nabla_{\boldsymbol{t}}\nabla_{\boldsymbol{t}}'\Phi_q\left(\boldsymbol{t};\boldsymbol{\nu},\Omega\right).$$

Simulating a CSN R.V.

$$\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_{n+m}\left(\begin{pmatrix} 0 \\ \nu \end{pmatrix}, \begin{pmatrix} \Sigma & -D^t\Sigma \\ -\Sigma D & \Delta + D^t\Sigma D \end{pmatrix} \right),$$

Then

$$\mu + (Y|X \leq 0) = CSN_{n,m}(\mu, \Sigma, D, \nu, \Delta)$$

Simulation algorithm

1. simulate a vector $X \sim N_m(\nu, \Delta + D^t\Sigma D)$, conditional on $X \leq 0$
2. simulate a vector Y conditionally on X, according to the bivariate model above
3. return $\mu + Y$

Temporal application: a weather generator

- One of the priority of INRA is to explore the impact of climate change on agriculture and forest
- GCM provide output variables at scale 50 km
- Series are not numerous

Need for very long/numerous series of weather variables at local scale

Building a stochastic weather generator WACS-gen

General principle of WACS-gen

Ph. D. thesis of Cédric Flecher (D. Allard and P. Naveau co-advisors)

- ✓ We consider five variables $X(t) = (R, T_n, T_x, RR, W)^t(t)$
- ✓ R is log–transformed
- ✓ Series are centered and standardized using medians a mean absolute deviation
- ✓ The following parameters are estimated independently for each season:
- ✓ K weather types are determined using MCLUST
- ✓ Weather types form a Markov Chain
- ✓ In each class residuals $\sim$ CSN (4 or 5)
- ✓ Temporal correlation is accounted for

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- ✓ Weather types form a Markov Chain
- ✓ In each class residuals $\sim$ CSN (4 or 5)
- ✓ Temporal correlation is accounted for

CSN vectors

Reminder

$CSN_{n,m}(\mu, \Sigma, D, \nu, \Delta)$:

$$f_{n,m}(y) = \frac{1}{\Phi_m(0; \nu, \Delta + D^t \Sigma D)} \phi_n(y; \mu, \Sigma) \Phi_m(D^t(y - \mu); \nu, \Delta)$$

In WACS-gen,

To simplify the model, we set $k = m = n$; $D = \Sigma^{-\frac{1}{2}} S$; $\Delta = I_k - S^2$; $S = \text{diag}(\delta_1, \ldots, \delta_K)^t$.

$$CSN_k^*(\mu, \Sigma, S): \quad f_{k,k}(y) = 2^{-k} \, \phi_k(y; \mu, \Sigma) \, \Phi_k(S\Sigma^{-\frac{1}{2}}(y - \mu); 0, I_k - S^2).$$

Hence

$$\tilde{X} = \Sigma^{-1/2}(X - \mu) \sim CSN_5^*(0, I_5, S).$$

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Estimation of the parameters

Estimation is done by weighted moments (Flecher, Allard and Naveau, 2009 Stat. Prob. Letters)

$$E[\Phi_n(Y,0,I_n)] = 2^n\Phi_{2n}\left(0;\begin{bmatrix} -mu \\ 0 \end{bmatrix},\begin{bmatrix} \Sigma + I_n & \lambda\Sigma^{1/2} \\ \lambda\Sigma^{1/2} & I_n \end{bmatrix}\right)$$

Bivariate example: $\mu_1 = \mu_2 = \sigma_1 = \sigma_2 = 1$; $\lambda = 0.89$ et $\rho = 0.8$.