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# APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES AND EMBEDDING THEOREMS

by

Sergey Mikhaylovich Nikol'skiy

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13. ABSTRACT

The theory of embeddings of classes of differentiable functions of several variables has been intensively expanded during the past two decades, and a number of its fundamental problems have been resolved. But till now these results are to be found in journal articles. This book presents the complete theory of embeddings of the main classes ($W^r_p$, $H^r_p$, $B^r_{p\theta}$, $L^r_p$) of differentiable functions given for the entire n-dimensional space $R_n$.

The reader will find in the book the inequalities between partial derivatives in the various contexts that have found application in mathematical physics. Emphasis is placed on problems of compactness, integral representations of functions of these classes, and problems of the isomorphisms of these classes.

In the book the author chiefly employs the method of approximation with exponential type integral functions and trigonometric polymonials. The theory of approximation suitably adapted for these ends is set forth at the outset of the volume. Use of the Bessel-Macdonald integral operator is also essential. The reader will even find in the book remarks given without proof on the embedding of classes of differentiable functions specified for the domains $G \subset R_n$.

The reader must be familiar with the fundamentals of Lesbesgue integral theory. The book widely employs the concept of the generalized function, but it is clarified with proofs to the extent that this is necessary.

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APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES AND EMBEDDING THEOREMS

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The theory of embeddings of classes of differentiable functions of several variables has been intensively expanded during the past two decades, and a number of its fundamental problems have been resolved. But till now these results are to be found in journal articles. This book presents the complete theory of embeddings of the main classes ($W_p^r$, $H_p^r$, $B_{p\theta}^r$, $L_p^r$) of differentiable functions given for the entire n-dimensional space $R_n$.

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## INTRODUCTION

In the past two decades the theory of embeddings of classes of differentiable functions of several variables, whose foundations were laid back in the 1930's by S. L. Sobolev, has experienced rapid growth. Presently, the solution of several fundamental problems in this theory has come to a head and the need to present them in compact form has arisen. I personally arrived at problems of embedding theory as part of a field that had long fascinated me, the concepts of the classical theory of approximating functions with polynomials, above all with trigonometric polynomials and their nonperiodic analogues -- exponential type integral functions.

These notions, which I had occasion to suitably ramify, served me as the starting-point for constructing a theory of embeddings of H-classes, where already in problems of the traces of functions not only did direct theorems emerge, but also their wholly inverse counterpart theorems. The latter can even be called theorems on the extension of functions into a space with the manifolds of the least number of measures pertaining to it. Here, not only is the isotropic case of functions with differential properties that are identical in different directions embraced, but also the anisotropic case.

Later, O. V. Besov constructed a similar theory of embeddings of the B-classes he introduced, also founded on methods of the theory of approximation with trigonometric polynomials or with exponential type integral functions. The B-classes are remarkable in that they, like the H-classes, are as we have said closed in upon themselves with respect to embedding theorems. By this we wish to state that the embedding theorems of interest to us (we will not actually formulate them here) are expressed in terms of B-classes and here possess to some extent the properties of transitivity and invertibility in the case of the problem of traces.

S. L. Sobolev proved his embedding theorems for the classes $W_p^l = W_p^l(\Omega)$ of functions that have on a sufficiently broad domain $\Omega$ of n-dimensional space $R_n$ derivatives up to the order $l$ inclusively that are integrable to the p-th degree ($1 \leq p \leq \infty$). Sobolev classes can be called discrete classes, because the parameter $l$ expressing the differentiable properties of the functions included in it ranges through the discrete sequence $l = 0, 1, 2, \ldots$. In fact, the classes

H and B are continuous in this sense. When the reader acquaints himself with chapter 9 of this book, he will become aware that those theorems of S. L. Sobolev with the corollaries to them that were contributed by V. I. Kondrashov and V. P. Il'in, which are accompanied by a change of the metric, are in a certain sense are terminal, and even, as far as this is permitted by the discreteness of the classes, transitive.

As pertains to the embedding theorem accompanied only by change of measure without metric change -- we call these the theorems of traces, here the situation is more involved. Of course, S. L. Sobolev theorems do supply an answer to the question as to which differential properties are possessed by the trace of the function of the class $W_p^l(\Omega)$ in the manifold $\Gamma \subset \Omega$, but the answer is furnished in terms of W classes. But now we know that generally speaking, if we omit the case p = 2, no final answer to this question is expressed in terms of the classes W.

The first conclusive results on the problems of traces of W-classes were obtained by Aronszajn, [1] and by L. N. Slobodetskiy [2]. In this case (p=2), fractional classes $W_2^l(\Omega)$ and $W_2^l(\Gamma)$ corresponding to any positive, but not necessarily integral parameter l, were introduced, and direct theorems of embedding and the inverse theorems wholly corresponding to them were obtained in terms of these classes. In the notation used in this book, $W_2^l = L_2^l = B_2^l$.

Further investigations of Gal'yardo [1], O. V. Besov [1, 2], P. I. Lizorkin [9], and S. V. Uspenskiy [1, 2] led to the complete solutions of the problem of traces of the functions of classes $W_p^l$ for any finite p> 1. The reader will find what this solution looks like in the same chapter 9 we referred to (by setting $W_p^l = L_p^l$). But as for now we can only state that traces of the functions f of the class $W_p^l$ when $p \neq 2$, generally speaking, belong not to W- but to B-classes. This indicates, on the one hand, the fact that the theorems of embedding of different measures (theorems of traces) cease to be closed with respect to W-classes; but on the other hand, this indicates that an intimate relationship holds between the classes W and B. This relationship is so close that at one time, when not everything about these problems was clear, it was held that $B_p^l$ classes for fractional l are the natural extensions of integral Sobolev classes and were denoted by $W_p^l$. Actually, these natural extensions are the so-called Liouville classes $L_p^r$. Chapter 9 therefore deals with them, in particular, also with the classes W, because we assume $W_p^l = L_p^l$ (l = 0, 1, ...). The reader must recall that in this book the notation $W_p^l$ is used only when l = 0, 1, ... Cf 4.3 on this matter.

S. L. Sobolev studied functions of his classes by means of the integral representations he introduced; these were greatly developed in the works of V. P. Il'in, and later O. V. Besov (cf. 6.10 below). Functions of the classes $L_p^l$ are defined on the entire space, and in their integral representation it is very important to see that the kernels of the latter rapidly enough decrease to zero at infinity. These are the familiar Bessel-Macdonald kernels. They were in fact adopted as the basis for representing functions of the classes $L_p^l$. We say as the basis because actually here anisotropic classes $L_p^l$ are what we are considering. The kernels of their integral representations constitute certain complications of the Macdonald kernels. I note that in writing chapter 9, I made heavy use of materials given me by my colleague, P. I. Kozorkin, who quite recently derived a complete system of embedding theorems for general anisotropic classes $L_p^r$, where r = any positive vectors. His results have thus far been published in the form of a brief note.

In the one-dimensional case (where the problem of traces does not come up), theorems of embedding of different measures for the classes $L_p^r$ and for nonintegral r for the classes $H_p^r$ were already obtained in the works of Hardy and Littlewood.

The $I_r$ operators defined by Bessel-Macdonald kernels are universal in character. In this book they are investigated and applied in a variety of contexts. We quite extensively use the concept of the generalized function, so the book contains a small section presenting with complete proofs only those deductions from the theory of the generalized function that the reader must know to understand the following treatment. I introduce the concept of a generalized function that is regular in the sense of $L_p$ by employing the $I_r$ operator. For regular functions, different proofs associated with multiplication by the generalized functions are greatly simplified. I make wide use of this because the generalized functions encountered in the book are regular.

The $I_r$ operator receives interesting applications also in chapter 8. It executes isomorphisms not only of the L-, but also of the B- and H-classes and can serve as a means for the integral representations of functions of these classes. These ideas which in the periodic one-dimensional case derive from the time of Hardy and Littlewood have quite recently been explored from different vantage points in the works of Aronshayn [also spelled Aronszajn] and Smith, Cal'deron, Teyblson, Lions, P. I. Lizorkin, the present author, and others.

Quite naturally, this book also takes up the foundations of the theory of approximation of functions of several functions with trigonometric polynometric polynomials and exponential type integral functions. These in themselves are of interest, but basically they play a subordinate role -- as tools of approximation theory. Further, theorems of embedding are proved for H- and B-classes and the representations of functions of these classes are also in terms of series in exponential type integral functions or in trigonometric polynomials. Bearing these goals in mind, along with the traditional inequalities, we also introduce and utilize other inequalities (of different measures and metrics).

We must note that in this book we furnish complete proofs of embedding theorems for the above-cited classes of functions defined on the entire n-dimensional space $R_n$. But these classes can be defined for the domains $\Omega \subset R_n$. These definitions are given in the book. Also formulated (without proof) are extremely wide-ranging theorems on the extensions of the functions of these classes on all space (with the preservation of class). This permits extending the theorems proven for the $R_n$ space to the case of the domains $\Omega \subset R_n$.

Finally, we note that recently investigations have been pursued (begun by L. D. Kudryavgsev) of more general classes -- weight classes. In this book, we confine ourselves only to some remarks about the relationship of weight classes with the nonqeight classes discussed here.

I note still further that for more than 10 years now a permanent seminar on the theory of differentiable functions of several variables has been held in the Mathematics Institute imeni V. A. Steklov, headed by V. I. Kondrashov, L. D. Kudryavtsev, and myself. Actively participating in it have been O. V. Besov, Ya. S. Bugrov, V. I. Burenkov, A. A. Vasharin, P. I. Lizorkin, S. V. Uspenskiy, G. N. Yakovlev, and other mathematicians. Many results presented in this book belong to the participants of this seminar and were discussed in it as they were taken shape.

In conclusion, I deem it my happy duty to express my deep gratitude to colleagues O. V. Besov, who read the book in manuscript, P. I. Lizorkin, who read chapters 8 and 9, and S. A. Telyakovskiy, who read several chapters. They have made many valuable observations, which in one way or another I have taken cognizance of.

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Finally, I am very thankful to my younger colleague V. I. Burenkov, the book's editor. Much of his advice pertaining not only to format, but also to substance of the exposition was taken into account.

## CHAPTER I PRELIMINARIES

### 1.1 Space $C(\mathcal{E})$ and $L_p(\mathcal{E})$

In this book we will discuss functions that are generally dependent on several variables.

The symbol will also signify the n-dimensional Euclidean space with points $x = (x_1, \ldots, x_n)$ with real coordinates. The length of the vector will be denoted thusly:

$$|x| = \sqrt{\sum_1^n x_k^2}. \tag{1}$$

If $\mathcal{E}$ is a closed set belonging to $R_n$ ($\mathcal{E} \subset R_n$), $C(\mathcal{E})$ will stand for the set of all (real or complex-valued) functions $f = f(x)$ uniformly continuous on $\mathcal{E}$.

We will set each function $f \in C(\mathcal{E})$ into correspondence with its norm (in the sense of $C(\mathcal{E})$):

$$\|f\|_{C(\mathcal{E})} = \sup_{x \in \mathcal{E}} |f(x)|. \tag{2}$$

In the case of a restricted (closed) set $\mathcal{E}$ sup can be replaced with max.

If p is a real number satisfying the inequalities $1 \leq p < \infty$ and in some measurable but not necessarily bounded set $\mathcal{E} \subset R_n$ belonging to $R_n$ a measurable real or complex-valued function f is given, such that the function $|f|^p$ is integrable (summable) in the Lesbesgue sense on $\mathcal{E}$, then we assume

$$\|f\|_{L_p(\mathcal{E})} = \left(\int_{\mathcal{E}} |f|^p \, d\mathcal{E}\right)^{1/p}. \tag{3}$$

The variable (3) is called the norm of the function f in the sense of $L_p(\mathcal{E})$. $L_p(\mathcal{E})$ will stand for the set of all functions that have the finite norm (3).

We will not distinguish between the two <u>equivalent</u> functions $f_1$ and $f_2 \in L_p(\mathcal{E})$, i.e., those differing in the set by the zero measure. We will assume them to be equal to the same element of the functional space $L_p(\mathcal{E})$ and write $f_1 = f_2$. In particular, if the function $f \in L_p(\mathcal{E})$ equals zero for almost all $x \in \mathcal{E}$, we will write $f = 0$, thus identifying this function with the function that is identically equal to zero on $\mathcal{E}$. In this way, from the equality $\| f_1 - f_2 \|_{L_p(\mathcal{E})} = 0$ it follows that $f_1 - f_2 = 0$ and $f_1 = f_2$.

The set $\mathcal{E}$ can have the measure m, that is smaller than n, and then the integral appearing in equality (3) is understood in the sense of a natural (m-dimensional) Lesbegue measure defined on the set $\mathcal{E}$. We do not need to discuss sets $\mathcal{E}$ that are structurally complex. Often $\mathcal{E}$ will coincide on the entire space $R_n$ or will be some one of its m-dimensional subspace or a m-dimensional cube or sphere belonging to $R_n$. Finally, $\mathcal{E}$ can be a smooth or piecewise-smooth hypersurface, consisting of sufficiently smooth pieces, and then the measure of the measurable subset $\mathcal{E}$, on the basis of which the integral appearing in the right-hand side of (3) is defined, is a generalization (extension) of the customary concept of the area of a hypersurface.

The definition (3) naturally extends also to the case $p = \infty$. Actually, if the function $f(x)$ is measurable and is substantially restricted to the bounded set $\mathcal{E}$, i.e., for it there exists the quantity $\sup\limits_{x \in \mathcal{E}} \mathrm{vrai} |f(x)| = M_f$, called the essential maximum*) $|f(x)|$ on $\mathcal{E}$, then the following equality obtains:

$$\lim_{p \to \infty} \| f \|_{L_p(\mathcal{E})} = M_f. \tag{4}$$

This equality is proven thusly. Let $\mu e$ stand for the measure of e. If $M_f = 0$ or $\mu \mathcal{E} = 0$, equality (4) is obvious. We will assume that $0 < M_f < \infty$. If $\mathcal{E}$ is a bounded measurable set, then

$$\left( \int_{\mathcal{E}} |f(x)|^p \, dx \right)^{1/p} \leqslant M_f (\mu \mathcal{E})^{1/p}.$$

Consequently,

$$\overline{\lim_{p \to \infty}} \| f \|_{L_p(\mathcal{E})} \leqslant M_f. \tag{5}$$

*) $M_f$ is the smallest number among the numbers M possessing the property that a set of all $x \in \mathcal{E}$ for which $|f(x)| > M$ has the measure zero. It is easy to see that it does exist.

If $\mathcal{E}$ is an infinite-measurable set, then the inequality (5), generally speaking, is not satisfied (for example, $\mathcal{E} = R_n$ and $f(x) = 1$). However, this inequality can be proven on the assumption that $f(x) \in L_p(\mathcal{E})$ for all sufficiently large p and that $\lim\limits_{p \to \infty} \|f\|_{L_p(\mathcal{E})} < \infty$. In this case

$$\left(\int\limits_{\mathcal{E}} |f(x)|^p\, dx\right)^{1/p} \leqslant M_f^{1/2}\left(\int\limits_{\mathcal{E}} |f(x)|^{p/2}\, dx\right)^{1/p},$$

therefore

$$\varlimsup_{p\to\infty}\left(\int\limits_{\mathcal{E}} |f(x)|^p\, dx\right)^{1/p} \leqslant M_f^{1/2}\left[\varlimsup_{p\to\infty}\left(\int\limits_{\mathcal{E}} |f(x)|^p\, dx\right)^{1/p}\right]^{1/2},$$

from whence derives the inequality (5).

On the other hand, from the definition of the essential maximum of a function follows the existence of the bounded set $\mathcal{E}_1$ with positive measure such that for all of its points the inequality

$$|f(x)| > M_f - \varepsilon,$$

is satisfied, where $0 < \varepsilon \leqslant M_f$. Therefore

$$\|f\|_{L_p(\mathcal{E})} \geqslant \left(\int\limits_{\mathcal{E}_1} (M_f - \varepsilon)^p\, d\mathcal{E}_1\right)^{1/p} = (M_f - \varepsilon)(\mu\mathcal{E}_1)^{1/p},$$

from whence

$$\varliminf_{p\to\infty} \|f\|_{L_p(\mathcal{E})} \geqslant M_f - \varepsilon.$$

Since $\varepsilon$ is arbitrary, then

$$\varliminf_{p\to\infty} \|f\|_{L_p(\mathcal{E})} \geqslant M_f. \tag{6}$$

Notice that inequality (6) is valid for any measurable set $\mathcal{E}$.

(4) follows from (5) and (6).

Thus, it has been proven that is the function f(x) is substantially confined to the bounded measurable set $\mathcal{E}$, the finite limit

$$\lim_{p\to\infty} \|f\|_{L_p(\mathcal{E})}, \tag{7}$$

exists, equal to the essential maximum f(x) on $\mathcal{E}$.

On the other hand, from the existence of the limit (7) follows the substantial confinement of f(x) on $\mathcal{E}$. Actually, if there were not so, then no matter how large the N a measurable and bounded subset $\mathcal{E}'$ of the set $\mathcal{E}$ with positive measure would exist, on which

$$|f(x)| > N.$$

Then for any $p > 1$

$$\|f\|_{L_p(\mathcal{E})} \geq N(\mu\mathcal{E}')^{1/p},$$

from whence

$$\lim_{p\to\infty} \|f\|_{L_p(\mathcal{E})} \geq N.$$

Since N is as large as we please, the limit (7) cannot be finite and we reach a contradiction.

These arguments point to the utility of the following notation:

$$\|f\|_{L_\infty(\mathcal{E})} = \sup_{x\in\mathcal{E}} \mathrm{vrai}\,|f(x)|, \tag{8}$$

supplementing the notation of (3) for $p = \infty$. In functional analysis, it is also customary to denote the norm (8) thusly:

$$\|f\|_{M(\mathcal{E})} = \sup_{x\in\mathcal{E}} \mathrm{vrai}\,|f(x)|. \tag{9}$$

We also will sometimes use this notation, assuming therefore that

$$\|f\|_{M(\mathcal{E})} = \|f\|_{L_\infty(\mathcal{E})}. \tag{10}$$

The symbol $M(\mathcal{E})$ will stand for the set of all functions f that have a finite norm in the sense of $M(\mathcal{E})$.

If $\mathcal{E}$ is a bounded closed set and the function f(x) is continuous on $\mathcal{E}$, the value of (8) will be equal to the usual maximum of the function $|f(x)|$ on $\mathcal{E}$. In this case

$$\|f\|_{L_\infty(\mathcal{E})} = \|f\|_{C(\mathcal{E})}. \tag{11}$$

1.1.1. For the case when the function $f(x) = f(x_1, \ldots, x_n)$ is periodic with a $2\pi$ period with respect to all variables, i.e., if for it the identity

$$f(x_1, \ldots, x_{l-1}, x_l + 2\pi, x_{l+1}, \ldots, x_n) = f(x_1, \ldots, x_n) \tag{1}$$

is satisfied for all or almost all x and $i = 1, \ldots, n$, then when this function is normed we will consider as the set $\mathcal{E}$ the n-dimensional cube

$$\Delta^{(n)} = \{0 \leqslant x_i \leqslant 2\pi;\ i = 1, \ldots, n\}$$

of the space $R_n$ and we will denote the corresponding norm thusly:

$$\|f\|_{L_p(\Delta^{(n)})} = \|f\|^{(n)}_{L_p^*},\ \|f\|_{M(\Delta^{(n)})} = \|f\|^{(n)}_{M^*},\ \|f\|_{C(\Delta^{(n)})} = \|f\|^{(n)}_{C^*}. \tag{2}$$

The asterisk will always indicate the fact that the function f is periodic and that its norm was computed with respect to the cube defining the period of the function.

When $n = 1$, as a rule, we will write $\|f\|_{L_p^*}$, $\|f\|_{M^*}$, and $\|f\|_{C^*}$ in place of, respectively, $\|f\|^{(1)}_{L_p^*}$, $\|f\|^{(1)}_{M^*}$, and $\|f\|^{(1)}_{C^*}$.

The set of all $2\pi$-periodic functions with finite norm $\|f\|^{(n)}_{L_p^*}$ defined on $R_n$ will be denoted by $L^{(n)}_{p*}$. The set of all continuous $2\pi$-periodic functions defined on $R_n$ will be denoted by the symbol $C^{(n)}_*$.

Incidentally, we will omit the index (n) in these symbols when possible.

Quite frequently we will consider the measurable set $\mathcal{E} = R_m \times \mathcal{E}' \subset R_n$, which is the topological product of the m-dimensional subspace $R_m$ $(m < n)$ of points $(x_1, \ldots, x_m)$ and the measurable set $\mathcal{E}' \subset R_{n-m}$, where $R_{n-m}$ is the subspace of the points $(x_{m+1}, \ldots, x_n)$.

Here the function space consisting of $\mathcal{E}$- measurable functions f(x), periodic with a period of $2\pi$ with respect to the variables $x_1, \ldots, x_m$ and summable to the p-th degree in the set $\Delta_m \times \mathcal{E}'$, where

$$\Delta_m = \{0 \leqslant x_k \leqslant 2\pi,\ k = 1, \ldots, m\},$$

we will denote by $L_p^*(\mathcal{E})$. The asterisk will indicate the existence of periodicity (with respect to $\Delta_m$) for the functions $f \in L_p^*(\mathcal{E})$ and the fact that the norm of the function $f \in L_p^*(\mathcal{E})$ is defined by the equality

$$\|f\|_{L_p^*(\mathcal{E})} = \left(\int_0^{2\pi} \cdots \int_0^{2\pi} \int_{\mathcal{E}'} |f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)|^p \times \right.$$
$$\left. \times\, dx_1 \ldots dx_m\, dx_{m+1} \ldots dx_n\right)^{1/p}.$$

1.1.2. We will make generous use of the fact that for a summable periodic function $\varphi$, i.e., belonging to $L_p^{(n)}$, the equality

$$\|\varphi(x+a)\|_{L_p^*} = \|\varphi(x)\|_{L_p^*}, \tag{1}$$

exists for any $a \in R_n$, just as does the equality

$$\|\varphi(x+a)\|_{L_p(R_n)} = \|\varphi(x)\|_{L_p(R_n)}, \tag{2}$$

for the functions $\varphi(x) \in L_p(R_n)$.

## 1.2 Linear Normed Spaces

1.2.1. Linear set. The set G of elements x, y, z, ... is called a linear set if by some law, to each two of its elements x and y there corresponds the element $z = x + y$ belonging to G, called the sum of x and y, and if to each real (complex) number $\alpha$ and to the element $x \in G$ there corresponds the element $\alpha x \in G$, called the product of the number $\alpha$ by the element x, and where the operations of addition and multiplication are subject to the following axioms:

1) $x+y=y+x$,
2) $(x+y)+z=x+(y+z)$,
3) from $x + y = x + z$ follows $y = z$,
4) $\alpha x+\alpha y=\alpha(x+y)$,
5) $\alpha x+\beta x=(\alpha+\beta)x$,
6) $\alpha(\beta x)=(\alpha\beta)x$, and
7) $1 \cdot x=x$.

The set G is a real or complex linear set, depending on whether the numbers $\alpha$ and $\beta$ appearing in it are real numbers or complex.

From the definition of a linear space it follows that in it there exists a unique element $\theta$, the zero element, such that for all $x \in G$, the following relationships are valid: $x+\theta=x$, $0\cdot x=\theta$.

Actually, let elements x and y belong to G. We will set $\theta = \theta_x = 0\cdot x$ and $\theta_y = 0\cdot y$, then

$$x+\theta_x=1\cdot x+0\cdot x=1\cdot x=x$$

and similarly

$$y+\theta_y=y.$$

From these equalities, based on the axioms it follows that

$$x+y+\theta_x = x+y+\theta_y,$$

from whence

$$\theta_x = \theta_y = \theta.$$

We postulate further that $-1 \cdot x = -x$, then $x + (-x) = \theta$. If $x$ and $y$ are arbitrary elements of G, the equation $x + z = y$ has the solution $z = y + (-x)$ that is unique by axiom 3), which is naturally called the difference of y and x and so we denote $z = y - x$. Thus, besides addition, the operation of subtraction is defined in G.

Linear set axioms give us the right, by using the operations of addition, subtraction, and multiplication by a number, to transform the finite sums of the type

$$\alpha x + \beta y + \ldots + \gamma z,$$

just as is done with letter-based algebraic expressions.

Any set $G_1 \subset G$ containing along with elements x and y, the element $x + y$, where $\alpha$ and $\beta$ are real (complex) numbers, obviously is in turn a linear set.

A finite system of elements $x_1, \ldots, x_n$ of G is called linearly independent, if from the equality

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = \theta$$

there follows $\alpha_k = 0$ $(k = 1, \ldots, n)$. Otherwise, this system is termed linearly dependent.

The set of functions $C(\mathcal{E})$ defined in section 1.1 is obviously a linear set. The zero element in $C(\mathcal{E})$ is a function identical equal to zero on $\mathcal{E}$.

The set $L_p(\mathcal{E})$ of functions f integrable in the p-th degree in the measurable set $\mathcal{E}$ is also a linear set with a zero element that is a function almost everywhere on $\mathcal{E}$ equal to zero (equivalent to zero).

1.2.2. Banach space. Spaces $L_p(\mathcal{E})$ and $C(\mathcal{E})$. A linear (real or complex) space E is termed a normed space if to each element $x \in E$ there is set in correspondence a nonnegative number $\|x\|$, called the norm of the element x (in the space E or in the sense of the space E) satisfying the following conditions:

1) if $\|x\| = 0$, then $x = \theta$,
2) $\|x + y\| \leqslant \|x\| + \|y\|$ $(x, y \in E)$, and
3) $\|\alpha x\| = |\alpha| \, \|x\|$,

where $x \in E$ and $\alpha$ is an arbitrary (real or complex) number.

From 2) follows the validity of the inequality

$$\big|\|x\| - \|y\|\big| \leqslant \|x - y\| \quad (x, y \in E). \tag{1}$$

The normed space E is called complete if from the fact that for the sequence $x_n \in E$ $(n = 1, 2, \ldots)$ the condition (Cauchy)

$$\lim_{n,\, m \to \infty} \|x_n - x_m\| = 0,$$

is satisfied, there follows the existence in E of the element $x_0$ for which the equality

$$\lim_{n \to \infty} \|x_n - x_0\| = 0. \qquad (2)$$

is satisfied.

The fact that property (2) is satisfied can also be written thusly:

$$\lim_{n \to \infty} x_n = x_0, \qquad (3)$$

which states that $x_n$ approaches $x_0$ with respect to the norm of the space E or in the sense of the metric of E.

A complete linear normed space is also called a Banach space or a Banachian space.

The function set $C(\mathfrak{E})$ is obviously a Banach space. It is also well known that the set of functions $L_p(\mathfrak{E})$ $(1 \leq p \leq \infty)$ defined in the same section is also a Banach space. Here $C(\mathfrak{E})$ and $L_p(\mathfrak{E})$ are real or complex spaces, depending on whether they consist of real or complex functions f. In the former case, f can be multiplied by real numbers, and in the latter -- by complex.

1.2.3. Finite-measurable space. The set $\mathfrak{M} \subset E$ is termed a subspace of the Banach space E if it is a closed (relative to the norm $\|\cdot\|$) linear set.

Let the elements $x_1, \ldots, x_n$ belonging to E form a linearly independent system. The set $\mathfrak{M}_n$ of elements of the type

$$y = \sum_1^n c_k x_k, \qquad (1)$$

where $c = (c_1, \ldots, c_n)$ is an arbitrary system of real (complex) numbers, is called an n-dimensional (finite-measurable) space. If $\mathfrak{M}_n$ is part of E, then $\mathfrak{M}_n$ is also called the n-dimensional space E, and the system of elements $x_1, \ldots, x_n$ is its basis. To justify this definition, we must show that $\mathfrak{M}_n$ is a closed linear set. The linearity of $\mathfrak{M}_n$ is self-evident, and the closure will be established below.

If along with the element $y$ defined by equality (1), still another element

$$y' = \sum_1^n c'_k x_k,$$

is given defined by the system $c' = (c'_1, \ldots, c'_n)$, then, obviously,

$$\|y - y'\| \leqslant \sum_1^n |c_k - c'_k| \|x_k\|.$$

Hence it follows that

$$\lim_{|c-c'| \to 0} \|y - y'\| = 0. \qquad (2)$$

Property (2) means that the element **y** depends (with respect to the norm) continuously on its defining coefficients $c_k$. By virtue of the inequality

$$|\|y\| - \|y'\|| \leqslant \|y - y'\|$$

from (2) it also follows that

$$\lim_{|c-c'| \to 0} \|y'\| = \|y\|. \qquad (3)$$

Thus, the norm

$$\|y\| = \Phi(c_1, \ldots, c_n) = \Phi(c)$$

is a continuous function of $c = (c_1, \ldots, c_n)$.

The minimum of this function in the (closed and bounded) set defined by the equation

$$|c| = \sqrt{\sum_1^n c_k^2} = 1,$$

is reached for some system of coefficients $c^0 = (c_1^{(0)}, \ldots, c_n^{(0)})$ and is equal to the number

$$\frac{1}{\lambda} = \Phi(c^{(0)}) > 0,$$

which is necessarily positive because the system $x_1, \ldots, x_n$ is linearly independent.

Let us take an arbitrary system of numbers $c = (c_1, \ldots, c_n)$ $(|c| > 0)$ and set

$$c' = \frac{c}{|c|}.$$

Then by virtue of $|c'| = 1$, the inequality

$$\frac{1}{\lambda} \leqslant \left\| \sum_1^n c'_k x_k \right\|.$$

will obtain, which after multiplication of the left and right sides by $\lambda|c|$ is transformed into the inequality

$$|c| \leqslant \lambda \left| \sum_1^n c_k x_k \right|. \qquad (4)$$

Now it is no longer hard to prove that the linear set $\mathfrak{M}_n$ is closed and, thus, is a space.

Actually, from the fact that

(5)

$$y_l = \sum_1^n c_k^{(l)} x_k \qquad (l = 1, 2, \ldots)$$

and

$$\| y_l - y_m \| \to 0 \qquad (l, m \to \infty),$$

it follows by (4) that

$$|c^{(l)} - c^{(m)}| \leqslant \lambda \| y_l - y_m \| \to 0 \qquad (l, m \to \infty),$$

where $c^{(l)} = (c_1^{(l)}, \ldots, c_n^{(l)})$ $(l = 1, 2, \ldots)$. Therefore, the limit

(6)

exists, from whence

$$\lim_{l \to \infty} c^{(l)} = c^{(0)}, \qquad (6)$$

$$\| y_l - y_0 \| \to 0 \;\; (l \to \infty), \qquad (7) \qquad (7)$$

where

$$y_0 = \sum_1^n c_k^{(0)} x_k \in \mathfrak{M}_n. \qquad (8)$$

(8)

Let us note yet another important property of the finite-measurable space $\mathfrak{M}_n$ stemming directly from inequality (4). It is that any bounded (normwise) set $\Omega \subset \mathfrak{M}_n$ is compact in $\mathfrak{M}_n$, i.e., from any sequence of elements $y_l \in \Omega$ $(l = 1, 2, \ldots)$ can be distinguished a sequence converging (normwise) to some element $\mathfrak{M}_n$. Actually, from the fact that elements $y_l$ defined by equalities (5) form a bounded set, it follows by (4) that the vectors $c^{(l)}$ are also bounded in the set. But then for some subsequence of natural number $l$ equality (6) will be satisfied for some vector $c^{(0)}$ and so will the relationships (7) and (8).

Note. In special courses on functional analysis it is proven that, conversely, if any bounded set belonging to a given Banach space $\mathfrak{M}$ is compact, then $\mathfrak{M}$ is a finite-measurable space, i.e., all its elements can be written in the form of a finite sum (1), where the elements $x_1, \ldots, x_n$ form a linearly independent system.

Since

$$|c_k| \leqslant \sqrt{\sum_1^n c_k^2},$$

then

$$\sum_1^n |c_k| \leqslant n|c|,$$

therefore, if we assume that

$$M \geqslant \|x_k\| \quad (k = 1, \ldots, n),$$

then, noting also (4), we get

$$\left\|\sum_1^n c_k x_k\right\| \leqslant M \sum_1^n |c_k| \leqslant Mn|c| \leqslant Mn\lambda \left\|\sum_1^n c_k x_k\right\|,$$

and we have proven that for any $c_k$ $(k = 1, \ldots, n)$ the inequalities

$$\frac{1}{\lambda}\left(\sum_1^n c_k^2\right)^{1/2} \leqslant \left\|\sum_1^n c_k x_k\right\| \leqslant Mn\left(\sum_1^n c_k^2\right)^{1/2}, \tag{9}$$

where $\lambda$ and $M$ are positive numbers dependent on the property of the norm defined in $\mathfrak{M}_n$.

If another norm $\|\cdot\|'$ is introduced into this n-dimensional set, and the new norm thus defines another space $\mathfrak{M}'_n$, we get the new inequalities

$$\frac{1}{\lambda'}\left(\sum_1^n c_k^2\right)^{1/2} \leqslant \left\|\sum_1^n c_k x_k\right\|' \leqslant M'n\left(\sum_1^n c_k^2\right)^{1/2}, \tag{10}$$

where $\lambda'$ and $M'$ are other positive numbers different in general from and $M$. From (9) and (10), it follows that

$$\frac{1}{\lambda M'n}\left\|\sum_1^n c_k x_k\right\|' \leqslant \left\|\sum_1^n c_k x_k\right\| \leqslant \lambda' Mn \left\|\sum_1^n c_k x_k\right\|'. \tag{11}$$

1.2.4. **Equivalent normed spaces.** If a linear set is normed by the two methods, which leads to two normed spaces $E_1$ and $E_2$, and if two positive constants $c_1$ and $c_2$ independent of $x \in E_1$, $E_2$ exists, such that

$$c_1 \|x\|_{E_1} \leqslant \|x\|_{E_2} \leqslant c_2 \|x\|_{E_1} \qquad (1)$$

for all $x \in E_1$, $E_2$, then the spaces $E_1$ and $E_2$ are termed equivalent.

As a rule, we will not distinguish between equivalent norms, i.e., we will use the same notation for equivalent norms.

It follows from inequality 1.2.3 that any two normings of an n-dimensional linear manifold lead to equivalent normed spaces.

In further discussion, sets of trigonometric or algebraic polynomials of one variable of given degree $\nu$ or of n variables with given degrees $\nu_1, \ldots, \nu_n$ or simply a system $\xi = \{\xi_1, \ldots, \xi_n\}$ of n numbers normed by any given method will usually figure as finite-measurable subspaces.

1.2.5. **Real Hilbert spaces.** Let H be a linear set and bring in correspondence to each two of its elements $f, \varphi$ a real number $(f, \varphi)$ -- the scalar product of f and $\varphi$, exhibiting the following properties:

1) $(f,f) \geq 0$; from $(f,f) = 0$ it follows that $f = \theta$, the zero element in H;
2) $(f, \varphi) = (\varphi, f)$; and
3) $(c_1 f_1 + c_2 f_2, \varphi) = c_1(f_1, \varphi) + c_2(f_2, \varphi)$, whatever be the real number $c_1$, $c_2$ and the elements $f, \varphi, f_1$, and $f_2 \in H$.

The norm

$$\|f\| = (f, f)^{1/2}$$

is introduced in H (it is not difficult to test whether this expression is actually is the norm). H is made into a normed space with this norm. If H is a complete space, H is called a Hilbert space (real).

Notice that for any real $\lambda$ and $f, \varphi \in H$

$$0 \leqslant (\lambda f + \varphi, \lambda f + \varphi) = \lambda^2 (f, f) + 2\lambda (f, \varphi) + (\varphi, \varphi),$$

therefore

$$|(f, \varphi)| \leqslant (f, f)^{1/2} (\varphi, \varphi)^{1/2} = \|f\| \, \|\varphi\|.$$

The space $L_2(\Omega)$ of real functions measurable on $\Omega$ and with integrable squares on $\Omega$ with the scalar product

$$(f, \varphi) = \int_\Omega f(x) \varphi(x)\, dx \quad (f, \varphi \in L_2(\Omega)),$$

serves as an important example of a real Hilbert space. We also come across other examples (cf, for example, 4.3.1 (4)).

We can easily see that for any f and $\varphi \in H$ the equality

$$\|f+\varphi\|^2+\|f-\varphi\|^2=2(\|f\|^2+\|\varphi\|^2), \tag{1}$$

is satisfied, recalled a familiar truth from geometry: the sum of the squares of the diagonals of a parallelogram equal to sum of the squares of its sides. The space $L_p(\Omega)$ when $p \neq 2$ is not Hilbertian, because functions $f, \varphi$ can be shown to belong to it for which equality (1) is not satisfied.

1.2.6. Distance from element to a subspace. Best approximation. Let $\mathfrak{M}$ be a subspace of a Banach space E, and let $y \in E$. The distance from y to $\mathfrak{M}$ will be the term for the lower bound

$$E(y)=\inf_{x \in \mathfrak{M}} \|y-x\|, \tag{1}$$

extended to all elements $x \in \mathfrak{M}$. We will frequently, following the conventions accepted in the theory of function approximation, called the number E(y) the best approximation of the element y by means of the elements $x \in \mathfrak{M}$.

It can be the case that in $\mathfrak{M}$ there exists the element $x_*$ such that for it the lower bound considered here is realized, i.e.,

$$E(y)=\min_{x \in \mathfrak{M}} \|y-x\|=\|y-x_*\|. \tag{2}$$

In this case the element $x_*$ is called the best element, approximating y by means of the elements $x \in \mathfrak{M}$.

It is important to note the quite general cases when it can be stated in advance that the best element in the problem (1) does exist. Moreover, another problem is of interest: whether the best element is unique for the given problem.

It is not difficult to see that if $\mathfrak{M}=\mathfrak{M}_n$ is a finite-measurable subspace of an arbitrary normed space E, then for any element $y \in E$ the best element approximating y by means of $x \in \mathfrak{M}_n$ will always exist. Actually, let

$$E(y)=\inf_{x \in \mathfrak{M}_n} \|y-x\|;$$

then there exists a (minimizing) sequence of elements $x^{(l)}$ $(l = 1, 2, \ldots)$, such that

$$\|y-x^{(l)}\|=E(y)+\varepsilon_l \quad (\varepsilon_l \geqslant 0,\ \varepsilon_l \xrightarrow[l\to\infty]{} 0).$$

This sequence is bounded and, therefore, compact, and thus, some one of its subsequences converges normwise to some element $x_* \in \mathfrak{M}_n$. It is not difficult to see that $x_*$ is the best element approximating y by means of $x_* \in \mathfrak{M}_n$. Generally speaking, it is not unique.

If $\mathfrak{M}$ is an infinite-measurable (not finite-measurable) subspace of the space E, then in the problem (1) the best element may not exist at all. These effects are found, for example, in the spaces $L_\infty(\xi)$ and $L_1(\xi)$. However, when p satisfies the inequalities $1 < p < \infty$, the existence of the best function occurs for any function $f \in L_p(\xi)$ and any subspace $\mathfrak{M} \subset L_p(\xi)$. Moreover, in this case the best function is always unique; these facts are proven below in 1.3.6. In the spaces $L_1(\xi)$ and $L_\infty(\xi)$, if the best element exists, then it is not always unique (cf. 1.2.7, examples 1) and 2)). Incidentally, cases of the uniqueness of the best function are found in the spaces L, $L_\infty$ and C; but these cases depend on the special properties of the subspaces $\mathfrak{M}$ and the approximable functions f. These questions are not taken up in this book.

1.2.7. Example 1. Let the function $f(x) = \operatorname{sign} x$. We will approximate it in the metric L(-1,1)*) by means of the constant functions $\varphi(x) = c$, i.e., we will search for the constant $\lambda$ for which the following minimum will be attained

$$\min_c \|f - c\|_{L(-1,+1)} = \min_c \int_{-1}^{1} |f(x) - c|\,dx = \int_{-1}^{1} |f(x) - \lambda|\,dx.$$

It is not difficult to see that the minimum is attained for any constant $\lambda$ that satisfies the inequalities $-1 \leq \lambda \leq 1$.

From the viewpoint of the notations that figure in the preceding section, it can be stated that we approximated the function $f \in L(-1, +1)$ by means of the constants at (-1, +1) of the functions $\varphi(x) = c$ forming a one-dimensional subspace of the space L(-1, +1). The best function did not prove to be unique.

Example 2. We will approximate the functions $f(x) = \operatorname{sign} x$ now in the metric $L_\infty(-1, +1) = M(-1, +1)$ by using the linear functions

$$\varphi(x) = Ax + B,$$

where A and B = arbitrary real numbers.

It is not difficult to see that

$$\min_{A,B} \|f(x) - Ax - B\|_{M(-1,+1)} = \min_{A,B} \max_{-1 \leq x \leq 1} |f(x) - Ax - B| = \|f(x) - \lambda x\|_{M(-1,+1)},$$

*) $L_p(a,b)$ stand for $L_p(\xi)$, where $\xi$ is the segment $[a, b]$.

where $\lambda$ can be any number satisfying the inequality $|\lambda| \leq 1$.

Thus, in this example as well the best function is not unique.

1.2.8. Linear operators. If E and E' are Banach spaces and there corresponds to each element $\mathbf{x} \in E$, by means of some law, the specific element

$$y = A(\boldsymbol{x}),$$

belonging to E', then we say that A is an operator reflecting E and E'. The operator A is linear if, whatever be the elements $\mathbf{x}_1$ and $\mathbf{x}_2 \in E$ and the numbers $c_1$ and $c_2$ (real or complex, depending on whether E and E' are real or complex spaces), the following equality holds:

$$A(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2) = c_1 A(\boldsymbol{x}_1) + c_2 A(\boldsymbol{x}_2).$$

The linear operator is called bounded, if there is a positive constant M such that the equality

$$\|A(\boldsymbol{x})\|_{E'} \leqslant M \|\boldsymbol{x}\|_E \qquad \text{for all } \mathbf{x} \in E \qquad (1)$$

obtains. The smallest constant M for which this inequality is satisfied for all $\mathbf{x} \in E$ is called the norm of the operator A and is denoted by the symbol $\|A\|$. The norm of an operator can also be defined as one of the upper bounds:

$$\|A\| = \sup_{\|\boldsymbol{x}\|_E} \frac{\|A(\boldsymbol{x})\|_{E'}}{\|\boldsymbol{x}\|_E} = \sup_{\|\boldsymbol{x}\|_E \leq 1} \|A(\boldsymbol{x})\|_{E'}.$$

The operator A is called wholly continuous if it maps any bounded set $\subset E$ onto a compact set belonging to E'. In other words, whatever be the bounded sequence $\{\mathbf{x}_l\}$ of elements E, it is possible to select from it such a subsequence $\{\mathbf{x}_{l_k}\}$ and such an element $\mathbf{y}_0 \in E'$ that

$$\lim_{k \to \infty} A(\boldsymbol{x}_{l_k}) = \boldsymbol{y}_0.$$

If the space E' is finite-measurable, any linear bounded operator A mapping E onto E' is a wholly continuous operator, since A maps any bounded set of E onto a bounded set of E', and the latter is compact by virtue of the finite-measurability of E'.

Let us look at an example. Let E as before stand for a Banach space and let $\mathfrak{M}$ be its finite-measurable subspace.

Further, let there be brought into correspondence with each element $\mathbf{x} \in E$ only one element $\mathbf{x}_* = A(\mathbf{x})$ that bests approximates $\mathbf{x}$ among the elements $\mathbf{u} \in \mathfrak{M}$, in other words, let $A(\mathbf{x})$ be the unique element of $\mathfrak{M}$ for which the equality

$$\min_{\boldsymbol{u} \in \mathfrak{M}} \|\boldsymbol{x} - \boldsymbol{u}\|_E = \|\boldsymbol{x} - A(\boldsymbol{x})\|_E.$$

is satisfied.

Then $A(x)$ is an operator mapping E onto $\mathfrak{M}$. This operator, generally speaking, is nonlinear (it is linear if E is a Hilbert space), but is wholly continuous, as is evident from the following argument. From the inequality

$$\| A(x)\|_E - \| x \|_E \leqslant \| x - A(x)\|_E \leqslant \| x \|_E$$

it follows that

$$\| A(x)\|_E \leqslant 2 \| x \|_E .$$

Hence, it follows that the operator A maps a bounded set of element of E onto a bounded set of elements of $\mathfrak{M}$. But the latter, by reason of the finite-measurability of $\mathfrak{M}$, is compact.

Note. The definition of the wholly continuous operator can be extended also to multi-valued operators mapping E onto E', i.e., such that to each $x \in E$ there corresponds, generally speaking, more than one element $y = A(x)$. The multivalued operator A is called wholly continuous if, from any bounded sequence of elements $x_1 \in E$ a subsequence $\{x_{1_k}\}$ and such specific values of the operator A that the sequence $\{A(x_{1_k})\}$ converges in E' can be separated out.

This example of the operator $A(x)$ of the best approximation of the element x by means of elements of a finite-measurable subspace $\mathfrak{M}$ in the general case yields a multivalued operator, which is wholly continuous in the above-indicated sense.

## 1.3 Properties of the space $L_p(\mathfrak{T})$

We have only formulated and explained a few of the properties of the space $L_p(\mathfrak{T})$, referring the reader for their proof to other sources (cf. notes to chapter 1 at the end of the book).

1.3.1. It was already pointed out in 1.2.2 that $L_p(\mathfrak{T})$ is a Banach (real or complex) space. Thus, the following properties are satisfied for elements of the space $L_p(\mathfrak{T})$:

1) the norm

$$\| f \|_{L_p(\mathfrak{T})} = \left( \int_{\mathfrak{T}} | f |^p \, d\mathfrak{T} \right)^{1/p}$$

of each function $f \in L_p(\mathfrak{T})$ is nonnegative and equal to zero only for the function $f_0$ equivalent to zero ($f_0 = 0$);

2) $$\| f_1 + f_2 \|_{L_p(\mathfrak{T})} \leqslant \| f_1 \|_{L_p(\mathfrak{T})} + \| f_2 \|_{L_p(\mathfrak{T})};$$

3) $$\| cf \|_{L_p(\mathfrak{T})} = | c | \| f \|_{L_p(\mathfrak{T})},$$

where c is an arbitrary (real or complex) number;

4) from the fact that $f_k \in L_p(\mathscr{E})$ and

$$\| f_k - f_l \|_{L_p(\mathscr{E})} \to 0 \quad (k, l \to \infty),$$

there follows the existence of the function $f_* \in L_p(\mathscr{E})$, for which

$$\lim_{k \to \infty} \| f_k - f_* \|_{L_p(\mathscr{E})} = 0. \tag{1}$$

Properties 1) and 3) are self-evident. Inequality 2) is called the Minkowski inequality. It can be converted into an equality if and only if the functions $f_1$ and $f_2$ are linearly dependent as elements of the space $L_p$. Property 4) is the theorem of the completeness of the space $L_p$.

We will write
$$\Psi(x) = u_0(x) + u_1(x) + \ldots \quad (x \in \mathscr{E})$$

and state that the series appearing in the right side of this equality converges in the sense of $L_p(\mathscr{E})$ to its sum $\Psi(x)$, if

$$\lim_{N \to \infty} \left\| \Psi - \sum_0^N u_k \right\|_{L_p(\mathscr{E})} = 0.$$

The Minkowski inequality is extended by induction to the case of N functions, and then it takes on the form

$$\left\| \sum_1^N f_k \right\|_{L_p(\mathscr{E})} \leqslant \sum_1^N \| f_k \|_{L_p(\mathscr{E})}. \tag{2}$$

From which it is also easy to derive the inequality

$$\left\| \sum_1^\infty f_k \right\|_{L_p(\mathscr{E})} \leqslant \sum_1^\infty \| f_k \|_{L_p(\mathscr{E})}, \tag{3}$$

corresponding to the case $N = \infty$. It is read thusly: if the functions $f_k \in L_p(\mathscr{E})$ ($k = 1, 2 \ldots$) and the series (of numbers) in the right side of (3) converge, then the series $f_1 + f_2 + \ldots$ converges in the sense of $L_p(\mathscr{E})$ to some function (belonging to $L_p(\mathscr{E})$), which is symbolized by $\sum_1^\infty f_k$ and inequality (3) holds.

Let us note yet another following fact we need to have. If the series

$$f(x) = f_1(x) + f_2(x) + \ldots$$

converges in the ordinary sense almost everywhere on $\mathcal{E}$ to the function f and, moreover, it converges to $f_*$ in the sense of $L_p(\mathcal{E})$, then $f(x) = f_*(x)$ almost everywhere on $\mathcal{E}$. Actually, from the condition, the sum $S_n(x)$ of the first n numbers of our series converges in the metric $L_p(\mathcal{E})$ to $f_*$. But then, as we know from the theory of functions of a real variable, there exists the subsequence of indexes $n_1, n_2, \ldots$ such that $S_{n_k}(x)$ converges in the usual sense almost everywhere to $f_*(x)$ on $\mathcal{E}$ and, since $S_{n_k}(x)$ almost everywhere also converges to $f(x) = f_*(x)$.

In the left side of inequality (2), at first an operation of summation with respect to the index k was carried out, and as a result the operation of taking the norm was used, while in the right side these two operations were interchanged. Below is derived a similar inequality, when the operation of summing over the index k is replaced by the operation of integrating over the variable k.

1.3.2. Generalized Minkowski inequality. For the function $k(u, y)$ given on a measurable set $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2 \subset R_n$, where $x = (u, y)$, $u = (x_1, \ldots, x_m)$, and $y = (x_{m+1}, \ldots, x_n)$, the inequality

$$\left(\int_{\mathcal{E}_2}\left|\int_{\mathcal{E}_1} k(u, y)\,dy\right|^p du\right)^{1/p} \leqslant \int_{\mathcal{E}_2}\left(\int_{\mathcal{E}_1} |k(u, y)|^p\,du\right)^{1/p} dy, \qquad 1 \leqslant p \leqslant \infty, \tag{1}$$

obtains, which must be understood in the sense that if its right side is rational, i.e., for almost all $y \in \mathcal{E}_2$ there exists an inner integral in $\mathcal{E}_1$ and there exists an outer integral in $\mathcal{E}_2$, then the left side also is rational; the left side does not exceed the right.

1.3.3. Inequality 1.3.2(1), in particular, will often be used in the following situations:

$$\begin{aligned}\left(\int\left|\int K(t-x)f(t)\,dt\right|^p dx\right)^{1/p} &= \\ = \left(\int\left|\int K(t)f(t+x)\,dt\right|^p dx\right)^{1/p} &\leqslant \\ \leqslant \int |K(t)|\left(\int |f(t+x)|^p dx\right)^{1/p} dt &= \\ = \int |K(t)|\,dt\left(\int |f(u)|^p du\right)^{1/p} &= \|K\|_{L(R)}\|f\|_{L_p(R)},\end{aligned} \tag{1}$$

где $1 \leqslant p \leqslant \infty$, $K \in L(R)$, $f \in L_p(R)$ *).

where

---

*) Here and in the treatment below $\int = \int_R$, $R = R_n$.

If functions $K(t)$ and $f(t)$ are periodic functions with a $2\pi$ period and if $K \in L(0, 2\pi)$ and $f \in L_p(0, 2\pi)$, then the analogous inequality

$$\left(\int_0^{2\pi}\left|\int_0^{2\pi} K(t-x)f(t)\,dt\right|^p dx\right)^{1/p} \leqslant \|K\|_{L(0,2\pi)}\|f\|_{L_p(0,2\pi)} \quad (2)$$

(2)

holds, or a similar inequality for periodic functions of n variables.

1.3.4. Hölder's inequality. If $f_1 \in L_p(\mathfrak{E})$, $f_2 \in L_q(\mathfrak{E})$ and $1/p + 1/q = 1$, where $1 \leq p \leq \infty$*), then Hölder's inequality

$$\int_{\mathfrak{E}} |f_1 f_2|\,d\mathfrak{E} \leqslant \|f_1\|_{L_p(\mathfrak{E})}\|f_2\|_{L_q(\mathfrak{E})}. \quad (1)$$

obtains. It converts to an equality if and only if $f_1$ and $f_2$ are linearly dependent.

*) When $p = \infty$, it is assumed that $1/p = 0$.

1.3.5. **Clarkson inequality**). Uniform convexity.** Let $f_1 \in L_p(\mathcal{E})$, $f_2 \in L_p(\mathcal{E})$, and $1/p + 1/q = 1$, where $1 \leqslant p \leqslant \infty$*), then the Hölder inequality

$$\left\| \frac{f_1+f_2}{2} \right\|^p_{L_p(\mathcal{E})} + \left\| \frac{f_1-f_2}{2} \right\|^p_{L_p(\mathcal{E})} \leqslant \frac{1}{2}\|f_1\|^p_{L_p(\mathcal{E})} + \frac{1}{2}\|f_2\|^p_{L_p(\mathcal{E})}. \qquad (1)$$

If however $1 < p \leqslant 2$, $1/p + 1/q = 1$, then

$$\left\| \frac{f_1+f_2}{2} \right\|^q_{L_p(\mathcal{E})} + \left\| \frac{f_1-f_2}{2} \right\|^q_{L_p(\mathcal{E})} \leqslant \left( \frac{1}{2}\|f_1\|^p_{L_p(\mathcal{E})} + \frac{1}{2}\|f_2\|^p_{L_p(\mathcal{E})} \right)^{\frac{1}{p-1}}. \qquad (2)$$

When $p = 2$, inequalities (1) and (2) convert to equalities (equalities of a parallelogram).

It is said that the Banach space E is uniformly convex, if from the fact that

$$\max_{0 \leqslant \alpha \leqslant 1} \left(1 - \|\alpha x_1^{(n)} + (1-\alpha) x_2^{(n)}\|\right) \xrightarrow[n\to\infty]{} 0,$$

where $\| x_1^{(n)} \| = \| x_2^{(n)} \| = 1$, it follows that

$$\| x_1^{(n)} - x_2^{(n)} \| \xrightarrow[n\to\infty]{} 0.$$

**) Cf, for example, the book by S. L. Sobolev [4].

From the Clarkson inequalities (1) and (2), it follows that the space $L_p(\mathcal{E})$ $(1 < p < \infty)$ is uniformly convex. Actually, let

$$\|\cdot\|_{L_p(\mathcal{E})} = \|\cdot\|.$$

and

$$\|f_1^{(n)}\| = \|f_2^{(n)}\| = 1.$$

Then, from (1) and (2) it follows that

$$\left\|\frac{f_1^{(n)} - f_2^{(n)}}{2}\right\|^{\lambda} \leqslant 1 - \left\|\frac{f_1^{(n)} + f_2^{(n)}}{2}\right\|^{\lambda}, \tag{3}$$

where $\lambda = p$ in the case of (1) and $\lambda = q$ in the case of (2). If now

$$\max_{0 < \alpha < 1} (1 - \|\alpha f_1^{(n)} + (1-\alpha) f_2^{(n)}\|) \xrightarrow[n\to\infty]{} 0,$$

then

$$1 - \left\|\frac{f_1^{(n)} + f_2^{(n)}}{2}\right\| \to 0.$$

But then the right side of (3) tends to zero, and with it, the left side as well. This means that

$$\|f_1^{(n)} - f_2^{(n)}\| \xrightarrow[n\to\infty]{} 0.$$

1.3.6. Theorem. Let E (in particular, $L_p(\mathcal{E})$, $1 < p < \infty$) be a uniformly convex Banach space, $\mathfrak{M}$ be its subspace, and $y \in E - \mathfrak{M}$.

Then there exists an element, that is unique, $u \in \mathfrak{M}$ best approximating y by means of elements from $\mathfrak{M}$:

$$\|y - u\| = \inf_{x \in \mathfrak{M}} \|y - x\|. \tag{1}$$

Proof. Let

$$\inf_{x \in \mathfrak{M}} \|y - x\| = d \qquad (d > 0);$$

then there exists a minimizing sequence of elements $x_n \in \mathfrak{M}$ for which

$$\|y - x_n\| = d + \varepsilon_n \qquad (\varepsilon_n \to 0,\ \varepsilon_n \geqslant 0).$$

We will assume that x and x' stand for any elements of $\mathfrak{M}$. Obviously, the elements

$$w_n = \frac{y - x_n}{d + \varepsilon_n}$$

are unique norms also for any $\alpha, \beta > 0, \alpha + \beta = 1$,

$$0 < 1 - \|\alpha w_n + \beta w_m\| = 1 - \left\|\left(\frac{\alpha}{d+\varepsilon_n} + \frac{\beta}{d+\varepsilon_m}\right) y - x\right\| =$$
$$= 1 - \left(\frac{\alpha}{d+\varepsilon_n} + \frac{\beta}{d+\varepsilon_m}\right)\|y - x\| <$$
$$< 1 - \left(\frac{\alpha}{d+\varepsilon_n} + \frac{\beta}{d+\varepsilon_m}\right) d = \eta_{nm} \xrightarrow[n, m \to \infty]{} 0,$$

that are uniform with respect to the $\alpha$ and $\beta$ considered.

For this case, by the definition of a uniformly convex space

$$\| w_n - w_m \| \xrightarrow[n, m \to \infty]{} 0,$$

But

$$\| w_n - w_m \| = \left\| y \left(\frac{1}{d+\varepsilon_n} - \frac{1}{d+\varepsilon_m}\right) - \left(\frac{x_n}{d+\varepsilon_n} - \frac{x_m}{d+\varepsilon_m}\right)\right\| =$$
$$= \left\| \frac{x_n}{d+\varepsilon_n} - \frac{x_m}{d+\varepsilon_m}\right\| + o(1) = \frac{1}{d}\|x_n - x_m\| + o(1) \ (n, m \to \infty),$$

since elements $x_n$ and $x_m$ are bounded with respect to the norm. We have proven that

$$\| x_n - x_m \| \xrightarrow[n, m \to \infty]{} 0.$$

Owing to the completeness of E and the closure of $\mathfrak{M}$, there exists the element $u \in \mathfrak{M}$ such that $x_n \longrightarrow u$ and obviously, (1) is satisfied.

Now let yet another element $u'$ exist for which (1) is satisfied. For $0 \leq \alpha \leq 1$, we have

$$d < \|\alpha u + (1-\alpha) u' - y\| < \alpha\|u - y\| +$$
$$+ (1-\alpha)\|u' - y\| = \alpha d + (1-\alpha) d = d.$$

Consequently, $\| \alpha u + (1 - \alpha) u' - y \| = d$. Thus,

$$\max_{0 < \alpha < 1} \left\| \alpha \frac{u - y}{d} + (1-\alpha) \frac{u' - y}{d} \right\| = 1,$$

where $\left\| \dfrac{u - y}{d} \right\| = \left\| \dfrac{u' - y}{d} \right\| = 1$. Due to the uniform convexity of the space E($x_1^{(n)} = \dfrac{u - y}{d}$, $x_2^{(n)} = \dfrac{u' - y}{d}$), we obtain

$$\frac{u - y}{d} = \frac{u' - y}{d},$$

i.e., $u = u'$.

1.3.7. We often even have to resort to using the following facts pertaining to the theory of the functions of a real variable.

Let $f, f_k \in L_p(\mathcal{E})$ $(k = 1, 2, \ldots)$ and

$$\lim_{k\to\infty} \|f - f_k\|_{L_p(\mathcal{E})} = 0 \quad (k \to \infty), \quad (1 \leqslant p \leqslant \infty). \tag{1}$$

Then there exists a subsequence $\{k_l\}$ of natural numbers such that

$$\lim_{k_l\to\infty} f_{k_l}(x) = f(x) \text{ почти для всех } x \in \mathcal{E}. \quad \text{for almost all } x \in \mathcal{E} \tag{2}$$

Thus, there exists a set $\mathcal{E}' \subset \mathcal{E}$, distinct from $\mathcal{E}$, on a set of zero measure, such that $f$ and $f_{k_l}$ $(l = 1, 2, \ldots)$ on $\mathcal{E}$ are finite and equality (2) is satisfied for all $x \in \mathcal{E}'$, whence it easily follows that if along with (1), for some $p_*$ there exists $\lim_{k\to} \| f_* - f_k \|_{L_{p*}(\mathcal{E})} = 0$, then $f_* = f$, i.e., the functions $f$ and $f_*$ are equivalent in $\mathcal{E}$.

If the measurable set $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2$ is the topological product of two measurable sets $\mathcal{E}_1$ and $\mathcal{E}_2$, i.e., each point $x \in \mathcal{E}$ can be represented in the form of a pair $x = (y, z)$, where $y \in \mathcal{E}_1$ and $z \in \mathcal{E}_2$, then we can assume that

$$f(x) = f(y, z), \quad f_k(x) = f_k(y, z) \quad (k = 1, 2, \ldots).$$

In this case,

В этом случае

$$\|f - f_k\|_{L_p(\mathcal{E})} = \left(\int_{\mathcal{E}_1} \Psi_k(y)\, dy\right)^{1/p} \quad (k = 1, 2, \ldots),$$

where

где

$$\Psi_k(y) = \int_{\mathcal{E}_2} |f(y, z) - f_k(y, z)|^p\, dz = \|f(y, z) - f_k(y, z)\|^p_{L_p(\mathcal{E}_2)}, \tag{3}$$

are $\mathcal{E}_1$-summable (belonging to $L(\mathcal{E}_1)$) nonnegative functions.

From equality (1) it follows that

$$\lim_{k\to\infty} \int_{\mathcal{E}_1} \Psi_k(y)\, dy = 0,$$

and, thus, applying to $\Psi_k$ the above-noted property (where we must assume $p = 1$ and replace $\mathcal{E}$ by $\mathcal{E}_1$), we arrive at the following lemma.

1.3.8. Lemma. From equality 1.3.7 (1), where the measurable set $\mathscr{E} = \mathscr{E}_1 \times \mathscr{E}_2$ is the topological product of the measurable sets $\mathscr{E}_1$ and $\mathscr{E}_2$, it follows that for some subsequence $\{k_l\}$ of natural numbers $k_l$ for almost all $y \in \mathscr{E}_1$, the equality

$$\lim_{k_l \to \infty} \| f(y, z) - f_{k_l}(y, z) \|_{L_p(\mathscr{E}_2)} = 0. \qquad (1)$$

is satisfied.

From the proven lemma and the note at the beginning of section 1.3.7 derives also the following lemma.

1.3.9. Lemma. Let the set $\mathscr{E} = \mathscr{E}_1 \times \mathscr{E}_2$ be defined, as in the preceding lemma and let the following equality be satisfied for the sequence of functions $f_k \in L_p(\mathscr{E})$ (k = 1, 2, 111) and the function $f(f \in L_p(\mathscr{E}))$

$$\lim_{k \to \infty} \| f - f_k \|_{L_p(\mathscr{E})} = 0 \quad (1 \leqslant p \leqslant \infty). \qquad (1)$$

Let, moreover, for some number $p'(1 \leq p' \leqslant \infty)$, in general distinct from p, and the function $f_*$, the equality

$$\lim_{k \to \infty} \| f_*(y, z) - f_k(y, z) \|_{L_{p'}(\mathscr{E}_2)} = 0 \qquad (2)$$

be satisfied for almost all $y \in \mathscr{E}_1$.

Then $f = f_*$, i.e., functions $f(x)$ and $f_*(x)$ are equivalent on $\mathscr{E}$.

Proof. By virtue of the preceding lemma for some subsequence of natural number $\{k_l\}$ and on some set $\mathscr{E}'_1 \subset \mathscr{E}'_1$, distinct from $\mathscr{E}_1$ on a set of measure (in the sense of $\mathscr{E}_1$) zero, equality 1.3.8 (1) holds for all $y \in \mathscr{E}'_1$. It can be assumed that equality (2) also obtains for all $y \in \mathscr{E}'_1$. And so, if $y \in \mathscr{E}'_1$, then (1) and (2) are satisfied for it simultaneously.

But then equality $f(y, z) = f_*(y, z)$, obtains for almost all $z \in \mathscr{E}_2$, i.e., almost everywhere in the measurable set $\mathscr{E} = \mathscr{E}_1 \times \mathscr{E}_2$.

Note also the following theorems.

1.3.10. **Theorem** (P. Fatou)*). If a sequence of measurable nonnegative functions $\{f_n\}$ almost everywhere on the measurable set $\mathcal{E} \subset R_n$ converge to the function $F(x)$, then

$$\int_{\mathcal{E}} F\,dx \leqslant \sup\left\{\int_{\mathcal{E}} f_n\,dx\right\}.$$

1.3.11. **Theorem****). From a sequence of functions $\{f_k\}$ bounded in the sense of $L_p(\mathcal{E})$ $(1 < p < \infty)$:

$$\|f_k\|_{L_p(\mathcal{E})} \leqslant M$$

we can separate the subsequence $\{f_{k_l}\}$ weakly convergent to some function $f \in L_p(\mathcal{E})$ with $\|f\|_{L_p(\mathcal{E})} \leq M$. This means that the equality

$$\lim_{k\to\infty} \int_{\mathcal{E}} f_k\varphi\,dx = \int_{\mathcal{E}} f\varphi\,dx$$

holds for any functions $\varphi \in L_p(\mathcal{E})$ $(1/p + 1/q = 1)$.

1.3.12. The function $f \in L_p(\mathcal{E})$ is called continuous in the whole in $L_p(\mathcal{E})$ if for any $\epsilon > 0$ a $\delta(\epsilon) > 0$ can be found such that

$$\|f(x+y) - f(x)\|_{L_p(\mathcal{E}_\delta)} < \varepsilon$$

only when $|y| < \delta$. (Here $\mathcal{E}_\delta$ is the set of such $x \in \mathcal{E}$, that $x + y$ for any $y$ satisfying the inequality $|y| < \delta$.)

**Theorem.** Any function $f(x) \in L_p(\mathcal{E})$, $1 \leqslant p < \infty$ is continuous in the whole in $L_p(\mathcal{E})$.

1.3.13. We will widely employ also the following inequalities for $1 \leqslant p \leqslant \infty$:

$$\left(\sum_1^\infty |a_k + b_k|^p\right)^{1/p} \leqslant \left(\sum_1^\infty |a_k|^p\right)^{1/p} + \left(\sum_1^\infty |b_k|^p\right)^{1/p}, \tag{1}$$

$$\sum_1^\infty |a_k b_k| \leqslant \left(\sum_1^\infty |a_k|^p\right)^{1/p} \left(\sum_1^\infty |b_k|^q\right)^{1/q}, \text{ if } 1/p + 1/q = 1 \tag{2}$$

*) Cf, for example, the book by I. P. Natanson [1].
**) Cf, for example, the book by V. I. Smirnov [1].

where $a_k$ and $b_k$ are arbitrary numbers. They are called, respectively, the Minkowski inequality and the Hölder inequality for sums.

It is established by (1) that a linear n-dimensional manifold of vectors $\xi = \{\xi_1, \ldots, \xi_n\}$ with the norm

$$|\xi|_{l_p^n} = \left(\sum_1^n |\xi_k|^p\right)^{1/p}, \qquad 1 < p < \infty,$$

is a normed space. In particular, from 1.2.4 it follows that for any p and p' ($1 \leq p < p' \leq \infty$)

$$c_1 \|\xi\|_{l_p^n} < \|\xi\|_{l_{p'}^n} < c_2 \|\xi\|_{l_p^n}, \tag{3}$$

where $c_1$ and $c_2$ are positive constants, independent of $\xi$. Of course, these inequalities can be derived directly, by establishing the exact constants $c_1$ and $c_2$.

$$\varkappa_n \int \psi(t)\,dt = 1,$$

## 1.4 Averaging of Functions According to Sobolev*)

Let

$$\sigma_\varepsilon = \{|x| \leq \varepsilon\}, \quad \sigma_1 = \sigma,$$

stand for a sphere in $R = R_n$ with radius $\delta$ and its center at the zero point.

Let $\psi(t)$ be an infinitely differentiable even nonnegative function of one variable t ($-\infty < t < \infty$) equal to zero for $|t| \geq 1$ such that

(1)

where $\varkappa_n$ is the area of the unit ((n-1)-dimensional) sphere in n-dimensional space.

We can take the function

$$\psi(t) = \begin{cases} \frac{1}{\lambda_n} e^{\frac{t^2}{t^2-1}}, & 0 \leq |t| < 1, \\ 0, & 1 \leq |t|, \end{cases}$$

as $\psi$ where the constant $\lambda_n$ is chosen so that condition (1) is satisfied.

*) S. L. Sobolev [4].

The function

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right), \quad \varphi(x) = \psi(|x|), \qquad \varepsilon > 0, \tag{2}$$

is infinitely differentiable on R (noting the evenness of $\psi$), has its carrier [nositel'] on $\sigma_\varepsilon$, and satisfies the condition

$$\int \varphi_\varepsilon(x)\,dx = \frac{1}{\varepsilon^n}\int \varphi\left(\frac{x}{\varepsilon}\right)dx = 1. \tag{3}$$

Let $g \subset R_n = R$ be an open set and $f \in L_p(g)$ $(1 \leq p \leq \infty)$. Assume $f = 0$ on $R - g$. The function

$$f_\varepsilon(x) = f_{g,\varepsilon}(x) = \frac{1}{\varepsilon^n}\int \varphi\left(\frac{x-u}{\varepsilon}\right)f(u)\,du = \\ = \frac{1}{\varepsilon^n}\int \varphi\left(\frac{u}{\varepsilon}\right)f(x-u)\,du \tag{4}$$

is called $\varepsilon$-averaged according to Sobolev. This is obviously an infinitely differentiable function on R.

Now we direct our attention to the following important property of f :

$$\|f_\varepsilon - f\|_p \to 0 \ (\varepsilon \to 0, \ \|\cdot\|_p = \|\cdot\|_{L_p(R)}, \ 1 \leq p < \infty). \tag{5}$$

It shows that for a finite p $(1 \leq p < \infty)$ a set of functions infinitely differentiable on R and everywhere compact in $L_p(g)$, i.e., regardless of how the open set g is constructed, for each function $f \in L_p(g)$ a family of functions $f_\varepsilon$ (its Sobolev averagings) infinitely differentiable on R can be specified such that (5) is fulfilled.

Actually, by (3)

$$f_\varepsilon(x) - f(x) = \frac{1}{\varepsilon^n}\int \varphi\left(\frac{x-u}{\varepsilon}\right)[f(u) - f(x)]\,du = \\ = -\int \varphi(v)[f(x-\varepsilon v) - f(x)]\,dv,$$

from whence, using the generalized Minkowski inequality and since $\varphi$ has a carrier in $\sigma$, we get

$$\|f_\varepsilon - f\|_p \leq \int_\sigma \varphi(v)\|f(x-\varepsilon v) - f(x)\|_p\,dv \leq \\ \leq \sup_{|v|<\varepsilon}\|f(x-v) - f(x)\|_p \to 0 \qquad (\varepsilon \to 0). \tag{6}$$

For the case $p = \infty$ property (5) is not satisfied. However, if we consider that $g = R$ and $f(x)$ is uniformly continuously on $R(f \in C(R))$, (6) can be rewritten as

$$\|f_\varepsilon - f\|_\infty \leq \sup_{|v|<\varepsilon} |f(x-v) - f(x)| \to 0 \qquad (\varepsilon \to 0).$$

Notice also the inequality

$$\|f_\varepsilon\|_p \leq \frac{1}{\varepsilon^n} \int \varphi\left(\frac{u}{\varepsilon}\right) \|f(x-u)\|_p \, du = \|f\|_p \quad (1 \leq p \leq \infty). \tag{7}$$

1.4.1. Finite functions. Let $g \subset R$ be an open set. The function (x) is called finite in $g$ if it is defined on $g$ and has a compact carrier lying on $g$. The carrier of a function is the term given to the closure of the set of all points, where it is not equal to zero.

Lemma. If $f \in L_p(g)$ $(1 \leq p < \infty)$, then there exists a sequence of functions $\varphi_l$ infinitely differentiable on $g$ for which the properties

$$\|f - \varphi_l\|_p \to 0 \qquad (l \to \infty),$$
$$|\varphi_l(x)| \leq \sup_{x \in g} \operatorname{vrai} |f(x)|.$$

are satisfied. If $f$ simultaneously belongs to $L_p$ and $L_{p'}$ $(1 \leq p, p' < \infty)$, the sequence $\{\varphi_l\}$ can be taken the function sought for.

Proof. Suppose $\eta > 0$ and select an open bounded set $\Omega \subset \bar{\Omega} \subset g$ such that

$$\|f\|_{L_p(g-\Omega)} < \frac{\eta}{2}.$$

Let d stand for the distance from $\Omega$ to the bound $g$ ($d > 0$; if $g$ is not bounded, then $d = \infty$). Let us further introduce the function

$$f_\Omega(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

Its $\varepsilon$-averaging $f_{\Omega,\varepsilon} = \varphi$ when $\varepsilon < d$ is a finite function infinitely differentiable on $g$, for which the inequalities

$$|f - f_{\Omega,\varepsilon}|_{L_p(g)} \leq |f - f_\Omega|_{L_p(g)} + |f_\Omega - f_{\Omega,\varepsilon}|_{L_p(g)} =$$
$$= \|f\|_{L_p(g-\Omega)} + |f_\Omega - f_{\Omega,\varepsilon}|_{L_p(g)} < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

are satisfied if and only if $\varepsilon$ is sufficiently small.

Further (cf 1.4 (7))

$$|f_{\Omega,\varepsilon}(x)| \leq \sup_{x \in R} \operatorname{vrai} |f_\Omega(x)| \leq \sup_{x \in R} \operatorname{vrai} |f(x)|.$$

Therefore, if $\eta = \eta_1 \to 0$, then assuming that $\varepsilon = \varepsilon_1$ and $\Omega = \Omega_1$, we obtain the result that the functions $\varphi_1 = f_{\Omega_1, \varepsilon_1}$ satisfy the requirements of the lemma. Here, if simultaneously $f \in L_p, L_{p'}$ then for both p and p' unique $\Omega_1$ and $\varepsilon_1$, and therefore, also $\varphi_1$ can be chosen.

1.4.2. Lemma. If $f \in L_\infty(g)$ (a measurable function substantially bounded in the open set $g \subset R$), then there exists a sequence of finitely differentiable functions $g_N$ finite in $g$ that satisfies the conditions

$$\lim_{l\to\infty} \varphi_l(x) = f(x) \quad \text{почти всюду на } g, \quad \text{almost everywhere in } g \tag{1}$$

$$|\varphi_l(x)| \leqslant \sup_{x \in g} \operatorname{vrai} |f(x)|. \tag{2}$$

Proof. Let $g_N$ stand for the intersection of $g$ with the sphere $|x| < N$, and let $\eta_N$ decrease monotonically to zero ($N = 1, 2, \ldots$). Since $f \in L(g_N)$ then we can specify a function $f_N$ finite in $g_N$ and therefore in $g$ such that

$$\|f - f_N\|_{L(g_N)} < \eta_N \tag{3}$$

and

$$|f_N(x)| \leqslant \sup_{x \in g} \operatorname{vrai} |f(x)|. \tag{4}$$

From (3) and (4) it follows that from the sequence $\{f_N\}$ can be separated a subsequence $\{\varphi_l\}$ subject to the requirements of the lemma.

## 1.5 Generalized Functions

Let us introduce the class S (L. Schwartz [1]) of fundamental functions $\varphi = \varphi(x)$. The function $\varphi$ of class S is defined on R, is complex-valued ($\varphi = \varphi_1 + i\varphi_2$, $\varphi_1$ and $\varphi_2$ are real), is infinitely differentiable on R, and is such that for any nonnegative number l (sufficiently integral l) and nonnegative integral vector $k = (k_1, \ldots, k_n)$

$$\sup_x (1 + |x|^l) |\varphi^{(k)}(x)| = \varkappa(l, k, \varphi) < \infty,$$

where

$$\varphi^{(k)} = \frac{\partial^{|k|}\varphi}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \qquad |k| = \sum_{j=1}^{n} k_j. \tag{1}$$

Assume $L_p = L_p(R)$. From (1), in particular, it follows that

$$|\varphi^{(k)}(x)| \leq \frac{1}{2} \varkappa(0, k, \varphi) < \infty,$$

i.e., the function $\varphi^{(k)} \in S$ is bounded ($\varphi^{(k)} \in L_\infty$). Further, $\varphi^{(k)} \in L_p$ $(1 \leq p < \infty)$, because

$$\int |\varphi^{(k)}(x)|^p \, dx \leq c_1 \int \frac{\left|\varphi^{(k)}(x)\left(1+|x|^{\frac{n+1}{p}}\right)\right|^p dx}{(1+|x|)^{n+1}} \leq$$

$$\leq c_1 \varkappa^p\left(\frac{n+1}{p}, k, \varphi\right) \int \frac{dx}{(1+|x|)^{n+1}} = c_2 \varkappa^p\left(\frac{n+1}{p}, k, \varphi\right) < \infty,$$

where n = dimensionality of R. Thus, for any

$$\varphi^{(k)} \in L_p \text{ и } \|\varphi^{(k)}\|_{L_p} \leq c\varkappa\left(\frac{n+1}{p}, k, \varphi\right). \quad (2)$$

Moreover,

$$|\varphi^{(k)}(x)| \leq \frac{\varkappa(1, k, \varphi)}{1+|x|} \to 0 \quad (|x| \to \infty). \quad (3)$$

If $\varphi_m, \varphi \in S$ (m = 1, 2, ...) and for any nonnegative integral number l and integral vector k

then we will write $\qquad \varphi_m \to \varphi (S).$

We will state the following about the function $\psi$ infinitely differentiable on R: it exhibits polynomial growth if for any nonnegative vector k there exists $l = l(k)$ such that

$$|\psi^{(k)}(x)| < c(1+|x|^l),$$

where c does not depend on x.

If $\varphi \in S$, then $\psi\varphi \in S$, because

$$(\psi\varphi)^{(k)} = \sum_{|s| \leq |k|} C_k^s \psi^{(s)} \varphi^{(k-s)}$$

$$(k = (k_1, \ldots, k_n),\ s = (s_1, \ldots, s_n),$$

$$C_k^s = \frac{k!}{s!(k-s)!}, \quad k! = \prod_{j=1}^n k_j!),$$

and if m is a natural number, then

$$|(1+|x|^m)\psi^{(s)}\varphi^{(k-s)}| \leqslant c_1|(1+|x|^m)(1+|x|^{l(s)})\varphi^{(k-s)}| \leqslant$$
$$\leqslant c_2|(1+|x|^{m+l(s)})\varphi^{(k-s)}| \leqslant c_2\varkappa(m+l(s),\ k-s,\ \varphi).$$

Moreover, these inequalities show that if

$\varphi_m,\ \varphi \in S$ and $\varphi_m \to \varphi(S)$, then $\psi\varphi_m \to \psi\varphi(S)$.

The Fourier transform of the function $\varphi$ will be denoted by:

$$\tilde{\varphi}(x) = \frac{1}{(2\pi)^{n/2}} \int \varphi(\lambda) e^{-i\lambda x} d\lambda, \quad \lambda x = \sum_1^n \lambda_j x_j,$$

and the transform that is inverse to it, as:

$$\hat{\varphi}(x) = \frac{1}{(2\pi)^{n/2}} \int \varphi(\lambda) e^{i\lambda x} d\lambda.$$

Let us show that if $\varphi \in S$, then $\tilde{\varphi}$, $\hat{\varphi} \in S$ and whatever be the non-negative number $l$ and the integral vector $k$,

$$(1+|x|^l)|\tilde{\varphi}^{(k)}(x)| \leqslant c_{l,k} \sum_{(l',k') \in \mathcal{E}_{lk}} \varkappa(l',\ k',\ \varphi), \tag{4}$$

where $c_{l,k}$ is a constant dependent on $(l, k)$ and $\mathcal{E}_{lk}$ dependent on $(l, k)$ is a finite set of pairs $(l', k')$. Here it follows that, in particular, if $\varphi_m$, $\varphi \in S$, $\varphi_m \to \varphi(S)$, then $\tilde{\varphi}_m \to \tilde{\varphi}(S)$ and $\hat{\varphi}_m \to \hat{\varphi}(S)$.

Actually,

$$\tilde{\varphi}^{(k)}(x) = \int \psi(\lambda) e^{-i\lambda x} d\lambda,$$

where

$$\psi(\lambda) = \frac{(-i\lambda)^k}{(2\pi)^{n/2}} \varphi(\lambda) \quad (\lambda^k = \lambda_1^{k_1} \dots \lambda_n^{k_n}).$$

Obviously, $\psi(\lambda) \in S$

$$(1+|x|) \leqslant (1+|x_1|+\dots+|x_n|) \tag{5}$$

and

$$|\tilde{\varphi}^{(k)}(x)| \leqslant c \int |\lambda|^{|k|} |\varphi(\lambda)| d\lambda \leqslant$$
$$\leqslant c \int \frac{|\lambda|^{|k|} d\lambda}{1+|\lambda|^{|k|+n+2}} \varkappa(|k|+n+2,\ 0,\ \varphi) \leqslant \tag{6}$$
$$\leqslant c_1 \varkappa(|k|+n+2,\ 0,\ \varphi).$$

Further, for $|x_j| \leq 1$

and for $|x_j| \geqslant 1$, assuming that $\Delta_N$ is part of R, where $|\lambda_j| < N$, and considering that (cf (3)) that $\psi \to 0$ when $\lambda_j = \pm N \to \infty$, we get

$$\bar{\varphi}^{(k)}(x) = \lim_{N \to \infty} \left\{ \psi(\lambda) \frac{e^{-i\lambda x}}{-ix_j} \Big|_{\lambda_j = -N}^{\lambda_j = N} + \frac{1}{ix_j} \int_{\Delta_N} \frac{\partial \psi}{\partial \lambda_j} e^{-i\lambda x} d\lambda \right\} =$$

$$= \frac{1}{ix_j} \int \frac{\partial \psi}{\partial \lambda_j} e^{-i\lambda x} d\lambda = \frac{c}{x_j} \int \left( \frac{\partial}{\partial \lambda_j} \left( \prod_{s=1}^{n} \lambda_s^{k_s} \right) \varphi(\lambda) + \right.$$

$$\left. + \prod_{s=1}^{n} \lambda_s^{k_s} \frac{\partial \varphi}{\partial \lambda_j} \right) e^{-i\lambda x} d\lambda. \tag{7}$$

Since $\left| \frac{\partial}{\partial \lambda_j} \left( \prod_{s=1}^{n} \lambda_s^{k_s} \right) \right| \leqslant c_1 |\lambda|^{|k|-1}$ (when $k = 0$, it must be assumed that $k - 1 = 0$), therefore

$$|x_j \bar{\varphi}^{(k)}(x)| \leqslant c_2 \int \frac{|\lambda|^{|k|-1} + |\lambda|^{|k|}}{1 + |\lambda|^{n+|k|+2}} (\varkappa(n + |k| + 2, 0, \varphi) + + \varkappa(n + |k| + 2, e_j, \varphi)) d\lambda, \tag{8}$$

where $e_j$ is the unit vector oriented along the $x_j$ axis.

From (5), (6), and (8) it follows that

$$(1 + |x|) |\bar{\varphi}^{(k)}(x)| \leqslant$$
$$\leqslant c_{1k} \left( \varkappa(n + |k| + 2, 0, \varphi) + \sum_{j=1}^{n} \varkappa(n + |k| + 2, e_j, \varphi) \right),$$

and we have proven inequality (4) for any $k$ and $l = 1$. For an arbitrary $l$, the proof is analogous; it is only necessary that integration by parts instead of once, $l$ times be carried out in equality (7).

Assume for $\varphi, \psi \in S$*) $(\varphi, \psi) = \int \varphi(x) \psi(x) dx.$

From the theory of Fourier integrals, it is known that

$$(\bar{\varphi}, \psi) = (\varphi, \bar{\psi}), \quad (\hat{\varphi}, \psi) = (\varphi, \hat{\psi}).$$

*) We direct the reader's attention to the fact that in the integrand $\psi$ is taken without the sign of complex conjugation (cf. V. S. Vladimirov [1]).

The functional $(f, \varphi)$ that is linear and continuous on S is called a generalized function (over S).

Thus, if $\varphi_1, \varphi_2, \varphi_m, \varphi$ S, $c_1$ and $c_2$ are complex numbers and $\varphi_m \to \varphi(S)$, then

$$(f, c_1\varphi_1 + c_2\varphi_2) = c_1(f, \varphi_1) + c_2(f, \varphi_2)$$
$$(f, \varphi_m) \to (f, \varphi) \quad (m \to \infty).$$

The set of all generalized (over S) functions f is symbolized by S'.

The derivative of $f \in S'$ with respect to the variable $x_j$ is defined as the linear functional

$$(f'_{x_j}, \varphi) = -(f, \varphi'_{x_j}).$$

If f(x) is an ordinary measurable function defined on R and such that the integral

$$(f, \varphi) = \int f(x)\varphi(x)\,dx \qquad (9)$$

exists for all $\varphi \in S$, which proves to be a linear functional over S, then the generalized function defined by equality (9) is identical with f(x). For example, if $f \in L_p (1 \leq p \leq \infty)$, then integral (9) is a linear functional over S. Actually,

$$\int |f(x)\varphi(x)|\,dx \leqslant \left(\int |f|^p\,dx\right)^{1/p} \left(\int |\varphi|^q\,dx\right)^{1/q} \leqslant$$
$$\leqslant c\varkappa\left(\frac{n+1}{q}, 0, \varphi\right)\left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

and therefore, integral (9) is finite for all $\varphi \in S$ and is continuous in S. The linearity of (9) is obvious.

If $f(x) \in S$, $a \in R$, and $c \neq 0$ is a real number, then $f(x + a)$ and $f(cx) \in S'$ are defined, respectively, as the functionals

$$(f(x+a), \varphi(x)) = (f(x), \varphi(x-a)),$$
$$f(cx) = \frac{1}{|c|}\left(f(x), \varphi\left(\frac{x}{c}\right)\right).$$

If f is a generalized function, and $\psi$ is an infinitely differentiable functions of polynomial growth, the functional over S defined by the equality

$$(f\psi, \varphi) = (f, \psi\varphi),$$

obviously is also a generalized function, denoted by $f\psi$ or by $\psi f (f\psi = \psi f)$.

If $\psi_1$ and $\psi_2$ are two infinitely differentiable functions with polynomial growth, their product exists the same property; here it is easy to see that if $f \in S'$, then

$$(\psi_1\psi_2)f = \psi_1(\psi_2 f).$$

Clearly, if f(x) is an ordinary function belonging to L, and $\psi(x)$ is an infinitely differentiable function with polynomial growth, the ordinary product $f(x)\psi(x)$ corresponds by the rule of identity to the generalized function $f\psi$ (to the product of the generalized function f and $\psi$).

The Fourier transform (direct and inverse) for $f \in S'$ is defined, respectively, by the equalities

$$(\tilde{f}, \varphi) = (f, \tilde{\varphi}), \quad (\hat{f}, \varphi) = (f, \hat{\varphi}) \; (\varphi \in S).$$

Since $\varphi_m \to \varphi \, (S)$ ($\varphi_m, \varphi \in S$) entails $\tilde{\varphi}_m \to \tilde{\varphi}$, $\hat{\varphi}_m \to \hat{\varphi} (S)$, therefore $\tilde{f}, \hat{f} \in S'$.

If $f(x) \in L_p$ ($1 \leq p \leq \infty$) is an ordinary functions summable to the p-th degree on R, then it, as we know, is a generalized function and has the Fourier transformation $\hat{f}$, which is, generally speaking, a generalized function. If $f \in L_2$, then, as we know, $\hat{f} \in L_2$ (Plansherel' theorem, cf. the book by N. I. Akhiyezer [1]),

$$\tilde{f}(x) = \frac{1}{(2\pi)^{n/2}} \lim_{N \to \infty} \int_{\Delta_N} f(t) e^{-ixt} \, dt,$$

$$\Delta_N = \{|x_j| < N; \; j = 1, \ldots, n\},$$

and the convergence is understood in the $L_2$-sense. Here $\int \hat{f} \varphi \, dx = \int f \hat{\varphi} \, dx$ (for all $\varphi \in S$), which shows that in this case the ordinary Fourier transform of the function coincides (identifies) with the generalized function.

Suppose $\varphi \in S$; then

$$\varphi^{(k)}(x) = \frac{1}{(2\pi)^{n/2}} \int (iu)^k \tilde{\varphi}(u) e^{ixu} \, du$$

and

$$\widetilde{\varphi^{(k)}} = (iu)^k \tilde{\varphi}(u) \qquad (u^k = u_1^{k_1}, \ldots, u_n^{k_n}).$$

Further

$$\tilde{\varphi}^{(k)}(x) = \frac{1}{(2\pi)^{n/2}} \int (-iu)^k \varphi(u) e^{-ixu} \, du = \widetilde{(-iu)^k \varphi(u)}.$$

The analogous equalities

$$\widetilde{f^{(k)}} = (iu)^k \tilde{f}, \quad \tilde{f}^{(k)} = \widetilde{(-iu)^k f}. \tag{10}$$

obtain for the generalized functions $f \in S'$.

Actually, if $f \in S'$ and $\varphi \in S$, then

$$(\widetilde{f^{(k)}}, \varphi) = (-1)^{|k|}(f, \tilde{\varphi}^{(k)}) = (-1)^{|k|}(f, \widetilde{(-iu)^k\varphi}) = ((iu)^k \tilde{f}, \varphi),$$

$$(\tilde{f}^{(k)}, \varphi) = (-1)^{|k|}(f, \widetilde{\varphi^{(k)}}) = (-1)^{|k|}(f, (iu)^k \tilde{\varphi}) = (\widetilde{(-iu)^k f}, \varphi).$$

Let as before $\varphi \in S$ and

$$\Delta_N = \{|x_j| < N;\ j = 1, \ldots, n\} \subset R.$$

Then

$$(\tilde{1}, \varphi) = (1, \tilde{\varphi}) = \frac{1}{(2\pi)^{n/2}} \int dx \int \varphi(t) e^{-ixt} dt =$$

$$= \frac{1}{(2\pi)^{n/2}} \lim_{N \to \infty} \int \varphi(t) dt \int_{\Delta_N} e^{-ixt} dx =$$

$$= (2\pi)^{n/2} \lim_{N \to \infty} \frac{1}{\pi^n} \int \varphi(t) \prod_{j=1}^{n} \frac{\sin Nt_j}{t_j} dt = (2\pi)^{n/2} \varphi(0).$$

The last equality follows from the ordinary Fourier integral theory.

Thus,

$$\tilde{1} = (2\pi)^{n/2} \delta(x),$$

where $\delta(x)$ is the ordinary delta-function, i.e., a generalized function defined by the equality

$$(\delta, \varphi) = \varphi(0) \qquad (\varphi \in S).$$

Hence, if $k = (k_1, \ldots, k_n)$ is a vector with integral nonnegative components, then

$$\widetilde{x^k} = i^k \widetilde{(-ix)^k \cdot 1} = i^k (2\pi)^{n/2} \delta^{(k)}(x). \tag{11}$$

Further

$$(\tilde{\delta}, \varphi) = (\delta, \tilde{\varphi}) = \frac{1}{(2\pi)^{n/2}} \int \varphi(t) dt,$$

i.e.,

$$\tilde{\delta} = \frac{1}{(2\pi)^{n/2}}. \tag{12}$$

We write for the functions $f, f_l \in S'$ $(l = 1, 2, \ldots)$

$$f_l \to f(S'), \text{ если } (f_l, \varphi) \to (f, \varphi) \tag{13}$$

for all $\varphi \in S$, and we state that $f_l$ tends to f in the S'-sense or even more weakly. If $f_l$ and f are ordinary integrable functions such that almost every

$$f_l(x) \to f(x) \quad (l \to \infty)$$

and

$$|f_l(x)| \leqslant \Phi(x) \in L \quad (l = 1, 2, \ldots),$$

where $\Phi$ does not depend on l, then obviously $f_l$, $f \in S'$ and, by the Lesbesgue theorem, $f_l \to f$, weakly.

From (13) it follows, obviously, that if $f_l \to f(S')$, then

$$\tilde{f}_l \to \tilde{f}, \quad \hat{f}_l \to \hat{f}\,(S'), \tag{14}$$
$$\lambda f_l \to \lambda f\,(S'), \tag{15}$$
$$f_l^{(s)} \to f^{(s)}\,(S'), \tag{16}$$

where $\lambda$ is an infinitely differentiable function with polynomial growth.

Let $\varphi \in S$, $\mu = (\mu_1, \ldots, \mu_n)$, and $t = (t_1, \ldots, t_n)$ be real vectors, then

$$\widetilde{e^{i\mu t}\hat{\varphi}} = \frac{1}{(2\pi)^{n/2}} \int e^{i\mu t} \frac{1}{(2\pi)^{n/2}} \int \varphi(u)\, e^{-iut}\, du\, e^{ixt}\, dt = \\ = \frac{1}{(2\pi)^n} \int e^{ixt}\, dt \int \varphi(\mu + v)\, e^{-ivt}\, dv = \varphi(\mu + x). \tag{17}$$

If here $f \in S'$, then, considering that the function $e^{i\mu t}$ is infinitely differentiable and is bounded together with its derivatives (of polynomial growth), we get for $\varphi \in S$

$$(\widetilde{e^{i\mu t}\hat{f}}, \varphi) = (f, \widehat{e^{i\mu t}\tilde{\varphi}}) = (f, \varphi(\mu + x)) = (f(x - \mu), \varphi(x)),$$

i.e.,

$$\widetilde{e^{i\mu t}\hat{f}} = f(x - \mu) \qquad (f \in S'). \tag{18}$$

Further

$$(\widehat{e^{i\mu t}\tilde{f}}, \varphi) = (f, \widetilde{e^{i\mu t}\hat{\varphi}}) = (f(x), \varphi(x - \mu)),$$

i.e.

$$\widehat{e^{i\mu t}\tilde{f}} = f(x + \mu) \quad (f \in S'). \tag{19}$$

1.5.1. Convolution. Multiplier. We will often have to deal with a situation in which some measurable function $\mu(x)$ is multiplied by $\hat{f}(x)$, where $f \in L_p = L_p(R)$ $(R = R_n)$. If $\hat{f}(x)$ is an ordinary function, we naturally assume that

$$\mu \tilde{f} = \mu(x) \tilde{f}(x).$$

(1)

However, even in this seemingly simple case difficulties can arise: along with the definition by equality (1), there can be another definition of $\mu\widetilde{f}$ as some generalized function (belonging to S'), and then we face the problem of identifying these two definitions.

In the case when $\tilde{f} \in S'$ is a generalized function, we still have available a unique definition of $\mu\tilde{f}$ on the assumption that $\mu$ is an infinitely differentiable function of polynomial growth. Specifically, $\mu\widehat{f}$ is defined as the functional

$$(\mu \tilde{f}, \varphi) = (\tilde{f}, \mu\varphi).$$

(2)

If $\tilde{f} \in L_p$, then this definition is in accord with formula (1), since here

$$(\mu \tilde{f}, \varphi) = \int [\mu(x) \tilde{f}(x)] \varphi(x)\, dx = \int \tilde{f}(x) [\mu(x) \varphi(x)]\, dx = (\tilde{f}, \mu\varphi).$$

Below we will introduce other definitions of $\mu\widetilde{f}$, where $\mu$ belongs to some class of functions measurable and bounded on $R = R_n$, and $f \in L_p$. This section deals with the important case when $\hat{\mu} = K \in L$. Here the function

$$\mu(x) = \frac{1}{(2\pi)^{n/2}} \int \hat{\mu}(u) e^{-ixu}\, du$$

is bounded and measurable on R, but naturally, it is not an arbitrary function continuous on R.

If $f \in S$, the functions $\widetilde{f}$, $\mu\widetilde{f}$, and $\widehat{\mu\widetilde{f}} \in S'$. But they also can be calculated by means of ordinary analysis (explanations given below):

$$\begin{aligned}
\widehat{\mu \tilde{f}} = \widehat{k \tilde{f}} &= \frac{1}{(2\pi)^{3n/2}} \int e^{ixu}\, du \int K(\xi) e^{-iu\xi}\, d\xi \int f(\eta) e^{-iu\eta}\, d\eta = \\
&= \frac{1}{(2\pi)^{3n/2}} \int e^{ixu}\, du \int K(\xi)\, d\xi \int f(\lambda - \xi) e^{-iu\lambda}\, d\lambda = \\
&= \frac{1}{(2\pi)^{3n/2}} \int e^{ixu}\, du \int e^{-iu\lambda}\, d\lambda \int K(\xi) f(\lambda - \xi)\, d\xi = \\
&= \frac{1}{(2\pi)^{n/2}} \int K(\xi) f(\lambda - \xi)\, d\xi = K * f.
\end{aligned}$$

(3)

Since $f \in S$, the integral in the third member in $\eta$ is a function of $u$ belonging to $S \subset L$; it is multiplied on the integral by $\xi$, which is a continuous bounded function; the product belongs to L, and therefore after its multiplication by $e^{ixu}$ and its integration in u, we get the continuous function Kf of x. Replacing the order of integration in $\xi$ and in $\lambda$ in the fourth equality is regular, since K, $f \in L$ (by the Fubini theorem).

The integral in the penultimate member of these relationships is called the convolution of K and f; here in the discussion where K is fixed, and $f \in L_p$ is an arbitrary function, K is called the kernel of the convolution, and we will call the function $\mu$ the multiplier in $L_p$.

The right-hand side of (3) is rational not only for $f \in S$, but also for $f \in L_p$. Whatever be the functions $K \in L$ and $f \in L_p(1 \leqslant p \leqslant \infty)$, their convolution

$$K * f = \frac{1}{(2\pi)^{n/2}} \int K(x-u) f(u)\, du =$$
$$= \frac{1}{(2\pi)^{n/2}} \int K(u) f(x-u)\, du, \tag{4}$$

is rational, satisfying the same important inequality (cf. 1.3.3 (1))

$$\|K * f\|_p \leqslant \frac{1}{(2\pi)^{n/2}} \|K\|_L \|f\|_p (\|\cdot\|_p = \|\cdot\|_{L_p(R)},$$
$$\|\cdot\|_L = \|\cdot\|_{L(R)}). \tag{5}$$

Equality (3), valid for the functions $f \in S$, serves as the basis for assuming by definition that

$$\mu \tilde{f} = \widetilde{K * f} \quad (\hat{\mu} = K \in L,\ f \in L_p). \tag{6}$$

Since, by (5), $K*f \in L_p \subset S'$, then $\widetilde{K*f} \in S'$, and we thus agree to let

f stand for this latter generalised function.

Let us show that for any function $f \in L_p(1 \leqslant p \leqslant \infty)$ there exists a sequence of infinitely differentiable finite functions $f_l$, not dependent on $\mu$ ($\hat{\mu} \in L$), such that when $l \to \infty$

$$\tilde{f}_l \to \tilde{f}(S') \quad \text{and} \quad \mu \tilde{f}_l \to \mu \tilde{f}(S'). \tag{7}$$

If p is finite, then we define (cf. section 1.4.1) the sequence of finite functions $f_l$ such that

$$\|f - f_l\|_p \to 0 \quad (l \to \infty),$$
$$\|(\hat{\mu} * f) - (\hat{\mu} * f_l)\|_p \leqslant \|\hat{\mu}\|_L \|f - f_l\|_p \to 0,$$

consequently, also in the weak sense $\tilde{f}_l \to \tilde{f}$ and $\mu \tilde{f}_l \to \mu \tilde{f}$. If however, $p = \infty$ then we define (cf. 1.4.2) the sequence of infinitely differentiable functions finite functions $f_l$, boundedly converging almost everywhere, as therefore, weakly converging in f. By virtue of the fact that $\hat{\mu} \in L$ and $f - f_l \in L_\infty$, the function

$$(\hat{\mu} * f) - (\hat{\mu} * f_l) = \frac{1}{(2\pi)^{n/2}} \int \hat{\mu}(x-t) [f(t) - f_l(t)]\, dt$$

of x is continuous and bounded on R. Based on the Lebesgue theorem on the limit under the sign of the integral, it boundedly tends to zero for all x, therefore, as a generalized function it weakly tends to zero, i.e., (7) holds.

If $\hat{\mu} = K \in L$ and at the same time $\mu$ is infinitely differentiable with polynomial growth, then we have available two definitions of the product $\mu \tilde{f}$ ($f \in L_p$). On the one hand, this is the functional

$$(\mu \tilde{f}, \varphi) = (\tilde{f}, \mu\varphi) \ (\varphi \in S),$$

and on the other, the functional (6). Let us show that these functionals are equal.

Suppose $\{f_l\}$ is a sequence of infinitely differentiable finite functions, for which $\mu \tilde{f}_l \to \tilde{\mu} f$ is weak. Then, if not only $\hat{\mu} \in L$, but even $\mu$ is infinitely differentiable with polynomial growth, then

$$(\mu \tilde{f}, \varphi) = \lim_{l \to \infty} (\mu \tilde{f}_l, \varphi) = \lim_{l \to \infty} (\tilde{f}_l, \mu\varphi) = (\tilde{f}, \mu\varphi), \tag{8}$$

and we have proven the equality of functions that is of interest to us.

Thus, definition (2) for the infinitely differentiable function of polynomial growth and definition (6), where $\hat{\mu} \in L$, do not contradict each other, whatever be the function $f \in L_p (1 \leq p \leq \infty)$.

If $\lambda$ and $\mu$ are differentiable functions of polynomial growth and $f \in S$, we know (cf. 1.5) that

$$\lambda(\mu f) = \mu(\lambda f) = (\lambda\mu) f. \tag{9}$$

If now $\hat{\lambda} = K_1 \in L$, $\hat{\mu} = K_2 \in L$, then both $\widehat{\lambda\mu} \in L$ and for all $f \in L_p (1 \leq p \leq \infty)$

$$\lambda(\mu \tilde{f}) = \mu(\lambda \tilde{f}) = (\lambda\mu) \tilde{f}. \tag{10}$$

Actually, it is easy to verify by ordinary analysis methods that under the specified conditions the function

$$K = K_1 * K_2 = \int K_1(x - u) K_2(u)\, du$$

belongs to L and that

$$K_1 * (K_2 * f) = K_2 * (K_1 * f) = (K_1 * K_2) * f, \tag{11}$$

obtains, which (by (6)) is equivalent to (10). We do not intend to examine in all its generality the case when the multiplier is the product $\lambda\mu$, where $\hat{\lambda} \in L$, and $\mu$ is an infinitely differentiable function of polynomial growth. We do not need this result in what follows below. But there is one case which we will find necessary, the case the multiplier $V^{-1}\mu V$, where $V$, $\hat{\mu} \in L$, and V is, moreover, a positive infinitely differentiable function of polynomial growth. If $f \in L_p$, then the operation

$$V^{-1}\mu V\tilde{f} = V^{-1}(\mu(V\tilde{f}))$$

is rational. Actually, $V\hat{f}$ can be understood in the sense of (1) or (6); this leads to the same outcome; in any case, the operation $\mu(V\tilde{f})$ (over $V\tilde{f}$) can be understood in the sense of (6) ($V\tilde{f} \in L_p$!) and, finally, the last operation

$V^{-1}(\mu V\tilde{f})$ (over $\mu V\tilde{f}$) in any case can be understood in the sense of (2); this only requires that we note that $\mu(V\tilde{f}) \in S'$, because $\widehat{\mu(V\tilde{f})} \in L_p$.

It is important that the equality

$$V^{-1}\mu V\tilde{f} = (V^{-1}\mu V)\tilde{f} = \mu\tilde{f} \qquad (12)$$

obtain for all $f \in L$. Actually, if $f$ is a finite function, it reduces to the corresponding obvious equality between ordinary functions. If however $f \in L_p$, then, as we know, we can select the sequence of finitely differentiable finite functions $f_1$ such that simultaneously $\mu\tilde{f}_1 \to \mu\tilde{f}$ and $(\mu V)\tilde{f}_1 \to (\mu V)\tilde{f}$ weakly ($\widehat{\mu V} \in L$!). But then, considering that $V^{-1}$ is an infinitely differentiable function of polynomial growth, $V^{-1}\mu V\tilde{f}_1 \to V^{-1}\mu V\tilde{f}$ weakly. Therefore, equality (12) can be obtained by the passage to limit when $1 \to \infty$ from the already established equality

$$(V^{-1}\mu V\tilde{f}_l,\ \varphi) = (\mu\tilde{f}_l,\ \varphi) \quad (\varphi \in S).$$

1.5.1.1. General definition of the multiplier in $L_p(1 \leq p \leq \infty)$. Suppose $\mu = \mu(x)$ is a bounded function measurable on $R = R_n$, therefore, $\mu \in S'$.

We emphasise that if $f \in S$, then $\tilde{f} \in S$ is an infinitely differentiable function of polynomial growth, and therefore, the product $\mu\tilde{f} \in S'$ is defined:

$$(\mu\tilde{f},\ \varphi) = (\mu,\ \tilde{f}\varphi), \qquad (1)$$

which is represented by the measurable function

$$\mu\tilde{f} = \mu(x)\tilde{f}(x).$$

By definition, the function $\mu$ is called the multiplier in $L_p(1 \leq p < \infty)$

if it is measurable and bounded (on R) and if for any infinitely differentiable finite function (or, which amounts to the same thing, for any function $f \in S$), the inequality

$$\|\widehat{\mu\tilde{f}}\|_p < c_p\|f\|_p, \qquad (2)$$

is satisfied, where the constant $c_p$ does not depend on f.

Now if $f \in L_p$ and $f_1$ are infinitely differentiable finite functions, for which $\|f - f_1\| \to 0 (1 \to \infty)$, then from (2) it follows that

$$\|\widehat{\mu\tilde{f}_k} - \widehat{\mu\tilde{f}_l}\|_p < c_p\|f_k - f_l\|_p \to 0 \qquad (k,\ l \to \infty).$$

Consequently, there exists the function $F \in L_p$ to which when $l \to \infty, \widehat{\mu \tilde{f}_l}$ tends in the $L_p$-sense. It is naturally denoted by

$$F = \widehat{\mu f} = \hat{\mu} * f, \tag{3}$$

calling $\hat{\mu} * f$ the convolution of the function (usually generalized) with f. The second member in (3) indicates that we have already defined f by the equality

$$\mu f = \overline{\hat{\mu} * f}, \tag{4}$$

where $\hat{\mu} * f$ is understood by the method described above. By this we have defined the product $\mu \tilde{f}$ for the functions $f \in L_p (1 \leq p < \infty)$. When $p = \infty$, this definition no longer obtains, because the function bounded on R cannot be approximated as closely as we would like in the metric $L_\infty$ by finite functions. But for our needs the definition of $p = \infty$ introduced in the preceding section when $\hat{\mu} = K \in L$ will be wholly adequate for the case $p = \infty$.

We will again call the multiplier $\mu$ (satisfying the property (2) the Marcinkievicz multiplier (cf. further 1.5.3).

Obviously,
$$\|\widehat{\mu f}\|_p \leqslant c_p \|f\|_p \tag{5}$$

for all $f \in L_p (1 \leq p < \infty)$, where $c_p$ is the same constant as in the corresponding inequality for $f \in S$.

The function $\mu$ for which $\hat{\mu} \in L$ obviously is the multiplier in the sense of the definition now advanced, because (cf 1.5.1) for infinitely differentiable finite functions f

$$\widehat{\mu f} = \frac{1}{(2\pi)^{n/2}} \int \hat{\mu}(x - u) f(u)\, du, \tag{6}$$

from whence (2) follows directly, where

$$c_p = \frac{1}{(2\pi)^{n/2}} \|\hat{\mu}\|_L.$$

This definition for $\hat{\mu} \in L$ is equivalent to the corresponding definition of the multiplier introduced in 1.5.1; this is to say that the function $\widehat{\mu f} = \hat{\mu} * f$ is definted (for $f \in L_p$, $1 \leq p < \infty$, and $\hat{\mu} \in L$) as the integral (6) or as the limit in the metric $L_p$ of the integral calculated for the infinitely differentiable finite function $f_1$ when $\|f - f_1\|_p \to 0$, which is obviously the same thing.

But here we generalized the concept of the multiplier and the convolution, because $\hat{\mu}$ cannot belong to L and can even be a generalized (not ordinary) function.

Notice that if $f \in L_p (1 \leq p < \infty)$ and $f_l$ are infinitely differentiable finite functions for which $\|f_l - f\|_p \to 0$ $(l \to \infty)$, and $\mu$ is the multiplier, then

$$\|\mu\tilde{f} - \mu\tilde{f}_l\|_p \leqslant c_p \|f - f_l\|_p \to 0,$$

from whence it follows that

$$\mu\tilde{f}_l \to \mu\tilde{f} \; (S'). \tag{7}$$

If $\mu$ is the Marcinkievicz multiplier and at the same time is an infinitely differentiable function of polynomial growth, then for $f \in L_p$ and the sequence $\{f_l\}$ of infinitely differentiable finite functions for which (7) is satisfied, we will have

$$(\mu\tilde{f}, \varphi) = \lim_{l \to \infty} (\mu\tilde{f}_l, \varphi) = \lim_{l \to \infty} (\tilde{f}_l, \mu\varphi) = (\tilde{f}, \mu\varphi). \tag{8}$$

In the first member of (8), $\mu\tilde{f}$ is understood in the sense of (4); in the second equality of (8) transferring $\mu$ beyond the comma is legitimate, because $\mu$ is infinitely differentiable and of polynomial growth; the last equality is based on the fact that $f_l \to f(S)$ and $\mu\varphi \in S$.

Equality (8) indicates that if $\mu$ is the Marcinkievicz multiplier and at the same time is an infinitely differentiable function of polynomial growth, then the definitions of $\mu\tilde{f}$ for $f \in L_p$ $(1 \leq p < \infty)$ corresponding to these facts do not contradict each other.

Let us show that together with $\lambda$ and $\mu$ the product $\lambda\mu$ is a multiplier in $L_p$ and that the equalities

$$\lambda\mu\tilde{f} = \mu\lambda\tilde{f} = (\lambda\mu)\tilde{f} \quad (f \in L_p, \; 1 \leqslant p < \infty). \tag{9}$$

obtain.

Actually, let us begin by providing an assertion that is interesting in its own right, to the effect that if the function (x) is measurable and bounded and the function F not only belongs to $L_p$, but also to $L_2$, then

$$\lambda\tilde{F} = \lambda(x)\tilde{F}(x),^{*} \tag{10}$$

i.e., the product $\lambda\tilde{F}$ understood in the sense of (4) is the ordinary production of the function $\lambda(x)$ and $\tilde{F}(x)$. Actually, since $F \in L_2$, there exists a sequence of infinitely differentiable finite functions $F_k$ $(k = 1, 2, \ldots)$ such that (cf. section 1.4.1)

$$\|F - F_k\|_p \to 0,$$
$$\|F - F_k\|_2 \to 0.$$

The relationship

$$\lambda \tilde{F}_k = \lambda(x)\tilde{F}_k(x) \tag{11}$$

holds for infinitely differentiable finite functions $F_k$ by definition. On the other hand,

$$\widehat{\lambda \tilde{F}_k} \to \widehat{\lambda \tilde{F}}\,(L_p),$$

consequently, in S' this also means that

$$\lambda \tilde{F}_k \to \lambda \tilde{F}\,(S'). \tag{12}$$

Due to the boundedness of $\lambda$ and based on the Parseval equality

$$\|\lambda(x)\tilde{F}_k(x) - \lambda(x)\tilde{F}(x)\|_2 \leqslant c\|\tilde{F}_k - \tilde{F}\|_2 = c\|F_k - F\|_2 \to 0,$$

hence it follows that

$$\lambda(x)\tilde{F}_k(x) \to \lambda(x)\tilde{F}(x)\,(S'). \tag{13}$$

From (11) - (13) it obviously follows that the statement (10) is necessary.

Let us now assume an arbitrary finite function $f \in S$. Suppose

$$F = \widehat{\mu \tilde{f}}.$$

Since $\tilde{f} \in L_2$, by virtue of the boundedness of $\mu$ we also have $\mu\tilde{f} \in L_2$ and $F \in L_2$. Hence, by (10)

$$\begin{aligned}\lambda\mu\tilde{f} &= \lambda(x)(\mu\tilde{f})(x) = \lambda(x)\mu(x)\tilde{f}(x) = \\ &= \mu(x)\lambda(x)\tilde{f}(x) = (\lambda(x)\mu(x))\tilde{f}(x) = (\lambda\mu)\tilde{f}.\end{aligned} \tag{14}$$

We have by this equality proven (9) also for $f \in S$. Therefore, for $f \in S$, since $\lambda$ and $\mu$ are multipliers.

$$\|\widehat{(\lambda\mu)\tilde{f}}\|_p = \|\widehat{\lambda(\mu\tilde{f})}\|_p \leqslant c\|\widehat{\mu\tilde{f}}\|_p \leqslant cc'\|f\|_p,$$

and we have proven that $\lambda\mu$ is also a multiplier. It remains to prove equality (9) for the arbitrary function $f \in L_p (1 \leqslant p < \infty)$. This necessitates that we take a sequence of infinitely differentiable finite functions $f_l$ converging to f in the metric $L_p$, that we replace f in (14) with $f_l$, that we apply to all members in (14) the operation $\wedge$, and that we make the passage to the limit when $l \to \infty$ in the $L_p$-sense.

1.5.1.2. **Lemma.** Suppose a $R_n$ be a fixed point. Then together with $\mu(x)$ the function $\mu(x-a)$ is a Marcinkiewicz multiplier and the equality

$$e^{iat}\mu(t)\widehat{\overline{e^{-iat}f}} = \widehat{\mu(x-a)f} \quad (\partial ля \text{ всех } f \in L_p),$$

$$(\text{for all } f \in L_p) \qquad (1)$$

is satisfied, from whence it follows that

$$\|\widehat{\mu(x-a)f}\|_p = \|\mu\widehat{e^{-iat}f}\|_p \leqslant c_p\|e^{-iat}f\|_p = c_p\|f\|_p. \qquad (2)$$

Thus, the constant $c_p$ in this inequality is the same as in the corresponding inequality for $(x)$.

Proof. Assume

$$f_\beta = e^{i\beta t}f(t) \qquad (\beta \in R_n). \qquad (3)$$

Then (cf 1.5 (18))

$$\hat{f} = \widehat{e^{-i\beta t}f_\beta} = \hat{f}_\beta(x+\beta).$$

Therefore (cf. again 1.5 (18))

$$\widehat{\mu(x-a)\hat{f}} = \widehat{\mu(x-a)\hat{f}_{-a}(x-a)} = e^{iat}\mu(x)\hat{f}_{-a}(x),$$

and by (3) we get (1).

1.5.2. **Periodic functions from $L_p$*.** The functions

$$\omega_n(x) = \operatorname{sign}\sin(2^{n+1}\pi x) \quad (0 \leqslant x \leqslant 1),$$

n = 0, 1, 2, ... form an orthogonal and normal (on [0, 1]) system (Rademacher).

The inequalities*)

$$\left(\sum a_{mn}^2\right)^{p/2} \ll \int_0^1\int_0^1 \left|\sum a_{mn}\omega_m(\theta)\,\omega_n(\theta')\right|^p d\theta\, d\theta' \ll \left(\sum a_{mn}^2\right)^{p/2} \qquad (1)$$

with constants not dependent on $a_{mn}$ are valid for any double sequence of real numbers $\{a_{mn}\}$ and $p > 0$.

Actually, if s is a natural number, then, using Newton's polynomial formula and the fact that $[\omega_n(\theta)]^l = \omega_n(\theta)$ for odd l, we get

*) Here and in the text below we will often write $A \ll B$ instead of $A \leqslant cB$, where c is a constant.

$$\int_0^1\int_0^1 \left|\sum a_{mn}\omega_m(\theta)\,\omega_n(\theta')\right|^{2s} d\theta\, d\theta' = $$
$$= \sum \frac{(2s)!}{(2\alpha_1)!\dots(2\alpha_{2s})!} a^{2\alpha_1}_{m_1n_1}\dots a^{2\alpha_{2s}}_{m_{2s}n_{2s}} <$$
$$< \frac{(2s)!}{s!\,2^s}\sum \frac{s!}{\alpha_1!\dots\alpha_{2s}!} a^{2\alpha_1}_{m_1n_1}\dots a^{2\alpha_{2s}}_{m_{2s}n_{2s}} =$$
$$= \frac{(2s)!}{s!\,2^s}\left(\sum a^2_{mn}\right)^s \quad (\alpha_1+\dots+\alpha_{2s}=s). \tag{2}$$

Therefore, for any $p>0$, if we select natural $s$ such that $2s\geqslant p$, we will have (using the Hölder inequality)

$$\left(\int_0^1\int_0^1 \left|\sum a_{mn}\omega_m(\theta)\,\omega_n(\theta')\right|^p d\theta\, d\theta'\right)^{1/p} <$$
$$< \left(\int_0^1\int_0^1 \left|\sum a_{mn}\omega_m(\theta)\,\omega_n(\theta')\right|^{2s} d\theta\, d\theta'\right)^{1/2s} \ll \left(\sum a^2_{mn}\right)^{1/2},$$

which proves the second inequality of (1). Further, if $p\geqslant 2$, then

$$\left(\sum a^2_{mn}\right)^{1/2} = \left(\int_0^1\int_0^1 \left|\sum a_{mn}\omega_m(\theta)\,\omega_n(\theta')\right|^2 d\theta\, d\theta'\right)^{1/2} <$$
$$\ll \left(\int_0^1\int_0^1 \left|\sum a_{mn}\omega_m(\theta)\,\omega_n(\theta')\right|^p d\theta\, d\theta'\right)^{1/p},$$

which proves the first inequality of (1). It remains to prove it when $p<2$. By (2)

$$\int_0^1\int_0^1 \left|\sum a_{mn}\omega_m(\theta)\,\omega_n(\theta')\right|^4 d\theta\, d\theta' \leqslant 3\left(\sum a^2_{mn}\right)^2.$$

Let $s^2 = \sum a^2_{mn}$, $S = \sum a_{mn}\omega_m(\theta)\,\omega_n(\theta')$. And the set of points $(\theta, \theta')$ such that $|S(\theta, \theta')| > s/2$, CA is supplementary to A in the unit square of the set and $|A|$ and $|CA|$ of their measure. Then

$$s^2 = \int_0^1\int_0^1 S^2\, d\theta\, d\theta' = \iint_A + \iint_{CA} \leqslant$$
$$\leqslant \frac{1}{4} s^2 |CA| + \sqrt{|A|}\left(\int_0^1\int_0^1 S^4\, d\theta\, d\theta'\right)^{1/2} \leqslant$$
$$\leqslant \left(\frac{1}{4} + \sqrt{3}\,|\sqrt{|A|}\right) s^2.$$

This means that $\quad 1 < \frac{1}{4} + 2\sqrt{|A|}$ или $|A| > \frac{1}{8}$.

or

So

$$\int_0^1\int_0^1 |S|^p\,d\theta\,d\theta' \geqslant \frac{1}{8}\cdot\frac{1}{2^p}s^p \geqslant \frac{1}{32}s^p,$$

which proves the first inequality of (1) when $p < 2$. Thus inequalities of (1) have been completely proven.

The inequalities corresponding to (2) are similarly proven in the n-dimensional case:

$$\left(\sum a_k^2\right)^{1/2} \ll \left(\int_0^1 \cdots \int_0^1 \left|\sum a_k \omega_k(\theta)\right|^p d\theta\right)^{1/p} \ll \left(\sum a_k^2\right)^{1/2}, \tag{3}$$

where $k = (k_1, \ldots, k_n)$ are all possible integral nonnegative vectors and

$$\omega_k(\theta) = \omega_{k_1}(\theta_1), \ldots, \omega_{k_n}(\theta_n). \tag{4}$$

Let $f(t) \in L^*_p = L_p(0, 2\pi)$ $(1 < p < \infty)$ be a function of the single variable t with period $2\pi$, expanding into a Fourier series of the form

$$f(t) = \sum_0^\infty c_k e^{ikt}.$$

It is known (cf. Zigmund [1], chapter VII) that when $1 < p < \infty$ this series converges to f in the $L_p^*$-sense.

Let us specify an increasing sequence of natural numbers

$$0 = n_0 < 1 = n_1 < n_2 < \ldots,$$

satisfying the condition

$$\frac{n_{k+1}}{n_k} > a > 1 \quad (k = 1, 2, \ldots), \tag{5}$$

and we introduce the functions

$$\delta_0(f) = c_0,\ \delta_k(f) = \sum_{n_{k-1}+1}^{n_k} c_\nu e^{i\nu t} \quad (k = 1, 2, \ldots).$$

Then the series

$$f(t) = \sum_0^\infty \delta_k(f)$$

converges to f in the $L^*_p$-sense. Suppose further that

$$f_1(t)=\sum_0^{\infty}\varepsilon_k\delta_k(f) \qquad (\varepsilon_k=\pm 1;\ k=0,\ 1,\ \ldots),$$

where the numbers $\varepsilon_k = \pm 1$ depend on some fashion on k. Then the following inequalities obtain (Littlewood and Paley [4], of Zigmund [1], chapters II and XV)

$$\|f\|_p \ll \|f_1\|_p \ll \|f\|_p \tag{6}$$

with constants dependent on $\alpha$, but not on f and the distribution $\{\varepsilon_k\}$ and with norms taken over the period. These statements are easily extends to functions of several variables

$$\begin{gathered} f(x)=\sum_{\nu\geqslant 0} c_\nu e^{i\nu x}=\sum_k \delta_k(f)\in L_p^* \\ (\nu=(\nu_1,\ \ldots,\ \nu_n);\ k=(k_1,\ \ldots,\ k_n)), \end{gathered} \tag{7}$$

where $\qquad \delta_k(f)=\delta_{k_1x_1}\ \ldots\ \delta_{k_nx_n}(f)$

and $\qquad \varepsilon_k=\varepsilon_{k_1}\ \ldots\ \varepsilon_{k_n},\quad f_1=\sum\varepsilon_k\delta_k(f).$

Here it is assumed that $\varepsilon_s$ $(s = 1, \ldots, n)$ can take on only the values $\pm 1$. For such f and $f_1$, the inequalities

$$\|f\|_p \ll \|f_1\|_p \ll \|f\|_p \quad (1<p<\infty), \tag{8}$$

also obtain, where the norms are already taken over the n-dimensional period $\{0 < x_j < 2\pi;\ j = 1, \ldots, n$ . We will assume that

$$\delta_{k'}(f)=\delta_{k_1x_1}\ \ldots\ \delta_{k_{n-1}x_{n-1}}(f),\ dx'=dx_1,\ \ldots,\ dx_{n-1},$$
$$\varepsilon_{k'}=\varepsilon_{k_1}\ \ldots\ \varepsilon_{k_{n-1}}.$$

Therefore, if we assume that inequalities (8) are valid for n - 1 and that the integrals are taken over the corresponding periods, we get

$$\begin{gathered} \|f\|_p^p=\int dx_n\int|f|^p\,dx' \ll \int dx_n\int\left|\sum\varepsilon_{k'}\delta_{k'}(f)\right|^p dx' = \\ =\int dx'\int\left|\sum\varepsilon_{k'}\delta_{k'}(f)\right|^p dx_n \ll \\ \ll \int dx'\int\left|\sum\varepsilon_k\delta_k(f)\right|^p dx_n \ll \int dx_n\int|f|^p\,dx'=\|f\|_p^p, \end{gathered}$$

i.e., (8), if we note that

$$\sum \varepsilon_{k_n} \delta_{k_n r_n} \sum \varepsilon_{k'} \delta_{k'}(f) = \sum \varepsilon_k \delta_k (f).$$

Finally, the inequalities

$$\|f\|_p \ll \|(\sum \delta_k^2(f))^{1/2}\|_p \ll \|f\|_p \qquad (1 < p < \infty) \tag{9}$$

with constants not dependent on f can be written for the functions (7).

Actually, setting $\Omega = \{0 \leq \theta_1 \leq 1\}$, we get

$$\begin{aligned} \|f\|_p^p = \int_\Omega \|f\|_p^p d\theta \ll \int_\Omega \left\| \sum \omega_k(\theta) \delta_k(f) \right\|_p^p d\theta = \\ = \left\| \int_\Omega \left| \sum \omega_k(\theta) \delta_k(f) \right|^p d\theta \right\|_p^p \ll \left\| \left( \sum \delta_k^2(f) \right)^{1/2} \right\|_p^p \ll \\ \ll \left\| \int_\Omega \left| \sum \omega_k(\theta) \delta_k(f) \right|^p d\theta \right\|_p^p = \int_\Omega \left\| \sum \omega_k(\theta) \delta_k(f) \right\|_p^p d\theta \ll \\ \ll \int_\Omega \|f\|_p^p d\theta = \|f\|_p^p. \end{aligned} \tag{10}$$

The passage from the second to the third and from the sixth to the seventh members is made on the basis of inequality (8) when $\varepsilon_k = \omega_k(\theta)$ and from the fourth to the fifth, and then to the sixth member -- on the basis of inequality (3).

From (9) it follows that if $f \in L_p{}^*$ is a function for which the Fourier coefficients $c_k$ do not equal zero unless $k \geqslant 0$ and $\mathcal{E}$ is an arbitrary set of vectors k, then the function

$$\sum_{k \in \mathcal{E}} \delta_k(f) = \varphi = \sum_k \delta_k(\varphi)$$

generates the Fourier series of some function $\varphi \in L_p{}^*$ for which the inequalities

$$\|\varphi\|_p \ll \left\| \left( \sum_{k \in \mathcal{E}} \delta_k^2(f) \right)^{1/2} \right\|_p \leqslant \left\| \left( \sum_k \delta_k^2(f) \right)^{1/2} \right\|_p \ll \|f\|_p.$$

are satisfied. Notice that if an arbitrary periodic function of one variable

$$f(t) = \sum_{-\infty}^{\infty} c_k e^{ikt} = \sum_{k>0} + \sum_{k<0} = f_+ + f_-$$

belongs to $L^*_p$ then the function (trigonometrically) conjugate to it (when $1 < p < \infty$)

$$f_*(t) = -i \sum_{-\infty}^{\infty} \operatorname{sign} k c_k e^{ikt} \tag{11}$$

(here sign 0 = 1), and together with it $if_* = f_+ - f_-$ also belongs to $L_p^*$ (Zigmund [1], chapter VII, 2.5). Consequently, $f_+ = \frac{1}{2}(f + if_*) \in L_p^{*p}$ and $\|f_+\| \ll \|f\|_p$. Hence, by induction it follows that also for the functions

$$f(x) = \sum c_k e^{ikx}$$

of several variables, if we assume

$$f_+ = \sum_{k \geqslant 0} c_k e^{ikx},$$

then from the fact that $f \in L_p^* = L_p^{*(n)}$, follows $\|f_+\|_p \ll f_p$.

Let $k = (k_1, \ldots, k_n)$ be an arbitrary (not necessarily nonnegative) integral vector. Let us set

$$\delta_k(f) = \sum_{\pm n_{|k_1|-1}+1}^{\pm n_{|k_1|}} \cdots \sum_{\pm n_{|k_n|-1}+1}^{\pm n_{|k_n|}} c_m e^{imx}, \tag{12}$$

where at the corresponding j-th site + or - is assigned, depending on whether $k_j > 0$ or $k_j < 0$; and when $k_j = 0$, we must assume $n_{|k_j|-1} + 1 = 0$. Based on the foregoing, it is obvious that along with inequalities (9), the inequalities

$$\|f\|_p \ll \|(\sum \delta_k(f)^2)^{1/2}\|_p \ll \|f\|_p \tag{13}$$

that are analogous to them also hold for an arbitrary periodic function $f(x)$ (not necessarily the same as the function for y which $c_k \neq 0$ only for $k \geqslant 0$).

It is easy to verify that inequalities (13) are also preserved for the functions

$$f(x) = \sum c_k e^{\frac{i\pi kx}{l}} = \sum \delta_k(f),$$
$$c_k = \frac{1}{l^n} \int_{\Delta_l} f(u) e^{-\frac{i\pi ku}{l}} du, \quad \Delta_l = \{|x_j| \leqslant l;\ j = 1, \ldots, n\}, \tag{14}$$

of arbitrary period 2l. Here we must of course suitably modify the definition of $\delta_k$ (replacing x by $\pi/l$ x in the right side of (12)).

It is important to note that the constants appearing in inequalities (13) do not depend on l.

1.5.2.1. Suppose

$$f = \sum_{-\infty}^{\infty} c_k e^{ikt} = \sum_{k \geq 0} + \sum_{k<0} = f_+ + f_- \in L_p, \quad (1 < p < \infty) \tag{1}$$

be a periodic function of one variable, and the sequence $\{n_k\}$ and the functions $\delta_k(f)(k = 0, 1, \dots)$ be defined as in 1.5.2.

Suppose further that

$$\delta_{-k}(f) = \sum_{n_{k-1}+1}^{n_k} c_{-\nu} e^{-i\nu t}, \tag{2}$$

$$\beta_k(f) = \delta_k(f) + \delta_{-k}(f), \quad k = 1, 2, \dots,$$
$$\beta_0(f) = \delta_0(f) = c_0. \tag{3}$$

Then from inequalities 1.5.2 (6) follow the inequalities analogous to them.

$$\|f\|_p \ll \|f_*\|_p \ll \|f\|_p, \quad 1 < p < \infty, \tag{4}$$

where

$$f_* = \sum_0^{\infty} \varepsilon_k \beta_k(f) \quad (\varepsilon_k = \pm 1), \tag{5}$$

with constants dependent on $\alpha$ and p, but not on f and the distribution $\{\varepsilon_k\}$. Actually,

$$\left(\sum \varepsilon_k \beta_k(f)\right)_+ = \sum \varepsilon_k \delta_k(f_+),$$
$$\left(\sum \varepsilon_k \beta_k(f)\right)_- = \sum \varepsilon_k \delta_k(f_-(-t)),$$

moreover, $\|f_+\|_p, \|f_-\|_p \ll \|f\|_p$, therefore by 1.5.2 (6)

$$\|f_+\|_p \ll \left\|\sum \varepsilon_k \delta_k(f_+)\right\|_p \ll \left\|\sum \varepsilon_k \beta_k(f)\right\|_p,$$
$$\|f_-\|_p = \|f_-(-t)\|_p \ll \left\|\sum \varepsilon_k \delta_k(f_-(-t))\right\|_p \ll \left\|\sum \varepsilon_k \beta_k(f)\right\|_p,$$

from whence follows the first equality in (4).

Further

$$\left\|\sum \varepsilon_k \beta_k(f)\right\|_p \leqslant \left\|\sum \varepsilon_k \delta_k(f_+)\right\|_p + \left\|\sum \varepsilon_k \delta_k(f_-(-t))\right\|_p \ll$$
$$\ll \|f_+\|_p + \|f_-\|_p \ll \|f\|_p,$$

i.e., the second inequality of (4).

From (4) follow the inequalities

$$\|f\|_p \ll \left\|\left(\sum \beta_k^2(f)\right)^{1/2}\right\|_p \ll \|f\|_p,$$

which is proven as in 1.5.2 (10) (replace $\delta$ by $\beta$ ).

The following statement is also valid, which in the one-dimensional case was proven in the book by Zigmund (chapter XV, 2.15) and can be extended by induction to the multidimensional case.

Suppose $f_1, f_2, \ldots \in L_p^*$ $(1 < p < \infty)$ be a sequence of functions of $\mathbf{x} = (x_1, \ldots, x_n)$ with period 2 and with Fourier coefficients $c_{\mathbf{k}}$ not equal to zero unless $\mathbf{k} \geqslant 0$, and let $S_n, k_n$ stand for the Fourier sum $f_n$ of order $\mathbf{k}_n$. Then there exists a constant $A_p$ not dependent on $f_n$ and N such that

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \left(\sum_{n=1}^{N} |S_{n,\mathbf{k}_n}|^2\right)^{p/2} dx_1 \ldots dx_n \leqslant$$
$$\leqslant A_p^p \int_0^{2\pi} \cdots \int_0^{2\pi} \left(\sum_1^N |f_n|^2\right)^{p/2} dx_1 \ldots dx_n. \tag{6}$$

1.5.3. Theorem on multipliers in the periodic case. Let us introduce the difference $\Delta\lambda_l = \lambda_{l+1} - \lambda_l$ for the numerical sequence $\{\lambda_l\}$ dependent on the single index l. For a multiple sequence $\{\lambda_{\mathbf{k}}\}$ dependent on the nonnegative integral vector $\mathbf{k} = (k_1, \ldots, k_n)$ we will examine the difference $\Delta_j \lambda_{\mathbf{k}}$ taken for each component $k_j$ and the multiple differences $\Delta_{j_1} \ldots \Delta_{j_m} \lambda_{\mathbf{k}}$ $(m \quad n)$.

Theorem (of Marcinkievicz). Assume that a multiple sequence $\{\lambda_{\mathbf{k}}\}$ subject to the inequalities

$$|\lambda_k| \leq M, \quad \sum_{\nu_1=\pm 2^{|k_1|-1}}^{\pm 2^{|k_1|}-1} \cdots \sum_{\nu_m=\pm 2^{|k_m|-1}}^{\pm 2^{|k_m|}-1} |\Delta_{j_1} \ldots \Delta_{j_m} \lambda_\nu| \leq M \tag{1}$$

is given for any collections of natural numbers $j_1, \ldots\ldots\ldots\ldots j_m$, such that $1 \leq j_1 < j_2 < \ldots < j_m \leq n$, where M is a constant not dependent on $\mathbf{k}$ and $j_1, \ldots, j_m$ (when $k_j = 0$, the corresponding sum is extended only to $\nu_j = 0$); + or - is assigned, depending on whether $k_j > 0$ or $< 0$.

Let us transform a function with period $2\pi$ of the form (cf 1.5.2 (7))

$$f(x) = \sum_k c_k e^{ikx} = \sum \delta_k(f) \in L_p, \quad (1 < p < \infty) \tag{2}$$

by means of the number $\lambda_k$ (Marcinkievicz multipliers):

$$F(x) = \sum_k \lambda_k c_k e^{ikx} = \sum \delta_k(F). \tag{3}$$

Then $F \in L_p$* and there exists a constant $c_p$ dependent only on p, such that

$$\|F\|_p \leq c_p M \|f\|_p. \tag{4}$$

Proof. Let us limit ourselves to the case n = 2. Moreover, we will assume that in (2) the series are extended only to $\mathbf{k} \geq 0$, which does not violate generality.

Setting

$$\sum_{\mu=2^{k-1}}^{s} \sum_{\nu=2^{l-1}}^{t} c_{\mu\nu} e^{i(\mu x+\nu y)} = r_{st} = r_{s,t,k,l} \tag{5}$$

and using the Abel transformation, we get

$$\delta_{kl}(F)=\sum_{2^{k-1}}^{2^k-1}\sum_{2^{l-1}}^{2^l-1}\lambda_{\mu\nu}c_{\mu\nu}e^{i(\mu x+\nu y)}=$$
$$=\sum_{2^{k-1}}^{2^k-2}\sum_{2^{l-1}}^{2^l-2}r_{ij}[\lambda_{i,j}-\lambda_{i,j+1}-\lambda_{i+1,j}+\lambda_{i+1,j+1}]+$$
$$+\sum_{2^{l-1}}^{2^l-2}r_{2^k-1,j}[\lambda_{2^k-1,j}-\lambda_{2^k-1,j+1}]+$$
$$+\sum_{2^{k-1}}^{2^k-2}r_{i,2^l-1}[\lambda_{i,2^l-1}-\lambda_{i+1,2^l-1}]+$$
$$+r_{2^k-1,2^l-1}\lambda_{2^k-1,2^l-1}=\sum r_{ij}\gamma_{ij}. \tag{6}$$

Let us use the Bunyakovskiy inequality and take (1) into account:

Therefore, based on 1.5.2 (13) there follows ($n_k = 2^k$):

$$\|F_p\|^p \ll \left\|\left(\sum_{k,l}\delta_{kl}^2(F)\right)^{1/2}\right\|_p^p \ll$$
$$\ll M^{p/2}\int_0^{2\pi}\int_0^{2\pi}\left\{\sum_{k,l}\sum\left(r_{ij}\sqrt{|\gamma_{ij}|}\right)^2\right\}^{p/2}dx\,dy. \tag{7}$$

The function (cf also (5))

$$r_{i,j,k,l}\sqrt{|\gamma_{ij}|} \tag{8}$$

appears within the parentheses in the right-hand side of (7) ( $\sqrt{|\gamma_{ij}|}$ is a coefficient not dependent on x and y). Obviously, it can be regarded as a segment of the Fourier series of the function

$$\delta_{kl}(f)\sqrt{|\gamma_{ij}|}. \tag{9}$$

Consequently, the sum $\sum_{k,l}\sum$ of the squares of segments of the Fourier series of the functions (9) appear within the braces of the right-hand side of (7).

Based on 1.5.2.1 (6), the integral (7) is majorized by the same integral, where the segments of the Fourier series of the functions are replaced by the functions themselves

$$\|F_p\|^p \ll M^{p/2} \int_0^{2\pi}\int_0^{2\pi} \left\{ \sum_{k,l} \delta_{kl}^2(f) \sum{}' \gamma_{il} \right\}^{p/2} dx\, dy \ll$$

$$\ll M^p \int_0^{2\pi}\int_0^{2\pi} \left\{ \sum_k \sum_l \delta_{kl}^2(f) \right\}^{p/2} dx\, dy \ll M^p \|f\|_p^p,$$

and we get inequality (4).

It is easy to verify that inequality (4) is preserved for functions with arbitrary period 2l with the same constant $c_p$.

1.5.4. Theorem on multipliers in the nonperiodic case. Let us assume a vector of the form

$$\mathbf{k} = (k_1, \ldots, k_n) \quad (k_j = 0, 1;\ j = 1, \ldots, n). \tag{1}$$

The set

$$e_k = \{j_1, \ldots, j_m\}$$

of those indexes j for which $k_j = 1$ is called the carrier of the vector $\mathbf{k}$.

Theorem. Suppose the function $\lambda(\mathbf{x})$ exhibiting the following properties be given on $R = R_n$.

Whatever be the vector $\mathbf{k}$ of the form (1), the derivative*)

$$D^k\lambda = \frac{\partial^{|k|}\lambda}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \tag{2}$$

exists and is continuous at any point $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_i \neq 0$, where $i \in e_k$ and is subject to the inequality

$$|x^k D^k \lambda| \leqslant M. \tag{3}$$

*) A certain generalization of this theorem in terms of generalized derivatives is possible.

Then $\lambda$ is the Marcinkiewicz multiplier. Specifically, there exists the constant $\varkappa_p$ not dependent on M and f, such that

$$\|\widehat{\lambda f}\|_p \leqslant \varkappa_p M \|f\|_p \qquad (1 < p < \infty) \tag{4}$$

for all f $\in$ $L_p$.

Notice that since satisfies the property indicated in the theorem for **k** = 0, therefore it is bounded on R and is continuous save for the points belonging to the coordinate planes. Therefore, $\lambda$ is a measurable function on $R_n$ and is at the same time generalized ( $\lambda \in S'$).

Proof. Let us confine ourselves to examining the two-dimensional case. Let f(x,y) be a finite, infinitely differentiable function. We will consider its carrier as belonging to the square

$$\Delta_{s_0} = \{|x|, |y| < s_0\}. \tag{5}$$

And suppose

$$f(x, y) = \sum_{\mu, \nu} c^s_{\mu\nu} e^{i\frac{\pi}{s}(\mu x + \nu y)} \qquad (s \geqslant s_0), \tag{6}$$

where

$$c^s_{\mu\nu} = \frac{1}{(2s)^2} \int\int_{\Delta_s} f(u, v) e^{-i\frac{\pi}{s}(\mu u + \nu v)} du\, dv \qquad (\mu, \nu = 0, \pm 1), \tag{7}$$

is its Fourier series. Let us set

$$u_s(x, y) = \sum_{\mu, \nu} \lambda\left(\frac{\mu\pi}{s}, \frac{\nu\pi}{s}\right) c^s_{\mu\nu} e^{i\frac{\pi}{s}(\mu x + \nu y)}. \tag{8}$$

By (3) (when **k** = 0)

$$\left|\lambda\left(\frac{\mu\pi}{s}, \frac{\nu\pi}{s}\right)\right| \leqslant M. \tag{9}$$

Now let k > 0, l $\geqslant$ 0, then

$$\sum_{\mu=2^{k-1}}^{2^k-1}\left|\lambda\left(\frac{(\mu+1)\pi}{s},\ \frac{l\pi}{s}\right)-\lambda\left(\frac{\mu\pi}{s},\ \frac{l\pi}{s}\right)\right|-$$

$$-\sum\left|\int_{\frac{\mu\pi}{s}}^{\frac{(\mu+1)\pi}{s}}\frac{\partial\lambda}{\partial x}\left(\xi,\ \frac{l\pi}{s}\right)d\xi\right|\leqslant$$

$$\leqslant\frac{s}{\pi\cdot 2^{k-1}}\sum\int_{\frac{\mu\pi}{s}}^{\frac{(\mu+1)\pi}{s}}\left|\xi\frac{\partial\lambda}{\partial\xi}\left(\xi,\ \frac{l\pi}{s}\right)\right|d\xi\leqslant\frac{1}{2^{k-1}}2^kM\leqslant 2M. \tag{10}$$

The continuity of $\partial\lambda/\partial x$ with respect to x when $x > 0$ and for any y was used in these computations. The resulting inequality is preserved also for $k = 0$ for any $l$:

$$\left|\lambda\left(\frac{\pi}{s},\ \frac{l\pi}{s}\right)-\lambda\left(0,\ \frac{l\pi}{s}\right)\right|\leqslant\left|\lambda\left(\frac{\pi}{s},\ \frac{l\pi}{s}\right)\right|+\left|\lambda\left(0,\ \frac{l\pi}{s}\right)\right|\leqslant 2M.$$

Similarly, using the continuity of $\partial\lambda/\partial y$ with respect to y for $y > 0$ and any x, we get

$$\sum_{\nu=2^{l-1}}^{2^l-1}\left|\lambda\left(\frac{k\pi}{s},\ \frac{(\nu+1)\pi}{s}\right)-\lambda\left(\frac{k\pi}{s},\ \frac{\nu\pi}{s}\right)\right|\leqslant 2M \qquad (k,\ l\geqslant 0). \tag{11}$$

Further, while for $k, l > 0$

$$\sum_{k=1}^{2^k-1}\sum_{l=1}^{2^l-1}\left|\lambda\left(\frac{(\mu+1)\pi}{s},\ \frac{(\nu+1)\pi}{s}\right)-\lambda\left(\frac{\mu\pi}{s},\ \frac{(\nu+1)\pi}{s}\right)-\right.$$

$$\left.-\lambda\left(\frac{(\mu+1)\pi}{s},\ \frac{\nu\pi}{s}\right)+\lambda\left(\frac{\mu\pi}{s},\ \frac{\nu\pi}{s}\right)\right|=$$

$$=\sum\sum\left|\int_{\frac{\mu\pi}{s}}^{\frac{(\mu+1)\pi}{s}}\int_{\frac{\nu\pi}{s}}^{\frac{(\nu+1)\pi}{s}}\frac{\partial^2\lambda}{\partial x\,\partial y}(\xi,\eta)\,d\xi\,d\eta\right|\leqslant$$

$$\leqslant\left(\frac{s}{\pi}\right)^2\frac{\lambda}{2^{k+l-2}}\sum\sum\int_{\frac{\mu\pi}{s}}^{\frac{(\mu+1)\pi}{s}}\int_{\frac{\nu\pi}{s}}^{\frac{(\nu+1)\pi}{s}}\left|\xi\eta\frac{\partial^2\lambda}{\partial x\,\partial y}(\xi,\eta)\right|d\xi\,d\eta\leqslant M.$$

(12)

Here we used the continuity of $\partial^2\lambda/\partial x\partial y$ when x, y $> 0$. For $k > 0$ and $l = 0$ this equality reduces to the following:

$$\sum_{k=1}^{2^k-1}\left|\lambda\left(\frac{(\mu+1)\pi}{s},\ \frac{\pi}{s}\right)-\lambda\left(\frac{\mu\pi}{s},\ \frac{\pi}{s}\right)-\lambda\left(\frac{(\mu+1)\pi}{s},\ 0\right)+\right.$$

$$\left.+\lambda\left(\frac{\mu\pi}{s},\ 0\right)\right|\leqslant 4M \qquad (13)$$

(here we consider inequality (10) valid for any $k, l \geqslant 0$), and when $k = 0$, $l = 0$ the sum in the left-hand reduces to a single member also not exceeding $4M$.

We have proven that the left-hand sides of (9) - (12) for any $k, l \geqslant 0$ do not exceed $4M$. Analogous inequalities are proven for the remaining three quadrants: 1) $k > 0$, $l \leqslant 0$; 2) $k \leqslant 0$, $l \geqslant 0$; and 3) $k, l \leqslant 0$.

This proves that the conditions of the Marcinkiewicz theorem are observed and therefore, a constant $c_p$ exists, not dependent on s (cf. Note at the end of section 1.5.3), M, and f, such that

$$\|u_s\|_{L_p(\Delta_s)}\leqslant c_pM\|f\|_{L_p(\Delta_s)}=c_pM\|f\|_p,\qquad 1<p<\infty. \qquad (14)$$

In this case the transformation of the function f by means of the multiplier $\lambda$ is written as the integral

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda(\xi, \eta) \tilde{f}(\xi, \eta) e^{i(x\xi + y\eta)} d\xi \, d\eta,$$

where

$$\tilde{f}(x, y) = \frac{1}{2\pi} \int_{\Delta_s} f(u, v) e^{-i(xu + yv)} du \, dv \qquad (s \geqslant s_0).$$

Obviously

$$c_{kl}^s = \frac{\pi}{2s^2} \tilde{f}\left(\frac{k\pi}{s}, \frac{l\pi}{s}\right). \tag{15}$$

Let us estimate in an arbitrarily specified square $\Delta_\mu (\mu > 0)$ the difference

$$u(x, y) - u_s(x, y) = r_1 + r_2 + r_3.$$

Here

$$r_1 = \frac{1}{2\pi} \int_{\Delta_N} \lambda(\xi, \eta) \tilde{f}(\xi, \eta) e^{i(x\xi + y\eta)} d\xi \, d\eta -$$

$$- \sum_{|k|, |l| \leqslant \alpha N} \lambda\left(\frac{k\pi}{s}, \frac{l\pi}{s}\right) c_{kl}^s e^{i\frac{\pi}{s}(kx + ly)} =$$

$$= \frac{1}{2\pi} \int_{\Delta_N} \lambda(\xi, \eta) \tilde{f}(\xi, \eta) e^{i(x\xi + y\eta)} d\xi \, d\eta -$$

$$- \frac{\pi}{2s^2} \sum_{|k|} \sum_{|l| \leqslant \alpha N} \lambda\left(\frac{k\pi}{s}, \frac{l\pi}{s}\right) \tilde{f}\left(\frac{k\pi}{s}, \frac{l\pi}{s}\right) e^{i\frac{\pi}{s}(kx + ly)}, \tag{16}$$

N is a natural number and s is chosen so that $\alpha = s/\pi$ is a natural number;

$$r_2 = \frac{1}{2\pi} \int_{R_2 - \Delta_N} \lambda(\xi, \eta) \tilde{f}(\xi, \eta) e^{i(x\xi + y\eta)} d\xi \, d\eta,$$

$$r_3 = -\frac{\pi}{2s^2} \sum{}' \lambda\left(\frac{k\pi}{s}, \frac{l\pi}{s}\right) \tilde{f}\left(\frac{k\pi}{s}, \frac{l\pi}{s}\right) e^{i\frac{\pi}{s}(kx + ly)},$$

where the sum $\sum'$ is extended over such pairs (k, l) that either $|k|$ or $|l|$ is larger than $\alpha N$. The function f, being an infinitely differentiable finite function, belongs to the main class S, therefore $\tilde{f} \in S$, which means

$$|\tilde{f}(\xi, \eta)| \leqslant (1 + \xi^2)^{-1} (1 + \eta^2)^{-1}$$

and

$$|r_2| \ll \int\int_{R_2 - \Delta_N} (1 + \xi^2)^{-1} (1 + \eta^2)^{-1} d\xi \, d\eta \to 0 \qquad (N \to \infty).$$

where $\alpha = s/\pi > 1$, a similar estimate exists for $r_3$:

$$|r_3| \ll \sum{}' \left[1+\left(\frac{k\pi}{s}\right)^2\right]^{-1}\left[1+\left(\frac{l\pi}{s}\right)^2\right]^{-1} \to 0 \qquad (N \to \infty).$$

Assigning $\varepsilon > 0$, we can indicate such an $N > 0$ that for all $s > s_0$

$$|r_2|,\ |r_3| < \varepsilon.$$

For this $N$ we can specify an $s_1 > s_0$ such that for all $s > s_1$ and for all $(x, y) \in \mathcal{U}$

$$|r_1| < \varepsilon.$$

We have proven that for any $\mu > 0$

$$\lim_{s \to \infty} u_s(x, y) = u(x, y) = \widehat{\lambda f}$$

is uniform on $\Delta_\mu$.

From (14) it follows that $\quad \|u_s\|_{L_p(\Delta_\mu)} \leqslant c_p \|f\|_p \qquad (\mu \leqslant s).$

Passing to the limit when $s \to \infty$, and then when $\mu \to \infty$ we obtain inequality (4) for the finite functions $f \in S$.

This proves that $\lambda$ is a Marcinkievicz multiplier.

1.5.4.1. If the function $u(\boldsymbol{x}) = u(x_1, \ldots, x_n)$ of $n$ variables is subject to the conditions of the theorem formulated in 1.5.4, then it obviously is also subject to the conditions of this theorem if it is considered as a function of $k$ variables $x_1, \ldots, x_k$ ($k$ $n$) and, therefore, is a multiplier with respect to them.

1.5.5. Examples of Marcinkievicz multipliers (in the $L_p$-sense, $1 < p < \infty$).

1. $\operatorname{sign} \boldsymbol{x} = \prod_1^n \operatorname{sign} x_j$.
2. $(1+|\boldsymbol{x}|^2)^{-\lambda}$ $(\lambda > 0)$.
3. $(1+x_j^2)^{r/2}(1+|\boldsymbol{x}|^2)^{-r/2}$ $(r > 0;\ j = 1, \ldots, n)$.
4. $(1+|\boldsymbol{x}|^2)^{r/2}\left(1+\sum_1^n |x_j|^r\right)^{-1}$ $(r > 0)$.
5. $\boldsymbol{x}^l(1+|\boldsymbol{x}|^2)^{-r/2}$ $(|l| \leqslant r,\ r > 0,\ l \geqslant 0)$.
6. $\boldsymbol{x}^l(1+x_s^2)^{\frac{\varkappa r_s}{2}}\left\{\sum_1^n (1+x_j^2)^{\frac{r_j}{2}}\right\}^{-1}$

$$\left(r > 0,\ l \geqslant 0,\ \varkappa = 1 - \sum_1^n \frac{l_j}{r_j} \geqslant 0\right).$$

7. $(1+|\boldsymbol{x}|^2)^{r/2}\left(1+\sum_1^n |x_j|\right)^{-r}$

7.

(r is an arbitrary real number)

8. $(1+|x|^2)^{-r/2}\Lambda_r^{-1}(x) \quad (r = r_1 = \ldots = r_n > 0).$

9. $(1+|x|^2)^{r/2}\Lambda_r(x) \quad (r = r_1 = \ldots = r_n > 0).$

10. $(1+x_i^2)^{\frac{r_i}{2}}\Lambda_r(x) \quad (i = 1, \ldots, n).$

11. $\left\{\sum_{j=1}^{n}(1+x_j^2)^{\frac{r_j}{2}}\right\}^{-1}\Lambda_r^{-1}(x).$

12. $$\left\{\sum_1^n (1+x_j^2)^{\frac{r_j\lambda}{2\sigma}}\right\}^{\sigma}\left\{\sum_1^n (1+x_j^2)^{\frac{r_j\delta}{2\sigma}}\right\}^{\sigma}\times$$
$$\times\left\{\sum_1^n (1+x_j^2)^{\frac{r_j(\lambda+\delta)}{2\sigma}}\right\}^{-\sigma} \quad (r_j>0;\ \sigma,\ \delta,\ \lambda>0).$$
$$\Lambda_r(x) = \left\{\sum_{j=1}^{n}(1+x_j^2)^{\frac{r_j\sigma}{2}}\right\}^{-\frac{1}{\sigma}}.$$

We let $\mu_i$ (i = 1, ..., 12) stand for these functions. They will be necessary to us in the treatment below. The proof that they are Marcinkievicz multipliers is reduced to the preceding theorem 1.5.4.

Its criterion for $\mu_1$ is trivially satisfied, since $\mu_1$ is a constant (+1 or -1) in each open coördinate junction.

The functions $\mu_i$ are continuous together with their partial derivatives of any order on $R = R_n$, with the exception of the functions $\mu_i$ (i = 4, 5, 6, 7) which are continuous on R, but their partial derivatives are generally discontinuous on the coordinate planes.

Below is given a proof that the Marcinkievicz criterion is satisfied for several $\mu_i$ functions. The problem reduces to verifying that the functions

$$x^k\mu_i^{(k)} \qquad (k=(k_1, \ldots, k_n),\ k_j = 0, 1)$$

are bounded on each coordinate junction of space R. Owing to the symmetry of these functions, it suffices for the verification to be made for a positive coordinate junction. All the functions considered, save for $\mu_6$ and $\mu_{12}$, are the products $\mu_i = \lambda_i\psi_i$ of the defined functions $\lambda_i$ and $\psi_i$. By the Leibnitz formula

$$x^k\mu_i^{(k)} = x^k\sum_{s\leq k} C_k^s\lambda_i^{(k-s)}\psi_i^{(s)},$$

where the sum is extended over all possible integral nonnegative vectors $\alpha \leq k$. The problem reduces to estimating functions of the form

$$x^k \lambda_i^{(k-\alpha)} \psi_i^{(\alpha)} = x^k \chi_i$$

on a positive coordinate junction.

Let us agree to write $A \asymp B$ instead of $|A| = cB$, where c is some constant. We have

$$x^k \chi_1 \approx x^k x^{k-1} (1 + |x|^2)^{r/2 - |k-\alpha|} x^{(r-1)\alpha} \left(1 + \sum_1^n x_j^r\right)^{-1-|\alpha|} =$$

$$= \frac{x^{\alpha r}}{\left(1 + \sum_1^n x_j^r\right)^{|\alpha|}} \frac{x^{2(k-\alpha)}}{(1 + |x|^2)^{|k-\alpha|}} \frac{(1 + |x|^2)^{r/2}}{1 + \sum_1^n x_j^r} \leqslant 1 \cdot 1 \cdot c < \infty.$$

When $r \prec 1$ the function $\chi_4$ is discontinuous, when one of the coordinates $x_j$, where $j \in e_a \subset e_k$ ($e_k$ is the carrier of the vector $\alpha$) is equal to zero. Then by theorem 1.5.4, it suffices that function $\chi_4$ be continuous for positive $x_j$ with $j \in e_k$ and for any remaining $x_j$, which obviously is satisfied in this case. When estimating $u_t = u v \omega$, we will have

$$\cdot_{(k)} m_{(\nu)} n_{(n)} n \sum {}_{\alpha} x = {}_{(\nu)} (m n n)_{\alpha} x \qquad (1)$$

where the sum is extended over all possible vectors $\alpha$, $\beta$, and $\nu$ with components equal to 1 or 0, such that $\alpha + \beta + \nu = k$. In estimating the derivative of the components of this sum, we will assume (otherwise it will equal zero) that $e_a \subset e_1$, and $\beta$ is a vector whose s-th component equals 1, while the remaining components are equal to zero. The problem reduces to estimating

$$x^s x^{l-s} (1 + x_s^2)^{\frac{\varkappa r_s}{2} - 1} x_s x^\nu \prod_{e_\nu} (1 + x_j^2)^{\frac{r_j}{2} - 1} \times$$

$$\times \left\{ \sum_1^n (1 + x_j^2)^{\frac{r_j}{2}} \right\}^{-1-|\nu|} = \frac{x^l (1 + x_s^2)^{\frac{\varkappa r_s}{2}}}{\sum_1^n (1 + x_j^2)^{\frac{r_j}{2}}} \frac{x_s^2}{1 + x_s^2} \times$$

$$\times \frac{x^{2\nu}}{\prod_{e_\nu} (1 + x_j^2)} \frac{\prod_{e_\nu} (1 + x_j^2)^{\frac{r_j}{2}}}{\left\{ \sum_1^n (1 + x_j^2)^{\frac{r_j}{2}} \right\}^{|\nu|}} \leqslant 1.$$

Let us provide an explanation to the estimate of the first multiplier in the second member of these relationships. Let

$$(1+x_{j_0}^2)^{r_{j_0}} = \max_j (1+x_j^2)^{r_j};$$

then

$$\frac{x^l (1+x_s^2)^{\frac{\varkappa r_s}{2}}}{\sum_1^n (1+x_j^2)^{\frac{r_j}{2}}} \leqslant (1+x_{j_0}^2)^{-\frac{r_{j_0}+\varkappa r_{j_0}}{2}} \prod_{j=1}^n (1+x_{j_0}^2)^{\frac{l_j r_{j_0}}{2r_j}} =$$

$$= (1+x_{j_0}^2)^0 = 1.$$

When $l_j > 1$, the estimated product without the multiplier $x^\kappa$ is discontinuous, when one of the coordinates $x_j$, where $j \in e_a \subset e_k$ equals zero, but by theorem 1.5.4 it is sufficient that this multiplier be continuous for $x_j > 0$, where $j \in e_k$ and for any remaining $x_j$, which in this case is obviously satisfied.

For $\mu_7$

$$x^k \chi_7 \approx x^k x^{k-s} (1+|x|^2)^{r/2-|k-s|} \left(1+\sum_1^n x_j\right)^{-r-|s|} =$$

$$= \frac{x^{2(k-s)}}{(1+|x|^2)^{|k-s|}} \frac{x^s}{\left(1+\sum_{j=1}^n x_j\right)^{|s|}} \left(\frac{(1+|x|^2)^{1/2}}{1+\sum_{j=1}^n x_j}\right)^r < c < \infty.$$

Here the inequalities

$$c_1\left(1+\sum_1^n x_j\right) \leqslant (1+|x|^2)^{1/2} < 1+\sum x_j,$$

are employed, the second when $r > 0$, and the first when $r < 0$.

The function $\chi_7$ is discontinuous on certain coordinate planes, but its limits on them within each coordinate junction do exist, therefore in each junction thus closed $\chi_7$ can be considered as continuous.

We will argue as follows for $\mu_8$. Let l be a vector with components equal to 1 or 0. Using the Leibnitz formula on the differentiation of the product of functions of several variables, omitting the constant coefficients and considering the vectors x with nonnegative coordinates, we get ($e_s$ is the carrier of the vector s)

$$x^l D^l\{(1+|x|^2)^{-r/2}\Lambda_r\} \ll$$
$$\ll \sum_{s<l} x^l x^{l-s}(1+|x|^2)^{-r/2-|l-s|}\Big(\sum(1+x_j^2)^{\frac{r\sigma}{2}}\Big)^{1/\sigma-|s|} \times$$
$$\times \prod_{j\in e_s}(1+x_j^2)^{\frac{r\sigma}{2}-1}x_j = \sum_{s<l}\frac{x^{2(l-s)}}{(1+|x|^2)^{|l-s|}}\frac{x^{2s}}{\prod_{j\in e_s}(1+x_j^2)}\times$$
$$\times\frac{\prod_{j\in e_s}(1+x_j^2)^{\frac{r\sigma}{2}}}{\Big\{\sum(1+x_j^2)^{\frac{r\sigma}{2}}\Big\}^{|s|}}\frac{\Big(\sum(1+x_j^2)^{\frac{r\sigma}{2}}\Big)^{1/\sigma}}{(1+|x|^2)^{r/2}} \ll 1$$

(учесть, что $(\sum u_j^\sigma)^{1/\sigma} \ll \sum u_j$, $\sigma>0$, $u_j \geqslant 0$).

(considering that

For $\mu_9$

$$x^l D^l\{(1+|x|^2)^{-r/2}\Lambda_r\} \ll$$
$$\ll \sum x^l x^{l-s}(1+|x|^2)^{r/2-|l-s|}\Big(\sum(1+x_j^2)^{\frac{r\sigma}{2}}\Big)^{\frac{1}{\sigma}-|s|} \times$$
$$\times \prod_{j\in e_s}(1+x_j^2)^{\frac{r\sigma}{2}-1}x^s = \sum_{s<l}\frac{x^{2(l-s)}}{(1+|x|^2)^{|l-s|}}\frac{x^{2s}}{\prod_{j\in e_s}(1+x_j^2)^{|s|}}\times$$
$$\times\frac{\prod_{j\in e_s}(1+x_j^2)^{\frac{r\sigma}{2}}}{\Big(\sum(1+x_j^2)^{\frac{r\sigma}{2}}\Big)^{|s|}}\frac{(1+|x|^2)^{r/2}}{\Big(\sum(1+x_j^2)^{\frac{r\sigma}{2}}\Big)^{\frac{1}{\sigma}}} \ll 1.$$

For $\mu_{10}$

$$x^l D^l\Big\{(1+x_i^2)^{\frac{r_i}{2}}\Lambda_r(x)\Big\} \ll \sum_{s<l} x^l\Big\{\sum_{j=1}^n(1+x_j^2)^{\frac{r_j\sigma}{2}}\Big\}^{-\frac{1}{\sigma}-|s|}\times$$
$$\times\prod_{j\in e_s}(1+x_j^2)^{\frac{r_j\sigma}{2}-1}x_j D^{(l-s)}(1+x_i^2)^{\frac{r_i}{2}} =$$
$$= \sum_{s<l}\frac{x^{2s}}{\prod_{j\in e_s}(1+x_j^2)^{|s|}}\frac{\prod_{j\in e_s}(1+x_j^2)^{\frac{r_j\sigma}{2}}}{\Big\{\sum_{j=1}^n(1+x_j^2)^{\frac{r_j\sigma}{2}}\Big\}^{|s|}}\frac{x^{l-s}D^{(l-s)}(1+x_i^2)^{\frac{r_i}{2}}}{\Big\{\sum_{j=1}^n(1+x_j^2)^{\frac{r_j\sigma}{2}}\Big\}^{1/\sigma}}.$$

The first two fractions in the right-hand side do not exceed the constant, and the third fraction also does not exceed the constant because its numerator $\psi \equiv 0$, if $j \neq i$ exists,

$$j \in e_{l-s};\ \psi = (1 + x_j^2)^{\frac{r_l}{2}}, \quad \text{if } l - s = 0; \quad \psi = 2x_i^2(1 + x_i^2)^{\frac{r_l}{2} - 1},$$

if the set $e_{l-s}$ consists only of one index i.

For $\mu_{11}$

$$x^l D^l \left\{ \Lambda_r^{-1}(x) \left( \sum_1^n (1 + x_j^2)^{\frac{r_l}{2}} \right)^{-1} \right\} \ll$$

$$\ll \sum_{s<l} x^l \frac{\prod_{j \in e_{l-s}} (1 + x_j^2)^{\frac{r_l}{2} - 1} x_j}{\left\{ \sum_1^n (1 + x_j^2)^{\frac{r_l}{2}} \right\}^{1+|l-s|}} \left\{ \sum_1^n (1 + x_j^2)^{\frac{r_j \sigma}{2}} \right\}^{1/\sigma - |s|} \times$$

$$\times \prod_{j \in e_s} (1 + x_j^2)^{\frac{r_j \sigma}{2} - 1} x_j =$$

$$= \sum_{s<l} \frac{\prod_{j \in e_s} (1 + x_j^2)^{\frac{r_j \sigma}{2}}}{\left\{ \sum_1^n (1 + x_j^2)^{\frac{r_j \sigma}{2}} \right\}^{|s|}} \frac{x^{2(l-s)}}{\left\{ \sum_1^n (1 + x_j^2)^{\frac{r_l}{2}} \right\}^{|l-s|}} \times$$

$$\times \frac{\left\{ \sum_1^n (1 + x_j^2)^{\frac{r_j \sigma}{2}} \right\}^{1/\sigma}}{\sum_1^n (1 + x_j^2)^{\frac{r_l}{2}}} \frac{x^{2s}}{\prod_{j \in e_s} (1 + x_j^2)} \ll 1.$$

For $\mu_{12}$, one of the members of the Leibnitz sum (1) is estimated as follows:

$$
\begin{aligned}
&u^{k}\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j\lambda}{2\sigma}}\right\}^{\sigma-|\alpha|}\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j\delta}{2\sigma}}\right\}^{\sigma-|\beta|}\times\\
&\times\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j(\lambda+\delta)}{2\sigma}}\right\}^{-\sigma-|\gamma|}\prod_{j\in e_\alpha}(1+u_j^2)^{\frac{r_j\lambda}{2\sigma}-1}u_j\times\\
&\times\prod_{j\in e_\beta}(1+u_j^2)^{\frac{r_j\delta}{2\sigma}-1}u_j\prod_{j\in e_\gamma}(1+u_j^2)^{\frac{r_j(\lambda+\delta)}{2\sigma}-1}u_j=\\
&=\frac{\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j\lambda}{2\sigma}}\right\}^{\sigma}\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j\delta}{2\sigma}}\right\}^{\sigma}}{\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j(\lambda+\delta)}{2\sigma}}\right\}^{\sigma}}\,\frac{u^{2k}}{\prod_{j\in e_k}(1+u_j^2)}\times\\
&\times\frac{\prod_{j\in e_\alpha}(1+u_j^2)^{\frac{r_j\lambda}{2\sigma}}\prod_{j\in e_\beta}(1+u_j^2)^{\frac{r_j\delta}{2\sigma}}\prod_{j\in e_\gamma}(1+u_j^2)^{\frac{r_j(\lambda+\delta)}{2\sigma}}}{\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j\lambda}{2\sigma}}\right\}^{|\alpha|}\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j\delta}{2\sigma}}\right\}^{|\beta|}\left\{\sum_{1}^{n}(1+u_j^2)^{\frac{r_j(\lambda+\delta)}{2\sigma}}\right\}^{|\gamma|}}\ll 1.
\end{aligned}
\tag{2}
$$

In the first fraction, if $\sigma$ is everywhere closed, the order will not be changed also if the exponents $\lambda$, $\delta$, and $\lambda+\delta$ are removed from the sign of the corresponding curved brackets.

For the proof in the case of the function $\mu_{12}^{-1}$ members appear that can be written as the right-hand side of (2). Only the reciprocal of its first fraction, which nonetheless will obviously be bounded, is changed.

It is easy to see that the functions $\mu_{12}$ and $\mu_{12}^{-1}$ remain Marcinkievicz multipliers, if in each of three of its multipliers the parameter $\sigma$ takes on different values $\sigma_1$, $\sigma_2$, and $\sigma_3$, or if in its first multiplier n is replaced by $m < n$. In the last case, in (2) we must assume that the carrier $e_a$ consists of indexes with numbers not exceeding m, otherwise the corresponding member of the Leibnitz sum equals zero. In the last member in (2), n in the first multiplier of the numerator must be replaced by m.

1.5.6. Extending inequality 1.5.2 (13) to the nonperiodic case. Our aim will be to prove that for any function $f \in L_p(R_n) = L_p$ $(1 < p < \infty)$ the inequalities

$$c_1 \| f \|_p \leqslant \| \{ \sum \delta_k (f)^2 \}^{1/2} \|_p \leqslant c_2 \| f \|_p, \tag{1}$$

are satisfied, where $c_1$ and $c_2$ are constants not dependent on f,

$$\delta_k (f) = \widehat{(1)_{\Delta_k} \tilde{f}} \quad \left( (1)_e = \begin{cases} 1, & x \in e, \\ 0, & x \notin e \end{cases} \right) \tag{2}$$

and for $k \geqslant 0$

$$\Delta_k = \Delta_{k_1, \dots, k_n} = \{ 2^{k_j - 1} \leqslant u_j \leqslant 2^{k_j};\ j = 1, \dots, n \} \tag{3}$$

($2^{k_j - 1}$ when $k_j = 0$ is replaced by 0), but for artibrary $k$ of the rectangles $\Delta_k$ there is a set of points $\{u_1 \operatorname{sign} k_1, \dots\dots, u_n \operatorname{sign} k_n\}$, where $u = (u_1, \dots, u_n) \in \Delta_{|k_1|, \dots, |k_n|}$.

Below it will be shown (cf 8.10.12) that if f is regular in the $L_p$-sense $(1 < p < \infty)$ the generalized function (cf. further 1.5.10), for which the norm appearing in the second member of (1) is finite, therefore $f \in L_p$.

Let us confine ourselves to considering the two-dimensional case. Let us specify an infinitely differentiable function $f(x, y)$ with a carrier belonging to

$$\Delta_{s_0} = \{ |x|, |y| < s_0 \}, \tag{4}$$

and the Fourier series 1.5.4 (6). By virtue of the fact $f \equiv 0$ outside $\Delta_{s_0}$ for $s > s_0$, we will have the inequalities

$$\| f \|_p \ll \| ( \sum \delta_{kl} (f)^2 )^{1/2} \|_{L_p(\Delta_s)} \ll \| f \|_p, \tag{5}$$

where

$$\delta_{kl} (f) = \sum_{\pm (n_{|k|-1}+1)}^{\pm n_{|k|}} \ \sum_{\pm (n_{|l|-1}+1)}^{\pm n_{|l|}} c_{\mu\nu} e^{i \frac{\pi}{s} (\mu x + \nu y)}$$

and where we this time assume that $n_0 = n_{-1} = 0$, $n_1 = 1$, $n_k = 2^{k-2} \beta$ $(k = 2, 3, \dots)$, and $s > s_0$ is selected so that $\beta = s/\pi > 2$ is integral. The sign + or - is assigned depending on whether k or l is positive or negative. Condition 1.5.2 (5) is observed

$$\frac{n_{k+1}}{n_k} \geqslant 2 \qquad (k = 1, 2, \ldots),$$

therefore the constants appearing in inequalities (5) do not depend on $s > s_0$.

Suppose
$$\delta_{kl}(f) = \frac{1}{2\pi} \int_{\Delta_{k,l}} f(u, v)\, e^{i(xu+yv)}\, du\, dv = \widehat{(1)_{\Delta_{kl}} f},$$

where $\Delta_{kl}$ is rectangle (3) (when $k_1 = k$, $k_2 = l$, and $n = 2$).

Let
$$a = \frac{k_1\pi}{s}, \qquad b = \frac{k_2\pi}{s}, \qquad c = \frac{l_1\pi}{s}, \qquad d = \frac{l_2\pi}{s};$$
$$\Delta = \{[a, b] \times [c, d]\};\ b - a,\ d - c \geqslant 1;$$
$$|f|,\ \left|\frac{\partial f}{\partial x}\right|,\ \left|\frac{\partial f}{\partial y}\right|,\ \left|\frac{\partial^2 f}{\partial x\, \partial y}\right| \leqslant M_\Delta,\ (x, y) \in \Delta.$$

We will use the Abel transformation for the sum

$$\begin{aligned}
\delta_\Delta = \sum_{k_1}^{k_2} \sum_{l_1}^{l_2} c_{kl} e^{i\frac{\pi}{s}(kx+ly)} &= \\
= \frac{\pi}{2s^2} \sum_{k_1}^{k_2} \sum_{l_1}^{l_2} f\left(\frac{k\pi}{s}, \frac{l\pi}{s}\right) e^{i\frac{\pi}{s}(kx+ly)} &= \\
= \frac{\pi}{2s^2} \sum_{l_1}^{l_2} e^{i\frac{\pi}{s}ly} \Bigg\{ \sum_{k_1}^{k_2-1} I_k(x)\, \Delta_x f\left(\frac{k\pi}{s}, \frac{l\pi}{s}\right) &+ \\
+ f\left(\frac{k_2\pi}{s}, \frac{l\pi}{s}\right) I_{k_2}(x) \Bigg\}&,
\end{aligned} \tag{6}$$

where

$$I_k(x) = \sum_{\nu=1}^{k} e^{i\frac{\pi}{s}\nu x} = \frac{e^{i\frac{\pi}{s}(k+1)x} - 1}{e^{i\frac{\pi}{s}x} - 1},$$
$$\Delta_x a_{kl} = a_{kl} - a_{k+1,l}.$$

A further Abel transformation leads to the equality

$$\delta_\Delta = \frac{\pi}{2s^2}\left\{\sum_{k_1}^{k_2-1}\sum_{l_1}^{l_2-1} I_k(x) I_l(y)\,\Delta^2_{xy} f\left(\frac{k\pi}{s},\ \frac{l\pi}{s}\right) + \right.$$
$$+ \sum_{k_1}^{k_2-1}\Delta_x f\left(\frac{k\pi}{s},\ \frac{l_2\pi}{s}\right) I_k(x) I_{l_2}(y) +$$
$$+ \sum_{l_1}^{l_2-1} I_{k_2}(x) I_l(y)\,\Delta_y f\left(\frac{k_2\pi}{s},\ \frac{l\pi}{s}\right) +$$
$$\left. + f\left(\frac{k_2\pi}{s},\ \frac{l_2\pi}{s}\right) I_{k_2}(x) I_{l_2}(y)\right\}. \tag{7}$$

If we consider that

$$|I_k(x)| \leqslant \frac{2}{\left|e^{i\frac{\pi}{s}x} - 1\right|} \ll \frac{s}{|x|} \qquad (|x| < s),$$

then from (6) and (7) follow four inequalities

$$|\delta_\Delta| \leqslant cM_\Delta|\Delta|\{1,\ |x|^{-1},\ |y|^{-1},\ |xy|^{-1}\},\ |x|,\ |y| < s, \tag{8}$$

where c does not depend on the series of standing multipliers and on s. The second inequality, for example, is obtained from (6) by means of the following computations:

$$|\delta_\Delta| \ll \frac{1}{s^2}(l_2 - l_1)\left\{(k_2 - k_1)\frac{s}{|x|}\frac{M_\Delta}{s} + M_\Delta\frac{s}{|x|}(b-a)\right\}$$

(the multiplier b - a $\geqslant$ 1 is added in the second member within the braces). The fourth inequality follows by means of similar computations from (7).

Let us set $\Phi_1(x, y) = \min c\,\{1,\ |x|^{-1},\ |y|^{-1},\ |xy|^{-1}\}$. Obviously, $\Phi_1 \in L_p$ $(1 < p \leqslant \infty)$ and from (8) it follows that

$$|\delta_\Delta| \leqslant M_\Delta|\Delta|\Phi_1(x,\ y) \quad ((x,\ y) \in \Delta_s). \tag{9}$$

Based on (9), since

$$\delta^s_{kl}(f) = \delta_{\Delta^s_{kl}},$$
$$\Delta^s_{kl} = \left\{\pm\frac{(n_{k-1}+1)\pi}{s} \lesseqgtr x \gtreqless \pm\frac{n_k\pi}{s};\ \pm\frac{(n_{l-1}+1)\pi}{s} \lesseqgtr y \gtreqless \pm\frac{n_l\pi}{s}\right\},$$

we get $\quad |\delta^s_{kl}(f)| \leqslant M_{\Delta^s_{kl}}|\Delta^s_{kl}|\Phi_1(x,\ y).$

Since the functions $\tilde{f}$, $\partial \tilde{f}/\partial x$, and $\partial \tilde{f}/\partial y$ decrease at infinity more rapidly than does $(1 + |x|^{\lambda} + |y|^{\lambda})^{-1}$, where $\lambda$ is as large as we wish, then obviously

$$\sum M_{\Delta^s_{kl}} |\Delta^s_{kl}| < A < \infty,$$

where the constant A does not depend on s. Therefore

$$\{\sum \delta^s_{kl}(f)^2\}^{1/2} \leqslant \sum |\delta^s_{kl}(f)| \leqslant A\Phi_1 = \\ = \Phi(x, y) \in L_p(R_n) \quad ((x, y) \in \Delta_s). \tag{10}$$

Confining ourselves for sake of simplicity to nonnegative k and l, we will have (s is selected so that $\beta = s/\pi$ is integral)

$$\delta^s_{kl}(f) = \frac{\pi}{2s^2} \sum_{2^{k-3}\beta+1}^{2^{k-2}\beta} \sum_{2^{l-3}\beta+1}^{2^{l-2}\beta} \tilde{f}\left(\frac{\mu\pi}{s}, \frac{\nu\pi}{s}\right) e^{i\frac{\pi}{s}(\mu x+\nu y)} \xrightarrow[s\to\infty]{} \\ \xrightarrow[s\to\infty]{} \frac{1}{2\pi} \int_{2^{k-3}}^{2^{k-2}} \int_{2^{l-3}}^{2^{l-2}} \tilde{f}(u, v)\, e^{i(ux+vy)}\, du\, dv = \delta_{k-2,\, l-2}(f)$$

is uniform relative to $(x, y) \in \Delta_N$ for k and $l \geqslant 2$ ($2^{-1}$ must be replaced by 0), whatever be the specified $N > 0$. If however one of the numbers (still nonnegative) be less than 2, the doubled sum is converted into a single sum or even (for $\delta^s_{00}$, $\delta^s_{10}$, and $\delta^s_{01}$) a sum that degenerates into a single member.

In these cases $\delta^s_{kl}(f) \to 0$ is uniform on $\Delta_N$, since under the integral appears

a function that is continuous with respect to x, y, u, and v. Similar arguments are also valid for the numbers k and l of any sign, therefore it has been proved that for any k and l

$$\delta^s_{kl}(f) \xrightarrow[s\to\infty]{} \delta_{k-2,\, l-2}(f) \text{ на } \Delta_N \qquad \text{on} \quad N$$

is uniform, whatever be $N > 0$.

From the second inequality (5), it follows for any N, $N_1 > 0$ that

$$\left\|\left(\sum_{|k|,|l|<N_1} \delta^s_{kl}(f)^2\right)^{1/2}\right\|_{L_p(\Delta_N)} \leqslant \|f\|_p$$

and after the passage to the limit when $s \to \infty$, then $N_1 \to \infty$ and then $N \to \infty$, we get

$$\|(\sum \delta_{kl}(f)^2)^{1/2}\|_p \leqslant \|f\|_p.$$

From (10) it follows that

$$\Big(\sum_{|k|,\,|l|<N} \delta_{kl}^{s}(f)^2\Big)^{1/2} \leqslant \Phi(x,\ y),$$

therefore, after the passage to the limit initially when $s \to \infty$, then $N_1 \to \infty$, we got

$$\{\sum \delta_{kl}(f)^2\}^{1/2} \leqslant \Phi(x,\ y).$$

Finally,

$$\begin{aligned}
&\Big| \|\{\sum \delta_{kl}^{s}(f)^2\}^{1/2}\|_{L_p(\Delta_s)}^{p} - \|\{\sum \delta_{kl}(f)^2\}^{1/2}\|_{L_p(R_2)}^{p} \Big| \ll \\
&\qquad \ll \Big| \|\{\sum_{|k|,\,|l|<N} \delta_{kl}^{s}(f)^2\}^{1/2}\|_{L_p(\Delta_N)}^{p} - \\
&\qquad - \|\{\sum_{|k|,\,|l|<N} \delta_{kl}(f)^2\}^{1/2}\|_{L_p(\Delta_N)}^{p} \Big| + \\
&+ \|\sum_{|k|,\,|l|<N} |\delta_{kl}^{s}(f)|\|_{L_p(\Delta_s-\Delta_N)} + \|\sum_{|k|,\,|l|<N} |\delta_{kl}(f)|\|_{L_p(R_2-\Delta_N)} + \\
&\quad + \|\sum{}' |\delta_{kl}^{s}(f)|\|_{L_p(\Delta_N)} + \|\sum{}' |\delta_{kl}(f)|\|_{L_p(\Delta_N)} = I_1 + \ldots + I_5,
\end{aligned}$$

where $\sum'$ is the sum over pairs of the numbers k and l, where at least one of these is not smaller than N.

Here
$$I_2,\ I_3 \leqslant \|\Phi\|_{L_p(R_2-\Delta_N)},$$
$$I_4 \leqslant \sum{}' M_{\Delta_{kl}^{s}} |\Delta_{kl}^{s}| \|\Phi\|_p \leqslant \varepsilon_N \|\Phi\|_p,$$

where $\varepsilon_N$ does not depend on s and tends to zero when N

$$I_5 \leqslant \varepsilon_N \to 0 \qquad (N \to \infty).$$

Thus, N can be taken to be so large that $I_2, \ldots, I_5$ is less than an assigned $\varepsilon > 0$, and then $s_0$ can be selected so that $I_1 < \varepsilon$ for all $s > s_0$.

We have proven that for any infinitely differentiable finite function f

$$\lim_{s\to\infty} \|\{\sum \delta_{kl}^{s}(f)^2\}^{1/2}\|_{L_p(\Delta_s)} = \|\{\sum \delta_{kl}(f)^2\}^{1/2}\|_p,$$

and then based on (5), where the constants in the inequality do not depend on s, we get (1) (still for infinitely differentiable finite functions).

Now if $f \in L_p$, then we select a sequence of infinitely differentiable finite functions $f_j$ ($j = 1, 2, \ldots$) such that

$$\|f-f_j\|_p \to 0 \ (j \to \infty). \tag{11}$$

This shows that for any $\varepsilon > 0$ and $\lambda$ is found such that for i and j

$$\left\| \left\{ \sum_{|k|, |l| < N} \left[ \widehat{(1)_{\Delta_{kl}} (\overline{f_i - f_j})} \right]^2 \right\} \right\|_p \ll \|f_i - f_j\|_p < \varepsilon,$$

after passage to the limit when $i \to \infty$ in these inequalities $f_i$ is replaced with f. But further passage to the limit when $N \to \infty$ leads to the inequality

$$\|\{\sum \delta_{kl} (f_j - f)^2\}^{1/2}\|_p < \varepsilon \quad (j > \lambda),$$

from whence it follows that

$$\|\{\sum \delta_{kl} (f_j - f)^2\}^{1/2}\|_p \to 0 \ (j \to \infty). \tag{12}$$

Inequalities (1) are satisfied for the functions $f_j$. But as a consequence of (11) and (12) in these inequalities it is legitimate to pass to the limit when $j \to \infty$, thereby obtaining inequalities (1).

1.5.6.1. By wholly analogous arguments, though simpler because we have in mind the one-dimensional case, it is proven that for the functions $f(x) \in L_p(-\infty, \infty) = L_p (1 < p < \infty)$ the inequalities

$$\|f\|_p \ll \|\{\sum \beta_l (f)^2\}^{1/2}\|_p \ll \|f\|_p, \tag{13}$$

obtain, where

$$\beta_l (f) = \widehat{(\overline{1})_{\Delta_l} f},$$

$\Delta_l = \{2^{l-1} \leq x \leq 2^l, \ l = 0, 1, \ldots; \ 2^{l-1}$ for $l = 0$ is replaced by zero $\}$,

and the constants in (13) do not depend on f. In the periodic case, 1.5.2.1 (4) must be selected as the original inequality.

1.5.7. Fourier transform of the function sign x. The function

$$\operatorname{sign} x = \prod_{j=1}^{n} \operatorname{sign} x_j$$

is a multiplier when $1 < p < \infty$ (cf section 1.5.5). The functional (explanation below) is

$$
\begin{aligned}
\widehat{(\operatorname{sign} x,}\ \varphi) &= (\operatorname{sign} x, \hat{\varphi}) = \frac{1}{(2\pi)^{n/2}} \int \operatorname{sign} u\, du \int e^{iut} \varphi(t)\, dt = \\
&= \frac{1}{(2\pi)^{n/2}} \int_{R_+} du \int \varphi(t) \prod_{j=1}^{n} (e^{it_j u_j} - e^{-it_j u_j})\, dt = \\
&= \left(\frac{2}{\pi}\right)^{n/2} i^n \lim_{N\to\infty} \int \varphi(t)\, dt \int_{\Delta_N^+} \prod_{j=1}^{n} \sin t_j u_j\, du_j = \\
&= \left(\frac{2}{\pi}\right)^{n/2} i^n \lim_{N\to\infty} \int \varphi(t)\, dt \prod_{j=1}^{n} \int_0^N \sin t_j u_j\, du_j = \\
&= \left(\frac{2}{\pi}\right)^{n/2} i^n \lim_{N\to\infty} \int \varphi(t) \prod_{j=1}^{n} \frac{1-\cos N t_j}{t_j}\, dt = \\
&= \left(\frac{2}{\pi}\right)^{n/2} i^n \lim_{N\to\infty} \int_{R_+} \Delta\varphi(t) \prod_{j=1}^{n} \frac{1-\cos N t_j}{t_j}\, dt = \\
&= \left(\frac{2}{\pi}\right)^{n/2} i^n \int_{R_+} \frac{\Delta\varphi(t)}{t}\, dt = \left(\frac{2}{\pi}\right)^{n/2} i^n \int \frac{\varphi(t)}{t}\, dt.
\end{aligned}
\tag{1}
$$

Here $R_+$ is the positive coordinate junction

$$
\begin{gathered}
\Delta_N = \{0 \leqslant x_j \leqslant N;\ i = 1, \ldots, n\}, \\
\Delta\varphi(t) = \Delta_1 \Delta_2 \ldots \Delta_n \varphi(t)
\end{gathered}
\tag{2}
$$

and

$$
\Delta_j \varphi(t) = \varphi(t) - \varphi(t_1, \ldots, t_{j-1}, -t_j, t_{j+1}, \ldots, t_n) \quad (j = 1, \ldots, n).
\tag{3}
$$

In the penultimate equality (1), when the product members are multiplied, the integrals

$$
\int_{R_+} \frac{\Delta\varphi(t)}{t} \prod_{j=1}^{k} \cos N t_j\, dt \to 0,
$$

appear, tending to zero when $N \to \infty$ due to the summability of $t^{-1}\Delta\varphi(t)$ on $R_+$ by virtue of a lemma well known in the theory of Fourier series. The integral in the last member in (1) is written in the Cauchy sense:

$$\int \frac{\varphi(t)}{t}\,dt = \lim_{\varepsilon\to 0}\int_{R^\varepsilon} \frac{\varphi(t)}{t}\,dt, \tag{4}$$

where $R^\varepsilon$ is a set of points $x \in R$, located from any coordinate planes by a distance greater than $\varepsilon > 0$. Functional (4) defines the generalized function, which is denoted by v. p. 1/t. And so, the equality

$$\widehat{\operatorname{sign} x} = \left(\frac{2}{\pi}\right)^{n/2} i^n \text{ v. p. } \frac{1}{t}.$$

For $f \in S$

$$\widehat{\operatorname{sign} x} * f = \widehat{\operatorname{sign} x \hat{f}} = \frac{1}{(2\pi)^n}\int \operatorname{sign} u \int f(t)\, e^{-itu}\,dt\, e^{iux}\,du =$$
$$= \frac{1}{(2\pi)^n}\int \operatorname{sign} u\,du \int f(x-t)\, e^{iut}\,dt = \left(\frac{i}{\pi}\right)^n \int \frac{f(x-t)}{t}\,dt, \tag{5}$$

where the last equality follows from the already proven equality between the third and last members of (1), if there we replace $\varphi(t)$ by $f(x - t)$. The last integral in (5) is understood in the Cauchy sense.

We will use the notation

$$\widehat{\operatorname{sign} x} * f = \widehat{\operatorname{sign} x \hat{f}} = \left(\frac{i}{\pi}\right)^n \int \frac{f(x-t)}{t}\,dt \tag{6}$$

for the case when $f \in L_p (1 < p < \infty)$, understanding the members of (6) to be limits to which the corresponding expressions for finite functions $f_1$ tend in the $L_p$-sense, where $\|f - f_1\|_p \to 0$. With respect to the first and

second members of (6), this was validated above (cf 1.5.1), because sign x is a multiplier in $L_p$ for $1 < p < \infty$. We have now provided the appropriate definition for the expression externally written in integral form. Actually, it can be proven (M, Ris [1] when n = 1) that for $f \in L_p(1 < p < \infty)$ this is a real integral in the Cauchy sense, existing for almost x, but we will not dwell on this matter.

Let $\mu = (\mu_1, \ldots, \mu_n)$, $a = (a_1, \ldots, a_n) > 0$ $(a_j > 0)$ be two specified

vectors and

$$\Delta_a = \{|x_j| < a_j;\ j = 1, \ldots, n\},$$
$$\Delta(\mu, a) = \{|x_j - \mu_j| < a_j;\ j = 1, \ldots, n\},\quad \Delta(0, a) = \Delta_a.$$

Thus, $\Delta(\mu, a)$ is the displacement $\Delta_a$ for the vector $\mu$. Notive that the characteristic function (of one variable t) on the interval (a, b) is

$$(1)_{(a,b)} = \frac{1}{2}[\operatorname{sign}(t-a) - \operatorname{sign}(t-b)].$$

Hence it follows that

$$(1)_{\Delta(\mu,a)} = \prod_{j=1}^{n} \frac{1}{2}[\operatorname{sign}(x_j - \mu_j + a_j) - \operatorname{sign}(x_j - \mu_j - a_j)] = \\ = \frac{(-1)^n}{2^n} \sum \operatorname{sign} \alpha \operatorname{sign}(x - \mu - \alpha), \tag{7}$$

where the sum is extended over all possible vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $|\alpha_j| = \alpha_j$, $j = 1, \ldots, n$.

We know that the function sign x is a multiplier:

$$\|\widehat{\operatorname{sign} x f}\|_p \leqslant \varkappa_p \|f\|_p \qquad (1 < p < \infty), \tag{8}$$

where $\varkappa_p$ does not depend on f (cf. 1.5.5), and sign (x = a) is also a multiplier with the same constant $_p$ in the corresponding inequality (cf 1.5.1.2), whatever be the vector a $\in$ R, therefore from (7) it follows that

$$\|\widehat{(1)_{\Delta(\mu,a)} f}\|_p \leqslant \frac{1}{2^n} \sum \varkappa_p \|f\|_p = \varkappa_p \|f\|_p, \qquad 1 < p < \infty, \tag{9}$$

because the sum is extended over $2^n$ terms. It is remarkable that the constant $\varkappa_p$ in (9) is the same as in (8) and, therefore, does not depend on $\mu$ and a.

From (7) it follows (cf 1.4(18)) that

$$\widehat{(1)}_{\Delta(\mu,a)} = \frac{(-1)^n}{2^n} \sum \operatorname{sign}\alpha\, e^{i(\mu+\alpha)x} \widehat{\operatorname{sign} x} = \\ = \frac{1}{2^n} e^{i\mu x} \prod_{j=1}^{n} (e^{ia_j x_j} - e^{-ia_j x_j}) \left(\frac{2}{\pi}\right)^{n/2} (-i)^n \text{ v. p. } \frac{1}{x} = \\ = \left(\frac{2}{\pi}\right)^{n/2} e^{i\mu x} D_a(x), \tag{10}$$

where

$$D_a(x) = \prod_{j=1}^{n} \sin a_j x_j \text{ v. p. } \frac{1}{x} = \prod_{j=1}^{n} \frac{\sin a_j x_j}{x_j}. \tag{11}$$

An ordinary function is obtained in (11) when an ordinary function is multiplied by a generalized function. For example, in the one-dimensional case this is proven thusly:

$$\left(\sin ax \,\mathrm{v.\,p.}\frac{1}{x},\ \varphi(x)\right)=\left(\mathrm{v.\,p.}\frac{1}{x},\ \sin ax\varphi(x)\right)=$$
$$=\int_0^\infty \frac{\Delta[\sin ax\varphi(x)]\,dx}{x}=\int \frac{\sin ax}{x}\varphi(x)\,dx, \tag{12}$$

where $\Delta F(x) = F(x) - F(-x)$. The integral in the right side of (12) can now be understood in the Lebesque sense.

The equality

$$\widehat{(1)_{\lambda(u,a)}}f=\frac{1}{\pi^n}\int e^{i(\mu,x-u)}D_a(x-u)f(u)\,du,$$

obtains for functions $f \in S$, in particular when $\mu = 0$

$$F(x)=\widehat{(1)_{\lambda_a}}f=\frac{1}{\pi^n}\int D_a(x-u)f(u)\,du, \tag{13}$$

where the integrals in the right sides are understood in the Lebesque sense. Let us dwell in greater detail on (13). Integral (13) is meaninglul also for any function $f \in L_p (1 \leq p < \infty)$ because $D_a(x) \in L_q (1/p + 1/q = 1)$ and

$$\int |D_a(x-a) f(u)| \, du \leqslant \| D_a \|_q \| f \|_p < \infty.$$

It is immediately clear that it is a continuous function of x (even uniformly continuous):

$$|F(x) - F(y)| \leqslant \| D_a(x-u) - D(y-u) \|_q \| f \|_p \to 0 \quad (x \to y).$$

If $f_l \in S$, $\| f_l - f \|_p \longrightarrow 0$, and $F_l$ is a result of substituting $f_l$ instead of f in (13), then

$$|F(x) - F_l(x)| \leqslant \| D_a \|_q \| f - f_l \|_p \to 0$$

is uniform. On the other hand, $(1)_{\Delta_a}$ is a Marcinkievicz multiplier, because $\| F_k - F_l \|_p \longrightarrow 0$ (k, l $\to$ 0). This shows that $F_l$ tends in the $L_p$-sense precisely

to the function F defined by integral (13) and that for $f \in L_p$ (13) is valid, where its right-hand side is a Lebesgue integral, and the left-hand side is understood in terms of the Marcinkievicz multiplier (cf 1.5.1).

In fact, F(x) is an analytic function, of the integral exponential type (cf further 3.6.2).

1.5.8. Functions $\varphi_\varepsilon$ and $\psi_\varepsilon$. The function $\varphi_\varepsilon$ is defined on $R = R_n$, depends on a small positive parameter $\varepsilon$ $(0 < \varepsilon < \varepsilon_0)$, and exhibits the following properties: $\varphi_\varepsilon(x)$ is infinitely differentiable and is nonnegative on R, and has a carrier on the cube

$$\Delta_\varepsilon = \{ |x_j| < \varepsilon; \; j = 1, \ldots, n \}$$

(i.e., $\varphi_\varepsilon = 0$ outside $\Delta_\varepsilon$) and, moreover, satisfies the equality

$$\int_{\Delta_\varepsilon} \varphi_\varepsilon(x) \, dx = 1 \qquad (0 < \varepsilon < \varepsilon_0). \tag{1}$$

It is important that $\varphi_\varepsilon \in S$ also have a compact carrier, i.e., is a finite function (cf 1.4.1).

If $\varphi$ is an arbitrary function continuous on R (even locally summable on R and continuous at the zero-point), then

$$\lim_{\varepsilon\to 0}\int \varphi_\varepsilon(x)\varphi(x)\,dx = \varphi(0), \tag{2}$$

because

$$\left|\int_{\Delta_\varepsilon}\varphi_\varepsilon(x)\varphi(x)\,dx - \varphi(0)\right| = \int_{\Delta_\varepsilon}\varphi_\varepsilon(x)[\varphi(x)-\varphi(0)]\,dx \leqslant$$
$$\leqslant \int_{\Delta_\varepsilon}\varphi_\varepsilon(x)\sup_{\Delta_\varepsilon}|\varphi(x)-\varphi(0)|\,dx =$$
$$= \sup_{\Delta_\varepsilon}|\varphi(x)-\varphi(0)| \to 0 \qquad \varepsilon\to 0.$$

If $\varphi \in S$, then equality (2) can be written thusly:

$$\lim_{\varepsilon\to 0}(\varphi_\varepsilon, \varphi) = (\delta, \varphi) = \varphi(0), \tag{3}$$

where $\delta = \delta(x)$ is a delta-function.

Let us suppose $\psi_\varepsilon(x) = (2\pi)^{n/2}\tilde{\varphi}_\varepsilon(x).$

Since $\varphi_\varepsilon \to \delta$ ( $\varepsilon \to 0$) weakly, then $\psi_\varepsilon \to (2\ )^{n/2}\ \tilde{\delta} = 1$ weakly. Moreover, $\psi_\varepsilon(x)$ as an ordinary function as $\varepsilon \to 0$ converges boundedly to 1 for all x:

$$\psi_\varepsilon(x) = \int \varphi_\varepsilon(t)e^{-ixt}\,dt \to 1, \tag{4}$$

$$|\psi_\varepsilon(x)| \leqslant \int \varphi_\varepsilon(t)\,dt = 1. \tag{5}$$

Below it will be shown that if $f \in L_p$, $g \in L$ and $\varepsilon \to 0$, then

$$\psi_\varepsilon f \to f, \tag{6}$$
$$\psi_\varepsilon g * f \to g * f, \tag{7}$$
$$g * \psi_\varepsilon f \to g * f. \tag{8}$$

weakly.

Further, if $f \in L_p$, $g \in L_p$, and $1/p + 1/q = 1$, then the convolution $g*f$ can be defined by means of the integral

$$g * f = \frac{1}{(2\pi)^{n/2}}\int g(u)f(x-u)\,du.$$

Obviously, $$|(g*f)(x)| \leqslant \frac{1}{(2\pi)^{n/2}} \|g\|_q \|f\|_p.$$

This convolution lies to one side of the generalization of this concept introduced in 1.5, where $g \in S'$ was such a function that $f \in L_p$ entails $g*f \in L_p$. But in the given case when $f \in L_p$, the function $g*f$ belong to the class $L_\infty = M$ of bounded (measurable) functions. However, a property analogous to (8)

$$g*\psi_\varepsilon f \to g*f \qquad (\varepsilon \to 0). \tag{9}$$

obtain for this convolution.

Proof for (6). By the Lebesgue theorem

$$(\psi_\varepsilon f, \varphi) = \int \psi_\varepsilon(t) f(t) \varphi(t)\, dt \to \int f\varphi\, dt = (f, \varphi).$$

Proof of (7).

$$(\psi_\varepsilon g*f, \varphi) = \frac{1}{(2\pi)^{n/2}} \iint \psi_\varepsilon(t) g(t) f(x-t) \varphi(x)\, dt\, dx \to$$
$$\to \frac{1}{(2\pi)^{n/2}} \iint g(t) f(x-t) \varphi(t)\, dt\, dx = (g*f, \varphi).$$

Since

$$\iint |g(t) f(x-t)|\, dt\, |\varphi(x)|\, dx \leqslant$$
$$\leqslant \left\| \int |g(t) f(x-t)|\, dt \right\|_p \|\varphi\|_q \leqslant \|g\|_L \|f\|_p \|\varphi\|_q \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Proof of (8).

$$(g*\psi_\varepsilon f, \varphi) = \frac{1}{(2\pi)^{n/2}} \iint \psi_\varepsilon(t) f(t) g(x-t) \varphi(x)\, dx\, dt \to$$
$$\to \frac{1}{(2\pi)^{n/2}} \iint f(t) g(x-t) \varphi(x-t)\, dx\, dt, \tag{10}$$

since $$\iint |f(t) g(x-t) \varphi(x)|\, dt\, dx \leqslant \left\| \int f(t) g(x-t)\, dt \right\|_p \|\varphi\|_q \leqslant$$
$$\leqslant \|g\|_L \|f\|_p \|\varphi\|_q.$$

Proof of (9). The same as the proof of (8), but we must take into consideration the inequality

$$\int\int |f(t)\, g(x-t)\, \varphi(x-t)|\, dx\, dt \leqslant \|f\|_p \|g\|_q \|\varphi\|_L.$$

1.5.9. Operation $I_r$ of the Liouville type. Let r be an arbitrary real number. The function

$$(1+|u|^2)^{r/2} = \left(1+\sum_{j=1}^{n} u_j^2\right)^{r/2} \tag{1}$$

is infinitely differentiable on R and has polynomial growth for any sign of r.

Let us suppose

$$G_r(u) = \widehat{(1+|u|^2)^{-r/2}}. \tag{2}$$

Since

$$\widetilde{G_r(u)} = (1+|u|^2)^{-r/2} \tag{3}$$

is an infinitely differentiable function with polynomial growth, then for any generalized function $f \in S'$, the convolution

$$F = G_r * f = \widehat{\widetilde{G}_r \tilde{f}} = \widehat{(1+|u|^2)^{-r/2} \tilde{f}} = I_r f, \tag{4}$$

defining the operation $I_r$ mapping $f \in S'$ onto $F \in S'$ is meaningful.

Obviously,

$$I_0 f = f. \tag{5}$$

If r and $\rho$ are arbitrary real numbers and $f \in S'$, then

$$I_{r+\rho} f = \widehat{(1+|\lambda|^2)^{-r/2}(1+|\lambda|^2)^{-\rho/2} \tilde{f}} = \widehat{(1+|\lambda|^2)^{-r/2} \widetilde{I_\rho f}} = I_r I_\rho f. \tag{6}$$

In particular, when $\rho = -r$

$$I_r I_{-r} f = I_0 f = f, \qquad (7)$$

i.e., the operation $I_r$ and $I_{-r}$ are mutually inverse.

It is not difficult also to see that the operation $I_r$ maps S onto S mutually single-valuedly and continuously: if $\varphi_m$, $\varphi \in S$, and $\varphi_m \to \varphi(S)$ as $m \longrightarrow \infty$, then

$$I_r \varphi_m \to I_r \varphi(S).$$

We can even introduce the operation $I_r^*$ defined by the formula

$$I_r^* = \overline{(1+|\lambda|^2)^{-r/2} f},$$

which we naturally call conjugate to $I_r$. Obviously, it exhibits all the properties established above for $I_r$, including continuity in the sense of convergence in S.

The connection between $I_r$ and $I_r^*$ is manifested in the equalities

$$(I_r f, \varphi) = (f, I_r^* \varphi),$$
$$(I_r^* f, \varphi) = (f, I_r \varphi) \quad (f \in S', \varphi \in S).$$

From these it immediately follows that the operations $I_r$ and $I_r^*$ are continuous on S' (weakly continuous), i.e., that from $f_m$, $f \in S'$, $m = 1, 2, \ldots,$ and

$$f_m \to f(S'),$$

it follows that

$$I_r f_m \to I_r f, \quad I_r^* f_m \to I_r^* f(S').$$

In fact, for example,

$$(I_r f_m, \varphi) = (f_m, I_r^* \varphi) \to (f, I_r^* \varphi) = (I_r f, \varphi).$$

Notice that when $r = -2$ the remarkable equality

$$I_{-2} f = \widehat{(1+|\lambda|^2) \tilde{f}} = f + \widehat{\sum_{j=1}^{n} \lambda_j^2 \tilde{f}} = f - \sum_{j=1}^{n} \widehat{(i\lambda_j)^2 \tilde{f}} = f - \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = f - \Delta f = (1-\Delta) f,$$

obtains, where $\Delta$ is a Laplace operator.

Consequently, for any natural l

$$I_{-2l}f = \overbrace{(1+|\lambda|^2)^l \tilde{f}} = (1-\Delta)^l f \qquad (f \in S'). \tag{8}$$

1.5.10. **Regular generalized functions. Further enlargement of the concept of convolution.** The operation $I_r$ can serve as a convenient means for enlarging the concept of convolution to the class of generalized functions, which we call regular.

By the definition, we will call the function $f \in S'$ regular in the $L_p$-sense and write $f \in S'_p$ if for some $\rho_0 > 0$

$$I_{\rho_0} f = F \in L_p. \tag{1}$$

obtains.

Let be a multiplier in $L_p$ ($\mu \in L$ when $p = 1$) (cf 1.5.1, 1.5.1.1). Let f further be a function in the $L_p$-sense for which the property (1) is satisfied.

Let us suppose for $\rho \geqslant \rho_0$ that

$$\hat{\mu} * f = I_{-\rho}(\hat{\mu} * I_\rho f). \tag{2}$$

(1) This definition does not depend on $\rho \geqslant \rho_0$. In fact, let -- along with

$$I_{\rho'} f = F_1 \in L_p \qquad (\rho' > \rho). \tag{3}$$

Then when $\rho' - \rho = r$, considering that $I_\rho f = F \in L_p$, we get

$$I_{-\rho'}(\hat{\mu} * I_{\rho'} f) = I_{-\rho} I_{-r}(\hat{\mu} * I_r I_\rho f) =$$
$$= I_{-\rho} \overbrace{(1+|x|^2)^{r/2} \mu (1+|x|^2)^{-r/2}} \widetilde{I_\rho f} = I_{-\rho} \widehat{\mu \widetilde{I_\rho f}} = I_{-\rho}(\hat{\mu} * I_\rho f)$$

(cf 1.5.1 (12) when $\mu \in L$ and 1.5.1.1 (9) when $1 \leqslant p < \infty$). In the third equality, we used a fact that will be proven later (cf 8.1) to the effect that

$$\overbrace{(1+|x|^2)^{-r/2}} \in L \qquad (r > 0),$$

and that the function $(1 + |x|^2)^\lambda$ for any real $\lambda$ is infinitely differentiable and of polynomial growth.

The equality $I_r \mathbf{x} = \mathbf{x} = x_1 \ldots x_n$ holds for any real r, showing that the function $\mathbf{x}$ does not belong to $S'_p$ $(1 \leqslant p \leqslant \infty)$, though it does belong to S'. This follows from 1.5 (12) when $\mathbf{k} = \omega = (1, \ldots, 1)$:

$$(1+|x|^2)^{-\frac{r}{2}}\tilde{x}=i^n(2\pi)^{\frac{n}{2}}(1+|x|^2)^{-\frac{r}{2}}\delta^{(\alpha)}(x)=$$
$$=i^n(2\pi)^{\frac{n}{2}}\delta^{(\alpha)}(x)=\tilde{x}.$$

It is important to note that for the generalized function f that is regular in the $L_p$-sense, the equality

$$I_{-\lambda}(\hat{\mu}*I_\lambda f)=\hat{\mu}*f \tag{4}$$

obtains for any $\lambda$ (positive and negative). In fact, for f there exists an $\rho > 0$ such that $I_\rho f \in L_p$. When $\lambda \geqslant \rho$, equality (7) was already proven above, while if $\lambda < \rho$, then we assume $\rho = \lambda + \sigma (\sigma > 0)$. Then the function $I_\lambda f$ is regular. Specifically, $I_\sigma I_\lambda f \in L_p$. Therefore,

$$I_{-\lambda}(\hat{\mu}*I_\lambda f)=I_{-\lambda}I_{-\sigma}(\hat{\mu}*I_\rho f)=I_{-\rho}(\hat{\mu}*I_\rho f)=\hat{\mu}*f.$$

It follows from (4) that for the functions f regular in the $L_p$-sense and for any real r

$$I_r(\hat{\mu}*f)=I_rI_{-r}(\hat{\mu}*I_rf)=\hat{\mu}*I_rf, \tag{5}$$

i.e., for the regular function f the operation $I_r$ can be taken under the sign of the convolution.

It follows from (5) that if $\mu$ is a Marcinkievicz multiplier and if f is a function regular in the $L_p$-sense, the convolution $\hat{\mu}*f$ is also regular. Actually, let $I_rf \in L_p$, then (5) obtains, where the right-hand side belongs to $L_p$.

Early the equalities 1.5.1.1 (9) were proven, which we wrote in terms of convolutions:

$$\hat{\lambda}*(\hat{\mu}*f)=\hat{\mu}*(\hat{\lambda}*f)=\widehat{\lambda\mu}*f, \quad f\in L_p(1\leqslant p<\infty). \tag{6}$$

They are valid if $\lambda$ and $\mu$ are Marcinkievicz multipliers, whence it follows that $(\lambda\mu)$ is also a Marcinkievicz multiplier. Now let f be a generalized function regular in the $L_p$-sense and let $I_\rho f \in L_p(\rho > 0)$. Then equalities

(6) will be satisfied, if $I_\rho f$ replaces f in them. But for regular f, the operation $I_\rho$ is validly removed from the signs of the convolution in all members of (6), but then the function appearing under the sign of $I_\rho$ are equal to each other and we have proven that (6) obtains for any generalized function that is regular in the $L_p$-sense.

## CHAPTER II TRIGONOMETRIC POLYNOMIALS

### 2.1 Theorem on Zeroes. Linear Independence

$$T_n(z) = \frac{\alpha_0}{2} + \sum_{k=1}^{n} (\alpha_k \cos kz + \beta_k \sin kz), \tag{1}$$

where $\alpha_k, \beta_k$ $(k = 0, 1, \ldots, n)$ are arbitrary complex numbers, and z is a complex or real variable, this function is called a trigonometric polynomial of n-th order. This definition does not exceed the case $\alpha_n = \beta_n = 0$.

The trigonometric polynomial is a function with period $2\pi$, and therefore in studying it it suffices to confine ourselves to examining variation of the independent variable $z = x + iy$ in an arbitrary vertical strip $a \leq x < a + 2\pi$ (or $a < x \leq a + 2\pi$), $-\infty < y < \infty$ of the complex plane of width 2 .

Using the equalities

$$\cos kz = \frac{e^{ikz} + e^{-ikz}}{2}, \quad \sin kz = \frac{e^{ikz} - e^{-ikz}}{2i} \tag{2}$$
$$(k = 0, 1, 2, \ldots)$$

the trigonometric polynomial (1) can be transformed to the more symmetrical form

$$T_n(z) = \sum_{k=-n}^{n} c_k e^{ikz},$$
$$c_k = \frac{\alpha_k - \beta_k i}{2},$$
$$c_{-k} = \frac{\alpha_k + \beta_k i}{2} \quad (k = 1, 2, \ldots), \ c_0 = \frac{\alpha_0}{2}. \tag{3}$$

It is clear from (3) that if coefficients $\alpha_k$ and $\beta_k$ of polynomial (1) are real, then coefficients $c_k$ and $c_{-k}$ for each k are pairwise complexly conjugate

$$c_{-k} = \bar{c}_k, \quad k = 0, 1, \ldots, n. \tag{4}$$

Conversely, it follows from (4) that the numbers $\alpha_k$ and $\beta_k$ are real.

The most important property of trigonometric polynomials is expressed by the following theorem.

2.1.1. Theorem. Trigonometric polynomial $T_n$ of order n, in which one of the coefficients $\alpha_n$ or $\beta_n$ in (1) is not equal to zero and has in any strip $a \leq x < a + 2\pi$ of the complex plane $z = x + iy$ exactly 2n zeroes, allowing for their multiplicity*).

If we represent them by $z_1, \ldots, z_{2n}$, then the equality

$$T_n(z) = A \prod_{k=1}^{2n} \sin \frac{z - z_k}{2}, \qquad (1)$$

obtains, where $A \neq 0$ is some constant. Conversely, equality (1) defines the trigonometric polynomial of order n.

Proof. Let us use the representation $T_n$ in the form of 2.1 (3). After substituting $Z = e^{iz}$, which transforms the mutually single-valuedly strip of the plane z considered here into the entire complex plane Z (except for Z = 0), we get

$$T_n(Z) = \sum_{k=-n}^{n} c_k Z^k = Z^{-n} P_{2n}(Z),$$

where

$$P_{2n}(Z) = c_{-n} + c_{-n+1} Z + \ldots + c_n Z^{2n}.$$

By the conditions of the theorem $c_n \neq 0$ and $c_{-n} \neq 0$ because the polynomial $P_{2n}(Z)$ of degree 2n has in the complex plane Z exactly 2n zeroes (with allowance for multiplicity) not equal to zero.

Hence it follows that the trigonometric polynomial $T_n$ has in the strip here considered exactly 2n zeroes (allowing for their multiplicity). Let us denote the zeroes of the polynomial $P_{2n}(Z)$ by $Z_k = e^{iz_k}$ ($k = 1, \ldots, 2n$), then

$$T_n(z) = c_n e^{-inz} \prod_{k=1}^{2n} (e^{iz} - e^{iz_k}) =$$

$$= c_n e^{\frac{i}{2} \sum_{k=1}^{2n} z_k} \prod_{k=1}^{2n} \left( e^{i \frac{z - z_k}{2}} - e^{i \frac{z_k - z}{2}} \right) = A \prod_{k=1}^{2n} \sin \frac{z - z_k}{2},$$

*) The number is called the zero of multiplicity m of the function f, if $f(a) = f'(a) = \ldots = f^{(m-1)}(a) = 0$, $f^{(m)}(a) \neq 0$.

where

$$A = c_n 2^{2n} (-1)^n e^{\frac{i}{2} \sum_{k=1}^{2n} z_k}.$$

Thus, the first part of the theorem has been proven. To verify that function (1) where the numbers $z_k$ ($k = 1, \ldots, 2n$) belong to some vertical (closed on one side) strip of the complex plane with width 2 is a trigonometric polynomial of order n, it suffices to make this transformation on the opposite side, starting from (1).

2.1.2. Linear independence. If the trigonometric polynomial $T_n(z)$ equals zero at more than 2n points of a vertical strip of width $2\pi$, then based on theorem 2.1.1 all its coefficients must equal zero. In particular, this occurs if a trigonometric polynomial of order n is identically or almost everywhere equal to zero at a real axis.

Hence it follows that the system of functions

$$1, \cos x, \sin x, \ldots, \cos nx, \sin nx \qquad (1)$$

is linearly independent in $C^*$ and $L_p^*$ (cf 1.1.1 and 1.2.1). We must consider that the zero element in $L_p^*$ is a function almost everywhere equal to zero.

The linear independence of system (1) also follows from the orthogonal properties of the trigonometric functions (m, n = 0, 1, 2, ...)

$$(m,\ n = 0, 1, 2, \ldots)$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

2.1.3. If $T_m$ and $T_n$ are trigonometric polynomials of, respectively, orders m and n and $m \geqslant n$, their sum and difference is a trigonometric polynomial of order not higher than m.

In fact, their product is a trigonometric polynomial of order not higher than m + n, which stems from equalities

$$\cos mx \cos nx = \frac{1}{2} [\cos (m-n) x + \cos (m+n) x],$$

$$\sin mx \sin nx = \frac{1}{2} [\cos (m-n) x - \cos (m+n) x],$$

$$\cos mx \sin nx = \frac{1}{2} [\sin (m+n) x - \sin (m-n) x].$$

2.1.4. It follows from the orthogonal properties of the system 2.1.2 (1) that if the trigonometric polynomial is even (an even function), then it contains as its members only the cosines ($\beta_k = 0$), and if it is odd, then only the sines $\alpha_k = 0$.

Inspecting the real parts of the equality

$$\cos nx + i\sin nx = (\cos x + i\sin x)^n,$$

we get получим

$$\cos nx = \cos^n x - C_n^2 \cos^{n-2} x\,(1-\cos^2 x) + C_n^4 \cos^{n-4} x\,(1-\cos^2 x)^2 + \ldots,$$

from whence it follows that any even ttigonometric polynomial of n-th order can be represented in the form of $P_n(\cos x)$, where

$$P_n(z) = a_0 + a_1 z + \ldots + a_n z^n$$

is some algebraic polynomial of n-th degree.

On the one hand, from the equality

$$\cos^n x = \frac{(e^{ix}+e^{-ix})^n}{2^n} = \frac{1}{2^n}\left(e^{inx} + C_n^1 e^{i(n-2)x} + \ldots + e^{-inx}\right)$$

it follows that

$$\cos^n x = \frac{1}{2^{n-1}}\left[\cos nx + C_n^1 \cos(n-2)x + \ldots + C_n^{\frac{n}{2}-1}\cos 2x + \frac{C_n^{\frac{n}{2}}}{2}\right] \tag{1}$$

for an even n, and

$$\cos^n x = \frac{1}{2^{n-1}}\left[\cos nx + C_n^1 \cos(n-2)x + \ldots + C_n^{\left[\frac{n}{2}\right]}\cos x\right]$$

for an odd n.

Thus, the function $P_n(\cos x)$ where $P_n(z)$ is an algebraic polynomial of n-th degree is an even polynomial of n-th order.

## 2.2 Important Examples of Trigonometric Polynomials

From the equality

$$\sum_{k=0}^{n} e^{ikx} = \frac{e^{i(n+1)x}-1}{e^{ix}-1} = \frac{e^{i\left(n+\frac{1}{2}\right)x}-e^{-i\frac{x}{2}}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}},$$

by inspecting in it, separately, the real and imaginary parts, we get

$$\frac{1}{2}+\sum_{k=1}^{n}\cos kx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} = D_n(x), \tag{1}$$

$$\sum_{k=1}^{n}\sin kx = \frac{\cos\frac{x}{2}-\cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} = D_n^*(x). \tag{2}$$

In particular, equality (1) shows that the polynomial $D_n(x)$ tends to zero at the points

$$x_k = \frac{2\pi k}{2n+1} \qquad (k = 1, \ldots, 2n)$$

of the interval (0, 2 ), therefore, it can also be written as the product

$$D_n(x) = A\prod_{k=1}^{2n}\sin\frac{x-x_k}{2},$$

where A is a constant. Assuming x = 0 in this equality, we get the relationship

$$\frac{2n+1}{2} = A\prod_{k=1}^{2n}\sin\frac{x_k}{2},$$

from which we can determine A.

The trigonometric polynomial $D_n(x)$ plays a large role in the theory of Fourier series. It is called the Dirichlet kernel.

We note that (explanations below)

$$\|D_n\|_{L*} = \int_0^\pi \left| \frac{\sin\frac{2n+1}{2}x}{\sin\frac{x}{2}} \right| dx = 2\int_0^\pi \frac{\left|\sin\frac{2n+1}{2}x\right|}{x} dx + O(1) =$$

$$= 2\int_0^{\frac{(2n+1)\pi}{2}} \frac{|\sin u|}{u} du + O(1) = 2\int_\pi^{n\pi} \frac{|\sin u|}{u} du + O(1) =$$

$$= 2\sum_{k=1}^{n-1}\int_0^\pi \frac{|\sin u|}{k\pi - u} du + O(1) = 2\sum_{k=1}^{n-1}\frac{1}{k\pi}\int_0^\pi \sin u\, du + O(1) =$$

$$= \frac{4}{\pi}\sum_{k=1}^{n-1}\frac{1}{k} + O(1) = \frac{4}{\pi}\ln n + O(1) \qquad (n = 1, 2, \ldots).$$

(3)

The variable $1/\pi \, \|D_n\|_{L*}$ is called the Lebesgue constants of the (n-th order) Fourier sum. Here $O(1)$ denotes some bounded function of a natural n. In the computation presented here we used the boundedness of the function $x^{-1}-(\sin x)^{-1}$ on $[0, \pi/2]$ and the fact that for $u \in [0, \pi]$

$$\sum_{k=1}^{n-1}\left(\frac{1}{k\pi} - \frac{1}{k\pi + u}\right) \leq c\sum_1^{n-1}\frac{1}{k^2} < c_1 < \infty.$$

For a finite $p > 1$ the norm $\|D_n\|_{Lp*}$ is bounded

$$\|D_n\|_{L_p^*}^p = 2\int_0^\pi \left| \frac{\sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}} \right|^p dx \leq \pi^p 2^{1-p}\int_0^\pi \left| \frac{\sin\frac{2n+1}{2}x}{x} \right|^p dx =$$

$$= \pi^p 2^{1-p}\int_0^{\frac{2n+1}{2}\pi} \left|\frac{\sin u}{u}\right|^p du \leq A_p < \infty, \; n = 1, 2, \ldots \qquad (4)$$

2.2.1. Separating the real and the imaginary parts of the equality

$$\sum_{k=0}^{n} e^{i\left(k+\frac{1}{2}\right)x} = \frac{e^{\frac{ix}{2}} - e^{i\left(n+\frac{3}{2}\right)x}}{1 - e^{ix}} = \frac{1 - e^{i(n+1)x}}{e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}},$$

we get

$$\sum_{k=0}^{n} \cos\left(k+\frac{1}{2}\right)x = \frac{\sin(n+1)x}{2\sin\frac{x}{2}}, \qquad (1)$$

$$\sum_{k=0}^{n} \sin\left(k+\frac{1}{2}\right)x = \frac{1-\cos(n+1)x}{2\sin\frac{x}{2}} = \frac{\sin^2\frac{n+1}{2}x}{\sin\frac{x}{2}}. \tag{2}$$

2.2.2. Using 2.2 (1) and 2.2.1 (2), we get

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{n+1-k}{n+1} \cos kx = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x) =$$

$$= \frac{1}{2(n+1)\sin\frac{x}{2}} \sum_{k=0}^{n} \sin\left(k+\frac{1}{2}\right)x =$$

$$= \frac{1}{(n+1)} \frac{\sin^2\frac{n+1}{2}x}{2\sin^2\frac{x}{2}} = F_n(x). \tag{1}$$

Let us note that the function

$$k_\nu(x) = \left(\frac{\sin\frac{\lambda x}{2}}{\sin\frac{x}{2}}\right)^{2\sigma}, \tag{2}$$

where $\lambda$ and $\sigma$ are natural numbers, is a trigonometric polynomial of order $\nu = \sigma(\lambda - 1)$, since it differs from the $\sigma$-th degree of the Fejer kernel $F_{\lambda-1}(x)$ only by the constant multiplier.

In the following it will be useful to estimate the exact order of variation of the variable

$$a_\nu = \int_{-\pi}^{\pi} k_\nu(x)\,dx = 2\int_{0}^{\pi} k_\nu(x)\,dx, \tag{3}$$

when $\nu = 1, 2, \ldots$

If we note that

$$\frac{2}{\pi}\alpha \leqslant \sin\alpha \leqslant \alpha \qquad \left(0 \leqslant \alpha \leqslant \frac{\pi}{2}\right), \tag{4}$$

then we will have

$$2^{2\sigma+1}\int_0^{\pi}\left(\frac{\sin\frac{\lambda x}{2}}{x}\right)^{2\sigma}dx\leqslant a_\nu\leqslant 2\pi^{2\sigma}\int_0^{\pi}\left(\frac{\sin\frac{\lambda x}{2}}{x}\right)^{2\sigma}dx.$$

But

$$\int_0^{\pi}\left(\frac{\sin\frac{\lambda x}{2}}{x}\right)^{2\sigma}dx=\left(\frac{\lambda}{2}\right)^{2\sigma-1}\int_0^{\frac{\lambda\pi}{2}}\left(\frac{\sin t}{t}\right)^{2\sigma}dt\sim\lambda^{2\sigma-1}\ *)$$
$$(\lambda=1,\ 2,\ \ldots).$$

Obviously, thus, for a fixed

$$a_\nu\sim\lambda^{2\sigma-1}\qquad(\lambda=1,\ 2,\ \ldots).\tag{5}$$

Let us further introduce the trigonometric polynomial

$$d_\nu(x)=\frac{1}{a_\nu}k_\nu(x)=\frac{1}{a_\nu}\left(\frac{\sin\frac{\lambda x}{2}}{\sin\frac{x}{2}}\right)^{2\sigma}\qquad(\nu=\sigma(\lambda-1)),\tag{6}$$

where $\sigma > 0$ is a specified integral number, $\lambda = 1, 2, \ldots$ and $a_\nu$ is a constant defined by equality (3).

## 2.3 Interpolational Lagrange Trigonometric Polynomial

If two trigonometric polynomials $T_n(x)$ and $Q_n(x)$ coincide at $2n + 1$ different points of the semiclosed interval $a \leqslant x < a + 2\pi$, their difference, being a polynomial of order n, equals zero at these points, and therefore is identically equal to zero, since a polynomial of n-th order not identically equal to zero can have no less than 2n zeroes in the period.

And thus, the trigonometric polynomial $T_n(x)$ of order n is wholly defined by its values

$$y_0,\ y_n,\ \ldots,\ y_{2n},$$

*) Everywhere in this book we assume that $a_\lambda \sim b_\lambda$ ($\lambda \in \mathcal{E}$), where $\mathcal{E}$ is some set of numbers $\lambda$, if there exists two positive constants $c_1$ and $c_2$ such that for all $\lambda \in \mathcal{E}$ the inequalities $c_1 a \leqslant b_\lambda \leqslant c_2 a_\lambda$ are satisfied.

corresponding to any 2n + 1 different points

$$x_0 < x_1 < \ldots < x_{2n} < x_0 + 2\pi$$

of the period.

It is not difficult to write an effective expression for it.

In fact, based on 2.1.1 the function

$$Q^{(m)}(x) =$$
$$= \frac{\sin\frac{x-x_0}{2}\ldots\sin\frac{x-x_{m-1}}{2}\sin\frac{x-x_{m+1}}{2}\ldots\sin\frac{x-x_{2n}}{2}}{\sin\frac{x_m-x_0}{2}\ldots\sin\frac{x_m-x_{m-1}}{2}\sin\frac{x_m-x_{m+1}}{2}\ldots\sin\frac{x_m-x_{2n}}{2}}$$
$$(m = 0, 1, \ldots, 2n)$$

is a trigonometric polynomial of order n, obviously exhibiting the property

$$Q^{(m)}(x_k) = \begin{cases} 1, & k = m, \\ 0, & k \neq m \end{cases} \qquad (k, m = 0, 1, \ldots, 2m).$$

Therefore, the unknown trigonometric polynomial $T_n(x)$ satisfying the conditions

$$T_n(x_k) = y_k \qquad (k = 0, 1, \ldots, 2n),$$

can be written as

$$T_n(x) = \sum_{m=0}^{2n} Q^{(m)}(x)\, y_m =$$
$$= \sum_{m=0}^{2n} \frac{\sin\frac{x-x_0}{2}\ldots\sin\frac{x-x_{m-1}}{2}\sin\frac{x-x_{m+1}}{2}\ldots\sin\frac{x-x_{2n}}{2}}{\sin\frac{x_m-x_0}{2}\ldots\sin\frac{x_m-x_{m-1}}{2}\sin\frac{x_m-x_{m+1}}{2}\ldots\sin\frac{x_m-x_{2n}}{2}}\, y_m.$$

The case of equidistant interpolation nodes is especially important, i.e., when

$$x_k = \frac{2k\pi}{2n+1} \qquad (k = 0, 1, \ldots, 2n).$$

In this case we can write a simple expression for $Q^{(m)}(x)$ if we note that the trigonometric polynomial

$$D_n(x) = \frac{1}{2} + \cos x + \ldots + \cos nx = \frac{\sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}} \qquad (1)$$

exhibits the properties

$$D_n(0)=\frac{2n+1}{2},\quad D_n(x_k)=0,\quad x_k=\frac{2k\pi}{2n+1}$$
$$(k=1,\ 2,\ \ldots,\ 2n).$$

Hence it follows that the polynomial

$$Q^{(m)}(x)=\frac{2}{2n+1}D_n(x-x_m)\qquad (m=0,\ 1,\ 2,\ \ldots)$$

satisfies the conditions

$$Q^{(m)}(x_k)=\begin{cases}1, & k=m,\\ 0, & k\neq m.\end{cases}$$

Thus, any trigonometric polynomial

$$T_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx) \tag{2}$$

can be written as

$$T_n(x)=\frac{2}{2n+1}\sum_{k=0}^{2n}D_n(x-x_k)\,T_n(x_k)=$$
$$=\frac{1}{2n+1}\sum_{k=0}^{2n}\frac{\sin\frac{2n+1}{2}(x-x_k)}{\sin\frac{x-x_k}{2}}\,T_n(x_k). \tag{3}$$

Substituting in this equality the corresponding sum for $D_n(x)$, we obtain*)

$$T_n(x)=\frac{2}{2n+1}\sum_{k=0}^{2n}\sum_{i=0}^{n}{}'\cos i(x-x_k)\,T_n(x_k)=$$
$$=\frac{2}{2n+1}\sum_{i=0}^{n}{}'\left[\left(\sum_{k=0}^{2n}\cos ix_kT_n(x_k)\right)\cos ix+\right.$$
$$\left.+\left(\sum_{k=0}^{2n}\sin ix_kT_n(x_k)\right)\sin ix\right].$$

*) We assume that $\sum_{k=0}^{n}{}' u_k=\frac{u_0}{2}+\sum_{k=1}^{n}u_k$.

Comparing the coefficients for cos ix and sin ix with the corresponding coefficients $T_n(x)$, we get

$$a_i = \frac{2}{2n+1} \sum_{k=0}^{2n} \cos ix_k T_n(x_k) \qquad (i = 0, 1, \ldots, n),$$

$$b_i = \frac{2}{2n+1} \sum_{k=0}^{2n} \sin ix_k T_n(x_k) \qquad (i = 1, 2, \ldots, n).$$

*) Мы считаем, что $\sum_{k=0}^{n}{}' u_k = \frac{u_0}{2} + \sum_{k=1}^{n} u_k.$

## 2.4 M. Riecz's Interpolational Formula*)

If $T_n(\theta)$ is a trigonometric polynomial

$$T_n(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta), \tag{1}$$

then the identity

$$T_n(\theta) = a_n \cos n\theta + \frac{\cos n\theta}{2n} \sum_{k=1}^{2n} (-1)^k \operatorname{ctg} \frac{\theta - \theta_k}{2} T_n(\theta_k), \tag{2}$$

is valid for it, where

$$\theta_k = \frac{2k-1}{2n}\pi \qquad (k = 1, 2, \ldots, 2n).$$

Let us prove it.

The points $\theta_k$ are zeroes of the polynomial of cos $n\theta$, therefore

$$\cos n\theta = A \prod_{k=1}^{2n} \sin \frac{\theta - \theta_k}{2}. \tag{3}$$

Hence the function

$$Q^{(m)}(\theta) = \frac{\cos n\theta}{2n} (-1)^m \operatorname{ctg} \frac{\theta - \theta_m}{2} = $$

$$= (-1)^{m+1} \frac{\cos n\theta}{2n} \frac{\sin \frac{\theta - (\pi + \theta_m)}{2}}{\sin \frac{\theta - \theta_m}{2}}$$

$$(m = 1, 2, \ldots, 2n)$$

*) M. Riecz [1].

is a trigonometric polynomial or order n, since it is a product of the form (3) in which the multiplier sin $(\theta - \theta_m)/2$ is replaced by the multiplier $\sin \dfrac{\theta - (\pi + \theta_m)}{2}$. This polynomial obviously equals zero at all points $\theta_k$, with the exception of point $\theta_m$, where it equals zero. We can verify the latter by using L'Hospital's rule. Thus.

$$Q^{(m)}(\theta_k) = \begin{cases} 1, & m = k, \\ 0, & m \neq k \end{cases} \quad (k,\ m = 1,\ 2,\ \ldots,\ 2n).$$

*) М. Рисс [1].

Hence it follows that the function

$$T_n^*(\theta) = \frac{\cos n\theta}{2n} \sum_{k=1}^{2n} (-1)^k \operatorname{ctg} \frac{\theta - \theta_k}{2} T_n(\theta_k)$$

is a trigonometric polynomial of order n coinciding with the original polynomial $T_n(\theta)$ at the zeroes of cos $n\theta$. In this case, based on theorem 2.1.1 on the zeroes of a trigonometric polynomial

$$T_n(\theta) = c \cos n\theta + T_n^*(\theta), \tag{4}$$

where c is a constant.

We still have to prove that

$$c = a_n. \tag{5}$$

In fact, the Fourier coefficient of the trigonometric polynomial cos $n\theta$ ctg $(\theta - \theta_k)/2$ corresponding to cos $n\theta$ is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 n\theta \operatorname{ctg} \frac{\theta - \theta_k}{2} d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 n(u + \theta_k) \operatorname{ctg} \frac{u}{2} du =$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 nu \operatorname{ctg} \frac{u}{2} du = 0,$$

since the integrand function in the last integral is odd. In this case the polynomial $Q^{(m)}(\theta)$, and consequently, also the polynomial $T_n^*(\theta)$ do not contain members in cos $n\theta$. Hence (5) follows from (1) and (4).

Identity (2) is proved. If we differentiate it and then set $\theta = 0$, we get

$$T_n'(0) = \frac{1}{4n}\sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{\sin^2\frac{\theta_k}{2}}T_n(\theta_k).$$

This latter equation is valid for any polynomial of order n; in particular, it is valid for the polynomial $T_n(u + \theta)$, where u is a variable and $\theta$ is arbitrarily specified. Thus, for any $\theta$

$$T_n'(\theta) = \frac{1}{4n}\sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{\sin^2\frac{\theta_k}{2}}T_n(\theta+\theta_k). \tag{6}$$

obtains. This then is M. Riecz's formula.

## 2.5 Bernshteyn's Inequality

If we assume in M. Riecz's formula 2.4 (6) that $T_n(\theta) = \sin n\theta$, then when $\theta = 0$ we get

$$n = \frac{1}{4n}\sum_{k=1}^{n}\frac{1}{\sin^2\frac{\theta_k}{2}}. \tag{1}$$

Therefore, from 2.5 (6) follows the inequality

$$\|T_n'\|_{L_p^*} \leqslant n\|T_n\|_{L_p^*} \qquad (1 \leqslant p \leqslant \infty), \tag{2}$$

$$\|f\|_{L_p^*} = \left(\int_0^{2\pi}|f|^p\,d\theta\right)^{1/p},$$

called Bernshteyn's inequality, for any trigonometric polynomial of order n.

It is exact in the sense that there exists a trigonometric polynomial for which it transforms into an equality. Specifically, this occurs for the polynomial

$$T_n(\theta) = A\sin(n\theta + \alpha),$$

where A and $\alpha$ are arbitrary real constants.

## 2.6 Trigonometric Polynomials in Several Variables

A function of the form

$$T_{\nu_1, \ldots, \nu_n}(z_1, \ldots, z_n) = \sum_{\substack{-\nu_l \le k_l \le \nu_l \\ l=1, \ldots, n}} c_{k_1, \ldots, k_n} e^{i(k_1 z_1 + \ldots + k_n z_n)}, \tag{1}$$

where $\nu_1, \ldots, \nu_n$ are natural numbers, $z_1, \ldots, z_n$ are complex variables, and $c_{k_1, \ldots, k_n}$ are constant coefficients that, generally speaking, are complex and dependent on integral $k_1, \ldots, k_n$, is called a trigonometric polynomial of orders $\nu_1, \ldots, \nu_n$, respectively, in the variables $z_1, \ldots, z_n$.

Using the vector notations

$$\nu = (\nu_1, \ldots, \nu_n), \qquad k = (k_1, \ldots, k_n),$$
$$z = (z_1, \ldots, z_n), \qquad kz = \sum_1^n k_l z_l,$$

we will write further

$$T = T_\nu(z) = \sum_{\substack{-\nu_l \le k_l \le \nu_l \\ l=1, \ldots, n}} c_k e^{ikz}$$

and assert that $T = T_\nu$ is a trigonometric polynomial in $z$ of order $\nu$.

If the coefficients satisfy the relationships

$$c_{-k} = \bar{c}_k, \tag{2}$$

i.e., if they vary for them the conjugate numbers when the sign is changed for all subscripts $k_l$, then for the real $z = (z_1, \ldots, z_n)$ the polynomial $T_\nu$ is a real function. In fact if $x = (x_1, \ldots, x_n)$ is a real point, then by (2)

$$\overline{T_\nu(x)} = \sum_{\substack{|k_l| \le \nu_l \\ l=1, \ldots, n}} \bar{c}_k e^{-ikx} = \sum_{\substack{-\nu_l \le -k_l \le \nu_l \\ l=1, \ldots, n}} c_{-k} e^{-ikx} = T_\nu(x).$$

We will mainly have to deal with polynomial satisfying condition (2), which we naturally call real trigonometric polynomials.

For complex $z$, the real polynomials $T_\nu(z)$ are not in general real, but they become real functions if they are considered as functions of real $x = (x_1, \ldots, x_n)$.

We have defined real trigonometric polynomials $T_v$ as linear combinations (1) of complex functions in $e^{ikx}$ with complex coefficients satisfying conditions (2) of conjugativity, but they can also be defined as linear combinations with real coefficients of real functions. Such functions all possible products of the form

$$\varphi_1(x_1)\varphi_2(x_2)\ldots\varphi_n(x_n), \tag{3}$$

where $\varphi_l(x_l)$ $(l = 1, \ldots, n)$ is either a function of $\sin kx_l$ $(1 \leq k \leq v_l)$ or a function of $\cos kx_l (0 \leq k \leq v_l)$.

Conversely, any linear combination of functions of the form (3) with real coefficients is a sum of the form (1) with coefficients satisfying the conjugativity condition (2), i.e., a real trigonometric polynomial of order $\mathbf{v} = (v_1, \ldots, v_n)$.

The trigonometric polynomials $T_v$ are continuous functions periodic in each variable and, therefore, they enter as elements on the space $C^{(n)}_*$ and all the more so on the space $L^{(n)}_{*p}$ (cf 1.1.1).

Different functions of the form (3) satisfy the condition of orthogonality for the rectangle

$$\Delta^{(n)} = \{0 < x_k < 2\pi;\quad k = 1, \ldots, n\}$$

and therefore form a linearly independent system on $C^{(n)}_*$ and on $L^{(n)}_{*p}$ $(1 \leq p \leq \infty)$.

As an example, we note that any real trigonometric polynomial of orders $\mu$ and $\nu$, respectively, in x and y can be written as

$$T_{\mu,\nu}(x, y) = \sum_{k=0}^{\mu}\sum_{l=0}^{\nu}(a_{kl}\cos kx\cos ly +$$
$$+ b_{kl}\cos kx\sin ly + c_{kl}\sin kx\cos ly + d_{kl}\sin kx\sin ly),$$

where $a_{kl}$, $b_{kl}$, $c_{kl}$, and $d_{kl}$ are real coefficients.

If all variables are specified in the polynomial $T_v(z)$, save one, for example, $z_1$, then we obviously get a trigonometric polynomial in the single variable $z_1$ of degree $v_1$, and to it are attributed all the properties of trigonometric polynomials in a single variable.

## 2.7 Trigonometric Polynomials With Respect to Several Variables

Suppose $\mathcal{E} = R_m \times \mathcal{E}' \subset R_n$ is a cylindrical set of points $x = (u, y)$, $u = (x_1, \ldots, x_m) \in R_m$, $y = (x_{m+1}, \ldots, x_n) \in \mathcal{E}'$, where $\mathcal{E}'$ is a measurable (n - m)-dimensional set. Let us separate from $\mathcal{E}$ a truncated cylinder

$$\mathcal{E}_* = \Delta^{(m)} \times \mathcal{E}',$$

where

$$\Delta^{(m)} = \{0 \leqslant x_k \leqslant 2\pi; \quad k = 1, \ldots, m\}$$

is an m-dimensional cube, and let us introduce the space $L_p^*(\mathcal{E})$ of functions $f = f(x)$ (real or complex), belonging to $L_p^*(\mathcal{E}_*)$, and that are for almost all $y \in \mathcal{E}'$ (in the sense of (n - m)-dimensional measure) periodic with period $2\pi$ in each of the variables $x_1, \ldots, x_m$. Obviously, $L_p^*(\mathcal{E})$ is a complete space.

Let us further denote by

$$T_v(x) = T_v(u, y) = T_v(x_1, \ldots, x_m, y)$$

functions such that each of them belong to $L_p^*(\mathcal{E})$ and for almost all $y \in \mathcal{E}'$ in $u = (x_1, \ldots, x_m)$ each is a trigonometric polynomial*) of the order $v = (v_1, \ldots, v_m)$.

The set of all such functions for a given $v$ is denoted by $\mathfrak{M}_{vp}^*(\mathcal{E})$. It obviously is linear.

Each function $T_v \in L_p(\mathcal{E}_*)$, therefore (Fubini's theorem) there exists a set $\mathcal{E}_1' \subset \mathcal{E}'$ of complete measure such that $T_v(u, y) \in L_p(\Delta^{(m)}) \subset L(\Delta^{(m)})$ in $u$ for all $y \in \mathcal{E}_1'$ ($\Delta^{(m)}$ is bounded!). At the same time we can consider that for all $y \in \mathcal{E}_1'$ there exists the representation

$$T_v(u, y) = \sum_{\substack{-v_l \leqslant k_l \leqslant v_l \\ l = 1, \ldots, n}} c_k(y) e^{iku}, \tag{1}$$

where $c_k(y)$ are certain functions dependent on $y$. The equalities

$$c_k(y) = \frac{1}{(2\pi)^m} \int_{\Delta^{(m)}} T_v(u, y) e^{iku} du, \tag{2}$$

*) It must be remembered that a function that is equivalent (relative to $\mathcal{E}$) to the function $T_v(x)$ is considered as equal to $T_v$.

are valid, by virtue of the orthogonal properties of $e^{iku}$, from which it follows, in particular, by Fubini's theorem that $c_k(y)$ are measurable functions on $\mathcal{E}'$, because $T_v$, since it belongs to $L_p(\mathcal{E}_*)$, is in any case locally summable (even if $\mathcal{E}'$ was unbounded). From (2), by using the generalized Minkowski inequality, and then Hölder's inequality, we get

$$\|c_k(y)\|_{L_p(\mathcal{E}')} \leqslant \frac{1}{(2\pi)^m} \int\limits_{\Delta^{(m)}} \|T_v(u, y)\|_{L_p(\mathcal{E}')}\, du \leqslant$$
$$\leqslant \frac{1}{(2\pi)^m} |\Delta^{(m)}|^{1/q} \|T_v\|_{L_p(\mathcal{E}_*)} = c\|T_v\|_{L_p(\mathcal{E}_*)} \left(\frac{1}{p} + \frac{1}{q} = 1\right), \tag{3}$$

where $|\Delta^{(m)}|$ is the (m-dimensional) volume $\Delta^{(m)}$ and c is a constant.

We have proven that each function $T_v \in \mathfrak{M}^*_{vp}(\mathcal{E})$ is representable in the form of (1), where $c_k(y)$ satisfy inequalities (3). The converse, obviously, is also valid.

Using this property of the functions $\mathfrak{M}^*_{vp}(\mathcal{E})$ and the fact that the space $L_p(\mathcal{E}')$ is complete, it is easy to see that the following lemma is valid.

2.7.1. Lemma. The set $\mathfrak{M}^*_{vp}(\mathcal{E})$ is a subspace in $L^*_p(\mathcal{E})$.

If $\mathcal{E} = R_n$, i.e., $\mathcal{E}'$ is empty, then $\mathfrak{M}^*_{vp}(\mathcal{E})$ is obviously a finite-measurable subspace as well. If however $\mathcal{E}'$ has a positive (n - m)-dimensional measure, then $\mathfrak{M}^*_{vp}(\mathcal{E})$ is not finite-measurable.

2.7.2. For the functions

$$T_v = T_v(x_1, y) \in \mathfrak{M}^*_{vp}(\mathcal{E}) = \mathfrak{M}^*_{vp}(R_1 \times \mathcal{E}'),$$

(which are trigonometric polynomials in $x_1$ of degree v) for almost all $y \in \mathcal{E}'$, the generalized Bernshteyn inequality

$$\left\|\frac{\partial T_v}{\partial x_1}\right\|_{L_p(\mathcal{E}_*)} \leqslant v\|T_v\|_{L_p(\mathcal{E}_*)} \tag{1}$$
$$(\mathcal{E}_* = [0, 2\pi] \times \mathcal{E}'; \quad x_1 \in [0, 2\pi], \quad y \in \mathcal{E}').$$

is satisfied.

In fact, $T_v(x_1, y)$ is a trigonometric polynomial in $x_1$ for all $y \in \mathcal{E}'_1$

$\subset \mathcal{E}'$, where $\mathcal{E}_1'$ is a set of complete measure in $\mathcal{E}'$. Therefore, based on 2.5 (2) when $1 \leq p < \infty$

$$\int_0^{2\pi} \left| \frac{\partial T_\nu(x_n, y)}{\partial x_1} \right|^p dx_1 \leqslant \nu^p \int_0^{2\pi} |T_\nu(x_1, y)|^p dx_1 \quad (y \in \mathcal{E}_1'). \tag{2}$$

Integrating both parts of this inequality in $y \in \mathcal{E}_1$ and raising it to the 1/p power, we get (1). When $p = \infty$, inequality (1) obviously derives from the corresponding inequality 2.5 (2).

2.7.3. Cf 3.3 and 3.4 for other inequalities for trigonometric polynomials which we will use extensively.

## CHAPTER III INTEGRAL FUNCTIONS OF THE EXPONENTIAL TYPE, BOUNDED ON $R_n$

### 3.1 Preliminaries

In this chapter we will examine several properties of integral functions of the exponential type, bounded on a real space $R_n = R$. We will see that they are very analogous to the corresponding properties of trigonometric polynomials. At the same time, the trigonometric polynomials are a good means for approximating periodic functions; integral functions of the exponential type can serve as a means of approximating*) nonperiodic functions assigned on an n-dimensional space. It may be that the reader uninitiated in these problems should begin this chapter by reading 3.1.1, where general information from the theory of multiple exponential series are additionally furnished.

Let us assume n nonnegative numbers $v_1, \ldots, v_n$ (not necessarily integral) or a nonnegative vector $v = (v_1, \ldots, v_n)$ 0.

The function

$$g = g_v(z) = g_{v_1, \ldots, v_n}(z_1, \ldots, z_n)$$

is called an exponential type integral function v if for it the following conditions are met:

1) it is an integral function in all variables, i.e., is expanded in the exponential series

$$g(z) = \sum_{k \geq 0} a_k z^k = \sum_{\substack{k_j \geq 0 \\ j=1, \ldots, n}} a_{k_1, \ldots, k_n} z_1^{k_1} \ldots z_n^{k_n} \tag{1}$$

*) Incidentally, while a trigonometric polynomial is defined by a finite number of numerical parameters (coefficients), the exponential type function, generally speaking, is essentially defined by an infinite (countable) number of parameters (for example, the coefficients of its Taylor series), therefore the need of approximating it with a simpler function would appear in practical computations.

(with constant coefficients $a_k = (a k_1, \dots k_n)$, converging absolutely for all complex $z = (z_1, \dots, z_n)$.

2) For any $\varepsilon > 0$ there exists a positive number $A_\varepsilon$ such that for all complex $z_k = x_k + iy_k$ ($k = 1, \dots, n$) the inequality

$$|g(z)| \leqslant A_\varepsilon \exp \sum_{j=1}^{n} (v_j + \varepsilon) |z_j|. \tag{2}$$

We will further assert that this function $g_v$ belongs to the class $E_v$.

Suppose $\rho = (\rho_1, \dots, \rho_n)$ ($\rho_j > 0$; $j = 1, \dots, n$) and let

$$|z_j| \leqslant \rho_j, \quad M(\rho) = \sup_{|z_j| \leqslant \rho_j} |g(z)|.$$

Then from property (2) obviously follows the space

$$M(\rho) \leqslant A_\varepsilon \exp \sum_{j=1}^{n} (v_j + \varepsilon) \rho_j,$$

and conversely, because

$$|g(z)| \leqslant M(|z_1|, \dots, |z_n|) \leqslant A_\varepsilon \exp \sum_{j=1}^{n} (v_j + \varepsilon) |z_j|.$$

A derivative order $k = (k_1, \dots, k_n)$ with respect to $g$ at the point $z = (z_1, \dots, z_n)$ can be written by the Cauchy formula

$$g^{(k)}(z) = \frac{k!}{(2\pi i)^n} \int_{C_1} \dots \int_{C_n} \frac{g(\zeta_1, \dots, \zeta_n)\, d\zeta_1 \dots d\zeta_n}{\prod_{j=1}^{n} (\zeta_j - z_j)^{k_j + 1}}, \tag{3}$$

where $C_j$ is a circle in the plane $\zeta_j$ with its center at $z = 0$. Therefore, if we assume that $z = 0$ and $C_j$ has the radius $\rho_j$, then we get the Cauchy inequality

$$|a_k| \leqslant \frac{M(\rho)}{\rho^k}.$$

Suppose

$$\rho_j = \frac{k_j}{v_j + \varepsilon}.$$

Then

$$|a_k| \leqslant A_\varepsilon \frac{e^{|k|} (v + \varepsilon)^k}{k^k} \qquad (\varepsilon = (\varepsilon, \dots, \varepsilon)). \tag{4}$$

We have proven that it follows from (2) that for any $\varepsilon > 0$, an $A_\varepsilon$ is found such that (4) is satisfied. By Stirling's formula

$$\frac{e^{|k|}}{k^k} = \frac{(\sqrt{2\pi})^n (k_1 \ldots k_n)^{1/2}}{k!} \prod_{j=1}^{n} (1+\varepsilon_{k_j})^{k_j} \quad (\varepsilon_{k_j} \to 0,\ k_j \to \infty),$$

therefore from (4) follows (2) (but generally with another constant $B_\varepsilon$ ):

$$M(\rho) \leqslant \sum_k |a_k| \rho^k \leqslant B_\varepsilon \sum_k \frac{(v+2\varepsilon)^k}{k!} \rho^k = B_\varepsilon e^{\sum_1^n (v_j + 2\varepsilon)\rho_j},$$

where $B_\varepsilon$ is a sufficiently large number dependent on .

From the foregoing it follows that if $g^{(\lambda)} \in E_v$, then any of its partial derivatives $g^{(\lambda)} \in E_v$. The issue is that it follows from (4) that the module of the $(k - \lambda)$-th coefficient of the exponential series $g^{(\lambda)}$ satisfied the inequality

$$\left| \frac{k!}{(k-\lambda)!} a_k \right| \leqslant A'_\varepsilon \frac{e^{|k-\lambda|} (v+2\varepsilon)^{k-\lambda}}{(k-\lambda)^{k-\lambda}},$$

where $A'_\varepsilon$ is sufficiently large.

From the above it follows that for the case of an integral function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

of one variable, the following two conditions, each of which express that f is of the exponential type of degree v, are equivalent:

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r)}{r} \leqslant v \tag{5}$$

and

$$\overline{\lim_{n \to \infty}} \frac{n}{e} \sqrt[n]{|a_n|} = \overline{\lim_{n \to \infty}} \sqrt[n]{n!\,|a_n|} \leqslant v. \tag{6}$$

Let us denote by $\mathfrak{M}_{\nu p}(R) = \mathfrak{M}_{\nu p}(1 \leq p \leq \infty)$ the collection of all integral functions of the exponential type $\nu$, which as functions of a real $x \in R = R_n$ belong to $L_p = L_p(R)$. Let us further suppose that $\mathfrak{M}_\nu = \mathfrak{M}_{\nu\infty}$, i.e., $\mathfrak{M}_\nu$ consists of all functions of the type $\nu$ bounded on R.

Let us here note what will be proved below, that (cf 3.2.5 or 3.3.5) for any p $(1 \leq p \leq \infty)$, $\mathfrak{M}_{\nu p} \subset \mathfrak{M}_\nu$. Moreover, it will be clear (cf 3.2.2 (10), $\mathcal{E} = R_n$, n = m) that for any function there exists a constant not dependent on z, such that

$$|g(z)| < Ae^{\sum_{j=1}^{n} \nu_j |y_j|} \quad (z_j = x_j + iy_j). \tag{7}$$

This inequality is stronger than inequality (2). It follows directly from it that g is bounded on $R_n$. Thus, $\mathfrak{M}_\nu$ can be defined as a class of integral functions f(z) for which (7) obtains.

The functions

$$e^{ikz}, \quad \cos kz = \frac{e^{ikz} + e^{-ikz}}{2}, \quad \sin kz = \frac{e^{ikz} - e^{-ikz}}{2i},$$

where k is a real number, obviously belongs to $\mathfrak{M}_{|k|}(R_1) = \mathfrak{M}_{|k|}$.

The trigonometric polynomial

$$T_\nu(z) = T_{\nu_1, \ldots, \nu_n}(z_1, \ldots, z_n) = \sum_{\substack{|k_j| \leq \nu_j \\ 1 \leq j \leq n}} c_k e^{ikz}$$

belongs to $\mathfrak{M}_\nu(R_n)$, but not to $\mathfrak{M}_{\nu p}$ $(1 \leq p < \infty)$.

The function $\sin z/z$ of a single variable z belongs to $\mathfrak{M}_{1p}(R_1)$ $1 < p \leq \infty$. In fact, as a function of a real x, it obviously belongs to $L_p$ with the stipulated restrictions on p. On the other hand, it is obviously an integral function; further, sin z is an integral function, and it is not difficult to see that for it some constant $c_1$, the inequality is satisfied. Therefore, $|\sin z| \leqslant c_1 e^{|y|}$.

$$\left|\frac{\sin z}{z}\right| \leqslant c_1 e^{|y|} \qquad (|z| \geqslant 1).$$

On the other hand, there exists a positive constant $c_2$ such that

$$\left|\frac{\sin z}{z}\right| \leqslant c_2 \qquad (|z| \leqslant 1).$$

But, since $1 \leq e^{|y|}$, therefore

$$\left|\frac{\sin z}{z}\right| \leqslant c_2 e^{|y|} \qquad (|z| \leqslant 1).$$

Thus,

$$\left|\frac{\sin z}{z}\right| \leqslant c e^{|y|} \qquad \text{obtains for all } z,$$

where

$$c = \max(c_1, c_2).$$

The function of $e^z$ belongs to $E_1(R_1)$, i.e., it is an integral function of the exponential type, but does not belong to $\mathfrak{M}_{1p}(R_1)$ for any p ($1 \leq p \leq \infty$). On the other hand, the function of $e^{iz}$ obviously belongs to $\mathfrak{M}_{1\infty} = \mathfrak{M}_1(R_1)$. The algebraic polynomial $P(z) = \sum_0^n a_k z^k$ is obviously a function of the 0 type, not belonging, however, to $\mathfrak{M}_{0p}(R_1)$ neither for any p ($1 \leq p \leq \infty$). From the following (cf footnote on text page 137 [translation page 125], it will be clear that if $f \in \mathfrak{M}_{0p}(R_1)$, then f is a constant (equal to 0, if $1 \leq p < \infty$).

Obviously, $\mathfrak{M}_{\nu p} \subset \mathfrak{M}_{\nu p'}$ if

If we consider that $g_\nu$ denotes some function of the class $\mathfrak{M}_\nu$, then obviously

where $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_j = \max(\nu_j, \nu_j')$.

$$\prod_{j=1}^{n} g_{\nu_j}(z_j) = g_\nu(z).$$

It is easy to see that if g is an integral function of the unity type of all variables and $\mu_j \neq 0$ ($j = 1, \ldots, n$), then $g(\mu_1 x_1, \ldots, \mu_n x_n)$ is an integral function of the type $|\mu_1|, \ldots, |\mu_n|$. The converse assertion is also valid. Perhaps by using the general properties stated above, other such functions can be constructed from the given integral functions of the exponential type. We here use the operations of addition and multiplication taken at a finite number. The process of integration by parameter (cf 3.6.2) is an important means of constructing integral functions of the exponential type.

3.1.1. Multiple power series. It suffices that all considerations be presented for the example of double series. The arguments are similar for series of higher multiplicity.

Under the sum of the series

$$S = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} u_{kl}, \tag{1}$$

where $u_{kl}$, in general, are complex numbers, subsumes the limit

$$\lim_{m, n \to \infty} \sum_{k=0}^{m} \sum_{l=0}^{n} u_{kl} = S \tag{2}$$

(if it exists), when natural numbers m and n increase unboundedly, independently of each other.

The series (1) converges absolutely, then its members are uniformly bounded, i.e., there exists a constant K such that

$$|u_{kl}| < K \quad (k, l = 0, 1, 2, \ldots).$$

However, if series (1) converges nonabsolutely, then its members are not necessarily uniformly bounded, as shown by the example of the series

$$\sum_{0}^{\infty} \sum_{0}^{\infty} a_{kl}, \tag{3}$$

where $a_{ol} = l!$, $a_{1l} = -l!$, and $a_{kl} = 0$ for the remaining natural k and l.

It converges to a sum equal to zero, but not absolutely and its members are not bounded in the set.

Let us examine the power series

$$f(\eta, \zeta) = \sum_{0}^{\infty} \sum_{0}^{\infty} c_{kl} \eta^k \zeta^l. \tag{4}$$

where $c_{kl}$ are the complex constants and $\eta$ and $\zeta$ are complex variables. Let this series absolutely*) converges at point $\eta_0$ and $\zeta_0$ where $\eta_0 \neq 0$ and $\zeta_0 \neq 0$. Then it also converges absolutely and uniformly for any $\eta$ and $\zeta$ satisfying the inequalities

$$|\eta| < \rho_1 |\eta_0|, \quad |\zeta| < \rho_2 |\zeta_0|, \quad 0 < \rho_1, \rho_2 < 1. \tag{5}$$

In fact, there exists a constant c such that

$$|c_{kl}\eta_0^k\zeta_0^l| < c \qquad (k, l = 0, 1, \ldots),$$

therefore for the specified $\eta$ and $\zeta$

$$|c_{kl}\eta^k\zeta^l| = |c_{kl}\eta_0^k\zeta_0^l| \left|\frac{\eta}{\eta_0}\right|^k \left|\frac{\zeta}{\zeta_0}\right|^l < c\rho_1^k\rho_2^l$$

and, therefore, the members of the series (1) in absolute value thus do not exceed the members of the converging series

$$c\sum\sum \rho_1^k\rho_2^l = \frac{c}{(1-\rho_1)(1-\rho_2)}.$$

Series (1) can be validly differentiated member by member for the indicated $\eta$ and $\zeta$ as many times as desired. Actually, after a single differentiation, for example, with respect to $\eta$, the common member of the resulting series for the indicated $\eta$ and $\zeta$ will satisfy the inequalities

$$|c_{kl}k\eta^{k-1}\zeta^l| = |c_{kl}\eta_0^k\zeta_0^l| \left|\frac{k}{|\eta_0|}\right| \left|\frac{\eta}{\eta_0}\right|^{k-1} \left|\frac{\zeta}{\zeta_0}\right|^l < \frac{ck}{|\eta_0|}\rho_1^{k-1}\rho_2^l.$$

Therefore, the differentiated series converges uniformly in domain (5), since the series

$$\sum\sum k\rho_1^{k-1}\rho_2^l < \infty.$$

converges. From the foregoing it follows that

$$c_{kl} = \frac{1}{k!\,l!}\frac{\partial^{k+l} f(0, 0)}{\partial\eta^k\,\partial\zeta^l}.$$

This, in particular, shows that the expansion of this function in the power series (1) is unique.

*) If we reject the word "absolutely", then this assertion is in general invalid. For example, if in (4) we take the coefficients $a_{kl}$ of series (3) as $c_{kl}$, then series (4) converges when $\eta = \zeta = 1$ and diverges when $\eta = 0$ and for any $\zeta \neq 0$, since it degenerates in this case into the divergent series

$$\sum_0^\infty a_{0l}\zeta^l = \sum_0^\infty l!\,\zeta^l.$$

The function $f(\eta,\zeta)$, representable in the form of an absolutely*) convergent power series (4) in the complex domain defined by the inequalities

$$|\eta|<\rho_1, \quad |\zeta|<\rho_2, \tag{6}$$

is called analytic in this domain.

Let $f(\eta,\zeta)$ be a function that is analytic in the domain (6). Then with specified $\eta\,(|\eta|<\rho_1)$ and arbitrary $\zeta\,(|\zeta|<\rho_2)$, the function

$$f(\eta,\zeta)=\sum_0^\infty\left(\sum_0^\infty c_{kl}\eta^k\right)\zeta^l$$

is expanded into a series convergent in powers of $\zeta$. Therefore, $f(\eta,\zeta)$ is an analytic function of $\zeta$ for $|\zeta|<\rho_2$. Similarly, $f(\eta,\zeta)$ for specified $\zeta\,(|\zeta|<\rho_2)$
is an analytic function of $\eta$ for $|\eta|<\rho_1$. Hence follows the representation of f in the form of the Cauchy integral

$$f(\eta,\zeta)=\frac{1}{2\pi i}\int_{C_1}\frac{f(u,\zeta)}{u-\eta}\,du=\frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2}\frac{f(u,v)\,du\,dv}{(u-\eta)(v-\zeta)}, \tag{7}$$

obtained by successive application of this representation in each of the variables $\eta$ and $\zeta$. Here $C_1$ and $C_2$ are circles in the complex planes $\eta$ and $\zeta$ with centers at the zero points and with radii $r_1<\rho_1$, $r_2<\rho_2$ and $|\zeta|<r_1$ and $|\zeta|<r_2$.

Since the series

$$\frac{1}{(u-\eta)(v-\zeta)}=\sum\sum\frac{\eta^k\zeta^l}{u^{k+1}v^{l+1}}$$

converges uniformly relative to $u\in C_1$, $v\in C_2$, $\eta$, and $\zeta$ satisfying the inequalities

$$|\eta|<r_1'<r_1, \quad |\zeta|<r_2'<r_2,$$

*) Here the word "absolutely" can be omitted, since we can show that from the convergibility of series (4) for all (!) $\eta$, $\zeta$, and with $|\eta|<\rho_1$ and $|\zeta|<\rho_2$ follows its absolute convergence for all specified $\eta$ and $\zeta$.

its substitution in (7) and memberwise integration leads to the original equality (4), from when

$$c_{kl} = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{f(u, v)}{u^{k+1} v^{l+1}} du\, dv. \tag{8}$$

If we had started from the arbitrary function f(u, v) continuous on $C_1$ and $C_2$, the integral (Cauchy type) appearing in the right-hand side of (7) would be equal to some function F(u, v), representable in the form of the series

$$F(u, v) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{kl} \eta^k \zeta^l, \tag{9}$$

absolutely and uniformly convergent for $|\eta| < r_1'$ and $|\zeta| < r_2'$, whatever the $r_1' < r_1$ and $r_2' < r_2$. Thus, the function F is analytic if $|\eta| < r_1$ and $|\zeta| < r_2$.

From the fact that the function f, analytic in the domain (6), is analytic with respect to each variable, follows the formula

$$f(\eta, \zeta) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(\eta + re^{i\theta}, \zeta + \rho e^{i\varphi})\, d\theta\, d\varphi, \tag{10}$$

$$0 < r < \rho_1 - |\eta|, \quad 0 < \rho < \rho_2 - |\zeta|,$$

which is obtained from the corresponding one-dimensional formula.

Let us also note the following property: if the sequence of functions $f_N(\eta, \zeta)$ analytic in domain (6) converges as $N \to \infty$ uniformly on the set

$$|\eta| < r_1 < \rho_1, \quad |\zeta| < r_2 < \rho_2$$

for any specified $r_1$ and $r_2$ to the function $f(\eta, \zeta)$, then the latter is analytic in the domain (6). To be convinced of this, let us substitute $f_N$ in (7) instead of f and make the passage to the limit as $N \to \infty$, then for the limit $f(\eta, \zeta)$, where $|\eta| < r_1$ and $|\zeta| < r_2$, (7) will be satisfied, which shows that it is analytic for $|\eta| < r_1$, $|\zeta| < r_2$ and as a consequence of the arbitrary status of $r_1 < \rho_1$, $r_2 < \rho_2$ is analytic in domain (6).

Let the set of power series $f_n(\eta, \zeta) = \sum_0^{\infty} \sum_0^{\infty} c_{kl}^{(n)} \eta^k \zeta^l \quad (n = 1, 2, \ldots).$

be specified, absolutely convergent for $\eta = \eta_0 \neq 0$ and $\zeta = \zeta_0 \neq 0$ and such that

$$\sum_0^\infty \sum_0^\infty |c_{kl}^{(n)} - c_{kl}^{(m)}| \, |\eta_0|^k |\zeta_0|^l \to 0 \quad n, m \to \infty.$$

Then it is obvious that

$$\lim_{n \to \infty} c_{kl}^{(n)} = c_{kl},$$

where $c_{kl}$ are certain numbers. Here the series

$$f(\eta, \zeta) = \sum_0^\infty \sum_0^\infty c_{kl} \eta^k \zeta^l$$

converges absolutely when $\eta = \eta_0$ and $\zeta = \zeta_0$, and for $|\eta| \leq \eta_0$ and $|\zeta| \leq \zeta_0$

$$|f(\eta, \zeta) - f_n(\eta, \zeta)| \leq \sum_0^\infty \sum_0^\infty |c_{kl} - c_{kl}^{(n)}| \, |\eta_0|^k |\zeta_0|^l \to 0$$

$$(n \to \infty),$$

from which it is clear that the equality

$$\lim_{n \to \infty} f_n(\eta, \zeta) = f(\eta, \zeta)$$

obtains in the domain

$$|\eta| < |\eta_0|, \quad |\zeta| < |\zeta_0|.$$

In this book we will work only with the integral functions

$$f(z) = \sum_{k \geq 0} a_k z^k, \tag{11}$$

i.e., with those functions for which series (11) absolutely converges for any complex $z$.

From the above-noted properties of the analytic functions, it follows that the power series (11) of any integral function converges uniformly on any bounded domain, just as do the series obtained by memberwise differentiation of (11), which is legitimately performed for any of the variables $z_1, \ldots, z_n$, any finite number of times. For specified $z_{m+1}, \ldots, z_m$, the function $f(z_1, \ldots, z_m, z_{m+1}, \ldots, z_n)$ is an integral function with respect to $z_1, \ldots, z_m$.

If function $f(z)$ is integral, then it can be expanded (uniquely) into the series

$$f(z) = \sum_{k \geq 0} c_k (z - z_0)^k$$

in powers of $(z - z_0)^k = (z_1 - z_{01})^{k_1} \ldots (z_n = z_{0n})^{k_n}$, absolutely convergent for all $z$. For example, for the case $n = 2$ this assertion follows from the fact that the formal identities

$$f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} z_1^k z_2^l =$$
$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \sum_{s=0}^{k} C_k^s z_{10}^{k-s} (z_1 - z_{10})^s \sum_{l=0}^{l} C_l^l z_{20}^{l-l} (z_2 - z_{20})^l =$$
$$= \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} c_{\mu\nu} (z_1 - z_{10})^{\mu} (z_2 - z_{20})^{\nu},$$

are essentially legitimate. The last equality is derived after reducing the same number of members with identical powers of $(z_1 = z_{1o})$ $(z_2 - z_{2o})$. To justify this, it suffices to show that its left-hand side is an absolutely convergent multiple series, i.e., that

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} | a_{kl} | \sum_{s=0}^{k} C_k^s x_0^{k-s} (x - x_0)^s \sum_{l=0}^{l} C_l^l y_0^{l-l} (y - y_0)^l < \infty$$
$$(x_0 = | z_1 |, \; y_0 = | z_{20} |, \; x - x_0 = | z_1 - z_{10} |, \; y - y_0 = | z_2 - z_{20} |).$$

But this is because all members of this series are nonnegative and its sum

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} | a_{kl} | x^k y^l < \infty$$

converges by the given condition.

3.1.2. Fourier transforms of class $\mathfrak{M}_{vp}$ functions. From 3.1 we know that an integral function of one variable

$$F(z) = a_0 + \frac{a_1}{1} z + \frac{a_2}{2!} z^2 + \ldots \tag{1}$$

of the type $\sigma > 0$ can be defined as an integral function possessing one of the following properties

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r)}{r} \leq \sigma \tag{2}$$

or

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \leq \sigma. \qquad (3)$$

As a result, it can be asserted that the function F(z) defined by series (1) is of the $\sigma$ type if and only if the series

$$f(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \ldots \qquad (4)$$

converges for $|z| > \sigma$.

The function f(z) is called the Borel transform of the function F(z). Associated with it is the following integral:

$$\int_0^{\infty(\theta)} F(\zeta) e^{-\zeta z} d\zeta = f(z, \theta), \qquad (5)$$

taken along the ray, ($\zeta = \rho e^{-i\theta}$, $0 \leq \rho < \infty$). Namely, it turns out (cf book by N. I. Akhiyezer [1], section 81) that if the integral function F is of the $\sigma$ type, then for it integral (5) uniformly and absolutely converges on any set that is internal with respect to the half-plane $\Delta_\theta$ which does contain the point z = 0 and whose boundary is a tangent to the circle $|z| = \sigma$ at the point $e^{i\theta}$. Here, the identity

$$f(z) = f(z, \theta) \quad (z \in \Delta_\theta)$$

obtains for any (real) $\theta$.

Assume that it is known that the function F(z) is not only an integral exponential type function, but also belongs to $L = L(-\infty, \infty)$ as a function of a real x, in other words, $F \in \mathfrak{M}_{\sigma 1}(R_1) = \mathfrak{M}_{\sigma 1}$. If we insert $\theta = 0, \pi$, and $z = x + iy$ in (5), then we get

$$f(x+iy) = \int_0^{\infty} F(\xi) e^{-\xi(x+iy)} d\xi \qquad (x \geqslant 0), \qquad (6)$$

$$f(x+iy) = -\int_{-\infty}^{0} F(\xi) e^{-\xi(x+iy)} d\xi \qquad (x \leqslant 0). \qquad (7)$$

Notice that based on the general considerations advanced above, we can state only that the integrals (6) and (7) converge for $x > \sigma$ and $x < -\sigma$. However in this case we are considering function $F \in L$. It is at once clear for it that integrals (6) and (7) converge in broader domains (respectively) $x \geqslant 0$ and $x \leqslant 0$.
The integrals obtained from (6) and (7) by formal differentiation with respect to $z = x + iy$ again, obviously, absolutely converge when $x \geqslant 0$ and $x \leq 0$. This shows that integrals (6) and (7) define analytic functions when $x > 0$ and $x < 0$, respectively. They therefore coincide on these indicated domains with the function f(z) — the Borel transform of the function F.

From (6) and (7), it follows for $\varepsilon > 0$ that

$$f(\varepsilon+iy)-f(-\varepsilon+iy)=\int_{-\infty}^{\infty} F(\xi)\,e^{-i\xi y}e^{-\varepsilon|\xi|}\,d\xi,$$

from whence, on passage to the limit as $\varepsilon \to 0$ we get

$$0=\int_{-\infty}^{\infty} F(\xi)\,e^{-i\xi y}\,d\xi \quad (|y|>\sigma),$$

i.e., the Fourier transform of the function $F \in \mathfrak{M}_{\sigma 1}$ is a function (continuous) identically equal to zero outside the segment $[-\sigma, \sigma]$.

If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a positive vector and $F \in \mathfrak{M}_{\sigma 1}(R_n) = \mathfrak{M}_{\sigma 1}$, then $\tilde{F}(x)$ is a function continuous on $R_n$. Since $F(u_1, \ldots, u_n) = F(u)$ is an integral function in $u_1$ of the $\sigma_1$ type, belonging to $L(R_1) = L(-\infty, \infty)$ for almost all $u' = (u_2, \ldots, u_n)$ from the corresponding $(n-1)$-dimensional space, then for such $u'$

$$\int F(u_1, u')\,e^{-ix_1u_1}\,du_1 = 0, \quad |x_1| > \sigma_1,$$

but then $\tilde{F}(x) = 0$ if $|x_1| > \sigma_1$. This argument can be pursued for all $x_j$ ($j = 1, \ldots, n$). We have thus proved the following assertion.

3.1.3. **Theorem.** If $F \in \mathfrak{M}_{\sigma 1}$, then $\tilde{F}(x)$ is a continuous function, equal to zero outside of

$$\Delta_\sigma = \{|x_j| \leqslant \sigma_j, \quad j = 1, \ldots, n\}.$$

3.1.4. It is known, and this consists of the Paley-Wiener theorem*), that if $F \in \mathfrak{M}_{\sigma 2}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$, then the function

$$\tilde{F}(x) = \frac{1}{(2\pi)^{n/2}} \int F(u)\,e^{-ixu}\,du, \tag{1}$$

where the integral is understood in the sense of convergence on the average

$$\left\| \tilde{F}(x) - \frac{1}{(2\pi)^{n/2}} \int_{\Delta_N} f(u)\,e^{-ixu}\,du \right\|_{L_2(\Delta_N)} \to 0 \quad (N \to \infty), \tag{2}$$

$$\Delta_N = \{|x_j| < N; \quad j = 1, \ldots, n\},$$

*) Paley and Wiener [1] for n = 1. For the proof in this case, cf, for example, the book by N. I. Akiyeser [1], and the book by Plansherel' and Fourier [1] when n    1.

not only belongs to $L_2(R_n)$, as follows from (2), but moreover, $\tilde{F}(x) = 0$ almost everywhere outside $\Delta_\sigma$. Conversely, if $\varphi$ is an arbitrary function from $L_2(\Delta_\sigma)$, then the function

$$F(x) = \frac{1}{(2\pi)^{n/2}} \int_{\Delta_\sigma} \varphi(u) e^{ixu} du \tag{3}$$

belongs to $\mathfrak{M}_{\sigma 2}$, and the function $\tilde{F} = \varphi$ defined by (1) equals $u$ almost everywhere.

It is easily verified that if it is assumed that $F(x) \in \mathfrak{M}_{\sigma 2}$ is a generalized function ($F \in S'$), then the function $\tilde{F}(x)$ is the transform $\tilde{F}$ (in the S-sense) and $F = \hat{\tilde{F}}$ (cf 1.5).

Thus, the Fourier transform of the function $F \in \mathfrak{M}_{\sigma 2}$ can be considered as a generalized and ordinary function, and incidentally one belonging to $L_2(\Delta_\sigma)$.

From the following it will be clear that if $1 \leq p \leq 2$ and $F \in \mathfrak{M}_{\sigma p}$, then $F \in \mathfrak{M}_{\sigma 2}$, therefore $\tilde{F}$ has a carrier on $\Delta_\sigma$ and $\tilde{F} \in L_2(\Delta_\sigma)$. But if $2 < p \leq \infty$, then the Fourier transform of the function $F \in \mathfrak{M}_{\sigma p}$ can prove to be an essentially generalized function. For example, $1 \in \mathfrak{M}_{\sigma\infty} = \mathfrak{M}_\sigma$, and $1 = (2\pi)^{n/2} \delta(x)$ (cf 1.5). Therefore, when $p > 2$, the assertion that $\tilde{F}$ has a carrier on $\Delta_\sigma$ can be formulated only in the idiom of generalized functions.

We will assume that the generalized function $\phi \in S'$ has a carrier on $\Delta_\sigma$ if for any fundamental function $\varphi$ ($\varphi \in S$) is such that $\varphi \equiv 0$ on $\Delta_{\sigma+\varepsilon}$, where $\sigma + \varepsilon = \{\sigma_1 + \varepsilon, \ldots, \sigma_n + \varepsilon\}$ then $(\phi, \varphi) = 0$ obtains.

3.1.5. Let us prove the following theorem belonging to L. Schwartz.

Theorem. If $g \in \mathfrak{M}_{\sigma,p}$ $(1 \leq p \leq \infty)$, then $\tilde{g}$ has a carrier on $\Delta_\sigma$.

Proof. We introduce the functions $\varphi_\varepsilon$ (cf 1.5.8) and $\psi_\varepsilon = (2\pi)^{n/2} \tilde{\varphi}_\varepsilon$. Since $\varphi_\varepsilon \in S$, then $\psi_\varepsilon \in S \subset L_q$ $(1/p + 1/q = 1)$, therefore $\psi_\varepsilon g \in L$. Moreover, the function $\varphi_\varepsilon$ is an integral exponential type $\varepsilon$ function, therefore $\psi_\varepsilon g$ is of the exponential type $\sigma + \varepsilon = (\sigma_1 + \varepsilon, \ldots, \sigma_n + \varepsilon)$ and, therefore, $\psi_\varepsilon g \in \mathfrak{M}_{\sigma+\varepsilon,1}$. This means that if $\varphi \in S$ and $\varphi = 0$ on $\Delta_{\sigma+\varepsilon}$, then (cf 3.1.3)

After passage to the limit as $\varepsilon \to 0$, we get (cf 1.5.8 (6)) $\quad (\tilde{g}, \varphi) = 0$.

which was required to be proved.

As for the inverse of this theorem, for our purposes it will be sufficient to know that the Fourier transform of the function (ordinary), equal to zero outside of $\Delta_\sigma$ and belonging to $L_2(\Delta_\sigma)$, is a function of the class $\mathfrak{M}_{\sigma 2}$ based on the Paley-Wiener theorem.

## 3.2. Interpolation Formula

Let (cf Civin [1]) $\omega_\nu(t)$ be a continuous function with $2\nu > 0$ for each of its variables and let $a = (a_1, \ldots, a_n)$ be a vector that the Fourier series

$$e^{iax}\omega_\nu(x) = \sum_k c_k^\nu e^{i\frac{k\pi}{\nu}x} \quad (|x_j| < \nu), \tag{1}$$

$$c_k^\nu = \frac{1}{(2\nu)^n}\int_{\Delta_\nu} \omega_\nu(u) e^{i\left(a-\frac{k\pi}{\nu}\right)u}\, du, \tag{2}$$

Converge absolutely, i.e.,

$$\sum_k |c_k^\nu| < \infty. \tag{3}$$

Let us show that if, moreover, $f, \tilde{f} \in L$ (thus, $f$ and $\tilde{f}$ are continuous and bounded on $R$), then

$$\begin{aligned}\widetilde{\omega_\nu(t)\tilde{f}(t)} &= \lim_{N\to\infty} \sum_{|k_j|<N} \widetilde{c_k^\nu e^{i\left(\frac{k\pi}{\nu}-a\right)t}\tilde{f}(t)} = \\ &= \lim_{N\to\infty} \sum_{|k_j|<N} c_k^\nu f\left(x + \frac{k\pi}{\nu} - a\right) = \sum_k c_k^\nu f\left(x + \frac{k\pi}{\nu} - a\right),\end{aligned} \tag{4}$$

where the series at the right uniformly converges relative to $x \in R$. The first equality in (4) follows from the fact that uniform convergence obtains as $N \to \infty$ of the partial sum

$$\lambda_\nu(x) = \sum_{|k_j|<N} c_k^\nu e^{i\left(\frac{k\pi}{\nu}-a\right)t}$$

to $\omega_\nu(x)$. In fact, considering that $\tilde{f} \in L$, we get

$$|\widehat{\omega f} - \widehat{\lambda_N f}| = |\widehat{(\omega_\nu - \lambda_N) f}| \leqslant$$
$$\leqslant \frac{1}{(2\pi)^{n/2}} \int |\omega_\nu - \lambda_N| \, |f| \, dt \to 0 \quad (N \to \infty).$$

the second equality in (4) follows from formula 1.5 (19). The third is self-evident.

3.2.1. **Theorem.** Let $\Omega(x) = \Omega(x_1, \ldots, x_n)$ be an infinitely differentiable function of polynomial growth, even or odd. If $\Omega(x)$ is even, then we will assume that for any $\nu > 0$ the function

$$\omega_\nu(x) = \Omega(x) \qquad (|x_j| < \nu; \; j = 1, \ldots, n) \tag{1}$$

that is periodic with the period $2\nu$ for each of its variables is expanded in $\Delta_\nu$ in an absolutely convergent Fourier series (3.2.(1) when $a = 0$). If however $\Omega(x)$ is an odd function, then we will assume that the series 3.2(1) when $a = a_\nu = (\pi/2\nu, \ldots, \pi/2\nu)$ converges absolutely.

We will further consider that

$$|c_k^\nu| \leqslant c_k \quad (\nu \leqslant \nu_0) \tag{2}$$

and

$$\sum_k c_k < \infty. \tag{3}$$

Further, let $g(x) \in \mathcal{M}_{\nu p}(R_n) = \mathcal{M}_{\nu p}$.

Then the equality

$$\widehat{\Omega(t)\hat{g}} = \sum_k c_k^\nu g\left(x - a_\nu + \frac{k\pi}{\nu}\right) \tag{4}$$

obtains ($a_\nu = 0$ for even $\Omega$, and $a_\nu = (\pi/2\nu, \ldots, \pi/2\nu)$ for odd $\Omega$), with the series converges in the $L_p$-sense.

It is not difficult to see that if $\Omega(x)$ is an odd function identically not equal to zero, then the periodic function $\omega_\nu(x)$ corresponding to it, generally speaking, is discontinuous and without being multiplied by $e^{iat}$ its Fourier series cannot converge absolutely, as we know.

Proof. Let $\varepsilon_1 > 0$ and let $0 < \varepsilon < \varepsilon_1$. We introduce functions $\varphi_\varepsilon$ and $\psi_\varepsilon$.

In 3.1.5 it was shown that $\psi_\varepsilon g \in \mathcal{M}_{\nu-\varepsilon, 1}$, therefore, $\widehat{\psi_\varepsilon g}$ is a continuous function with carrier on $\Delta_{\nu\varepsilon}$. Thus, $\widetilde{\psi_\varepsilon g} \in L$ and it is legitimate to apply formula 3.2(4) to $\psi_\varepsilon g$.

We have

$$\Lambda_\varepsilon(x) = \widehat{\Omega(t)\widetilde{\psi_\varepsilon g}} = \widehat{\omega_{\nu+\varepsilon_1}\widetilde{\psi_\varepsilon g}} =$$
$$= \sum_k c_k^{\nu+\varepsilon_1}\psi_\varepsilon\left(x - a_{\nu+\varepsilon_1} + \frac{k\pi}{\nu+\varepsilon_1}\right) g\left(x - a_{\nu+\varepsilon_1} + \frac{k\pi}{\nu+\varepsilon_1}\right) \tag{5}$$

($a_{\nu+\varepsilon_1} = 0$ for an even function $\Omega$ and $a_{\nu+\varepsilon_1} = (\pi/2(\nu+\varepsilon_1), \ldots, \pi/2(\nu+\varepsilon_1))$ for an odd function $\Omega$).

Ordinary functions figure everywhere in these relationships; the first equality obtains because $\Omega = \omega_{\nu+\varepsilon}$ on $\Delta_{\nu+\varepsilon}$ -- the carrier of the function $\tilde{\psi}_\varepsilon g$; the second equality is valid by virtue of 3.2(4).

Since $\widetilde{\psi_\varepsilon g} \in L$, then $\psi_\varepsilon g$ is a function bounded on R and series (5) converges uniformly on R to its sum, which we designated by $\Lambda_\varepsilon(x)$. On the other hand,

$$|\psi_\varepsilon(x)| = \left|\int_{\Delta_\varepsilon} \varphi_\varepsilon(t) e^{-ixt}\,dt\right| < \left|\int_{\Delta_\varepsilon} \varphi_\varepsilon\,dt\right| = 1$$

and under the condition $g \in L_p$, therefore series (5) converges to $\Lambda_\varepsilon(x)$ also in the $L_p$-sense. Let us understand the convergence of series (5) in an exactly this manner.

Notice that

$$\psi_\varepsilon(x) = \int_{\Delta_\varepsilon} \varphi_\varepsilon(t) e^{-ixt}\,dt \to 1 \qquad (\varepsilon \to 0),$$

therefore

$$\lim_{\varepsilon\to 0} \Lambda_\varepsilon(x) = \sum_k c_k^{\nu+\varepsilon} g\left(x - a_{\nu+\varepsilon_1} + \frac{k\pi}{\nu+\varepsilon_1}\right) \tag{6}$$

in the $L_p$-sense. In fact,

$$\left\|\Lambda_\varepsilon(x) - \sum_k c_k^{\nu+\varepsilon} g\left(x - a_{\nu+\varepsilon_1} + \frac{k\pi}{\nu+\varepsilon_1}\right)\right\|_{L_p} <$$
$$< \sum_{|k_j|<N} |c_k^{\nu+\varepsilon}| \left\|\left[\psi_\varepsilon\left(x - a_{\nu+\varepsilon_1} + \frac{k\pi}{\nu+\varepsilon_1}\right) - 1\right] \times\right.$$
$$\left.\times\, g\left(x - a_{\nu+\varepsilon_1} + \frac{k\pi}{\nu+\varepsilon_1}\right)\right\|_{L_p} + 2\sum{}' |c_k^{\nu+\varepsilon}| \|g\|_{L_p},$$

where the stroke in the second sum denotes that the sum is extended over all k which did not enter into the first sum. N can always be taken to be large enough that the second sum will be less than a preselected $\eta > 0$, and then $\varepsilon_0$ can be selected to be so small that the first sum will be less than $\eta$ for positive $\varepsilon < \varepsilon_0$, which is possible by the Lebesgue theorem. Both sides of equality (6) actually do not depend on $\varepsilon_1$ — this is clear from (5). From (6), after the passage to the limit as $\varepsilon_1$ --- 0, it finally follows that

$$\lim_{\varepsilon\to 0} \Lambda_\varepsilon(x) = \sum_k c_k^\nu g\left(x - a_\nu + \frac{k\pi}{\nu}\right), \tag{7}$$

where the series on the right converges in the $L_p$-sense, and the limit in the left-hand side, as noted above, is also understood in the $L_p$-sense. In fact, the norm in $L_p$-sense of the difference of the right-hand sides of (6) and (7) does not exceed

$$\sum_{|k_j|<N} |c_k^{\nu+\varepsilon_1} - c_k^\nu| \|g\|_{L_p} + \sum_{|k_j|<N} |c_k^\nu| \left\| g\left(x - a_{\nu+\varepsilon_1} + \frac{k\pi}{\nu+\varepsilon_1}\right) - \right.$$
$$\left. - g\left(x - a_\nu + \frac{k\pi}{\nu}\right)\right\|_{L_p} + \|g\|_{L_p} \cdot 2\sum{}' |c_k|.$$

where equalities (2) and (3) are taken into account, in which we must assume that $\nu_0 = \nu + \varepsilon^0$ ($\varepsilon_1 < \varepsilon^0$). N can here again be taken large enough so that the third sum will be smaller than $\eta$, where N of the first and second sums will for sufficiently small $\varepsilon_1$ also be smaller than $\eta$, because $a_{\nu+\varepsilon_1}$ --- $a_\nu$ and $c_k^{\nu+\varepsilon_1}$ --- $c_k^\nu$ ( $\varepsilon_1$ --- 0)(it is taken into account that $\Omega$ is an infinitely differentiable function, therefore one that is summable on $\Delta_{\nu+\varepsilon_1}$).

Thus, (7) has been proven.

On the other hand (cf (5)),

$$(\Lambda_\varepsilon, \varphi) = (\Omega\widetilde{\psi_\varepsilon g}, \varphi) = (\psi_\varepsilon g, \widetilde{\Omega\varphi}) \xrightarrow[\varepsilon\to 0]{} (g, \widetilde{\Omega\varphi}) = (\widehat{\Omega g}, \varphi),$$

and we have proven the interpolation formula (4).

3.2.2. Interpolation formula for the derivative of an exponential type integral function. This formula will be derived as a particular case of the general formula 3.2.1 (4). Let $g_\nu(x) = g(x) \in \mathfrak{M}_{\nu p}(R_1) = \mathfrak{M}_{\nu p}$, i.e., let there be an integral function of one variable of the three $\nu$ bounded on a real axis. The formula (1.5(10))

$$g'(x) = \widehat{itg}. \tag{1}$$

obtains for its derivative. The function it is infinitely differentiable, odd, and has polynomial growth. Let us examine the function

$$e^{ia|t|}it = \sum_{k=-\infty}^{\infty} c_k^{\nu} e^{\frac{ik\pi}{\nu}t} \quad \left\{|t| < \nu,\ a = \frac{\pi}{2\nu}\right\},$$

$$c_k^{\nu} = \frac{i}{2\nu}\int_{-\nu}^{\nu} ue^{i\left(\frac{\pi}{2\nu}-\frac{k\pi}{\nu}\right)u}\,du =$$

$$= -\frac{1}{\nu}\int_0^{\nu} u\sin\left(k-\frac{1}{2}\right)\frac{\pi}{\nu}u\,du = \frac{\nu(-1)^{k-1}}{\pi^2\left(k-\frac{1}{2}\right)^2}.$$

that is periodic with period $2\nu$. It is clear that

$$|c_k^{\nu}| < |c_k^{\nu_0}| \qquad (0 < \nu < \nu_0)$$

and

$$\sum_k |c_k^{\nu_0}| < \infty.$$

Therefore function it satisfies all the requirements that were imposed on $\Omega(t)$ in 3.2.1. By virtue of 3.2.1(4) the interpolation formula

$$g'(x) = \widehat{itg} = \frac{\nu}{\pi^2}\sum_{-\infty}^{\infty}\frac{(-1)^{k-1}}{\left(k-\frac{1}{2}\right)^2}\, g\left(x+\frac{\pi}{\nu}\left(k-\frac{1}{2}\right)\right). \tag{2}$$

is valid, where the series converges in the $L_p$-sense. It can be considered as the analog of the M. Riesz formula for trigonometric polynomials.

In the following it will be clear that $\mathfrak{M}_{\nu p} \subset \mathfrak{M}_{\nu\infty} = \mathfrak{M}_{\nu}$ and, thus, in fact series (2) converges not only in the $L_p$-sense, but also uniformly.

If we introduce in (2) $g(x) = \sin x \in \mathfrak{M}_{1\infty}$, and then substitute $x = 0$, we get

$$1 = \frac{1}{\pi^2}\sum_{-\infty}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2}. \tag{3}$$

Therefore, for any function $g \in \mathfrak{M}_{\nu p}$ $(1 \leq p \leq \infty)$ the inequality

$$\|g'\|_{L_p} \leq \nu\|g\|_{L_p}. \tag{4}$$

obtains, which is called Bernshteyn's inequality*). It was proven by S. N. Bernshteyn ([1], pp 269 - 270) when $p = \infty$.

If $z = x + iy$ is an arbitrarily complex number, then

$g(z) =$ $g^{(s)}(x)$, whence то $g(z) = \sum_{0}^{\infty} \frac{(iy)^s}{s!} g^{(s)}(x)$, откуда

$$\|g(x+iy)\|_{L_p} \leqslant e^{\nu|y|} \|g\|_{L_p}, \tag{5}$$

where the norm on the left-hand side is taken over $x \in R_1$.

Since the function $g(u + iy)$ for any fixed $y$ with respect to $u$ is again an integral exponential type $\sigma$ type formula, from (5) it follows that if $g(z) \in \mathfrak{M}_{\nu p}$, then $g(u + iy) \in \mathfrak{M}_{\nu p}$ also obtains, but then equality (2) is valid if we replace $x$ in it with the arbitrarily complex $z$:

$$g'(z) = \frac{\nu}{\pi^2} \sum_{-\infty}^{\infty} \frac{(-1)^{k-1}}{\left(k-\frac{1}{2}\right)^2} g\left(z + \frac{\pi}{\nu}\left(k - \frac{1}{2}\right)\right), \tag{6}$$

where the convergence is understood to occur with respect to $x$ ($z = x + iy$) in the $L_p(R_1)$-sense. We have already warned the reader that in the following it will be proven (cf 3.3.5) that $\mathfrak{M}_{\nu p} \subset \mathfrak{M}_{\nu\infty} = \mathfrak{M}_\nu$ from whence it directly follows that series (6) converges uniformly with respect to $x$($z = x + iy$), and by (5) it also easily follows that it converges uniformly on any strip $\{y_1 < y < y_2\}$, where $y_1$ and $y_2$ are arbitrary real numbers.

Let $g(x) = g(x_1, x')$ be a function defined on a measurable set $\mathcal{E} = R_1 \times \mathcal{E}'$ ($x_1 \in R_1$, $x' \in \mathcal{E}'$) belonging to $L_p(\mathcal{E})$, that is integral and of the exponential type $\nu$ with respect to $x_1$ for almost all (in the ($n-1$)-dimensional measure sense) $x' \in \mathcal{E}'$. By virtue of Fubini's theorem, it can be asserted that for the specified $x'$ the function $g(x_1, x') \in \mathfrak{M}_{\nu p}(R_1)$ with respect to $x_1$, and because of (4)

$$\left\|\frac{\partial g}{\partial x_1}\right\|^p_{L_p(R_1)} \leqslant \nu^p \|g\|^p_{L_p(R_1)} \quad (1 \leqslant p < \infty).$$

After integrating both parts of this inequality with respect to $x' \in \mathcal{E}'$ and raising it to the power $1/p$, we get

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*) Inequality (4) is valid also when $\nu = 0$: in fact, from (4) and the fact that $g_0 \in \mathfrak{M}_{0p} \subset \mathfrak{M}_{\nu p}$, follows that $\|g'\|_p = 0$ and $g$ is a constant that is equal to, obviously, zero for finite $p$.

$$\left\| \frac{\partial g}{\partial x_1} \right\|_{L_p(\mathcal{E})} \leq \nu \| g \|_{L_p(\mathcal{E})} \qquad (1 \leq p \leq \infty). \tag{7}$$

We have also assigned the obvious case $p = \infty$.

If the function $g(x) = g(x_1, \ldots, x_n) \in \mathfrak{M}_{\nu p}(R_n)$, then since any of its partial derivatives $g^{(\lambda)}$ is an integral function of the exponential type $\nu$ (cf 3.1), we easily get, based on (7) ($E = R_n$), the inequality

$$\| g^{(\lambda)} \|_{L_p(R_n)} \leq \nu^{\lambda} \| g \|_{L_p(R_n)}. \tag{8}$$

It can be generalized further, by assuming that $g(x) = g(u, x') = g(x_1, \ldots, x_m, x) \in L_p(\mathcal{E})$, $\mathcal{E} = R_m \times \mathcal{E}' \subset R_n$ is a measurable set, $m < n$ and $g$ is an integral exponential type $\nu = (\nu_1, \ldots, \nu_m)$ function for almost all $x' \in E'$ over $x_1, \ldots, x_m$. Then, if $\lambda = (\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)$ we get

$$\| g^{(\lambda)} \|_{L_p(\mathcal{E})} \leq \nu^{\lambda} \| g \|_{L_p(\mathcal{E})}. \tag{9}$$

If it is assumed that almost for all $x'$

$$g(u + iy, x') = g(z) = g(u + iy) = \sum_{\lambda \geq 0} \frac{g^{(\lambda)}(u)}{\lambda!} (iy)^{\lambda},$$

then from (9) we easily obtain the result that

$$\| g(u + iy, x') \|_{L_p(\mathcal{E})} \leq \| g(x) \|_{L_p(\mathcal{E})} e^{\sum_{j=1}^{m} \nu_j |y_j|}. \tag{10}$$

From (2) we can derive the M. Riesz formula as a particular case, proven in 2.4. We need only consider that a trigonometric polynomial of order is an integral function bounded on a real axis $\nu (T_\nu \in \mathfrak{M}_\nu)$, therefore formula (2) is applicable to it. We must consider further that $T_\nu$ is a periodic function with period $2\pi$.

3.2.3. Inequality 3.2.2(4) can be extended for more general norms*). Let $E$ be a Banach space of functions $f(x, w)$ defined and measurable on $\mathcal{E} = R_1 \times \mathcal{E}_1$, exhibiting the following properties:

1) addition of two function $E$ and multiplication of a function by number is defined thusly. Two functions $f_1$ and $f_2$, equal to each other almost everywhere

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*) cf note to 3.2.3 at end of book.

on $\mathcal{E}$, are assumed to be equal ($f_1 = f_2$) as elements of E;

2) if $f = f(x, w) \in E$, then $f_{x_0} = f(x + x_0, w) \in E$ for any real value $x_0$ and $\|f(x, w)\| = \|f(x + x_0, w)\|$;

3) from the fact that $f_n \in E (n = 1, 2, \ldots)$, $f \in E$, $\|f_n - f\| \to 0$, and $f_n(x, w) \to \psi(x, w) (n \to \infty)$ for $x \in R_1$ and for almost all $w \in \mathcal{E}_1$, it follows that $\psi = f$.

If the function $g_\nu(x, w) \in E$ and for almost all w relative to x it is a bounded integral function of the type $\nu$, then for it the equality

$$g'(x, w) = \frac{\nu}{\pi^2} \sum_{-\infty}^{\infty} \frac{(-1)^{k-1}}{\left(k - \frac{1}{2}\right)^2} g\left(x + \frac{\pi}{\nu}\left(k - \frac{1}{2}\right), w\right) \tag{1}$$

obtains for almost all w in the sense of ordinary convergence. On the other hand, as a consequence of property 2) sum of norms of members of series (1) does not exceed $\nu \|g_\nu\|$, and, thus, the right-hand side of series (1) converges according to the norm we are considering to some function $\psi \in E$. But function $\psi$ in the sense of property 3) must be equal to $\partial g_\nu / \partial x$. This is substantiated by the inequality

$$\left\| \frac{\partial g_\nu}{\partial x} \right\| \leqslant \nu \|g_\nu\|. \tag{2}$$

3.2.4. A generalized inequality analogous to 3.2.3(2) can be obtained, based on 2.7.2, also for trigonometric polynomials. To do this, it is sufficient to assume that E consists of functions $f(x, w)$ with period $2\pi$ in x with a norm subject only to properties 1) and 2).

3.2.5. Theorem*) Let $1 < p < \infty$ and the integral function

$$g = g_\nu(z) = g_\nu(x + iy)$$

of the exponential type $\nu = (\nu_1, \ldots, \nu_n) > 0$ belong to class $L_p(R_n)$. Then

$$\lim_{|x| \to \infty} g_\nu(x) = 0. \tag{1}$$

Hence it follows, in particular, that $g(x)$ is bounded on $R_n$.

Proof. It is sufficient to prove the theorem for the case $\nu_1 = \ldots \nu_n = 1$, to which we can reduce our function by replacing it with the following

*) Plansherel' and Polya [1].

$g(z_1/\nu_1, \ldots, z_n/\nu_n)$. Let us limit ourselves to the two-dimensional case, when $n > 2$ the proof is analogous.

And so, let an integral function $g(z_1, z_2) = g$ of the type $(1,1)$ be assumed, belonging to $L_p(R_2)$, where $1 \le p < \infty$. As always, we will assume $x_1$ and $x_2$ to be real.

The inequality (cf 3.1.1)

$$g(x_1, x_2) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} g(x_1 + \rho_1 e^{i\theta_1}, x_2 + \rho_2 e^{i\theta_2})\, d\theta_1\, d\theta_2,$$

obtains where $\rho_1$ and $\rho_2 > 0$. Let us multiply both of its sise by $\rho_1 \rho_2$ and integrate the results over the rectangle $0 \le \rho_1, \rho_2 \le \sigma$. Then we get

$$g(x_1, x_2)\frac{\delta^4}{4} = \frac{1}{(2\pi)^2} \int_\sigma \int_\sigma g(x_1 + \xi_1 + i\eta_1, x_2 + \xi_2 + i\eta_2)\, d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2,$$

where $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$ are Cartesian coordinates and $\sigma$ is a circle with radius $\delta$ with its center at origin of coordinates.

Hence

$$|g(x_1, x_2)| \le \frac{1}{\delta^4 \pi^2} \int_\sigma \int_\sigma |g(x_1 + \xi_1 + i\eta_1, x_2 + \xi_2 + i\eta_2)|\, d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2 \le$$

$$\le \frac{2}{\delta^4 \left(1 - \frac{1}{q}\right)\pi^2} \left( \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} d\eta_1\, d\eta_2 \int_{x_1-\delta}^{x_1+\delta} \int_{x_2-\delta}^{x_2+\delta} |g(\xi + i\eta)|^p\, d\xi_1\, d\xi_2 \right)^{1/p},$$

$$\xi + i\eta = (\xi_1 + i\eta_1, \xi_2 + i\eta_2). \tag{2}$$

Let us prove that the integral

$$I(g) = \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} d\xi_1\, d\xi_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(\xi + i\eta)|^p\, d\eta_1\, d\eta_2$$

is finite, from whence it will follow that the right-hand side of (2) tends to zero as $|x| \to \infty$, and the theorem will be proved.

In fact (cf 3.2.2(10)),

$$I(g)=\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\|g(\xi+i\eta)\|^{p}_{L_p(R_2)}\,d\eta_1\,d\eta_2<$$

$$<\|g(\xi)\|^{p}_{L_p(R_2)}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}e^{(|\eta_1|+|\eta_2|)p}\,d\eta_1\,d\eta_2<\infty.$$

when $p=\infty$, this theorem is invalid, as shown by the example of the function sine $z \in \mathcal{M}_{1,\infty}(R_1)$.

3.2.6. Integral functions of the exponential spherical type. We will state of the integral function

$$g(z)=g(z_1,\ldots,z_n)$$

that it is an exponential spherical type $\sigma\geqslant 0$ formula if for any $\varepsilon>0$ we can specify a constant $A_\varepsilon>0$ such that

$$|g(z)|<A_\varepsilon\exp\left\{(\sigma+\varepsilon)\sqrt{\sum_1^n|z_j|^2}\right\} \tag{1}$$

for all z. The collection of all such functions of the given types $\sigma\geqslant 0$ will be denoted by $SE_\sigma$. Since

$$\frac{1}{\sqrt{n}}\sum_{j=1}^n|z_j|<\sqrt{\sum_{j=1}^n|z_j|^2}<\sum_{j=1}^n|z_j|,$$

then

$$E_{\sigma/\sqrt{n}}\subset SE_\sigma\subset E_\sigma.$$

The set of functions $g\in SE_\sigma$ which as functions of a real vector $x\in R_n$ belong to $L_p(R_n)=L_p$ we will denote with $S\mathcal{M}_{\sigma p}$.

Let $\omega=(\omega_1,\ldots,\omega_n)$ be an arbitrary unit of vector (real). We will let

$$D_\omega f(x)=f'_\omega(x)=\sum_{j=1}^n\frac{\partial f}{\partial x_j}(x)\,\omega_j$$

stand for the derivative with respect to f at point x in the direction $\omega$ and we will let the notation

$$f^{(l)}_\omega(x)=D_\omega f^{(l-1)}_\omega(x)=\sum_{|k|=l}f^{(k)}(x)\,\omega^k\quad(l=1,2,\ldots)$$

be a derivative of order l with respect to f at point x at direction $\omega$.
Let us introduce the transformation

$$x=(x_1,\ldots,x_n)\rightleftarrows(\xi_1,\ldots,\xi_n)=\xi,$$

where $\xi_1, \ldots, \xi_n$ are coordinates of x in the new orthogonal system of coordinates (real), which is selected so that the increment in $\xi_1$ for specified $\xi_2, \ldots, \xi_n$ leads to the translation of the point x in the direction $\omega$. The transformation of coordinates

$$x_k = \sum_{s=1}^{n} a_{ks}\xi_s \qquad (k = 1, \ldots, n) \tag{2}$$

is defined by a real orthogonal matrix. This matrix also defines the transformation

$$z_k = \sum_{s=1}^{n} a_{ks} w_s$$

of the complex systems $w = (w_1, \ldots, w_n)$ into the systems $z = (z_1, \ldots, z_n)$. Here, obviously, the equality

$$\sum_{s=1}^{n} |z_s|^2 = \sum_{s=1}^{n} |w_s|^2.$$

will be satisfied.

Let us assume $g(z) = g(z_1, \ldots, z_n) = g_*(w_1, \ldots, w_n) = g_*(w)$,

and let $g \in S\mathfrak{M}_{\sigma p}$; then $g_* \in S\mathfrak{M}_{\sigma p}$ as well because $g_*$ is obviously an integral function and

$$|g_*(w)| = |g(z)| < A_\varepsilon e^{(\sigma+\varepsilon)\sqrt{\sum_{j=1}^{n}|z_j|^2}} = A_\varepsilon e^{(\sigma+\varepsilon)\sqrt{\sum_{j=1}^{n}|w_j|^2}}.$$

From this inequality it is clear that $g_*$ is a function of the type with respect to $w_1$ and the inequality 3.2.2(4) is applicable to it.

Further

$$g_\omega^{(l)}(x) = \frac{\partial^l}{\partial \xi_1^l} g_*(\xi),$$

therefore $\|g_\omega^{(l)}(x)\|_{L_p} = \left\| \frac{\partial^l g_*(\xi)}{\partial \xi_1^l} \right\|_{L_p} \leqslant \sigma^l \|g_*(\xi)\|_{L_p} = \sigma^l \|g(x)\|_{L_p}.$

(3)

Setting $z = (x_1 + iy_1, \ldots, x_n + iy_n)$, we get

$$g(z) = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{|k|=l} g^{(k)}(x)(iy)^k = \sum_{l=0}^{\infty} \frac{1}{l!} g_{\omega}^{(l)}(x)(i|y|)^l,$$

$$\left(|k| = \sum_{j=1}^{n} k_j, \quad |y|^2 = \sum_{j=1}^{n} |y_j|^2, \quad \omega = \frac{y}{|y|}\right).$$

where, thus, $g^{(l)}$ is a derivative of order l in the direction $\omega = y/|y|$ or $y$.

But then

$$\|g(x+iy)\|_{L_p} \leq \|g(x)\|_{L_p} \sum_{l=0}^{\infty} \frac{(\sigma|y|)^l}{l!} =$$

$$= \|g(x)\|_{L_p} \exp\left(\sigma \sqrt{\sum_{1}^{n} y_j^2}\right). \qquad (4)$$

The Fourier transform $\tilde{g}$ of the function $g \in S\mathcal{M}_{\sigma p}$ has a carrier belonging to the sphere $v_\sigma \subset R_n$ with radius $\sigma$ and its center at the zero point (L. Schwartz [1]).

Actually, if $g \in L_1$, then considering that in an orthogonal transformation of coordinates (2)

$$x \rightleftarrows \xi, \quad u \rightleftarrows v,$$

$xu = \xi v$ and $du = dv$ obtains, and considering that $v' = (v_2, \ldots, v_n)$ we get

$$\tilde{g}(x) = \frac{1}{(2\pi)^{n/2}} \int g(u) e^{-ixu} du = \frac{1}{(2\pi)^{n/2}} \int g_*(v) e^{-i\xi v} dv =$$

$$= \frac{1}{(2\pi)^{n/2}} \int e^{-i\xi' v'} dv' \int g_*(v_1, v') e^{-iv_1\xi_1} dv_1 = \tilde{g}_*(\xi).$$

that $g_*(\xi_1, \xi')$ is of the type $\sigma$ with respect to $\xi_1$ and for almost all $\xi'$ belongs to $L(R_1)$, therefore (cf 3.1.3) $\tilde{g}_*(\xi)$ outside the strip $|\xi_1| < \sigma$ for any choice of coordinates $(\xi_2, \ldots, \xi_n)$, but then $\tilde{g}(x) = 0$ outside the sphere $v_\sigma$.

Our assertion, if $g \in S\mathcal{M}_{\sigma 1}$, is proven. If $g \in S\mathcal{M}_{\sigma p}$, then we introduce the functions $\varphi_\varepsilon(x)$ and $\psi_\varepsilon(x)$ and argue as in 3.1.5.

If f is a generalised function and $e \subset R$ is an open set, then we will write

$$(f)_e = 0, \qquad (5)$$

if $\quad (f, \varphi) = 0$

for all $\varphi \in S$ that have a carrier belonging to e.

Let $0 < \lambda < \sigma$ and v be as before a sphere with its center at the zero point and with radius $\sigma$. Let us show that if $f \in L_p (1 \leq p \leq \infty)$ and

$$(f)_{v_\sigma} = 0,$$

then for the integral function g of spherical power $\lambda$ belonging to L the equality

$$g * f = \frac{1}{(2\pi)^{n/2}} \int g(x-u) f(u)\, du = 0. \qquad (6)$$

obtains.

Let us introduce the functions $\varphi_\varepsilon$ and $\psi_\varepsilon = (2\pi)^{n/2}$ defined in 1.5.8. For $\varphi \in S$ and such an $\varepsilon > 0$ that $\lambda + \varepsilon < \sigma$,

$$(\psi_\varepsilon g * f, \varphi) = (\widetilde{\psi_\varepsilon g} f, \hat{\varphi}) = (f, \widetilde{\psi_\varepsilon g} \hat{\varphi}) = 0, \qquad (7)$$

because g together with its derivatives is a bounded infinitely differentiable function, $\psi_\varepsilon \in S$, $\psi_\varepsilon g \in S$, and $\widetilde{\psi_\varepsilon g} \in S$, therefore, $\widetilde{\psi_\varepsilon g}\, \hat{\varphi} \in S$, and it has a carrier belonging to $v_\sigma$. After passage to the limit in equality (7) as $\varepsilon \to 0$ (cf 1.5.8(7)), we get (6).

## 3.3. Equalities of Different Metrics for Integral Functions of the Exponential Type

In this section we will be interested in classes of integral functions $\mathfrak{M}_{\nu p}(R_n)$.

Here prominence will be given to inequalities of different metrics, by means of which the norm of the function $g_\nu(x)$ in the matrix $L_{p'} = L_{p'}(R_n)$ is estimated in terms of its norm in matrix $L_p (1 \leq p \leq p' \leq \infty)$ and the product of several powers of $\nu_1, \ldots, \nu_n$. This inequality will play a substantial role in the following when we study differentiable functions of more general classes.

Obviously, $\mathfrak{M}_{\nu p}(R_n)$ is a linear set. It is infinite-measurable. For example, the functions $\sin^2 \frac{x}{2k} \Big/ x^2$ ($k = 1, 2, \ldots$) belong to $\mathfrak{M}_{1p}(R_1)$, $1 \leq p \leq \infty$, and exhibit a linearly independent system. Therefore, even from general considerations of function analysis can be concluded that the unit theorem $\mathfrak{M}_{\nu p}(R_n)$ is not compact in the metric $L_p(R_n) = L_p$. However, we will see that it is compact in the weak sense (cf 3.3.6).

3.3.1. Theorem. Let $1 \leq p \leq \infty$, $h > 0$, $x_k = kh$ ($k = 0, \pm 1, \pm 2, \dots$), and $g_\nu = g_\nu(z)$ be an integral function of a single variable of the type $\nu$ and u

$$((g_\nu))_{L_p} = \sup_u \left(h \sum_{-\infty}^{\infty} |g_\nu(x_k - u)|^p\right)^{1/p} < \infty$$

or $\|g_\nu\|_{L_p} < \infty$. Then the inequality

$$\|g_\nu\|_{L_p} \leq ((g_\nu))_{L_p} \leq (1 + h\nu)\|g_\nu\|_{L_p}. \tag{1}$$

obtain.

Proof. When $p = \infty$, the theorem is trivial. Let $1 \leq p < \infty$ and $\|g_\nu\|_{L_p} < \infty$. Then

$$\int_{-\infty}^{\infty} |g_\nu|^p\, dx = \sum_{-\infty}^{\infty} \int_{x_k}^{x_{k+1}} |g_\nu|^p\, dx = h \sum_{-\infty}^{\infty} |g_\nu(\xi_k)|^p,$$

where the numbers $\xi_k$ satisfy the inequalities $x_k < \xi_k < x_{k+1}$. Using the generalized Bernshteyn's inequality, the Hölder inequality, and also the inequality $|\,\|x\| - \|y\|\,| \leq \|x - y\|$, we get

$$\left|\left(h \sum_{-\infty}^{\infty} |g_\nu(\xi_k)|^p\right)^{1/p} - \left(h \sum_{-\infty}^{\infty} |g_\nu(x_k)|^p\right)^{1/p}\right| \leq$$
$$\leq \left(h \sum_{-\infty}^{\infty} |g_\nu(\xi_k) - g_\nu(x_k)|^p\right)^{1/p} \leq \left[h \sum_{-\infty}^{\infty} \left|\int_{x_k}^{x_{k+1}} g'_\nu(t)\, dt\right|^p\right]^{1/p} \leq$$
$$\leq \left(h \sum_{-\infty}^{\infty} \int_{x_k}^{x_{k+1}} |g'_\nu|^p\, dt h^{p/q}\right)^{1/p} = h\|g'_\nu\|_{L_p(R_1)} \leq h\nu\|g_\nu\|_{L_p(R_1)}.$$

Therefore

$$\left(h \sum_{-\infty}^{\infty} |g_\nu(x_k)|^p\right)^{1/p} =$$
$$= \left[\left(h \sum_{-\infty}^{\infty} |g_\nu(x_k)|^p\right)^{1/p} - \left(h \sum_{-\infty}^{\infty} |g_\nu(\xi_k)|^p\right)^{1/p}\right] +$$
$$+ \left(h \sum_{-\infty}^{\infty} |g_\nu(\xi_k)|^p\right)^{1/p} \leq (1 + h\nu)\|g_\nu\|_{L_p(R_1)}. \tag{2}$$

If we note that for any specified u the function $g_\nu(x - u)$, considered as a function of x, is an integral function of the type $\nu$, then from the equality (2), after replacing $g(x)$ with $g_\nu(x - u)$ in it, we get the second inequality of (1).

On the other hand, if $((g))_{L_p} < \infty$, then

$$\int_{-\infty}^{\infty} |g_\nu|^p \, dx = \sum_{-\infty}^{\infty} \int_{x_k}^{x_{k+1}} |g_\nu|^p \, dx = \sum_{-\infty}^{\infty} \int_0^h |g_\nu(x_{k+1} - u)|^p \, du =$$
$$= \int_0^h \sum_{-\infty}^{\infty} |g_\nu(x_{k+1} - u)|^p \, du \leq h \sup_u \sum_{-\infty}^{\infty} |g_\nu(x_k - u)|^p, \tag{3}$$

where the substitution of the order of summation and integration is legitimate by virtue of the fact that we are dealing with nonnegative functions. Thus we have proved the first inequality in (1).

3.3.2. Theorem*) Let $1 \leq p \leq \infty$, $x_k^{(l)} = kh_l$ ($l = 1, \ldots, n$, $k = 0, \pm 1, \pm 2, \ldots$), $g = g$ be an integral function of the type $\nu = (\nu_1, \ldots, \nu_n)$,

$$((g))_p^{(n)} = \sup_{u_l} \Big( \prod_{m=1}^{n} h_m \sum_{l_1=-\infty}^{\infty} \cdots$$
$$\cdots \sum_{l_n=-\infty}^{\infty} |g(x_{l_1}^{(1)} - u_1, \ldots, x_{l_n}^{(n)} - u_n)|^p \Big)^{1/p} < \infty \tag{1}$$

or

$$\|g\|_{L_p(R_n)} = \|g\|_p < \infty.$$

Then

$$\|g_\nu\|_{L_p(R_n)} \leq ((g_\nu))_p^{(n)} \leq \prod_1^n (1 + h_l \nu_l) \|g_\nu\|_{L_p(R_n)}. \tag{2}$$

Proof. When $p = \infty$, inequalities (2) are trivial. Let $1 \leq p < \infty$, then

*) S. M. Nikol'skiy [3].

$$\int |g|^p dx =$$

$$= \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_n=-\infty}^{\infty} \int_0^{h_1} \cdots \int_0^{h_n} |g(x_{l_1}^{(1)} - u_1, \ldots, x_{l_n}^{(n)} - u_n|^p du =$$

$$= \int_0^{h_1} \cdots \int_0^{h_n} \sum \cdots \sum |g(x_{l_1}^{(1)} - u_1, \ldots, x_{l_n}^{(n)} - u_n)| du < ((g))_p^{(n)},$$

and we have proven the first inequality of (2) on the assumption that the second member of (2) is finite. Now let the third member of (2) be finite. To prove the second inequality in (2), we note that $g(z - u) = g(z_1 - u_1, \ldots, z_n - u_n)$ for any specified $u_1$ is an integral function of the type with respect to $z$ for which $\|g(x - u)\|_p = \|g(x)\|_p$. Therefore, it suffices to prove the inequality

$$\left(\prod_{l=1}^{n} h_l \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_n=-\infty}^{\infty} |g(x_{l_1}^{(1)}, \ldots, x_{l_n}^{(n)})|^p\right)^{1/p} <$$

$$< \prod_{l=1}^{n} (1 + h_l \nu_l) \|g\|_p.$$

Body has already been proved in the preceding theorem for the case $n = 1$. Let us assume that its validity has been established for $m = n - 1$. Then by virtue of fact that for any specified $x_1$ in the function $g$ is an integral function of the type $\nu_2, \ldots, \nu_n$, respectively, for $x_2, \ldots, x_n$, we will have

$$\prod_{l=1}^{n} (1 + h_l \nu_l)^p \int \cdots \int |g(x_1, x_2, \ldots, x_n)|^p dx_2 \ldots dx_n \geqslant$$

$$\geqslant \prod_{l=2}^{n} h_l \sum_{l_2=-\infty}^{\infty} \cdots \sum_{l_n=-\infty}^{\infty} |g(x_1, x_{l_2}^{(2)}, \ldots, x_{l_n}^{(n)})|^p,$$

from whence after integration with respect to $x_1$ and raising to the power $p^{-1}$ we get

$$\prod_{i=2}^{n}(1+h_i\nu_i)\|g\|_p \geqslant \left(\prod_{i=2}^{n} h_i\right)^{1/p} \times$$

$$\times\left(\sum_{l_2=-\infty}^{\infty}\cdots\sum_{l_n=-\infty}^{\infty}\left(\int |g(x_1, x_{l_2}^{(2)}, \ldots, x_{l_n}^{(n)})|^p\, dx_1\right)\right)^{1/p} \geqslant$$

$$\geqslant \frac{1}{1+h_1\nu_1}\left(\prod_{i=1}^{n} h_i \sum_{l_1=-\infty}^{\infty}\cdots\sum_{l_n=-\infty}^{\infty} |g(x_{l_1}^{(1)}, \ldots, x_{l_n}^{(n)})|^p\right)^{1/p}.$$

The last inequality holds by virtue of the second inequality 3.3.1(1), since $g$ is an integral function of the type $\nu_1$ with respect to $x_1$.

3.3.3. **Lemma***). For any $a_k \geqslant 0$

$$\left(\sum_1^{\infty} a_k^{p'}\right)^{1/p'} \leqslant \left(\sum_1^{\infty} a_k^{p}\right)^{1/p} \qquad (1 \leqslant p \leqslant p' \leqslant \infty). \tag{1}$$

Proof. It is sufficient to hold that

$$\sum_1^{\infty} a_k^p = 1,$$

then

$$a_k \leqslant 1, \quad \sum_1^{\infty} a_k^{p'} \leqslant \sum_1^{\infty} a_k^p = 1,$$

from whence follows inequality (1) for $1 \leqslant p < p' < \infty$. To get (1) for $p' = \infty$, it is suffice to pass the limit as $p' \to \infty$.

3.3.4. **Theorem.** Under the conditions of theorem 3.3.2, the inequality

$$((g))_{p'}^{(n)} \leqslant \left(\prod_{i=1}^{n} h_i\right)^{\frac{1}{p'}-\frac{1}{p}} ((g))_p^{(n)} \qquad (1 \leqslant p \leqslant p' \leqslant \infty). \tag{1}$$

obtains.

It follows directly from the definition of $((g))_p^{(n)}$ and the preceding lemma.

*) cf Hardy, Littlewood, and Polya [1].

3.3.5. Theorem*) If $1 \le p \le p' \le \infty$, then for an integral function of the exponential type $g = g_\nu \in L_p(R_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, the inequality (of different metrics)

$$\| g_\nu \|_{L_{p'}(R_n)} \le 2^n \left( \prod_1^n \nu_k \right)^{\frac{1}{p} - \frac{1}{p'}} \| g_\nu \|_{L_p(R_n)}. \tag{1}$$

obtains.

For specified n and arbitrary $\nu_k$, this inequality is an exact in the sense of order.

Proof. Based on 3.3.2(2) and 3.3.4, and setting $\omega = 1/p - 1/p'$, we get

$$\begin{aligned} \| g \|_{L_{p'}(R_n)} &\le ((g))_{p'}^{(n)} \le \left( \prod_{l=1}^n h_l \right)^{-\omega} ((g))_p^{(n)} \le \\ &\le \prod_1^n \frac{1 + h_l \nu_l}{h_l^\omega} \| g \|_{L_p(R_n)} = \\ &= \prod_1^n \frac{1 + a_l}{a_l^\omega} \left( \prod_1^n \nu_l \right)^\omega \| g \|_{L_p(R_n)}, \quad (a_l = h_l \nu_l). \end{aligned} \tag{2}$$

Function

$$\psi(a) = \frac{1 + a}{a^\omega}$$

along the semiaxis $0 < \alpha < \infty$ reaches its minimum equal to

$$\lambda_\omega = \frac{1}{\omega^\omega (1 - \omega)^{1 - \omega}} \le 2. \tag{3}$$

Thus, we can write

$$\| g \|_{L_{p'}(R_n)} \le (\lambda_\omega)^n \left( \prod_1^n \nu_l \right)^\omega \| g \|_{L_p(R_n)},$$

whence by (3) follows (1).

---

*) S. M. Nikol'skiy [3], cf notes 3.3 - 3.4.3 at the end of the book.

To prove the second assertion of the theorem, let us examine the function

$$F_\nu = \prod_1^n \frac{\sin^2 \frac{\nu_k z_k}{2}}{z_k^2}, \tag{4}$$

which obviously belongs to $L_p(R_n)$ for any p satisfying the inequalities $1 \leq p \leq \infty$ and which is an integral function of the type $\nu = (\nu_1, \ldots, \nu_n)$. Its norm is

$$\|F_\nu\|_{L_p(R_n)} = \left(2^n \prod_1^n \int_0^\infty \left|\frac{\sin^2 \frac{\nu_k t}{2}}{t^2}\right|^p dt\right)^{\frac{1}{p}} = c_p \left(\prod_1^n \nu_k\right)^{2-\frac{1}{p}}$$

$$(1 \leq p \leq \infty),$$

where $c_p$ is a positive constant not dependent on $\nu_1$. Consequently,

$$\|F_\nu\|_{L_{p'}(R_n)} = \frac{c_{p'}}{c_p}\left(\prod_1^n \nu_k\right)^{\frac{1}{p}-\frac{1}{p'}} \|F_\nu\|_{L_p(R_n)},$$

which was what we set out to prove.

3.3.6. Theorem on compactness*). From any sequence of functions $g_{(k)} \in \mathfrak{M}_{\nu p}(R_n)$ $(1 \leq p \leq \infty,\ k = 1, 2, \ldots)$ bounded on the metric $L_p(R_n)$, we can separate a subsequence $g_{(k_l)}$ $(l = 1, 2, \ldots)$ and define such function $g \in \mathfrak{M}_{\nu p}(R_n)$ that the inequality

$$\lim_{k_l \to \infty} g_{(k_l)}(z) = g(z)$$

obtains, uniformly on any bound set.

Proof. By the given condition there exists the constant $A_1$ such that

$$\|g_{(k)}\|_{L_p(R_n)} \leq A_1, \quad k = 1, 2, \ldots \tag{1}$$

Hence, by 3.3.5(1)

*) $p = \infty$ and $n = 1$, S. N. Bernshteyn [1], pp 269-270.

$$|g_{(k)}(x)| \leqslant 2^n \left(\prod_1^n \nu_s\right)^{\frac{1}{p}} \|g_{(k)}\|_{L_p(R_n)} < A, \qquad (2)$$

where the constant A also does not depend on k.

Let us expand $g_{(k)}(z)$ into a Taylor series:

$$g_{(k)}(z) = \sum_{\alpha \geqslant 0} \frac{c_\alpha^{(k)} z^\alpha}{\alpha!},$$

where $a = (a_1, \ldots, a_n)$ are systems of nonintegral integers and

$$c_\alpha^{(k)} = \frac{\partial^{\alpha_1 + \ldots + \alpha_n} g_{(k)}(0)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

By means of (2) and Bernshteyn's inequality (3.2.2(8))

$$|c_\alpha^{(k)}| \leqslant A \nu^\alpha, \qquad k = 1, 2, \ldots. \qquad (3)$$

Thus, the coefficients $c_a^{(k)}$ $k = 1, 2, \ldots$ are uniformly bounded for any specified system a and, it is possible, by using the diagonal process, to get a subsequence of natural numbers $k_1, k_2, \ldots$ such that

$$\lim_{k_s \to \infty} c_\alpha^{(k_s)} = c_\alpha. \qquad (4)$$

Suppose that

$$g(z) = \sum_{\alpha \geqslant 0} \frac{c_\alpha}{\alpha!} z^\alpha. \qquad (5)$$

then

$$|g(z)| \leqslant A \sum_{\alpha \geqslant 0} \frac{\nu^\alpha |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n}}{\alpha!} = A e^{\sum_1^n \nu_j |z_j|},$$

because the numbers $c_a$ satisfy the inequality

$$|c_\alpha| \leqslant A \nu^\alpha. \qquad (6)$$

Consequently, $g(z)$ is an integral function of the exponential type .

Further, considering that $|\alpha|^2 = \sum_{j=1}^n \alpha_j^2$, we get

$$|g(z)-g_{(k_s)}(z)|\leq\sum_{|\alpha|<N}\frac{|c_\alpha-c_\alpha^{(k_s)}|}{\alpha!}|z_1|^{\alpha_1}\dots|z_n|^{\alpha_n}+$$
$$+\sum_{|\alpha|>N}\frac{|c_\alpha|+|c_\alpha^{(k_s)}|}{\alpha!}|z_1|^{\alpha_1}\dots|z_n|^{\alpha_n}=\sigma_1+\sigma_2.$$

But by (3) and (6) for $\sqrt{\sum_1^n|z_j|^2}<K$

$$\sigma_2\leq 2A\sum_{|\alpha|>N}\frac{|v_1z_1|^{\alpha_1}\dots|v_nz_n|^{\alpha_n}}{\alpha!}<\varepsilon$$

for sufficiently large N. If this sufficiently large N is specified, then by (4) we can specify such an $s_0$ that $|\sigma_1|<\varepsilon$ for all $s>s_0$ and $|z|\leq K$.

Thus,

$$\lim_{k_s\to\infty}g_{k_s}(z)=g(z) \tag{7}$$

is uniform for all $z$ satisfying the inequality $|z|\leq K$, where K is any positive number.

Finally, if $V_\rho\subset R_n$ is a sphere with radius $\rho$ and its center at the origin of coordinates, then by (7) and (1)

$$\|g\|_{L_p(V_\rho)}=\lim_{s\to\infty}\|g_{(k_s)}\|_{L_p(V_\rho)}\leq A_1,$$

from whence after passage to the limit as $\rho\longrightarrow\infty$, we get

$$\|g\|_{L_p(R_n)}\leq A_1$$

and $g\in\mathfrak{M}_{\nu p}(R_n)$.

3.3.7. Example of the application of theorem 3.3.5. Let us assume the numbers $1\leq p_1, p_2, \dots, p_n\leq\infty$ and examine the space $L_{(p_1,\dots,p_n)}(R_n) = L_p(R_n)$ of $R_n$-measurable functions $f(x)=f(x_1,\dots,x_n)$, for which the norm

$$=\left\{\int\left[\dots\left(\int\left(\int|f|^{p_n}dx_n\right)^{\frac{p_{n-1}}{p_n}}dx_{n-1}\right)^{\frac{p_{n-2}}{p_{n-1}}}\dots\right]^{\frac{p_1}{p_2}}dx_1\right\}^{\frac{1}{p_1}}, \tag{1}$$

if finite, where all integral taken from $-\infty$ to $+\infty$.

Here, if $p_{n'} = p_{n'+1} = \dots = p_n = \infty$, then we must assume

$$\left\{\int\left[\dots\left(\left(\int |f|^{p_n}\,dx_n\right)^{\frac{p_{n-1}}{p_n}} dx_{n-1}\right)^{\frac{p_{n-2}}{p_{n-1}}}\right]^{\frac{p_{n'}}{p_{n'+1}}} dx_{n'}\right\}^{\frac{1}{p_{n'}}} = \sup_{x_{n'},\dots,x_n} |f(x_1,\dots,x_{n'-1},x_{n'},\dots,x_n)| \tag{2}$$

for specified arbitrary $x_1, \dots, x_{n'-1}$.

Suppose initially $1 \leq p \leq p_1 \leq p_2 \dots \leq p_n$ and $g_\nu = g_{\nu_1}, \dots, {}_{\nu_n} = g$ is an integral function of exponential type $\nu$ bounded by $R_n$. Considering it as a function only of the variable $x_n$, we can write

$$\left(\int |g|^{p_n}\,dx_n\right)^{\frac{1}{p_n}} \leq 2\nu_n^{\frac{1}{p_{n-1}}-\frac{1}{p_n}}\left(\int |g|^{p_{n-1}}\,dx_n\right)^{\frac{1}{p_{n-1}}},$$

where everywhere we agree to take the integrels within infinite limits $(-\infty, \infty)$.

Hence

$$\left(\int\left(\int |g|^{p_n}\,dx_n\right)^{\frac{p_{n-1}}{p_n}} dx_{n-1}\right)^{\frac{1}{p_{n-1}}} \leq$$

$$\leq 2\nu_n^{\frac{1}{p_{n-1}}-\frac{1}{p_n}}\left(\int\int |g|^{p_{n-1}}\,dx_{n-1}\,dx_n\right)^{\frac{1}{p_{n-1}}} \leq$$

$$\leq 2^{1+2}\nu_n^{\frac{1}{p_{n-1}}-\frac{1}{p_n}}(\nu_{n-1}\nu_n)^{\frac{1}{p_{n-2}}-\frac{1}{p_{n-1}}}\left(\int\int |g|^{p_{n-2}}\,dx_{n-1}\,dx_n\right)^{\frac{1}{p_{n-2}}}.$$

Here the first inequality follows from theorem 3.3.5 when $n = 1$ and $p = p_{n-1}$, and $p' = p_n$, but the second inequality of theorem 3.3.5 holds when $n = 2$ and $p = p_{n-2}$ and $p' = p_{n-1}$. Extending this process to the end, we get the inequality*)

$$\|g_\nu\|_{L_{(p_1,\dots,p_n)}(R_n)} \leq 2^{\frac{n(n+1)}{2}}\nu_n^{\frac{1}{p_{n-1}}-\frac{1}{p_n}}(\nu_{n-1}\nu_n)^{\frac{1}{p_{n-2}}-\frac{1}{p_{n-1}}}\times\dots$$

$$\dots\times(\nu_1\dots\nu_n)^{\frac{1}{p}-\frac{1}{p_1}}\|g_\nu\|_{L_p(R_n)} = 2^{\frac{n(n+1)}{2}}\prod_1^n \nu_k^{\frac{1}{p}-\frac{1}{p_k}}\|g_\nu\|_{L_p(R_n)}. \tag{3}$$

*) S. M. Nikol'skiy [5, 13, 14].

To get this inequality, essentially we used inequality 3.3.5(1) n times in the corresponding particular cases.

In order to prove inequality (3) in the general case $1 \leq p \leq p_1, \ldots, p_n \leq \infty$, it is sufficient to note that

$$\|f\|_{L_{(p_1, \ldots, p_n)}} \leq \|f\|_{L_{(q_1, \ldots, q_n)}}, \tag{4}$$

where $q_1, \ldots, q_n$ is the permutation of the numbers $p_1, \ldots, p_n$ in the non-descending order. Inequality (4) stems from the generalized Minkowski inequality (cf 1.3.2). For example, when $n = 2$ and $p_1 \leq p_2$, we have

$$\left[\int\left(\int |f(x_1, x_2)|^{p_2} dx_2\right)^{\frac{p_1}{p_2}} dx_1\right]^{\frac{1}{p_1}} =$$

$$= \left[\int\left(\int (|f(x_1, x_2)|^{p_1})^{\frac{p_2}{p_1}} dx_2\right)^{\frac{p_1}{p_2}} dx_1\right]^{\frac{1}{p_1}} =$$

$$= \left(\int \||f(x_1, x_2)|^{p_1}\|_{\frac{p_2}{p_1}, x_2} dx_1\right)^{\frac{1}{p_1}} \leq \left\|\int |f(x_1, x_2)|^{p_1} dx_1\right\|^{\frac{1}{p_1}}_{\frac{p_2}{p_1}, x_2} =$$

$$= \left[\int\left(\int |f(x_1, x_2)|^{p_1} dx_1\right)^{\frac{p_2}{p_1}} dx_2\right]^{\frac{1}{p_2}}.$$

Inequality (3) in the order sense is exact, which can be verified for the functions F (cf 3.3.5(4)).

## 3.4. Inequalities of Different Measures for Integral Functions of the Exponential Type

This inequality will also be very significant for the following: using them the norm of an integral function of the exponential type computed for the subspace $R_m \subset R_n$ $(m < n)$ is estimated in terms of its norm computed for the entire space $R_n$. We will subsequently see that inequalities of different measures serve as basis of studying stable boundary properties of differentiable functions.

3.4.1. Let $\mathcal{E} = R_m \times \mathcal{E}' \subset R_n$ be a cylindrical measurable set of points $x = (u, y)$,

$$u=(x_1,\ \dots,\ x_m)\in R_m,$$

$$y=(x_{m+1},\ \dots,\ x_n)\in\mathscr{E}'\subset R_{n-m}$$

and $$\nu=(\nu_1,\ \dots,\ \nu_m).$$

By the definition of the function $g(x)\in\mathfrak{M}_{\nu p}(\mathscr{E})$, if it belongs to $L_p(\mathscr{E})$, and for almost all $y\in\mathscr{E}'$ with respect to $u$ is a function of the exponential type $\nu$.

For the functions $g=g_\nu\in\mathfrak{M}_{\nu p}(\mathscr{E})=\mathfrak{M}_{\nu p}(R_m\times\mathscr{E}')$

the inequality

$$\left\|\,\|g_\nu(u,\ y)\|_{L_{p'}(R_m)}\right\|_{L_p(\mathscr{E}')}\leqslant 2^m\left(\prod_1^m\nu_k\right)^{\frac{1}{p}-\frac{1}{p'}}\|g_\nu\|_{L_p(\mathscr{E})}, \tag{1}$$
$$1\leqslant p\leqslant p'\leqslant\infty,$$

is satisfied, where in the left-hand side the interior norm is computed with respect to the variable $u\in R_m$, and the exterior with respect to the variable $y\in\mathscr{E}'$. In fact, based on the inequality of different measures (3.3.5(1)), which is used for almost all $y\in\mathscr{E}'$

$$\left(2^m\left(\prod_1^m\nu_k\right)^{\frac{1}{p}-\frac{1}{p'}}\|g(u,\ y)\|_{L_p(\mathscr{E})}\right)^p=$$
$$=\int_{\mathscr{E}'}\left(2^m\left(\prod_1^m\nu_k\right)^{\frac{1}{p}-\frac{1}{p'}}\|g(u,\ y)\|_{L_p(R_m)}\right)^p dy\geqslant$$
$$\geqslant\int_{\mathscr{E}'}\|g(u,\ y)\|^p_{L_{p'}(R_m)}\,dy,$$

from whence, by raising both sides of the resulting inequality to the power $1/p$, we get (1).

Let us set $p'=\infty$ in formula (1) and consider that for some set $\mathscr{E}_1'\subset\mathscr{E}'$ of complete measure the following property obtains: for any $y$ $\mathscr{E}_1'$ function $g(u,\ y)$ is of the type $\nu$ with respect to $u$ and the norm

$$\|g(u,\ y)\|_{L_\infty(R_m)}=\sup_{u\in R_m}\operatorname{vrai}|g(u,\ y)|=$$
$$=\lim_{\rho\to\infty}\max_{u\in V_\rho}|g(u,\ y)|\geqslant|g(u,\ y)|\qquad(u\in R_m). \tag{2}$$

is finite, where $V_\rho$ denotes a sphere with its center at the origin of the radius $\rho$, belonging to $R_m$.

Inequality (2) thus is valid for all $y \in \mathcal{E}_1'$ and $u \in R_m$, therefore

$$\|g(u, y)\|_{L_p(\mathcal{E})} < \| \|g(u, y)\|_{L_\infty(R_m)} \|_{L_p(\mathcal{E})},$$

and we get, by taking (1) into account, the following inequality:

$$\|g(u, y)\|_{L_p(\mathcal{E})} < 2^m \left(\prod_1^m \nu_k\right)^{\frac{1}{p}} |g_\cdot|_{L_p(\mathcal{E})}. \qquad (3)$$

3.4.2. Theorem*). If $1 \leq p \leq \infty$ and $1 \leq m < n$, then for any integral function $g(x) = g_{\nu_1, \ldots, \nu_n}(x_1, \ldots, x_n) \in L_p(R_n)$ of the exponential type $\nu$ the inequality (of different measures)

$$\left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |g(u_1, \ldots, u_m, x_{m+1}, \ldots, x_n)|^p \, du_1 \ldots du_m\right)^{\frac{1}{p}} <$$
$$< 2^{n-m} \left(\prod_{m+1}^{n} \nu_k\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |g|^p \, du_1 \ldots du_n\right)^{\frac{1}{p}}. \qquad (1)$$

obtains.

For specified u and m and arbitrary $\nu = (\nu_1, \ldots, \nu_n)$, this inequality is exact in the sense of order.

Proof. The space $R_n$ can be considered as the topological product

$$R_n = R_{n-m} \times R_m,$$

where $(x_1, \ldots, x_m) \in R_m$, $(x_{m+1}, \ldots, x_n) \subset R_{n-m}$. If now we assume in inequality 3.4.1(3) that $\mathcal{E} = R_n$ and $\mathcal{E}_1 = R_m$ and for case $R_m$ with $R_{n-m}$, we get the inequality we seek.

The exactness of inequality (1) in the sense of sense of order relative to $\nu$ can be verified for the functions $F_\nu$ (of 3.3.5(4)), which have

*) S. M. Nikol'skiy [3].

already served a similar purpose in 3.3.5.

**Note.** Setting in 3.3.7(3) $p_1 = \dots = p_n = p$, and $p_{m+1} = \dots = p_n = \infty$, we get the inequality

$$\left(\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \sup_{x_{m+1},\dots,x_n} |g(u_1,\dots,u_m,x_{m+1},\dots,x_n)|^p du_1\dots du_m\right)^{\frac{1}{p}} <$$

$$< 2^{\frac{n(n+1)}{2}}\left(\prod_{m+1}^{n}\nu_k\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}|g(u)|^p du_1\dots du_n\right)^{\frac{1}{p}},$$

which refines inequality (1) in the sense that on the left-hand side instead of $|g(u_1,\dots,u_m,x_{m+1},\dots,x_n)|^p$ appearing in the integral, we have

$$\sup_{x_{m+1},\dots,x_n} |g(u_1,\dots,u_m,x_{m+1},\dots,x_n)|^p.$$

3.4.3. **Inequalities of different metrics and measures for trigonometric polynomials.** There are analogous to the corresponding inequalities for integral functions of the exponential type.

Let $T = T(x) \in \mathfrak{M}^*_\nu(R_n)$, i.e., $T$ is a trigonometric polynomial with respect to n variables, and

$$((T))_p^{(n)} = \max_{u_l}\left(\prod_{l=1}^{n} h_l \sum_{l_1=1}^{N_1}\dots\right.$$
$$\left.\dots\sum_{l_n=1}^{N_n}|T(x_{l_1}^{(1)}-u_1,\dots,x_{l_n}^{(n)}-u_n)|^p\right)^{\frac{1}{p}} \tag{1}$$
$$\left(h_l=\frac{2\pi}{N_l},\ N_l=1,2,\dots;\ l=1,\dots,n,\ 1\le p\le\infty\right).$$

Then the inequalities*)

$$\|T_\nu\|_{L_p^*}^{(n)} \le ((T_\nu))_p^{(n)} \le \prod_1^n(1+h_l\nu_l)\|T_\nu\|_{L_p^*}^{(n)},$$

$$\|T_\nu\|_{L_{p'}^*}^{(n)} \le 3^n\left(\prod_1^n\nu_l\right)^{\frac{1}{p}-\frac{1}{p'}}\|T_\nu\|_{L_p^*}^{(n)}, \tag{2}$$

$$\left(\int_0^{2\pi}\dots\int_0^{2\pi}|T(u_1,\dots,u_m,x_{m+1},\dots,x_n)|^p du_1\dots du_m\right)^{\frac{1}{p}} < \tag{3}$$

*) See note 3.3-3.4.3 at the end of the book.

$$< 3^{n-m}\left(\prod_{m+1}^{n} v_i\right)^{\frac{1}{p}}\left(\int_0^{2\pi}\cdots\int_0^{2\pi}|T|^p\,du_1\ldots du_m\right)^{\frac{1}{p}}. \tag{4}$$

obtain. They are analogous to the corresponding inequalities proved above for integral functions of the exponential type and are similarly proven. Here already in our proof all the sum ($\Sigma$) are extended, as in (1), over a finite number of summands ($N_1, \ldots, N_m$), the integrals are taken over periods,

and we use Bernshteyn's inequality for trigonometric polynomials. However, if we pursue the argument on analogy with what was done for functions of the exponential type, we get a (rounded) constant 3 instead of 2, which is because in the periodic case we have to seek for the minimum of $\psi(\alpha)$ among discrete values of $\alpha$. But, of course, those cases be reduced constants are over-stated.

Analogs of still other inequalities presented in 3.3 can be obtained for trigonometric polynomials.

The exactness of these inequalities in the sense of order is verified in this case for the Fejer kernels (cf 2.2.2).

## 3.5. Subspaces of Functions of a Given Exponential Type

Theorem. The space $\mathfrak{M}_{\nu p}(\mathfrak{E}) = \mathfrak{M}_{\nu p}(R_m \times \mathfrak{E}')$ (cf 3.4.1) is a subspace of the space $L_p(\mathfrak{E})$, i.e., a set linearly closed on $L_p(\mathfrak{E})$.

Proof. The linearity of $\mathfrak{M}_{\nu p}(\mathfrak{E})$ is obvious.

Suppose let the condition

$$\lim_{k, l \to \infty} \|g_k - g_l\|_{L_p(\mathfrak{E})} = 0. \tag{1}$$

is satisfied for the sequence $g_k = g_{\nu k} \in \mathfrak{M}_{\nu p}(\mathfrak{E})$ ($k = 1, 2, \ldots$). Then the exists of function $f \in L_p(\mathfrak{E})$ such that

$$\lim_{k \to \infty} \|f - g_k\|_{L_p(\mathfrak{E})} = 0. \tag{2}$$

Obviously, we can specify this set $\mathcal{G}'_1 \subset \mathcal{G}'$ of complete measure just as for all $k = 1, 2, \ldots$ such that $g_k(u, y)$ will be integral with respect to u and the exponential type $\nu$ for all $y \in \mathcal{G}'_1$. At the same time we can maintain that $\mathcal{G}'_1$ also exhibits the property

$$\lim_{k_s \to \infty} \|f(u, y) - g_{k_s}(u, y)\|_{L_p(R_m)} = 0 \text{ для всех } y \in \mathcal{G}'_1, \qquad \text{for all } y \quad '_1 \tag{3}$$

where $k_s$ is some subsequence of natural numbers that is same for all $y \in \mathcal{G}'_1$ (this falls for (2) on the basis of lemma 1.3.8). Further, from (3), by virtue of inequality 3.2.2(10) ($p = \infty$) and the inequality of different matrics it follows that ($y \in \mathcal{G}'_1$)

$$\begin{aligned} &|g_{k_s}(u + iv, y) - g_{k_{s'}}(u + iv, y)| \leqslant \\ &\leqslant \sup_u |g_{k_s}(u, y) - g_{k_{s'}}(u, y)| e^{\sum_{j=1}^m \nu_j |v_j|} \leqslant \\ &\leqslant 2^m \prod_{j=1}^m \nu_j^{\frac{1}{p}} \|g_{k_s}(u, y) - g_{k_{s'}}(u, y)\|_{L_p(R_m)} e^{\sum_{j=1}^m \nu_j |v_j|} \to 0 \\ &\qquad s, s' \to \infty. \end{aligned} \tag{4}$$

This shows that $g_{k_s}(z, y)$ for any specified $y \in \mathcal{G}'_1$ as $s \to \infty$ uniformly on any bounded set of complex $z$ tends to some function $g(z, y)$ that obviously is integral with respect to $z$. Suppose $\Delta_N = \{|x_j| \leqslant N;\ j = 1, \ldots, n\}$.

From the foregoing it follows that $g_{k_s}(x) \to g(x)$ ($s \to \infty$) almost everywhere on $\mathcal{G}\Delta_N$, and from (2) it then follows that $g(x) = f(x)$ almost everywhere on $\mathcal{G}\Delta_N$, and consequently (by virtue of the arbitrary status of N), also on $\mathcal{G}$.

Finally, from an inequality analogous to (4),

$$|g_{k_s}(z, y)| \leqslant 2^m \prod_{j=1}^m \nu_j^{\frac{1}{p}} \|g_{k_s}(u, y)\|_{L_p(R_m)} \quad (y \in \mathcal{G}'_1),$$

passing to it at the limit as $s \to \infty$, we obtain the same inequality, but now for $g$, which shows that $g$ for any $y \in \mathcal{G}'_1$ is of the exponential type $\nu$ with respect to $u$.

We have proven that the function f appearing in (2) can be modified on a set of n-dimensional measure zero such that for almost all y it will be integral and of the exponential type $\nu$ with respect to u, and since $f \in L_p(R_n)$, then $f \in \mathcal{M}_{\nu p}(\mathcal{E})$. The theorem stands proven.

## 3.6. Convolutions With Integral Functions of the Exponential Type

3.6.1. **Lemma.** Let

$$g(t)=\sum_{0}^{\infty} c_{2k}t^{2k} \tag{1}$$

be an even integral function of one variable of the exponential type $\nu$. Then the function

$$g_*(x)=g(|x|)=\sum_{0}^{\infty} c_{2k}|x|^{2k} \tag{2}$$

is integral and of the spherical type $\nu$.

Proof. Series (1) converges absolutely for any t, and the polynomial

$$|x|^{2k}=(x_1^2+\ldots+x_n^2)^k$$

has positive coefficients. Therefore series (2), after removal of the parentheses, is in each of its members a power series in powers of $x_1, \ldots, x_n$ converging absolutely for any $x = (x_1, \ldots, x_n)$. Consequently, $g_*(z)$ is an integral function. It is an exponential, spherical type $\nu$ function, because

$$|g_*(z)|=\left|g\left(\sqrt{\sum_{1}^{n}|z_j|^2}\right)\right|<A_\varepsilon e^{(\nu+\varepsilon)\sqrt{\sum_{1}^{n}|z_j|^2}}.$$

3.6.2. **Theorem.** Suppose g is an integral function of the exponential type $\nu_j$ with respect to $z_j$ $(j = 1, \ldots, n)$ (or of the spherical type $\nu$), belonging to $L_q(R_n)$, $1 \leq q \leq \infty$, and $f \in L_p(R_n)$ $(1/p + 1/q = 1)$. Then the function

$$\omega(x)=\int g(x-u)f(u)\,du$$

belongs to $\mathcal{M}_{\nu\infty}$ (respectively, to $S\mathcal{M}_{\nu\infty}$), i.e., is an integral exponential type $\nu_j$ function with respect to $z_j$ (correspondingly, of spherical degree $\nu$) bounded on $R = R_n$.

If $g \in L$, and $f \in L_p(1 \leq p \leq \infty)$, then $\omega \in L_p(1.3.3)$, therefore $\omega \in \mathcal{M}_{\nu p}(S\mathcal{M}_{\nu p})$.

Proof. The boundedness of $\omega$ on R follows from the inequality

$$|\omega(x)| \leqslant \|g(x-u)\|_q \|f(u)\|_p = \|g\|_q \|f\|_p. \tag{1}$$

Because $g$ is an integral function, the Taylor series expansion

$$g(z-u) = \sum_{k \geq 0} \frac{g^{(k)}(-u)}{k!} z^k, \tag{2}$$

obtains, absolutely convergent for any $u \in R$ and any complex $z = (z_1, \ldots, z_n)$.

We have

$$\sum_{|k|<N} \left| \frac{g^{(k)}(-u) f(u) z^k}{k!} \right| < \sum_{k \geq 0} \left| \frac{g^{(k)}(-u) f(u) z^k}{k!} \right| = \Phi(u),$$

then $(1/p + 1/q = 1)$

$$\int \Phi(u)\, du < \sum_k \frac{|z^k|}{k!} \|g^{(k)}\|_q \|f\|_p <$$

$$< \|g\|_q \|f\|_p \sum_k \frac{|z^k| \nu^k}{k!} = |g|_q |f|_p e^{\sum_{j=1}^{n} \nu_j |z_j|}.$$

This inequality shows that equality (2) after its multiplication by $f(u)$ can be legitimately (based on the Lebesgue theorem) integrated memberwise:

$$\omega(z) = \int g(z-u) f(u)\, du = \sum_k \frac{c_k}{k!} z^k,$$

$$c_k = \int g^{(k)}(-u) f(u)\, du,$$

and here the inequality

$$|\omega(z)| \leqslant \|g\|_q \|f\|_p e^{\sum_{j=1}^{n} \nu_j |z_j|}$$

obtains.

If $g$ is not only of the type $\nu$ with respect of each of the variables, but also of the spherical type $\nu$, then it can be further proven that $\omega$ is also of the spherical type $\nu$.

In fact, for real x, u, and y

$$g(x+iy-u)=\sum_{k\geq 0}\frac{g^{(k)}(x-u)}{k!}(iy)^k=$$
$$=\sum_{l=0}^{\infty} i^l \sum_{|k|=l}\frac{g^{(k)}(x-u)}{k!}y^k=\sum_{l=0}^{\infty} i^l \frac{g_y^{(l)}(x-u)}{l!}|y|^l,$$

where $g_y^{(1)}$ is a derivative of $g$ of the order 1 in the direction $y$. Therefore, reasoning as in the derivation of (3), we will have

$$\omega(x+iy)=\sum_{l=0}^{\infty}\frac{\int g_y^{(l)}(x-u)f(u)\,du}{l!}(i|y|)^l;$$

considering the inequalities

$$\left|\int g_y^{(l)}(x-u)f(u)\,du\right|\leq |g_y^{(l)}|_q\|f\|_p\leq \nu^l\|g\|_q\|f\|_p,$$

which can be derived based on 3.2.6(3), we get

$$|\omega(x+iy)|\leq\|g\|_q\|f\|_p\sum_{l=0}^{\infty}\frac{(\nu|y|)^l}{l!}=\|g\|_q\|f\|_p e^{\nu|y|}.$$

We have thus proven that $\omega\in\mathfrak{M}_{\nu p}$

3.6.3. Theorem. Let

$$\omega(z)=\int k(|z-u|)f(u)\,du,\quad \int=\int_{R_m}, \tag{1}$$

where

$$k(t)=\left(\frac{\sin\frac{t}{\lambda}}{t}\right)^{\lambda}, \tag{2}$$

is a natural even number satisfying the inequalities

$$0<w-b(n-\gamma) \tag{3}$$

and

$$\left(\frac{1}{p}+\frac{1}{q}=1\right)$$

$$\int |f(u)(1+|u|)^{-\mu}|^p \, du < \infty \quad (\mu > 0). \tag{4}$$

Then (z) is an integral function of the exponential spherical type 1.

Proof. Suppose

$$A=\left(\sum_{j=1}^{m}|z_j|^2\right)^{1/2}.$$

Using Hölder's inequality, we obtain

$$|\omega(z)| \leqslant I(z)\|f(u)(1+|u|)^{-\mu}\|_{L_p(R_m)}, \tag{5}$$

where

$$I(z)=\left(\int |k(|z-u|)|^q(1+|u|)^{q\mu}\,du\right)^{1/q} <$$
$$< \left(\int_{|u|<1}\right)^{1/q}+\left(\int_{|u|<3A}\right)^{1/q}+\left(\int_{|u|>1,\,3A}\right)^{1/q}=I_1+I_2+I_3. \tag{6}$$

Notice that

$$|z-u|^2=\left|\sum z_j^2-2\sum z_ju_j+\sum u_j^2\right|>$$
$$>|u|^2-2A|u|-A^2=(|u|-A)^2-2A^2=\psi(|u|). \tag{7}$$

Functions $(\sin t/\lambda)^\lambda$ and $k(t)$ of the single variable t are integral functions of the type 1, bounded on a real axis, therefore $(\sin |z|/\lambda)^\lambda$ and $k(|z|)$ are integral functions of the spherical type 1 bounded on $R_m$(3.6.1). For this case, considering that $u \in R_m$ are real points and $z = (x_1 + iy_1,\ldots, x_m + iy_m)$, and that (3.6.1) and 3.2.6(4) obtain, we have

$$k(|z-u|) \ll \exp|y| \leqslant \exp A, \tag{8}$$

$$\left(\sin\frac{|z-u|}{\lambda}\right)^\lambda \ll \exp|y| \leqslant \exp A. \tag{9}$$

Therefore, by (8)

$$I_1 \ll \exp A \left( \int_{|u|<1} (1+|u|)^{q\mu} du \right)^{1/q} \ll \exp A, \tag{10}$$

$$I_2 \ll \exp A \left( \int_{|u|<3A} (1+|u|)^{q\mu} du \right)^{1/q} \ll \exp\{(1+\varepsilon) A\}, \tag{11}$$

where $\varepsilon > 0$ is an arbitrary small number, the constant in the second inequality (11) depends on $\varepsilon$, and by (7) and (9) (explanations below)

$$I_3 \ll \exp A \left( \int_{\rho > 1, 3A} \frac{\rho^{\mu q + m - 1}}{\psi(\rho)^{\frac{\lambda q}{2}}} d\rho \right)^{1/q} \ll \exp A \quad (\rho = |u|). \tag{12}$$

The function

$$\frac{\rho^2}{\psi(\rho)} = \varphi(\rho) \qquad (\rho \geqslant 3A)$$

is bounded, because it is positive and its derivative is negative. Consequently, assuming

$$\nu = \frac{1}{2}(\lambda q - \mu q - m + 1)$$

and $\rho = tA$, we get the result at the integral appearing in (12) can, with an accuracy to constant multiplier, not surpass

$$\int_{\rho > 3A, 1} \frac{d\rho}{\psi(\rho)^{\nu}} = \frac{1}{A^{2\nu-1}} \int_{t > 3, \frac{1}{A}} \frac{dt}{\psi(t)^{\nu}} \ll$$

$$\ll \frac{1}{A^{2\nu-1}} \int_{t>3, \frac{1}{A}} \frac{dt}{t^{2\nu}} \ll \begin{cases} \int_3^{\infty} t^{-2\nu} dt \ll 1 & (A \geqslant 1), \\ \frac{1}{A^{2\nu-1}} A^{2\nu-1} = 1 & (A < 1), \end{cases}$$

i.e., that the second inequality (12) is valid. From the estimate obtained it follows that for any $\varepsilon > 0$ there exists a constant $c_\varepsilon$ not dependent on f, such that

$$|\omega(z)| \leqslant c_\varepsilon \| f(u)(1+|u|)^{-\mu} \|_{L_p(R_m)} \exp\{(1+\varepsilon) A\}. \tag{13}$$

It remains to prove that $\omega(z)$ is an integral function. Suppose $f_N = f$ for $|u| < N$ and $f_N = 0$ for $|u| > N$ and

$$\omega_N(z) = \int k(|z-u|) f_N(u)\,du.$$

Let us assign an arbitrary number $\Lambda > 0$ and let $\mathcal{E}_\Lambda$ be the set of points $z = (z_1, \ldots, z_m)$ for which $|z_j| < \Lambda$. For such points

$$A = \sqrt{\sum_1^m |z_j|^2} < m\Lambda$$

and by (13)

$$|\omega(z) - \omega_N(z)| < c_\varepsilon m\Lambda \| (f - f_N)(1+|u|)^{-\mu} \|_{L_p(R_m)} \to 0, \quad N \to \infty$$

i.e., $\omega_N(z) \to \omega(z)$ and $N \to \infty$ uniformly on any $\mathcal{E}_\Lambda$, and therefore, $\omega(z)$ is an integral (3.1.1). The assertion has been proven (cf further 4.2.2).

## CHAPTER IV FUNCTION CLASSES W, H, AND B

### 4.1. Generalized Derivative

Let us assign in the space $R = R_n$ the open set $g$ and let $g_1$ stand for its orthogonal projection on the hyperplane $x_1 = 0$. Let the real (complex) measurable function $f(x) = f(x_1, y)$ be given on $g$. For a specified $y$ it is a function of $x_1$ determined on the corresponding open single-dimensional set. This function f is absolutely continuous on any close finite segment belonging to this set, then we will state that it is locally absolutely continuous with respect to $x_1$ for a specified $y$.

By definition, function f has a generalized derivative $\partial f/\partial x_1$ (with respect to $x_1$), if f is measurable on $g$ and if there exists a function $f_1$ equivalent to it (relative to $g$) and locally absolutely continuous for almost all admissible $y$ (i. e., $y \in g_1$). The function $f_1$ will have almost everywhere on $g$ (in the sense of the n-dimensional measure) for ordinary partial derivative $\partial f_1/\partial x_1$. We will then call any function equivalent to it (in the sense of the n-dimensional measure) the generalized derivative of f on $g$ with respect to $x_1$, and refer to it with $\partial f/\partial x_1$.

If $\varphi(t)$ is a function of the single variable t and $\Omega$ is an open set of points t, then the fact that $\varphi$ has on $\Omega$ the generalized derivative $\varphi'(t)$ can be expressed thusly: there exists a function $\varphi_1$ equivalent to $\varphi$ (with respect to $\Omega$), and locally absolutely continuous on $\Omega$. Then $\varphi_1$ has, as we know, almost everywhere on $\Omega$ the ordinary derivative $\varphi_1'(t)$. Any function equivalent to $\varphi'_1(t)$ is therefore by definition the generalized derivative $\varphi'(t)$ on $\Omega$.

In order that there be no confusion, let us explain in greater detail why under the specified conditions the ordinary partial derivative $\partial f_1/\partial x_1$ exists almost everywhere on $g$.

The projection $g_1$ of the open set $g$ on the subspace of points $y = (0, x_2, \dots, x_n)$ is obviously also an open set. To each specified point

$y \in g_1$ there corresponds a one-dimensional open (in the one-dimensional sense) set $e_y$ of points of the form $(x_1, y) \in g$. The set $g$ can be regarded as the theoretic-set sum

$$g = \bigcup_{y \in g_1} e_y.$$

Under the condition, the function $f_1(x_1, y)$ for almost all $y \in g_1$ is absolutely continuous on $x_1$ for each closed segment of variation of $x_1$ belonging to $e_y$. Hence it follows that for almost all points $y \in g_1$, the function $f_1(x_1, y)$ has for almost all $x_1 \in e_y$ the ordinary partial derivative $\partial f_1/\partial x_1 = f_{1x_1}$. Let $g'$ stand for the set of all points $x = (x_1, y) \in g$ for which the partial derivative $f_{1x_1}$ does not exist. The set $g'$ is measurable, since it is complementary to the set of all points $x \in g$ for which there exists the limit of the relationship

$$\lim_{h \to 0} \frac{f_1(x_1 + h, y) - f_1(x_1, y)}{h} = f_{1x_1}(x_1, y),$$

which is a measurable function for each h (f is measurable on $g$ according to the given condition!). We must bear in mind that the set of points of the convergence of the sequence of measurable functions on the (measurable) set $\mathscr{E}$ is measurable[1]).

On the other hand,

$$g' = \bigcup_{y \in g_1} e'_y,$$

where for almost all $y \in g_1$ in the sense of the $(n - 1)$-dimensional measure, $\mu e'_y = 0$. Hence (by Fubini's theorem) $g'$ has the n-dimensional measure $\mu g' = 0$, and thus, the function $f_1$ has almost everywhere on $g$ the ordinary partial derivative $\partial f_1/\partial x_1$, which we called the generalized partial derivative of f with respect to $x_1$.

The function $\partial f/\partial x_1$ (measurable on the open[2]) set $g$) can in turn have a generalized partial derivative with respect to $x_1$, i.e., it can be that there exists a function equivalent to it (in the sense of n-dimensional measure),

---

1) Let $F_k$ be a sequence of measurable functions given on the measurable set $\mathscr{E}$, $e_{nm} = \{x : | F_k(x) - F_l(x) | < 1/m \}$ for any k and $l \geqslant n$; n and m = 1, 2, ... Then $A = \bigcap_m \bigcup_n e_{nm}$ is a set of points of convergence $\{F_k\}$, which is obviously measurable.
2) on following page.

defined on $g$, and absolutely continuous with respect to $x_1$ for any closed segment of variation of $x_1$ for almost all $y \in g_1$. The ordinary derivative with respect to $x_1$ of $\varphi$, existing almost everywhere, or the function equivalent to it is denoted by $\partial^2 f/\partial x_1^2$. Similarly, $\partial^k f/\partial x_1^k = f_{x_1}^{(k)}$ ($k = 0, 1, 2, \ldots$, $f_{x_1}^{(0)} = f$) is defined by induction. It is not difficult to see that if there exists on $g$ the generalized derivative $\partial^k f/\partial x^k$, then the function $f$ can always be brought into correspondence with the function $_2$ equivalent to it and defined on $g$ such that the derivative $\partial\varphi/\partial x_1, \partial^2\varphi/\partial x_1^2, \ldots, \partial^k\varphi/\partial x_1^k$ exists in the ordinary sense almost everywhere on $g$ and here $\partial^i\varphi/\partial x_1^i$ ($i = 0, 1, \ldots, k-1$) are absolutely continuous with respect to $x_1$ on any closed segment of variation of $x_1$ for all $y$ belonging with the same set $g_1' \subset g_1$, distinct from $g_1$ on a set of zero measure.

The derivatives $^k f/\ x_i^k$ ($i = 2, \ldots, n$) generalised on $g$ are similarly defined. Mixed derivatives of the second and higher orders are defined inductively. For example, the derivative $\partial^2 f/\partial x_1 \partial x_2$ is defined by the equality

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_2}.$$

Obviously, the fact that the function $f(x)$ of one variable has a generalized derivative of the order $k$ on $(a, b)$ reduces to the fact that it (after variation on the zero-measure set) has ordinary derivatives up to the order $k-1$ inclusively, that are absolutely continuous on any closed segment $[c, d]$ $(a, b)$, which further entail the existence of the derivative $f^k(x)$ the order of $k$ almost everywhere on the interval $(a, b)$.

2) In the definition of $\partial f/\partial x_1$, we can take instead of the set $g \subset R_n$ by this open on $R_n$, the measurable set $\mathcal{E} \subset R_n$, which is open with respect to the variable $x_1$. More precisely, we can take the measurable set $\quad R_n$, whose projection $_1$ on the subspace of points $y = (0, x_2, \ldots, x_n)$ is measurable in the $(n - 1)$-dimensional sense, such that

$$\mathcal{E} = \bigcup_{y \in \mathcal{E}_1} e_y$$

where $e_y$ are open one-dimensional sets of points of the form $(x_1, y)$ with the variable number $x_1$. In particular, the measurable set of the form $\mathcal{E} = R_1 \times \mathcal{E}_1 \subset R_n$, where $\mathcal{E}_1$ is a set measurable in the sense of the $(n - 1)$-dimensional measure is such a set.

Throughout this book we will be dealing with generalised derivatives and therefore we will often call them derivatives without adding the word "generalised".

Though the definition of the generalized derivative given above is extremely general, even as it presently stands as it is quite effective in applications for integration by parts, let function f have on g the generalized derivative $\partial f/\partial x_1$. Here we will consider f to be already modified on the set of n-dimensional zero measure, as it is required by definition. Suppose, moreover, that $\varphi(x)$ is a function that is continuous on g together with its derivative $\frac{\partial \varphi}{\partial x}$. Then, almost all $y = (x_2, \ldots, x_n)$, whatever be the segment $[a, b] \times y$ belonging to g, integration by parts is legitimate:

$$\int_a^b f(x_1, y) \frac{\partial \varphi}{\partial x_1}(x_1, y)\, dx_1 = f(b, y)\varphi(b, y) -$$

$$- f(a, y)\varphi(a, y) - \int_a^b \frac{\partial f}{\partial x_1}(x_1, y)\varphi(x_1, y)\, dx. \qquad (1)$$

Often it becomes necessary that this expression be integrated with respect to y, but for this the measurability of $f(x) = f(x_1, y)$ on g is insufficient, since auxiliary conditions on f are necessary. Summability for local summability of f and $\partial f/\partial x_1$ or of only $\partial f/\partial x_1$ on g can be these effective conditions.

We find the concept of the generalised derivative in the works of Beppo Levi [1], who considered generalized derivatives with an integrable square on g. Subsequently many mathematicians came to this concept, often independently of their predecessors.

S. L. Sobolev [1, 2] arrived at the definition of the generalized derivative from the viewpoint of the concept of the generalised function that he introduced. Sobolev's definition consists of the following. Suppose f and $\lambda$ are functions locally summable on the open set g. If for any infinitely differentiable function $\varphi$ finite on g, the equality

$$\int \lambda\varphi\, dx = (-1)^{|s|} \int f\varphi^{(s)}\, dx,$$

is fulfilled, then $\lambda$ is the generalised derivative $f^{(s)}$ of f.

If the function f is locally summable on g together with its derivative $\partial f/\partial x_1$ on the sense of the first definition, then for an infinitely differentiable function $\varphi$ that is finite on g we will have (cf (1)) and we have proven that the second definition followed from the first definition of the derivative $\partial f/\partial x_1$. The converse is also true. It is more convenient for us to present the proof of this assertion later (cf 4.5.2), while we will operate from the first definition. Notice that both definitions of a nonmixed generalized derivative $\partial^\rho f/\partial x_1^\rho$ also coincide. But this is no longer true for a mixed derivative. From the first definition stemmed the second, but not vice versa, as shown by the example of the function $f(x_1, x_2) = \varphi(x_1) + \psi(x_2)$, where $\varphi(x_1)$ and $\psi(x_2)$ are continuous nowhere-differentiable functions. In the sense of S. L. Sobolev, $\partial^2 f/\partial x_1 \partial x_2 = 0$, but in the sense of the first definition the derivative $\partial^2 f/\partial x_1 \partial x_2$ does not exist.

The coincidence of both definitions of $f^{(s)}$ obtains in any case when $f^{(s)} = f^{(s_1, \dots s_n)}$, also

$$\int_g f \frac{\partial \varphi}{\partial x_1} dx = \int_{g_1} dy \int f(x, y) \frac{\partial \varphi}{\partial x_1}(x_1, y) dx_1 =$$

$$= - \int_{g_1} dy \int \frac{\partial f}{\partial x_1}(x_1, y) \varphi(x_1, y) dx_1 = - \int \frac{\partial f}{\partial x_1} \varphi \, dx,$$ and $f$ a locally summable. This obtains for the function classes W, H, B, and L, which we will study in this book (cf for example, 4.4.6).

Let us present a typical problem that naturally leads to the concept of the generalized derivative.

Let the sequence of continuously differentiable functions $f_k(x) (k = 1, 2, \dots)$ and exhibiting the following property be given on $g$: whatever the bounded domain $\Omega \subset \bar{\Omega} \subset g$

$$\| f_k - f_l \|_{L_p(\Omega)} \to 0 \qquad (k, l \to \infty), \tag{2}$$

$$\left\| \frac{\partial f_k}{\partial x_1} - \frac{\partial f_l}{\partial x_1} \right\|_{L_p(\Omega)} \to 0 \qquad (k, l \to \infty). \tag{3}$$

It is required to characterize the local properties of the function $f$ to which $f_k$ tends on the average (locally):

$$\lim_{k \to \infty} \| f_k - f \|_{L_p(\Omega)} = 0. \tag{4}$$

These properties consist of the fact that (cf 4.4.5) the function $f$ has the generalized derivative $\partial f/\partial x_1$ on $g$ and $f$ and $\partial f/\partial x_1$ are locally summable on $g$ in the p-th degree.

Let us present yet another problem intimately related with our goals, leading to the concept of the generalized function.

Let $\Omega_h$ stand for the set of points $x \in \Omega$ situated from the boundary of the open set $\Omega$ by the distance greater than $h>0$, and let

$$M_\alpha[f] = \sup_h \frac{\|f(x_1+h, y) - f(x_1, y)\|_{L_p(\Omega_h)}}{|h|^\alpha} \qquad (0<\alpha\leq 1). \tag{5}$$

Further, let there be assigned on $\Omega$ a sequence of continuously differentiable function $f_k(x)$ exhibiting property (5) and such that

$$M_\alpha\left[\frac{\partial f_k}{\partial x_1} - \frac{\partial f_l}{\partial x_1}\right] \to 0 \qquad (k,\ l \to \infty). \tag{6}$$

It is required to characterise the properties of the function f for which (4) obtains. These properties consist in fact that (cf 4.7) f has the generalised derivative $\partial f/\partial x_1$ on $\Omega$, and that the value of $M_\alpha(\partial f/\partial x_1)$ is finite.

## 4.2. Finite Differences and Continuity Modules

Let $g \subset R_n$ be an open set and $h = (h_1, \ldots, h_n) \in R_n$ is an arbitrary vector. We let $g_h$ stand for the set of points $x \in g$ such that along with x, any point $x + th$ belong to g, where $0 \leq t \leq 1$, i.e., any point of the segment connecting x and $x + h$.

We will also use the symbol $g_\delta$, where $\delta > 0$, and the set of points $x \in g$ situated from the boundary of g by a distance greater than $\delta$. The sets $g_h$ and $g_\delta$ can be empty. Obviously, $g_{|h|} \subset g_h$.

Let f be a function defined on g. If $x \in g_h$, then the (first) difference

$$\Delta_h f = \Delta_h f(x) = f(x+h) - f(x)$$

of the function f at point x with (vector) pitch h has a meaning.

By induction, we introduce the concept of the k-th difference of function f at point x with pitch h:

$$\Delta_h^k f = \Delta_h^k f(x) = \Delta_h \Delta_h^{k-1} f(x) \quad (\Delta_h^0 f = f,\ \Delta_h^1 = \Delta_h,\ k = 1, 2, \ldots).$$

In any case, it is defined on the set $g_{kh}$.

Obviously,

$$\Delta_h^k f(x) = \sum_{l=0}^{k} (-1)^{l+k} C_k^l f(x+lh) \qquad (k = 0, 1, \ldots). \tag{1}$$

If s is a natural number, then obviously

$$\Delta_{sh} f(x) = \sum_{l=0}^{s-1} \Delta_h f(x+lh)$$

and (by induction)

$$\Delta_{sh}^{k} f(x) = \sum_{l_1=0}^{s-1} \dots \sum_{l_n=0}^{s-1} \Delta_h^k f(x + l_1 h + \dots + l_n h). \tag{2}$$

We turn the module of continuity of order k of function f in the metric $L_p(g)$ in the direction h the variable

$$\omega^k(\delta) = \omega_h^k(f, \delta) = \sup_{|t| \leq \delta} \|\Delta_{th}^k f(x)\|_{L_p(g_{kth})}, \tag{3}$$
$$\omega(\delta) = \omega_h(f, \delta) = \omega_h^1(f, \delta).$$

(If $\mathcal{E}$ is an empty set, then we assume $\|\cdot\|_{L_p(\mathcal{E})} = 0$.) For the variable (3) to have a meaning, it is necessary that the norm under the sign sup be finite which will obtain, for example, if $f \in L_p(g)$. Below we will dwell on several representative properties of the modules $\omega^k(\delta)$.

It is well known (cf 1.3.12) that if the function $f \in L_p(g)$ and $1 \leq p < \infty$, then

$$\lim_{t \to 0} \omega(t) = 0. \tag{4}$$

When $p = \infty$ this property does not, generally speaking, obtain. However, it is satisfied trivially, if f is uniformly continuous on g.

The inequalities

$$0 \leqslant \omega(\delta_2) - \omega(\delta_1) \leqslant \omega(\delta_2 - \delta_1) \qquad (0 < \delta_1 < \delta_2). \tag{5}$$

obtain. The first of these is obvious. The second can be demonstrated thusly. If $\delta_1$ and $\delta_2 \geqslant 0$, then any t with $|t| \leq \delta_1 + \delta_2$ can be represented in the form $t = t_1 + t_2$, where $t_1$ and $t_2$ are of the same sign with t and $|t_1| \leq \delta_1$, $|t_2| \leqslant \delta_2$.

Therefore,

$$\begin{aligned}\omega(\delta_1 + \delta_2) &= \sup_{\substack{|t'| \leq \delta_1 \\ |t''| \leq \delta_2}} \|f(x + (t' + t'')h) - f(x)\|_{L_p(g_{th})} \leqslant \\ &\leqslant \sup_{\substack{|t'| \leq \delta_1 \\ |t''| \leq \delta_2}} \|f(x + (t' + t'')h) - f(x + t''h)\|_{L_p(g_{th})} + \\ &\quad + \sup_{|t''| \leqslant \delta_2} \|f(x + t''h) - f(x)\|_{L_p(g_{th})} \leqslant \\ &\leqslant \sup_{|t'| \leqslant \delta_1} \|f(x + t'h) - f(x)\|_{L_p(g_{t'h})} + \omega(\delta_2) = \omega(\delta_1) + \omega(\delta_2).\end{aligned}$$

Substituting $\delta_2 - \delta_1$, $\delta_2$ for $\delta_2$, $\delta_1 + \delta_2$, respectively, in this inequality, we get (5).

From (4) and (5) it follows that the function (t) (when $1 \leq p < \infty$ and $p = \infty$, if f is uniformly continuous on g) is continuous for any $t \geqslant 0$.

Yet another property follows from the second inequality of (5):

$$\omega(l\delta) \leqslant l\omega(\delta) \qquad (\delta > 0;\ l = 1, 2, \ldots).$$

It can be obtained also, and in a more general form, from equality (2) (s = 1):

$$\omega^k(l\delta) \leqslant l^k \omega^k(\delta) \qquad (k, l = 1, 2, \ldots). \tag{6}$$

Obviously,

$$\omega^k(\delta) \leqslant \omega^k(\delta') \qquad (0 < \delta < \delta'). \tag{7}$$

Inequality (6) is generalized for arbitrary, not necessarily integral $l > 0$. To do this, let us select a natural m such that $m \leq l < m + 1$; then

$$\begin{aligned} &\omega^k(l\delta) \leqslant \omega^k[(m+1)\delta] \leqslant \\ &\leqslant (m+1)^k \omega^k(\delta) \leqslant (l+1)^k \omega^k(\delta) \quad (l > 0,\ k = 1, 2, \ldots). \end{aligned} \tag{8}$$

Let us note still further that

$$\sup_{|t| \leqslant \delta} \| \Delta_{th}^{s+k} f(x) \|_{L_p(g_{t(s+k)h})} \leqslant \sup_{|t| \leqslant \delta} 2^s \| \Delta_{th}^k f(x) \|_{L_p(g_{tkh})}, \tag{9}$$

and consequently,

$$\omega^{s+k}(\delta) \leqslant 2^s \omega^k(\delta). \tag{10}$$

Let $1 \leq m \leq n$, $x = (u, y)$, $u \in (x_1, \ldots, x_m) \in R_m$, and $y = (x_{m+1}, \ldots x_n)$. We will also use $R_m$ to stand for this set of points $(x_1, \ldots, x_m, 0, \ldots 0)$ of the space $R_n (R_m \subset R_n)$.

Let us introduce the variable

$$\Omega_{R_m}^k(f, \delta)_{L_p(g)} = \sup_{h \in R_m} \omega_h^k(f, \delta)_{L_p(g)}, \tag{11}$$

which we will call the module of continuity of order k of function f in direction (of subspace) $R_m \subset R_n$. If g is a bounded set and d is its diameter, then it is easy to see that for $\delta > d$ the function $\Omega^k_{R_m}(f, \delta)$ is constant.

Now let function f have any derivatives with respect to $u \in R_m$ of order $\rho$. Then the derivative with respect to the direction of any unit vector $h \in R_m$ on g has meaning for it:

$$f_h^{(\rho)} = \sum_{|s|=\rho} f^{(s)} h^s \tag{12}$$

$$\begin{gathered}(h = (h_1, \ldots, h_m, 0, \ldots, 0), \ |h| = 1, \\ h^s = h_1^{s_1} \ldots h_m^{s_m} = h_1^{s_1} \ldots h_m^{s_m} 0^0 \ldots 0^0, \ 0^0 = 1).\end{gathered}$$

Let us assume

$$\Omega_{R_m}(f^{(\rho)}, \delta) = \sup_{h \in R_m} \omega_h^k (f_h^{(\rho)}, \delta). \tag{13}$$

We will call this variable the module of continuity of derivatives (all) of order $\rho$ of function f.

Since by (8) $\quad \omega_h^k (f_h^{(\rho)}, l\delta) \leqslant (1 + l)^k \omega_h^k (f_h^{(\rho)}, \delta),$

then the upper limits of these variables with respect to $h \in R_m$ remain in the same relation

$$\Omega^k_{R_m}(f^{(\rho)}, l\delta) \leqslant (1 + l)^k \Omega^k_{R_m}(f^{(\rho)}, \delta). \tag{14}$$

Inequalities (8) and (14) show that the finiteness of the continuity module for small $\delta$ entails their continuity for large $\delta$.

Since $f_h^{(\rho)}$ is a finite linear combination of the derivatives $f^{(s)}$, $|s| = \rho$ (in the coordinate directions) with bounded coefficients $h^s$ ($|h^s| \leq 1$) not dependent on x,

$$\begin{aligned}\Omega^k_{R_m}(f_h^{(\rho)}, \delta) &= \sup_{\substack{|h|=1 \\ h \in R_m}} \sup_{|t| \leqslant \delta} \| \Delta^k_{th} f_h^{(\rho)}(x) \|_{L_p(g_{th})} \leqslant \\ &\leqslant \sup_h \sup_t \sum_{|s|=\rho} \| \Delta^k_{th} f^{(s)}(x) \|_{L_p(g_{th})} = \\ &= \sup_h \sum_{|s|=\rho} \omega_h^k (f^{(s)}, \delta) = \sum_{|s|=\rho} \Omega^k_{R_m}(f^{(s)}, \delta),\end{aligned} \tag{15}$$

obtains, where the sum $\sum$ is extended over all coordinate derivatives $f^{(s)}$ of order $\rho$ with $s = (s_1, \ldots, s_m, 0, \ldots, 0)$.

4.2.1. If $\mathcal{E} = R_1 \times \mathcal{E}' \subset R_n$ ($x = (x_1, y)$, $x_1 \in R_1$, and $y \in \mathcal{E}'$ is a cylindrical measurable set and f is a function with period $2\pi$ with respect to $x_1$ defined on $\mathcal{E}$, then in this case the norm of the function f in $L_p^*(\mathcal{E})$.

$$\|f\|_{L_p^*(\mathcal{E})} = \left(\int_{\mathcal{E}_*} |f|^p dx\right)^{1/p},$$

where $\mathcal{E}_* = \{(0, 2\pi) \times \mathcal{E}'\}$. Therefore in this case

$$\omega_{*x_1}^k(t) = \sup_{|h| \leq t} \|\Delta_{x_1,h}^k f(x)\|_{L_p(\mathcal{E}_*)},$$

where h is the increment of $x_1$ (number).

Properties of the continuity modules $\omega_*^k(t)$ are analogous to the properties $\omega^k(t)$.

4.2.2. Growth of a function with a bounded difference. Let $\mathcal{E} = R_m \times \mathcal{E}'$ be a cylindrical set of points $x = (u, y)$, $u = (x_1, \ldots, x_m)$, $y = (x_{m+1}, \ldots, x_n)$, $u \in R_m$, and $y \in \mathcal{E}'$. For brevity we will write (in this section)

$$\|\cdot\|_\Lambda = \|\cdot\|_{L_p(\Lambda \times \mathcal{E}')},$$
$$\|\cdot\| = \|\cdot\|_{R_m} = \|\cdot\|_{L_p(\mathcal{E})}.$$

Let us assign a natural k and a positive number $\delta > 0$.

Let a function f(x) satisfying the conditions

$$\|f(x)\|_{\{|u| < \delta(k+m)\}} < A, \tag{1}$$
$$\|\Delta_h^k f(x)\| < B \tag{2}$$

be assigned on $\mathcal{E}$ for any $h \in R_m$ and $|h| = \delta$.

Let us assume that

$$\sigma_N = \{N < |u| < N+1,\ y \in \mathcal{E}'\}. \tag{3}$$

We will prove the existence of a constant $c = c_k$ for which the inequality

$$\|f\|_{\sigma_N} < cN^{\frac{m-1}{p}} (A + (A+B) N^k), \qquad N = 1, 2, \ldots \tag{4}$$

is satisfied.

Notice that for $\varepsilon > 0$, from (4) it follows that

$$\frac{\|f\|_{\omega_N}}{N^{m+(k+\varepsilon)p}} < \frac{c_1}{N^{1+\varepsilon_1}},$$

where $c_1$ does not depend on N = 1, 2, ... and $\varepsilon_1 > 0$ depends on $\varepsilon$, from whence when $\varepsilon > 0$, taking (1) into account, we get the inequality

$$\left\| \frac{f(x)}{\left(1+|x|^{\frac{m}{p}+k+\varepsilon}\right)} \right\|_{L_p(R)} < \infty, \tag{5}$$

in which we cannot assume $\varepsilon = 0$, as is shown by the example of the function of a single variable $x^k$ (k = 1, 2, ...).

In the proof, for simplicity we will take $\delta = 1$.

Let us assign an arbitrary unit vector $u' \in R_m$ and define on $R_m$ (m - 1)-dimensional cube that is orthogonal to $u'$ with its center at the zero point and that has edges of unit length. On this cube as the base and with the vector $u'$ as the height we will construct the unit cube $\omega = \omega_{u'} \in R_m$.

Let us further specify a natural number N, and let $\omega_N = \omega_{Nu'}$ represent the unit cube consisting of points of the form $Nu' + u$, where $u$ run through $\omega$.

Notice that for the function $\psi(x)$ locally summable in the p-th degree (when $p = \infty$, it is locally bounded),

$$\psi(Nu' + u, y) = \psi(u, y) + \sum_{j=0}^{N-1} \Delta\psi(ju' + u, y),$$
$$\Delta\psi(ju' + u, y) = \psi((j+1)u' + u, y) - \psi(ju' + u, y),$$

obtain, from whence

$$\|\psi\|_{\omega_N} \leqslant \|\psi\|_{\omega} + \sum_{j=0}^{N-1} \|\Delta\psi\|_{\omega_j}. \tag{6}$$

Let us prove the inequality

$$\|\Delta_{u'}^{k-s} f\|_{\omega_{Nu'}} < c(A + (A+B)N^s) \tag{7}$$
$$(c = c_{k,s};\quad N = 0, 1, \ldots;\quad s = 0, 1, \ldots, k).$$

when s = 0 it directly follows from (2) ( $\delta = 1$). Let (7) be valid for s, and let us demonstrate its validity for s + 1. We will assume

$$\psi(x) = \Delta_{u'}^{k-s-1} f(x).$$

then

$$\|\psi\|_{\omega} < \left| \sum_{l=0}^{k-s-1} (-1)^{l+k-s-1} C_{k-s-1}^{l} f(u + lx', y) \right|_{\omega} <$$

$$< A \sum_{l=0}^{k-s-1} C_{k-s-1}^{l} = 2^{k-s-1} A,$$

$$\|\Delta_{u'} \psi\|_{\omega_j} < \|\Delta_{u'}^{k-s} f\|_{\omega_j} < c(A + (A+B) j^s).$$

Therefore, based on (6)

$$\|\Delta_{u'}^{k-s-1} f\|_{\omega_N} < 2^{k-s-1} A + c \sum_{j=0}^{N-1} (A + (A+B) j^s) <$$

$$< c_1 (A + (A+B) N^{s+1}).$$

We have proven (7). Inserting s = k in (7), we get

$$\|f\|_{\omega_{Nu'}} < c(A + (A+B) N^k). \tag{8}$$

From (8) and (3) follows the existence of the constants $c_2$ such that

$$\|f\|_{\sigma_N}^{p} < c_2 N^{m-1} (A + (A+B) N^k)^p \qquad (N = 1, 2, \ldots)$$

or (4). The concern here is that the domain $\sigma_N$ cannbe covered by cubes of the form $\omega_{Nsu'}$, where s = N - 1, N, and N + 1, whose number is of the order of $N^{m-1}$.

## 4.3. Classes W, H, and B

Let us begin with the definition of the embedding concept widely employed in this book.

If E and E' are two normed bases, $E \subset E'$, and here there exists the constant c not dependent on x such that

$$\|x\|_{E'} \leqslant c\|x\|_{E}, \tag{1}$$

where $\|\cdot\|_{E'}$ and $\|\cdot\|_{E}$ are the norms, respectively, in the E'- and E-sense, then we will assert that the embedding $E \rightarrow E'$ obtains. If $E \rightarrow E'$ and $E' \rightarrow E$, then we will write $E \rightleftarrows E'$.

If the elements of the same linear set are normed in the sense of different metrics E and $E_1$ and $E \rightleftarrows E_1$, then we often write: $E = E_1$ and even $\|x\|_E = \|x\|_{E_1}$, adding in the cases when there can be confusion that this inequality obtains with an accuracy to equivalency.

Let $R_n$ be considered as the direct product $R_n = R_m \times R_{n-m}$ of coordinate subspaces $R_m$ and $R_{n-m}$, $1 \leq m \leq n$. Then the arbitrary point $x \in R_n$ can be written in the form $x = (u, y)$, where $u \in R_m$ and $y \in R_{n-m}$. In particular, $x = u$ when $m = n$. Further, let $g \subset R_n$ be an open set and $1 \leq p \leq \infty$. In this section the classes

$$W^l_{up} = W^l_{up}(g) \quad (l = 0, 1, \ldots; \quad W^0_{up}(g) = L_p(g)),$$

$$H^r_{up} = H^r_{up}(g) \quad (r > 0),$$

$$B^r_{up\theta} = B^r_{up\theta}(g) \quad (r > 0, \ 1 \leqslant \theta < \infty; \quad B^r_{upp} = B^r_{up}).$$

are defined.

When $m = n$ in these notations, we will omit the letter $u$ and then we get: $W^l_p$, $H^r_p$, $B^r_{p\theta}$, and $B^r_p$ $(\theta = p)$. In another important case when $R_m = R_{x_j}$ $(j = 1, \ldots, m)$, we will write $W^l_{x_jp}$, $H^r_{x_jp}$, $B^r_{x_jp\theta}$, and $B^r_{x_jp}$.

We will call these classes isotropic with respect to the $R_m$ directions, because their differential properties along any $R_m$ directions are identical, or simply isotropic, if $m = n$. For the integral vector $l = (l_1, \ldots, l_m) \geqslant 0$ $(l_j \geqslant 0)$ and the vector $p = (p_1, \ldots, p_m)$, where $1 \leqslant p_j \leqslant \infty$, we will additionally define the class (for different $l_j$ or $p_j$, and isotropic)

$$W^l_p(g) = \bigcap_{j=1}^{n} W^{l_j}_{x_jp_j}(g) \quad (W^l_p = W^l_p \text{ при } p = (p, \ldots, p))$$

when $p = (p, \ldots, p)$)

as the intersection of the classes $W^{l_j}_{x_jp}(g)$. The classes

$$H^r_p(g) = \bigcap_{j=1}^{n} H^{r_j}_{x_jp_j}(g), \quad B^r_{p\theta}(g) = \bigcap_{j=1}^{n} B^{r_j}_{x_jp_j\theta}(g),$$

где $r = (r_1, \ldots, r_n) > 0 \quad (H^r_p = H^r_p, \quad B^r_{p\theta} = B^r_{p\theta}$
$p = (p, \ldots, p))$.

are analogously defined, where $r = (r_1, \ldots, r_n) > 0$ ($H^r_p = H^r_p$, $B^r_{p\theta} = B^r_{p\theta}$ when $p = (p, \ldots, p)$).

The classes (n-dimensional) $W^l_p(g)$ ($l = 0, 1, \ldots$) are called Sobolev classes, named after S. L. Sobolev*), who studied their fundamental properties and was the first to obtain for them the fundamental theorems of embedding as applied to domains $g$, star-shaped relative to a certain sphere, and to finite sums of these domains. These classes consist of functions integrable in the p-th degree on $g$ together with their partial derivatives (generalized) of order $l$.

The classes (n-dimensional) $h^r_p(g)$ and $H^r_p(g)$ are defined for any $r > 0$ or $r_j > 0$. They consist of the functions belonging to $L_p(g)$ and that have on $g$ partial derivatives of specific orders satisfying in the $L_p$ matrix Hölder's condition (Lipshits' condition when $p = \infty$) or (for integral $r$ and $r_j$), the thusly generalized condition (Zigmund's condition) in which the first difference is replaced by a higher-order difference.

H-classes were completely defined in the works of S. M. Nikol'skiy**), who obtained embedding theorems for them. It turns out that these theorems form close system and, in particular, the embedding theorems of different measures (cf below) are completely invertible.

The classes $B^r_{p\theta}$ and $B^r_{p\theta}$ were determined in all completeness by O. V. Besov***), who obtained a close system of embedding theorems for them. The embedding theorems of different measures for these classes are also invertible.

In the following equivalent definitions in terms of the best approximations of exponential type functions will be given for the classes H and B. As applied to classes B there will be broader, encompassing the case $\theta = \infty$. We will see that it is natural to assume that

$$B^r_{up\infty} = H^r_{up}.$$

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*) S. L. Sobolev [3, 4]. cf S. M. Nikol'skiy [10] for anisotropic Sobolev classes $W^l_p$.

**) S. M. Nikol'skiy [3, 5, 10].

***) O. V. Besov [2, 3]. cf V. P. Il'yin and V. A. Solonnikov [1, 2] for the embedding theorem for classes $B^r_{p\theta}$.

In the following it will be shown (cf 9.3) that for sufficiently general domains ($\theta = p$)*)

$$B^l_{up} \to W^l_{up} \quad (1 \leqslant p \leqslant 2), \tag{2}$$

$$W^l_{up} \to B^l_{up} \ (2 \leqslant p \leqslant \infty, \ B^l_{u\infty} = H^l_{u\infty}) \quad (l = 1, 2, \ldots). \tag{3}$$

In particular, therefore,

$$B^l_{u2} = W^l_{u2} \quad (l = 1, 2, \ldots). \tag{4}$$

Equality (4) indicates a certain relation between classes B and W, appearing when $p = 2$. But there is also another relation, appearing for any p. It stems from the properties of traces of the functions of these classes (cf 9.1).

Historically, the existence of these relations was the occasion to call**) classes which are here denoted by $B^r_p$ and $B^r_p$($\theta = p$), for cases of fractional (not integral) r and r by the classes $W^r_p$ and $W^r_p$, respectively, assuming obviously that it is precisely these classes that are the natural extensions of the Sobolev (with integral l, l) classes $W^l_p$ and $W^l_p$. Of course, the issue does not lie in notation, but even now when all the fundamental problems of the interrelations of these classes have been thoroughly clarified, it is clear that the natural (if we like, true) extensions of the Sobolev classes in the n-dimensional case are the other so-called Liouville classes constructed on the basis of the direct generalization of the concept of the fractional derivative in the Liouville sense (or in the Weyl sense for the periodic case). We will talk about the n-dimensional case because in the one-dimensional case it was always held that the problem of traces does not arise.

And so, we will begin with the following notation. There exists the Sobolev classes $W^l_p$ defined for integral $l = 0, 1, \ldots$; they are "buried" in fractional Liouville classes, denoted by $L^l_p$(l is a real number); thus, $W^l_p = L^l_p$($l = 0, 1, \ldots$). We see that the classes $L^r_p$ are merged by the fact that the functions belonging to them have a unified integral representation (in terms of convolutions of the Bessel-Macdonald kernels with the functions $f \in L_p$, cf 9.1). We also become aware that the classes $L^r_p$ form a closed system with respect to the embedding theorems of different metrics The closeness is

*) O. V. Besov [3, 5].
**) L. N. Slobodetskiy [1].

manifested in that the embedding theorems of different metrics for the classes $L_p^r$ are wholly expressed in terms of these classes and where the theorems exhibit the property of transitivity (cf further 7.1). However the classes $L_p^r$ when $p \neq 2$ do not form a closed system with respect to the embedding theorems of different measures, and here there is no difference between integral and nonintegral $r$.

The exact embedding theorems of different metrics for the classes $L_p^r$ when $p \neq 2$ no longer are expressed in terms of these classes. To express them, it becomes necessary to involve the classes $B_p^r$. However, an exception is found in this case $p = 2$, studied in the works of Aronszajn [1] and L. N. Slobodetskiy *). The embedding theorem of different metrics for the classes $B_2^r$ (in the notation of L. N. Slobodetskiy, $W_2^r$) where $p = 2$ is not changed, are self-closed. The classes $B_p^r$ of themselves form a closed system with respect to the embedding theorems of different metrics and the measures (and several others) and have a unified integral representation in terms of Macdonald kernels (cf 8.9.1), but at the same time these classes play a service role in the problem on traces of functions of the classes $L_p^r$ (or $W_p^l$ when $r = l$, a natural number), which is solved by the embedding theorems of different measures. Here lies the relation between the classes L and B; another relation, as noted above, is the fact that $L_2^l = B_2^l$ ($l = 0, 1, \ldots$). These relationships also obtain for the corresponding isotropic classes.

After the foregoing, it would be sounder either to assume that $W_p^r$ for fractional $r$ denotes a Liouville class, and in general not to use the symbol $L_p^r$, or else to continue only with the notation $L_p^r$ for all $r$, discarding the special notation $W_p^l$ for the Sobolev classes. But I did not do this in this book, because I feared I would be like a person who became aware of the soundness of renaming a street, did so, but did not seek the views of the residents living on the street about this change.

We will see (cf 6.1) that for any $\varepsilon > 0$ the embedding

*) L. N. Slobodetskiy [1, 2]; cf also V. M. Babich and L. N. Slobodetskiy [1].

$$H_{up}^{r+\varepsilon} \to B_{up\theta}^{r} \to H_{up}^{r} \qquad (r>0,\ 1\leqslant \theta \leqslant \infty), \tag{5}$$

$$H_{up}^{l+\varepsilon} \to W_{up}^{l} \to H_{up}^{l} \qquad (l=1,\ 2,\ \ldots). \tag{6}$$

obtain.

These classes are linear normed spaces. This will be immediately eminent from their definitions. As will be clear in the following, they are complete, therefore, Banach spaces (cf 4.7).

We would see that the norm in the sense W, H, and B is comprised of two numbers

$$\|f\|_W = \|f\|_{L_p} + \|f\|_w,\ \|f\|_H = \|f\|_{L_p} + \|f\|_h,\ \ldots, \tag{7}$$

where the second term (which we will call the seminorm) characterises purely differential properties of f. Seminorm can be considered the norm in the corresponding space w, h, and b where functions distinct from each other by polynomials of specific degrees (with respect to $x_1, \ldots, x_m$) are not distinguish from each other.

In the following (cf 8.9.2 and 9.2) the specified classes will be defined for the case $g = R_n$ and for zero and negative values of r, but they will in general consist of generalised functions (regular in the $L_p$-sense).

4.3.1. Class W. Let $g \in R_n$ be an open set, l be an integral nonnegative number, $1 \leq p \leq \infty$ and $x = (u, y)$, $u = (x_1, \ldots, x_m)$ $R_m$, and $y = (x_{m+1}, \ldots, x_n)$; $R_m$ will also refer to the subspace of points of the form $(u, 0)$.

By definition $f \in W_{up}^{l}(g)(W_{up}^{l}(g) = W_p^{l}(g)$*) when m = n and $W_{up}^{0}(g) = L_p(g))$, if the norm

$$\|f\|_{W_{up}^{l}(g)} = \|f\|_{L_p(g)} + \|f\|_{w_{up}^{l}(g)} \qquad (l=1,\ 2,\ \ldots), \tag{1}$$

$$\|f\|_{W_{up}^{0}(g)} = \|f\|_{L_p(g)},$$

$$\|f\|_{w_{up}^{l}(g)} = \sum_{|s|=l} \|f^{(s)}\|_{L_p(g)}$$

$$\left(s=(s_1,\ \ldots,\ s_m,\ 0,\ \ldots,\ 0),\quad |s| = \sum_1^m s_j\right), \tag{2}$$

*) S. L. Sobolev [3, 4].

is finite, where, thus, the sum is extended over all derivatives(generalized), mixed and nonmixed, of order $l$ with respect to $m$. Thus it is assumed that for $f$ there exist generalized derivatives with respect to $m$ of orders less than $l$, but a priori it is not assumed that they belong to $L_p(g)$. But we will see that in any case they are locally summable on $g$; moreover, they, including derivatives of order $l$ and do not depend on the order in which the differentiation is performed (cf 4.5.1).

We can consider the space $w^l_{up}(g)$ (when $m = n$ $w^l_p(g)$) of functions $f$ for which the seminorm (2) is finite, i.e., we can assume that $w^l_{up}(g)$ consists of measurable functions $f$ that may not belong to $L_p(g)$, but such that for these the generalized derivatives on $g$ of order $l$ belonging to $L_p(g)$ are meaningful. Obviously, $w^l_{up}(g)$ is a linear set. It will be a normed space if it is assumed that the two functions $f_1$ and $f_2 \in w^l_{up}(g)$, differing by the polynomial of degree $l - 1$, defined the same element of the space $w^l_p(g)$; in other words, the zero element in $w^l_p(g)$ is the arbitrary polynomial

$$P_{l-1}(x) = \sum_{|k| \leq l-1} a_k x^k, \qquad k = (k_1, \ldots, k_m; 0, \ldots, 0),$$

of degree $l - 1$ with coefficient $a_k = a_k(y)$ dependent on $y = (x_{m+1}, \ldots, x_n)$.

The norm (1) equivalent to the following norm:

$$\|f\|_{W^l_{up}(g)} = \left(\int_g \left(|f|^p + \sum_{|s|=l} |f^{(s)}|^p\right) dx\right)^{1/p}, \tag{3}$$
$$s = (s_1, \ldots, s_m, 0, \ldots, 0).$$

The advantage of this latter expression is that when $p = 2$ it is Hilbertian. The scalar product generating this norm when $p = 2$ is of the form

$$(f, \varphi) = \int_g \left(f\varphi + \sum_{|s|=l} f^{(s)}\varphi^{(s)}\right) dx. \tag{4}$$

We can also talk about classes $W^l_{x_j p}(g)$ of functions $f$ for which the norm

$$\|f\|_{W^l_{x_j p}} = \|f\|_{L_p(g)} + \left\|\frac{\partial^{l_j}}{\partial x_j^{l_j}}\right\|_{L_p(g)} \qquad (j = 1, \ldots, n), \tag{5}$$

finite and classes*)

$$W^{r}_{up}(g)=\bigcap_{j=1}^{n} W^{r_j}_{x_j p_j}(g)\,\big(W^{r}_{up}=W^{r}_{up} \quad \text{when } p = p_1 = \dots = p_n\big), \tag{6}$$

$$r=(r_1, \dots, r_n)>0,\ p=(p_1, \dots, p_n), \qquad 1\leqslant p_j\leqslant \infty,$$

with the norm

$$\|f\|_{W^{r}_{up}(g)}=\sum_{1}^{m}\left(\|f\|_{L_{p_j}(g)}+\left\|\frac{\partial^{r_j} f}{\partial x_j^{r_j}}\right\|_{L_{p_j}(g)}\right). \tag{7}$$

Let us further introduce another class $'W^{1}_{up}$: the function $f \in {}'W^{1}_{up}(g)$, if for it the norm

$$\|f\|_{W^{l}_{up}(g)}=\|f\|_{L_p(g)}+\sup_{u\in R_m}\|f^{l}_{u}\|_{L_p(g)}, \tag{8}$$

is meaningful, where

$$f^{\rho}_{u}=\sum_{|s|=\rho} f^{(s)}u^{s}\ \ \big(u^{s}=(u_1^{s_1}\dots u_m^{s_m}),\ |u|=1\big), \tag{9}$$

is the derivative of f of order $\rho$ in the direction u.

In the following it will be shown (cf 9.2) that

$$W^{l,\dots,l}_{p}(R_n)\to W^{l}_{p}(R_n).$$

If the domain of g is such that for it the theorem on extension obtains (cf note at end of book to 4.3.6)

$$W^{l,\dots,l}_{p}(g)\to W^{l,\dots,l}_{p}(R_n),$$

then

$$'W^{l}_{p}(g)\to W^{l,\dots,l}_{p}(g)\to W^{l,\dots,l}_{p}(R_n)\to W^{l}_{p}(R_n)\to W^{l}_{p}(g),$$

where the first embedding is because the derivative $f^{1}_{x_j}$ is at the same time a derivative in the direction $x_j$. The inverse embedding $W^{l}_{p}(g)\to {}'W^{l}_{p}(g)$,

*) S. M. Nikol'skiy [10].

obviously, is also valid, therefore given the presence of the theorem on extension

$$W_p^l(g) \rightleftarrows {}'W_p^l(g).$$

as will be seen from the following, for many quite "good" sets g, it automatically follows from the fact that $f \in W_{up}^{\rho}(g)$ all partial derivatives of f with respect to u up to the order $\rho - 1$ inclusively belong to $L_p(g)$. However, this is generally invalid for an arbitrary open set g.

4.3.2. Example. The function f(x) of the one variable x is assigned on the set $g = \sum_1^\infty \sigma_k$, which is a theoretic-set sum of the integrals $\sigma_k = (a_k < x < b_k)$ of length $\delta_k = k^{-2}$.

Let

$$f(x) = \frac{(x - a_k)}{\delta_k^{\alpha}} \qquad (k = 1, 2, \ldots).$$

Then, if 1/2p 1,

$$\|f\|_{L_p(g)} = \left(\sum_1^\infty \int_{\sigma_k} \left(\frac{x - a_k}{\delta_k^\alpha}\right)^p dx\right)^{1/p} =$$

$$= \frac{1}{(p+1)^{1/p}} \left(\sum_1^\infty \delta_k^{p(1-\alpha)+1}\right)^{1/p} = \frac{1}{(p+1)^{1/p}} \left(\sum_1^\infty \frac{1}{k^{2[p(1-\alpha)+1]}}\right)^{1/p} < \infty,$$

$$\|f^{(l)}\|_{L_p(g)} = 0$$

when $l \geqslant 2$

$$\|f'\|_{L_p(g)} = \left(\sum_1^\infty \int_{\sigma_k} \left|\frac{1}{\delta_k^\alpha}\right|^p dx\right)^{1/p} = \sum_1^\infty \frac{1}{k^{2(1-\alpha p)}} = +\infty.$$

Here the condition $\delta_k = k^{-2}$ shows that the set g can be bounded.

Thus, $f \in W_p^{(l)}(g)$ $(l \geqslant 2)$ but the norm in the first derivative in the metric $L_p(g)$ is equal to $+\infty$.

4.3.3. Classes H. The notations introduced at the beginning of 4.3.1 remain in force. Let $1 \leq p \leq \infty$, $r > 0$, and the numbers $k$ and $\rho$ be integral nonnegative, satisfying the inequalities $k > r - \rho > 0$. We will call these pairs $(k, \rho)$ admissible.

By definition the function $f \in H^r_{up}(g)$*), if it belongs to $L_p(g)$ and if for it the derivatives $f^{(s)}$ of order $s = (s_1, \ldots, s_m, 0, \ldots, 0)$ with $|s| = \sum_1^m s_j = \rho$ are meaningful and if for these the inequalities

$$\|\Delta_h^k f^{(s)}(x)\|_{L_p(g_{kh})} \leqslant M |h|^{r-\rho}, \tag{1}$$

are satisfied, where M does not depend on $h \in R_m$, or the inequalities equivalent to it

$$\Omega^k(f^{(s)}, \delta) = \sup_{h \in R_m} \omega_h^k(f^{(s)}, \delta) \leqslant M\delta^{r-\rho}. \tag{1'}$$

Here let us assume

$$\|f\|_{H^r_{up}(g)} = \|f\|_{L_p(g)} + \|f\|_{h^r_{up}(g)}, \tag{2}$$

where the seminorm

$$\|f\|_{h^r_{up}(g)} = M_f = \inf M \tag{3}$$

is the lower bound of all M for which inequality (1) is satisfied for all $h \in R_m$ and any indicated $s$.

This definition actually depends on the admissible pair $(r, \rho)$, but it will be proven (5.5.3) that the norms (2) (but in general not (3)) for the measurable set $g = R_m \times g'$ and different admissible pairs are pairwise equivalent, and for other sets $g$ the equivalence will depend on the possibility of extending the functions beyond the limits of $g$ on $R_n$ with the preservation of the corresponding norms (cf notes at the end of the book to 4.3.6).

*) When $p = \infty$, here we have the situation in which the function f is equivalent to some function again denoted by f for which (1) is satisfied.

Let $r = \bar{r} + \quad$, where $\bar{r}$ is integral and $0 < \alpha \leq 1$. If $\alpha < 1$, then by selecting the numbers $\rho = \bar{r}$ and $k = 1$ as the admissible pair, to get the particular form of inequality (1):

$$\|\Delta_h f^{(\bar{r})}(x)\|_{L_p(g_h)} \leq M|h|^\alpha \quad (0 < \alpha < 1). \tag{4}$$

If however $\alpha = 1$, then this pair is not suitable, but we can take $\rho = \bar{r}$, $k = 2$ as the admissible pair, and then inequality (1) will obtain:

$$\|\Delta_h^2 f^{(\bar{r})}(x)\|_{L_p(g_{2h})} \leq M|h|. \tag{5}$$

Usually definitions*) (4) and (5) or simply one definition

$$\|\Delta_h^2 f^{(\bar{r})}(x)\|_{L_p(g_{2h})} \leq M|h|^\alpha \quad 0 < \alpha \leq 1, \tag{6}$$

suitable for any of the $\alpha$ considered are used.

It is possible that the modification of these definitions consisting in the fact that the lower bound of such M for which (1) is satisfied for all $h \in R_m$ satisfying the inequality $|h| \leq \eta$ where $\eta$ = given positive number is taken as the seminorm $M_f$. Thus, the modified norm is also, as we will see, equivalent to the above-defined norms in any case for domains of the form $g = R_m \times g'$.

Finally, yet another definition is possible: the function $f \in H^r_{up}(g)$, if for it the derivatives $f_h^\rho$ of order $\rho$ in any direction $h \in R_m$ and

$$\|\Delta_h^k f_h^\rho\|_{L_p(g_{k|h|})} \leq M|h|^{r-\rho}, \tag{7}$$

where $(k, \rho)$ is an admissible pair and M does not depend on $h \in R_m$ are meaningful for it. This inequality is equivalent to the following:

$$\Omega^k(f^\rho; \delta) = \sup_{|h|=1} \sup_{|t| \leq \delta} \|\Delta_{th}^k f_h^\rho\|_{L_p(g_{kh})} \leq M\delta^{r-\rho}. \tag{8}$$

The norm f is defined analogously to (2).

If $R_m (m = 1)$ is the coordinate axis $x_j$, then we will refer to the

*) S. M. Nikol'skiy [5].

corresponding class $H^r_{up}(g)$ with $H^r_{x_j p}(g)$ $(j = 1, \ldots, m)$ and the norm as

$$\|f\|_{H^r_{x_j p}(g)} = \|f\|_{L_p(g)} + M_{x_j f}, \tag{9}$$

$$M_{x_j f} = \|f\|_{h^r_{x_j p}(g)}. \tag{10}$$

Finally, if $r = (r_1, \ldots, r_m)$, $p = (p_1, \ldots, p_m)$ $(r_j > 0,\ 1 \leq p_j \leq \infty;$ $j = 1, \ldots, m \leq n)$, then we postulate*)

$$H^r_{up}(g) = \bigcap_{j=1}^{m} H^{r_j}_{x_j p_j}(g) \left(H^r_{up} = H^r_{up} \qquad \text{where } p = p_1 = \ldots = p_m\right) \tag{11}$$

with the norm

$$\|f\|_{H^r_{up}(g)} = \max_{1 \leq j \leq m} \|f\|_{L_{p_j}(g)} + \|f\|_{h^r_p(g)}, \tag{12}$$

$$\|f\|_{h^r_{up}(g)} = \max_{1 \leq j \leq m} \|f\|_{h^{r_j}_{x_j p_j}(g)}. \tag{13}$$

In (13) we can replace $\max_j$ with $\sum_j$, obtaining the equivalent norm.

4.3.4. Classes B. Let us preserve the notations introduced at the beginning of 4.3.1, and introduce an additional parameter $\theta$, where $1 \leq \theta < \infty$. Let $r > 0$ and the numbers $k$ and $\rho$ (forming the admissible pair) be integral nonnegative, satisfying the inequalities $k > r - \rho > 0$.

By definition, function $f$ belongs to the class $B^r_{up\theta}(g)$*) (when $m = n$, simply $B^r_{p\theta}(g)$), if $f \in L_p(g)$, there exist generalized partial derivatives with respect to $u \in R_m$ of $f$ of orders $s = (s_1, \ldots, s_m, 0, \ldots, 0)$ $(|s| \leq \rho)$, and one of the following seminorms is finite:

$$\|f\|_{b^r_{up\theta}(g)} = \sum_{|s|=\rho} \left(\int_0^\infty t^{-1-\theta(r-\rho)} \Omega^k_{R_m}(f^{(s)}, t)^\theta_{L_p(g)}\, dt\right)^{1/\theta}, \tag{1}$$

$$\|f\|_{b^r_{up\theta}(g)} \left(\int_0^\infty t^{-1-\theta(r-\rho)} \Omega^k_{R_m}(f^{\rho}, t)^\theta_{L_p(g)}\, dt\right)^{1/\theta}. \tag{2}$$

*) cf. note to text page 189 [translation page 174]

$$
{}^{3}\|f\|_{b^r_{up\theta}(g)} = \sum_{|s|=\rho} \left( \int_{R_m} |u|^{-m-\theta(r-\rho)} \|\Delta_u^k f^{(s)}(x)\|^\theta_{L_p(g_{ku})} du \right)^{1/\theta}, \tag{3}
$$

$$
{}^{4}\|f\|_{b^r_{up\theta}(g)} = \left( \int_{R_m} |u|^{-m-\theta(r-\rho)} \|\Delta_u^k f_u^{\rho}(x)\|^\theta_{L_p(g_{ku})} du \right) \tag{4}
$$

(cf 4.2 (12), (13)). Here we stipulate that

$$
{}^{j}\|f\|_{B^r_{up\theta}(g)} = \|f\|_{L_p(g)} + {}^{j}\|f\|_{b^r_{up\theta}(g)} \quad (j = 1, 2, 3, 4). \tag{5}
$$

All four of the seminorms (1) - (4) presented still depend on the admissible pairs $k, \rho$; moreover, they can be modified by taking the integrals written here over bounded domains (respectively, over $0 \leq t \leq \eta$ or $|u| \leq \eta$), and in fact all of the norms determined by means of these integrals proved to be equivalent in any case for domains of the form $g = R_m \times g' \subset R_n$ (cf, further, 5.6) and, therefore, for the domains $g$ with which the functions are extensible on $R_n$ with preservation of the indicated norms.

Often these norms are specified in the following situation**). For a given $r > 0$, an integral $\bar{r}$ is defined such that $r = \bar{r} + \alpha$ and $0 < \alpha \leq 1$. If $\alpha < 1$, then it suffices to take the admissible pair $\rho = \bar{r}$, $k = 1$; if however $\alpha = 1$, then $\rho = \bar{r}$, $k = 2$, or else, in order to combine these two classes, we can take $\rho = \bar{r}$, $k = 2$.

If $g$ is a bounded set and $d$ is its diameter, then for $t > d$ each of the functions $\Omega$ in (1) and (2) are equal to some constants $c$ and the residue of the $\int_d^\infty$ integrals appearing in the right-hand sides of (1) and (2) are finite (in fact, $\theta$, $r - \rho > 0$). Therefore the finiteness of the seminorms (1) and (2) depend exclusively on the properties of the indicated modules for $t$.

The classes $B^r_{x_j p}$ (j = 1, ..., m) correspond to the case when $R_m$ is replaced by the coordinate axis $x_j$.

**) O. V. Besov [3, 5]. The norms (1) and (3) are examined in these works.

Let us suppose*)

$$\|f\|_{B^r_{up\theta}(\mathscr{E})} = \sum_{j=1}^{m} \|f\|_{B^{r_j}_{x_j p \theta}(\mathscr{E})} \quad (B^r_{up\theta} = B^r_{p\theta} \text{ when } m = n).$$

We note the simplest inequalities between these seminorms (1) - (4) (for the same pair k, $\rho$ ):

$$^3\|f\|_b \ll {}^1\|f\|_b, \quad ^4\|f\|_b \ll {}^2\|f\|_b, \quad ^2\|f\|_b \ll {}^1\|f\|_b. \tag{6}$$

The last inequality follows from inequality 4.2(15). The first two are obtained directly if we introduce the polar coordinates $u = (t, \sigma)$, $t = |u|$, and $du = t^{m-1} dt\, d\sigma$ and taking into account the inequalities

$$\|\Delta_u^k f^{(s)}(x)\|_{L_p(\mathscr{E}_{ku})} \leqslant \Omega^k_{R_m}(f^{(s)}, t)_{L_p(\mathscr{E})}, \tag{7}$$

$$\|\Delta_u^k f^{\rho}(x)\|_{L_p(\mathscr{E}_{ku})} \leqslant \Omega^k_{R_m}(f^{\rho}, t)_{L_p(\mathscr{E})}. \tag{8}$$

4.3.5. Periodic classes. The periodic classes $W^l_{*x_j p}(\mathscr{E})$, $H^r_{*x_j p}(\mathscr{E})$, and $B^r_{*x_j p\theta}(\mathscr{E})$ are defined on the set $\mathscr{E} = R_j \times \mathscr{E}^j \subset R_n$, where $R_j$ is the real axis $x_j$ ($j = 1, \ldots, n$). These are classes of functions $f(x_j, y^j) y^j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ with period $2\pi$ with respect to $x_j$. There defined exactly just as the corresponding classes $W^l_{x_j p}(\mathscr{E}), \ldots$ of periodic functions, but with the only difference that everywhere the norm $\|\cdot\|_{L_p(\mathscr{E})}$ must be replaced with the norm $\|\cdot\|_{L_p(\mathscr{E}_*)}$, where $\mathscr{E}_* + [0, 2\pi] \times \mathscr{E}^j$. The periodic classes $W^r_{*up}(\mathscr{E})$, $H^r_{*up}(\mathscr{E})$, and $B^r_{*up}(\mathscr{E})$, when $\mathscr{E} = R_m \times \mathscr{E}'$ $R_n$ ($1 \leq m \leq n$); when $m = n$, we omit the subscript $u$.

*) O. V. Besov [3, 5], case $p_1 = \ldots = p_n$; V. P. Il'yin and V. A. Solonnikov [1, 2], general case.

4.3.6. Extension of functions with class preserved. Let us make yet another important remark. Let $\Lambda$(g) denote one of the classes W(g), H(g), and B(g) with given parameters of r, p, ... If the domain $g \subset R_n$ is such that any function $f \in \Lambda(g)$ can be brought into correspondence with a function $\bar{f}$ defined on $R_n$ such that $\bar{f} = f$ on g and

$$\|\bar{f}\|_{\Lambda(R_n)} \leqslant c\|f\|_{\Lambda(g)},$$

where c does not depend on f, then we will state that functions f of class $\Lambda(g)$ can be extended from g onto $R_n$ with preservation of class (or norm).

We will further assert in this case that the embedding

$$\Lambda(g) \to \Lambda(R_n).$$

holds.

Our classes are constructed so that if the function $f \in \Lambda(R_n)$, its values on g form the function $f \in \Lambda(g)$ and

$$\|f\|_{\Lambda(g)} \leqslant \|f\|_{\Lambda(R_n)}.$$

Accordingly; it is stated that the embedding $\Lambda(R_n) \to \Lambda(g)$.

obtains. Now we would assume that for some domain g the two classes $\Lambda(g)$ and $\Lambda'(g)$ are given and that

$$\Lambda(g) \to \Lambda(R_n) \to \Lambda'(R_n). \tag{1}$$

then

$$\Lambda(g) \to \Lambda'(g). \tag{2}$$

In this book we will place our principal emphasis on the study of these classes for the case when $g = R_n$ or $g = R_m \times g'$, where $1 \leqslant m < n$ and g' is a measurable (n - m)-dimensional set. In the notes at the end of the book to 4.3.6 the reader will find the formulation of several general theorems on extension with class preserved. The presence of embeddings (1) automatically entails the embedding (2).

## 4.4. Representation of the Intermediate Derivative by a Higher-Order Derivative and Functions. Corollaries

In this section several modified Taylor's formulas will be introduced, on the basis of which certain inequalities will be derived.

4.4.1. Let us consider on the finite integral (a, b) the function f(x) that has on any segment interior with respect to (a, b) absolutely continuous derivatives of the order ($\rho$- 2) inclusively, and, therefore, almost everywhere on [a, b] the derivative of order $\rho$- 1. For it there exists for almost all $x_0$ the purely formal Taylor's formula

$$f(x)=\sum_{j=0}^{\rho-1} f^{(j)}(x_0)\frac{(x-x_0)^j}{j!}+R(x, x_0) \qquad (a<x,\ x_0<b) \tag{1}$$

because under the specified conditions we can state nothing about the behavior of the residual term $R(x, x_0)$.

Let us denote the rectangle $\{a < x,\ x_0 < b\}$ by $\Delta$. Let us divide the segment [a, b] into $2\rho$ equal partial segments

$$\Delta_0, \ldots, \Delta_{2\rho-1}$$

and select for each segment a $\Delta_{2k}$ with an even subscript, respectively, for the point $x_k$. Let g stand for the $\rho$-dimensional cube of points $(x_1, \ldots, x_\rho)$, whose $x_k$ coordinates correspondingly belong to the partial segments $\Delta_{2k}$:

$$x_k\in\Delta_{2k} \qquad (k=1, \ldots, \rho).$$

Transferring $R(x, x_0)$ in (1) to the left side and substituting in place of x the numbers $x_1, \ldots, x$ , we obtain a linear system of $\rho$ equation

$$\sum_{j=0}^{\rho-1}\frac{(x_k-x_0)^j}{j!}f^{(j)}(x_0)=f(x_k)-R(x_k, x_0) \qquad (k=1, \ldots, \rho) \tag{2}$$

with $\rho$ unknown $f^{(j)}(x_0)$ and the determinant

$$W=W(x_1-x_0, \ldots, x_\rho-x_0)=$$
$$=\begin{vmatrix} 1 & (x_1-x_0) & \ldots & \frac{(x_1-x_0)^{\rho-1}}{(\rho-1)!} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 1 & (x_\rho-x_0) & \ldots & \frac{(x_\rho-x_0)^{\rho-1}}{(\rho-1)!}\end{vmatrix}=$$
$$=\sum_{k=1}^{\rho}a_{jk}(x_1-x_0, \ldots, x_\rho-x_0)\frac{(x_k-x_0)^j}{j!}=\sum_{k=1}^{\rho}a_{jk}\frac{(x_k-x_0)^j}{j!}, \tag{3}$$

where $\alpha_{jk}$ is the algebraic complement to determinant W corresponding to its element $(x_k - x_0)^f(j!)^{-1}$.

From (2) and (3) it follows that

$$f^{(j)}(x_0) = \frac{1}{W}\sum_{k=1}^{\rho} a_{jk}[f(x_k) - R(x_k, x_0)] \quad (j = 0, 1, \ldots, \rho - 1). \tag{4}$$

Function W differs only by the constant multiplier from the Vandermonde determinant equal to the product of all possible multipliers of the form $(x_k - x_l)$ where k  1, k, and l = 1, ..., , and since different $x_k$ and $x_l$ lie at a distance greater than the positive constant, then the function 1/W is bounded. Functions $\alpha_{jk}$ are also bounded, therefore from (4) follows the inequality

$$|f^{(j)}(x_0)| \leqslant c_1\left(\sum_{k=1}^{\rho}|f(x_k)| + |R(x_k, x_0)|\right) \tag{5}$$
$$(x_0 \in [a, b], \quad (x_1, \ldots, x_n) \in g).$$

Since the left side of (5) does not depend on $x_k$(k = 1, ..., ), therefore obviously

$$|f^{(j)}(x_0)| \leqslant c_2\sum_{k=1}^{\rho}\left(\|f(x_k)\|_{L_p(g)} + \|R(x_k, x_0)\|_{L_p(g)}\right) \leqslant$$
$$\leqslant c_3\left(\|f\|_{L_p(a, b)} + \|R(x, x_0)\|_{L_{p, x}(a, b)}\right) \quad (j = 0, 1, \ldots, \rho - 1), \tag{6}$$

where the sign of $L_{p,x}$ signifies that the norm is computed with respect to variable x.

Finally, from (6) follows

$$\|f^{(j)}\|_{L_p(a, b)} \leqslant c\left(\|f\|_{L_p(a, b)} + \|R\|_{L_p(\Delta)}\right) \quad (j = 0, 1, \ldots, \rho - 1). \tag{7}$$

when p = ∞, this is obvious, but when p is finite this is obtained if the left and right sides of (6) are raised to the power p, and if to the right side we apply the inequality

$$a + b \leqslant 2^{1 - \frac{1}{p}}(a^p + b^p)^{1/p} \quad (a, b > 0, 1 \leqslant p < \infty),$$

integrating both sides of the inequality with respect to $x_0$ and, finally, raising them to the power 1/p.

4.4.2. Let us note that if $\|R\|_{L(\Delta)} < \infty$, then by substituting expression 4.4.1(4) the derivatives $f^{(j)}(x_0)$ in equality 4.4.1(1) and integrating both its parts over the cube g of points $(x_0, \ldots, x_\rho)$ and dividing by the value of its volume $\varkappa$, we get the formula*)

$$f(x) = P(x) + F(x), \tag{1}$$

where

$$P(x) = \sum_{j=0}^{\rho-1} \sum_{k=1}^{\rho} \frac{1}{\varkappa} \int_g \frac{a_{jk}(x_1 - x_0, \ldots, x_\rho - x_0)}{W(x_1, \ldots, x_\rho)} \times \\ \times [f(x_k) - R(x_k, x_0)] \frac{(x - x_0)^j}{j!} dg \tag{2}$$

is a polynomial of degree $\rho - 1$ and

$$F(x) = \frac{1}{\varkappa} \int_g R(x, x_0)\, dg. \tag{3}$$

Formula (1) shows that function f can be represented as the sum of some polynomial P(x) of degree $\rho - 1$ and the residue F(x). Here P and F are explicitly expressed only in terms of the natural function f and its resudual Taylor's term R.

The residue R is usually given in terms of the derivative $f^{(\rho)}$ of the function f of order $\rho$.

Thus, no explicit intermediate derivatives $f^{(1)}, \ldots, f^{(\rho-1)}$ appear at all in the right side of formula (1), which enables us to estimate the norms of these derivatives in terms of the norms f and $f^{(\rho)}$.

4.4.3. Let us consider important particular cases of formulas 4.4.1 (6) and (7).

If function $f \in W_p^\rho(a, b)$, then it is equivalent to the wholly determined continuous function which we again will denote by f. The Taylor's formula 4.4.1(1) with residual term

$$R(x, x_0) = \frac{1}{(\rho - 1)!} \int_{x_0}^{x} (x - u)^{\rho-1} f^{(\rho)}(u)\, du, \tag{1}$$

where $f^{(\rho)} \in L_p(a, b)$ is valid for it.

*) S. M. Nikol'skiy [11].

From (1) follows the inequality

$$|R(x, x_0)| < c_1 \|f^{(\rho)}\|_{L_p(a, b)}, \qquad a < x, \ x_0 < b. \tag{2}$$

And further

$$\|R\|_{L_p(\Delta)} < c_2 \|f^{(\rho)}\|_{L_p(a, b)}, \tag{3}$$

where constant $c_2$ depends on b - a, p, and $\rho$. In this case, from 4.4.1(6) and (7) follows, respectively, the inequalities

$$|f^{(j)}(x_0)| < c_3 \left(\|f\|_{L_p(a, b)} + \|f^{(\rho)}\|_{L_p(a, b)}\right) = c_3 \|f\|_{W_p^{(\rho)}(a, b)}, \tag{4}$$

$$\|f^{(j)}\|_{L_p(a, b)} < c_3 \|f\|_{W_p^{(\rho)}(a, b)} \qquad (j = 0, 1, \ldots, \rho). \tag{5}$$

Both inequalities derived are directly extended to the case of the class of functions $W_{xp}(\mathcal{E})$, where $\mathcal{E} = [a, b] \times \mathcal{E}_1$ ($x \in [a, b]$, $y \in \mathcal{E}_1$, and $\mathcal{E} \subset R_n$) is the measurable set

$$\|f_{x_0}^{(j)}(x_0, y)\|_{L_p(\mathcal{E}_1)} < c_4 \|f\|_{W_{xp}^{\rho}(\mathcal{E})}, \tag{6}$$

$$\|f_x^{(j)}\|_{L_p(\mathcal{E})} = \left(\int_{\mathcal{E}_1} \|f_x^{(j)}(x, y)\|_{L_p(a, b)}^p dy\right)^{\frac{1}{p}} <$$

$$< c_3 \left(\int_{\mathcal{E}_1} \|f\|_{W_{xp}^{\rho}(a, b)}^p dy\right)^{\frac{1}{p}} < c_4 \|f\|_{W_{xp}^{\rho}(\mathcal{E})}$$

$$(j = 0, 1, \ldots, \rho). \tag{7}$$

If $f \in H_{xp}^r(\mathcal{E})$ and $\bar{r} = \rho - 1$, then f is written by formula 4.4.1(1), where

$$R(x, x_0) = \frac{1}{(\rho - 2)!} \int_{x_0}^{x} (u - x_0)^{\rho - 2} \left[f_x^{(\rho-1)}(u, y) - f_x^{(\rho-1)}(x_0, y)\right] du.$$

Hence

Hence

$$\int_a^b |R|^p dx_0 < c\left(\left|\int_a^x \int_{x_0}^x |f_x^{(p-1)}(u, y) - f_x^{(p-1)}(x_0, y)|^p du\, dx_0\right| + \right.$$

$$\left. + \left|\int_x^b \int_x^{x_0} |f_x^{(p-1)}(u, y) - f_x^{(p-1)}(x_0, y)|^p du\, dx_0\right|\right) =$$

$$= c\left(\left|\int_a^x \int_0^{x-x_0} |f_x^{(p-1)}(x_0+h, y) - f_x^{(p-1)}(x_0, y)|^p dh\, dx_0\right| + \right.$$

$$\left. + \left|\int_x^b \int_0^{x_0-x} |f_x^{(p-1)}(x_0-h, y) - f_x^{(p-1)}(x_0, y)|^p dh\, dx_0\right|\right) =$$

$$= c\left|\int_0^{x-a} \int_a^{x-h} |f_x^{(p-1)}(x_0+h, y) - f_x^{(p-1)}(x_0, y)|^p dh\, dx_0\right| +$$

$$+ c\left|\int_0^{b-x} \int_{x+h}^b |f_x^{(p-1)}(x_0-h, y) - f_x^{(p-1)}(x_0, y)|^p dx_0\, dh\right|$$

and

$$\left(\int_{\gamma_1} \int_a^b |R|^p dx_0\, dy\right)^{\frac{1}{p}} <$$

$$< \left[\int_0^{x-a} \left|\int_{\gamma_1} \int_a^{x-h} |f_x^{(p-1)}(x_0+h, y) - f_x^{(p-1)}(x_0, y)|^p dx_0\, dy\right| dh\right]^{\frac{1}{p}} +$$

$$+ \left[\int_0^{b-x} \left|\int_{\gamma_1} \int_{x+h}^b |f_x^{(p-1)}(x_0-h, y) - f_x^{(p-1)}(x_0, y)|^p dx_0\, dy\right| dh\right]^{\frac{1}{p}}.$$

Hence, noting that when $h > 0$

$$\left(\int_{\gamma_1} \int_a^{b-h} |f_x^{(p-1)}(x_0+h, y) - f_x^{(p-1)}(x_0, y)|^p dx_0\, dy\right)^{\frac{1}{p}} < Mh^a,$$

where

$$M = \|f\|_{h_p^r(g)},$$

we get

$$\|R\|_{L_p(\mathscr{E})} = \left(\int_a^b \int_{\mathscr{E}_1} \int_a^b |R|^p \, dx_0 \, dy \, dx\right)^{\frac{1}{p}} <$$

$$< \left(\int_a^b \int_0^{x-a} M^p h^{\alpha p} \, dh \, dx\right)^{\frac{1}{p}} + \left(\int_a^b \int_0^{b-x} M^p h^{\alpha p} \, dh \, dx\right)^{\frac{1}{p}} < c_2 M.$$

Therefore, from 4.5.1(7) follows

$$\|f_x^{(j)}\|_{L_p(\mathscr{E})} \leq c_3 (\|f\|_{L_p(\mathscr{E})} + \|R\|_{L_p(\mathscr{E})}) <$$
$$< c_4 (\|f\|_{L_p(\mathscr{E})} + M) < c_4 \|f\|_{H_p^r(\mathscr{E})} \qquad (j = 1, \ldots, \bar{r}). \tag{8}$$

Inequalities (4) and (5), just as (8), are valid also when $a = -\infty$ and $b = +\infty$. This is obvious for the case (4). But in the case (5) and (8), this follows from 6.1 (2) and (8); in the case when (5) $(1 < p < \infty)$, it follows from 9.2.2. The corresponding inequality for the interval $(a, \infty)$ reduces to the preceding by application of the extension theorem 4.3.6.

4.4.4. Let us note that in the determination of functions of classes $W_p^\rho(\mathscr{E})$ and $H_p^r(\mathscr{E})$, it was assumed that there exists on $\mathscr{E}$ generalized partial derivatives $f_x^{(j)}$ of orders $j = 1, \ldots, \rho - 1$ $(\bar{r} - 1)$, but it was not assumed that they have a finite norm in the $L_p(\mathscr{E})$-sense.

Inequalities 4.4.3 (7) and (8) show that the finiteness of the norms of these derivatives stems from the definition of the corresponding classes. But then derivatives $f_x^{(j)}$ $(j = 0, 1, \ldots, \rho - 1)$ are absolutely continuous on the closed segment $[\bar{a}, \bar{b}]$ with respect to the variable x for almost all $y \in \mathscr{E}_1$. Thus, the expansion of f by Taylor's formula

$$f(a, y) = \sum_0^{\rho-1} \frac{f_x^{(k)}(a, y)}{k!}(x-a)^k +$$
$$+ \frac{1}{(\rho-1)!} \int_a^x f_x^{(\rho)}(u, y)(x-u)^{\rho-1} \, du \tag{1}$$

obtains for almost all $y \in \mathcal{E}_1$ in the neighborhood of the endpoint a of segment $[a, b]$, as does the corresponding expansion in the neighborhood of the other endpoint b. Let us note the inequality

$$\|\Delta^{\rho}_{x_j, h} f\|_{L_p(g_{x_j \rho |h|})} \leq |h|^{\rho} \left\| \frac{\partial^{\rho} f}{\partial x_j^{\rho}} \right\|_{L_p(g)}, \tag{2}$$

that can be similarly interpreted: if the right-side of the inequality (2) is meaningful, then so is the left, and inequality (2) itself obtains.

The inversion of inequality (2) when $\rho = 1$ was obtained in 4.8.

Proof. First let $\rho = 1$; then by virtue of the equality

$$\Delta_{x_1, h} f(x) = \int_0^h f'_{x_1}(x_1 + t, y)\, dt, \quad x = (x_1, y) \in g_{|h|},$$

which holds for almost all admissible $y = (x_2, \ldots, x_n)$ and for all $x_1$ and h admissible for any y thus defined, we get (cf 1.3.2)

$$\|\Delta_{x_1 h} f\|_{L_p(g_{|h|})} \leq \left| \int_0^h \|f'_{x_1}(x_1 + t, y)\|_{L_p(g_{|h|})} dt \right| \leq$$
$$\leq \left| \int_0^h \|f'_{x_1}\|_{L_p(g)} dt \right| = |h| \|f'_{x_1}\|_{L_p(g)}.$$

Therefore for arbitrary $\rho$

$$\|\Delta^{\rho}_{x_1, h} f\|_{L_p(g_{x_1 \rho |h|})} = \|\Delta_{x_1, h} \Delta^{\rho-1}_{x_1, h} f\|_{L_p(g_{x_1 \rho |h|})} \leq$$
$$\leq |h| \|\Delta^{\rho-1}_{x_1, h} f'_{x_1}\|_{L_p(g_{x_1 (\rho-1)|h|})} \leq$$
$$\leq |h|^2 \|\Delta^{\rho-2}_{x_1, h} f''_{x_1}\|_{L_p(g_{x_1 (\rho-2)|h|})} \leq \ldots \leq |h|^{\rho} \|f^{(\rho)}_{x_1}\|_{L_p(g)}.$$

Corollary 1. Inequality*)

$$\left|\Delta^{\rho}_{x_1,h} g_\nu\right|_{L_p(\mathfrak{G})} < |h|^{\rho}\left|\frac{\partial^{\rho} g_\nu}{\partial x_1^{\rho}}\right|_{L_p(\mathfrak{G})} < (\nu h)^{\rho}\left|g_\nu\right|_{L_p(\mathfrak{G})} \tag{3}$$

obtains for the function $g_\nu(x) = g_\nu(x_1, \ldots, x_n) \in \mathfrak{M}_{x,\nu p}(\mathfrak{G})$ and $(\mathfrak{G}) = R_1 \times \mathfrak{G}_1$ (i.e., belonging to $L_p(\mathfrak{G})$ and of integral degree $\nu$ with respect to $x_1$, cf 3.4.1)(cf 3.2.2(7)). We must also consider that $\mathfrak{G}_{x_1,\delta} = \mathfrak{G}$, since $\mathfrak{G}$ is a set cylindrical in the $x_1$ direction.

Corollary 2. If $r > 0$ is integral, then

$$W^{(r)}_{x_j p}(g) \to H^{(r)}_{x_j p}(g). \tag{4}$$

This follows from the fact that

$$\frac{1}{h}\left|\Delta^2_{x_j h} f^{(r-1)}_{x_j}\right|_{L_p(g_{x_j 2|h|})} < \left|\Delta_{x_j h} f^{(r)}_{x_j}\right|_{L_p(g_{x_j|h|})} < 2\left|f^{(r)}_{x_j}\right|_{L_p(g)}.$$

4.4.5. Lemma. Let the sequence of functions $f_l (l = 1, 2, \ldots)$ belonging to $W^{\rho}_{x_1,p}(g)$, where $g \subset R_n$ is an open set, the given.

If for the two functions $f$ and $\varphi \in L_p(g)$

$$\|f - f_l\|_{L_p(g)} \to 0, \quad l \to \infty, \tag{1}$$

$$\left|\varphi - \frac{\partial^{\rho} f_l}{\partial x_1^{\rho}}\right|_{L_p(g)} \to 0, \quad l \to \infty, \tag{2}$$

then (in the generalized sense)

$$\varphi = \frac{\partial^{\rho} f}{\partial x_1^{\rho}} \quad \text{on } g \tag{3}$$

*) Inequality (3) is valid also for the trigonometric polynomials $g_\nu$ of order $\nu$ with respect to $x_1$ if we substitute $L_p^*(\mathfrak{G})$ for $L_p(\mathfrak{G})$.

Proof. First let $g = [a, b]$. From the fact that $f_l \in W^{\rho}_{xp}[a, b]$ ($l = 1, 2, \ldots$) it follows that for it or some function equivalent to it, again refer to by $f_l$, there obtains the expansion of $f_l$ by Taylor's formula

$$f_l(x) = \sum_0^{\rho-1} \frac{f_l^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{(\rho-1)!}\int_{x_0}^{x}(x-t)^{\rho-1} f_l^{(\rho)}(t)\,dt. \tag{4}$$

for any $x$ and $x_0 \in [a, b]$. By 4.4.3(4) and the conditions of the lemma

$$|f_k^{(l)}(x_0) - f_l^{(l)}(x_0)| \leqslant$$
$$\leqslant c\left[\|f_k - f_l\|_{L_p(a,b)} + \|f_k^{(\rho)} - f_l^{(\rho)}\|_{L_p(a,b)}\right] \to 0 \quad k, l \to \infty,$$

i.e., the uniform convergence

$$\lim_{l\to\infty} f_l^{(j)}(x_0) = \lambda_j(x_0) \quad (a \leqslant x_0 \leqslant b;\ j = 0, 1, \ldots, \rho - 1)$$

obtains on the segment $[a, b]$. But then after the passage to the limit in (4) as $l \to \infty$, we get

$$f(x) = \sum_0^{\rho-1} \frac{\lambda_k(x_0)}{k!}(x - x_0)^k + \frac{1}{(\rho-1)!}\int_{x_0}^{x}(x-t)^{\rho-1}\varphi(t)\,dt,$$

i.e.

$$\lambda_j(t) = f^{(j)}(t), \qquad j = 0, 1, \ldots, \rho - 1,$$
$$\varphi(t) = f^{(\rho)}(t) \qquad [a \leqslant t \leqslant b],$$

and the lemma stands as proven.

In the general case the lemma will obviously be proven if the validity of equality (3) stands proven for an arbitrary rectangular parallelipiped $\Delta \subset g$.

We will assert that $\Delta = [a, b] \times \Delta_1$, where $x_1 \in [a, b]$, $y \in \Delta_1$. By virtue of the conditions posed on function $f_l$ and the fact that there they are countable, we can take them to the modifications on a set of measure zero such that there exists that $\Delta_1' \subset \Delta_1$ of complete measure so that for all $y \in \Delta_1'$, all functions $f_l$ are locally absolutely continuous with respect to x. It follows from (1) and (2) that for almost any $y \in \Delta_1'$ for which (dependent on $y$) the subsequence of $l_s$ holds (cf 1.3.8)

$$\|f - f_{l_s}\|_{L_p(a,b)} \to 0,$$
$$\left\|\varphi - \frac{\partial^\rho f_{l_s}}{\partial x_1^\rho}\right\|_{L_p(a,b)} \to 0.$$

But then for the specified $y$ $\quad \varphi(x_1, y) = \frac{\partial^\rho f}{\partial x_1^\rho}(x_1, y)$

for almost all $x_1 \in [a, b]$. This in fact leads to the confirmation of the lemma.

4.4.6. Theorem. Let $g \subset R_n$ be an open set and $g_1$ be another open bounded set such that $g_1 \subset \bar{g}_1 \subset g$. Then, if $f \in W_p^l(g)$, then

$$\|f^{(s)}\|_{L_p(g_1)} \leqslant c_{g_1} \|f\|_{W_p^l(g)} \quad (|s| \leqslant l), \tag{1}$$

where $c_{g_1}$ is a constant dependent on p, l, and $g_1$, but not f.

This theorem easily follows by induction from inequality 4.4.3(7). Considering that $g_1$ can be covered with a finite number of cubes $\Delta \subset g$ with edges parallel to the coordinate axes, it is sufficient that the theorem be proven for one of them.

4.4.7. Lemma. Let there be given the sequence of functions

$$f_k = f_k(x_1, \ldots, x_n) = f_k(x) \quad (k = 1, 2, \ldots),$$

integrable on $g$ in the p-th degree ($1 \leqslant p \leqslant \infty$) together with their partial derivatives of to order $\rho$ inclusively appearing below in (1) and, moreover, that the functions

$$f, f_{\alpha_1}, f_{\alpha_1\alpha_2}, \ldots, f_{\alpha_1 \ldots \alpha_s}$$

be given ($\alpha_1 + \ldots + \alpha_s \leqslant \rho$, $\alpha_j$ = positive integers, $1 \leqslant s \leqslant n$) are such that

$$\lim_{k \to \infty} \|f - f_k\|_{L_p(g)} = 0,$$
$$\lim_{k \to \infty} \left\|f_{\alpha_1} - \frac{\partial^{\alpha_1} f_k}{\partial x_1^{\alpha_1}}\right\|_{L_p(g)} = 0, \tag{1}$$

$$\lim_{k\to\infty}\left\| f_{\alpha_1\ldots\alpha_s} - \frac{\partial^{\alpha_1+\ldots+\alpha_s} f_k}{\partial x_1^{\alpha_1}\ldots\partial x_s^{\alpha_s}}\right\|_{L_p(g)} = 0.$$

Then (in the generalized sense)

$$f_{\alpha_1} = \frac{\partial^{\alpha_1} f}{\partial x_1^{\alpha_1}},\ f_{\alpha_1\alpha_2} = \frac{\partial^{\alpha_1+\alpha_2} f}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}},\ \ldots,\ f_{\alpha_1\ldots\alpha_s} = \frac{\partial^{\alpha_1+\ldots+\alpha_s} f}{\partial x_1^{\alpha_1}\ldots\partial x_s^{\alpha_s}}. \tag{2}$$

This is lemma 4.4.5 for the case $s = 1$. The transition to the general case is made without difficulty by induction.

4.4.8. If the functions $f_k$ and their partial derivatives of the corresponding orders considered in 4.4.7 are continuous on $g$, then these partial derivatives do not depend on the order of integration, therefore the generalized derivatives 4.4.7(2) also do not depend on the order of integration almost everywhere on $g$.

4.4.9. Theorem. Let functions $f_1$, $f_2$, ... be continuous together with their partial derivatives up to order inclusively and together with the function $f$ satisfies the conditions of lemma 4.4.7 and, where equalities (1) are met for any $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leqslant \rho$. Let, moreover, domain $g$ of variation of the variables $x_1, \ldots, x_n$ be mutually uniquely mapped onto domain $\tilde{g}$ of variables $t_1, \ldots, t_n$ by means of the functions

$$x_j = \varphi_j(t_1, \ldots, t_n) \tag{1}$$

that are continuous, with partial derivatives that are continuous and bounded on $\tilde{g}$, of orders not exceeding $\rho$ and such that the Jacobian

$$D(t) = \frac{D(x_1, \ldots, x_n)}{D(t_1, \ldots, t_n)} > k > 0.$$

Then the function $F(t_1, \ldots, t_n) = f(\varphi_1, \ldots, \varphi_n)$

is integrable in the p-th degree on $\tilde{g}$ together with its partial derivatives of order up to $\rho$ inclusively, where these partial derivatives are computed by classical formulas, just as if function $f$ had continuous partial derivatives.

Proof. Acuually, it follows from the conditions of the lemma that $f^{(s)} \in L_p(g)$ for all $s$ with $|s| \leqslant \rho$ that the following relation

$$\int_{g} |f^{(s)}(x)|^p\,dx = \lim_{k\to\infty} \int_{g} |f_k^{(s)}(x)|^p\,dx =$$
$$= \lim_{k\to\infty} \int_{\tilde g} |f_k^{(s)}(\varphi_1, \ldots, \varphi_n)|^p D(t)\,dt =$$
$$= \int_{\tilde g} |f^{(s)}(\varphi_1, \ldots, \varphi_n)|^p D(t)\,dt,$$

obtains, and since $D(t)$ is bounded from below by a positive constant, then $f^{(s)}(\varphi_1, \ldots, \varphi_n) \in L_p(\tilde g)$. Let us set

$$F_k(t) = f_k(\varphi_1, \ldots, \varphi_n), \quad F(t) = f(\varphi_1, \ldots, \varphi_n).$$

By the classical formula the derivative of $F_k$ of order $l = (l_1, \ldots, l_n)$ is of the form

$$F_k^{(l)}(t) = \sum_{|s| \leq |l|} \alpha_s f_k^{(s)}(\varphi_1, \ldots, \varphi_n), \tag{2}$$

where $\alpha_s$ are functions continuous and bounded on $\tilde g$, defined by transformations (1). Since $f_k^{(s)} \to f^{(s)}$ $(k \to \infty)$ in the $L_p(g)$-sense, then based on the following $f_k^{(s)}(\varphi_1, \ldots, \varphi_n) \to f^{(s)}(\varphi_1, \ldots, \varphi_n)$ in the $L_p(\tilde g)$-sense and from (2) it follows that after passage to the limit as $k \to \infty$,

$$F^{(l)}(t) = \sum_{|s| \leq |l|} \alpha_s f^{(s)}(\varphi_1, \ldots, \varphi_n) \tag{2'}$$

for almost all $t \in \tilde g$. Generalised derivatives, in the main, appear in formula (2). If the latter are continuous, then (2) is the classical formula.

In these considerations we assume that $p$ is finite $(1 \leq p < \infty)$. When $p = \infty$, nothing novel emerges from the lemma conditions, because then $f_k^{(s)}$ $(|s| \leq \rho)$ converges uniformly to $f^{(s)}$.

## 4.5. More on Sobolev Averaging*)

Let $g \subset R = R_n$ be an open set, $1 \leq p \leq \infty$, the function $f \in L_p(g)$, and

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int \varphi\left(\frac{x-u}{\varepsilon}\right) f(u)\,du \qquad (f = 0 \text{ on } R - g) \tag{1}$$

*) S. L. Sobolev [4].

is its $\varepsilon$-averaging (cf 1.4).

Obviously, $f_\varepsilon(x)$ is infinitely differentiable on R and

$$f_\varepsilon^{(s)}(x)=\frac{1}{\varepsilon^{n+|s|}}\int\varphi^{(s)}\left(\frac{x-u}{\varepsilon}\right)f(u)\,du \tag{2}$$

for any integral vector $s=(s_1, \ldots, s_n)\geqslant 0$.

4.5.1. Let us, as usual, use $g_\varepsilon$ to stand for the set of points $x\in g$ situated from the boundary of $g$ by a distance greater than $\varepsilon>0$.

Suppose $f\in L_p(g)$ and $\partial f/\partial x_1\in L_p(g)$. If $x\in g_\varepsilon$, then in the equality

$$\frac{\partial}{\partial x_1}f_\varepsilon(x)=\frac{1}{\varepsilon^{n+1}}\int\varphi'_{x_1}\left(\frac{x-u}{\varepsilon}\right)f(u)\,du$$

under the integer, which we can assume to be distributed over a sphere of radius $\varepsilon$ with center at x is included in function f, absolutely continuous with respect to $u_1$ for almost all $(u_2, \ldots, u_n)$, therefore this integral can be integrated by parts with respect to $u_1$ (when $x\notin g_\varepsilon$ this generally speaking is not so, and f can be essentially discontinuous in this sphere).

Considering that

$$\frac{\partial}{\partial u_1}\varphi\left(\frac{x-u}{\varepsilon}\right)=-\frac{1}{\varepsilon}\varphi'_{x_1}\left(\frac{x-u}{\varepsilon}\right)$$

and that $\varphi\in 0$ outside the indicated sphere, we get

$$\frac{\partial}{\partial x_1}f_\varepsilon(x)=-\frac{1}{\varepsilon^n}\int\frac{\partial}{\partial u_1}\varphi\left(\frac{x-u}{\varepsilon}\right)f(u)\,du=$$
$$=\frac{1}{\varepsilon^n}\int\varphi\left(\frac{x-u}{\varepsilon}\right)\frac{\partial f}{\partial x_1}(u)\,du=\left(\frac{\partial f}{\partial x_1}\right)_\varepsilon(x).$$

Generally, if we consider the functions f, $\partial f/\partial x_{j_1}$, $\partial^2 f/\partial x_{j_1}\partial x_{j_2}$, ..., then, reasoning by induction, we get

$$D^s(f_\varepsilon)=(D^s f)_\varepsilon\ (x\in g_\varepsilon),\qquad D^s=\frac{\partial^{|s|}}{\partial x_1^{s_1}\ldots\partial x_n^{s_n}}. \tag{1}$$

In the definition of class $W_p^1(g)$ it was assumed that any function f belonging to it belongs to $L_p(g)$ together with its partial derivatives $f^{(1)}$

of order $l$. As for the subsumed derivatives $f^{(s)}$ ( $|s| < l$), then they, naturally, are assumed to exist (in the generalised sense) on $g$, but need not necessarily be summable in the p-th degree on $g$.

In 4.4.6 it was shown that if $f \in W_p^l(g)$ and $\sigma \subset g$ is an arbitrary n-dimensional sphere, then $f^{(s)} \in L_p(\sigma)$ ($|s| \leq l$). But then for a sufficiently small $\varepsilon > 0$, equality (1) obtains on $\sigma$, for which by 1.4(4) it follows that for $1 \leq p < \infty$ (or when $p = \infty$, on the assumption that $D^s f$ is uniformly continuous on $R_n$) that

$$\|D^s(f_\varepsilon) - D^s f\|_{L_p(\sigma)} = \|(D^s f)_\varepsilon - D^s f\|_{L_p(\sigma)} \to 0 \quad (\varepsilon \to 0, |s| \leq l). \tag{2}$$

Considering that $f_\varepsilon$ is an infinitely differentiable function and, therefore, for it the result of the operation $D^s f_\varepsilon$ does not depend on the order of differentiation (with respect to $s_1, \ldots, s_n$) and that $\sigma \subset g$ is an arbitrary sphere, we arrive at the following conclusion.

If $f \in W_p^l(g)$, then for the indicated $s$ the derivatives $f^{(s)}$ almost everywhere are independent of the order of differentiation.

4.5.2. Theorem. Let $f$ and $\lambda$ be functions locally summable on $g$. If the function $\lambda$ is a derivative with respect to $x_1$ on $g$ in the Sobolev sense, then it is also the derivative

$$\lambda = \frac{\partial f}{\partial x}, \tag{1}$$

in the sense we employed (cf beginning of section 4.1).

Proof. Let

$$f_\varepsilon(x) = \int \varphi_\varepsilon(x-u) f(u)\, du, \tag{2}$$

where $\varepsilon$ is the averaging of $f$; then

$$\begin{aligned} f_\varepsilon'(x) &= \int \frac{\partial}{\partial x_1} \varphi_\varepsilon(x-u) f(u)\, du = \\ &= -\int \frac{\partial}{\partial u_1} \varphi_\varepsilon(x-u) f(u)\, du = \\ &= \int \varphi_\varepsilon(x-u) \lambda(u)\, du \quad (x \in g_\varepsilon). \end{aligned} \tag{3}$$

The last equality obtains by virtue of $\lambda$ being a derivative of f with respect to $x_1$ on g in the Sobolev sense, and the fact that $\varphi_\varepsilon(x - u)$ for a fixed $x \in g_\varepsilon$ is a finite function of u.

Since f and $\lambda$ are locally summable on g, then from (2) and (3) it follows that (cf 4.5.1(2))

$$\|f_\varepsilon - f\|_{L(\sigma)} \to 0, \ \varepsilon \to 0,$$
$$\|f'_\varepsilon - \lambda\|_{L(\sigma)} \to 0, \ \varepsilon \to 0,$$

on any close sphere $\sigma \subset g$, but then by lemma 4.4.5,(1) obtains.

## 4.6 Estimates of Increment in Direction

Let us consider the linear transformation

$$x_l = \sum_{k=1}^{n} a_{lk} t_k \qquad (l = 1, \dots, n) \tag{1}$$

with a determinant not equal to zero and mapping mutually uniquely points $x = (x_1, \dots, x_n) \in g$ into points $t = (t_1, \dots, t_n) \in \tilde{g}$. It satisfies the requirements of theorem 4.4.9. If $f \in W_p^1(g)$, then we already know that 4.5.1(2) is satisfied for each sphere $\sigma \subset \bar{\sigma} \subset g$, then by theorem 4.4.9, the function $\tilde{f}(t) = f(x_1(t), \dots, x_n(t))$ transformed by means of (1) has on any sphere $\sigma \subset g$, consequently, on g as well all derivatives $\tilde{f}^{(s)}(t)$ with respect to t, where $|s| \leq 1$, calculated moreover by the classical roles. Clearly, $\tilde{f} \in W_p^1(\tilde{g})$.

In order to define the derivative of function $f \in W_p^1(g)$ in the direction of vector $h \in R_n$, let us introduce the orthogonal transformation (1) such that change of $t_1$ in the positive direction for fixed $t_2, \dots, t_n$ entails change of x in the direction h. We will assert that the derivative of f in the direction of h is defined by the equalities

$$\frac{\partial f}{\partial h} = \frac{\partial f}{\partial t_1} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cos(h, x_j), \tag{2}$$

$$\frac{\partial^s f}{\partial h^s} = \frac{\partial^s f}{\partial t_1^s} = \sum_{|s|=s} f^{(s)} h^s$$
$$(s = (s_1, \dots, s_n)\colon |s| \leq l, \ h^s = h_1^{s_1} \dots h_n^{s_n}, \ |h| = 1). \tag{3}$$

Obviously, this definition does not depend on the choice of the orthogonal transformation (1) subject to the indicated requirements.

The equality

$$\Delta_h f(x) = f(x+h) - f(x) = |h| \int_0^1 f'_h(x+th)\,dt, \tag{4}$$

obtains, from whence

$$\|\Delta_h f(x)\|_{L_p(g_h)} \leq |h| \int_0^1 \|f'_h(x+th)\|_{L_p(g_h)}\,dt = |h| \|f'_h\|_{L_p(g)}, \tag{5}$$

where h is an arbitrary vector. It is also easy to derive a more general inequality (in particular, one containing the relation

$$\|\Delta_h^\rho f(x)\|_{L_p(g_{\rho h})} \leq |h|^\rho \|f_h^{(\rho)}\|_{L_p(g)}. \tag{6}$$

in analogous to 4.4.2(2)).

## 4.7. Completeness of Spaces W, H, and B

Theorem. Whatever the open set $g \subset R_n$, the spaces

$$W^l_{up}(g),\ W^l_{up}(g),\ H^r_{up}(g),\ H^r_{up}(g),\ B^r_{up\theta}(g),\ B^r_{up\theta}(g)$$

are complete.

There are different variants of the definitions of these classes. Generally, there are not equivalent for an arbitrary open set g. We will prove completeness for one of the variants: (4.3.1(1) for spaces $W^r_{up}$, 4.3.3(5) for $H^r_{up}$, and 4.3.4(2) or 4.3.4(4) for $B^r_{up\theta}$. The proof for not materially differ for the other variants.

Proof. We will assert that $g_1 \subset \tilde{g}_1 \subset g$ is a bounded open set. Let there be specified the sequence of functions $f_k \in W^l_{up}(g)$ $(k = 1, 2, \ldots)$ satisfying the Cauchy condition in the matrix $W^l_{up}(g)$.

Then (cf 4.4.6 and 4.4.7)

$$\|f_k^{(s)} - f_j^{(s)}\|_{L_p(g_1)} \leqslant c_{g_1} \|f_k - f_j\|_{W^l_{up}(g)} \to 0$$
$$k,\ j \to \infty, \quad s = (s_1, \ldots, s_m, 0, \ldots, 0),\ |s| \leqslant l, \tag{1}$$

there exists the function f for which

$$\|f_k^{(s)} - f^{(s)}\|_{L_p(g_1)} \to 0 \qquad (k \to \infty). \tag{2}$$

It belongs to $W^l_{up}(g_1)$, because $f_k \in W^l_{up}(g_1)$. Assigning $\varepsilon > 0$, we can specify $N > 0$ such that for k, j > N

$$\|f_k - f_j\|_{L_p(g_1)} + \sum_{|s|=l} \|f_k^{(s)} - f_j^{(s)}\|_{L_p(g_1)} \leqslant \|f_k - f_j\|_{W^l_{up}(g)} < \varepsilon \tag{3}$$

for any $g_1$. Passage to the limit in the first term of (3) as $j \to \infty$ by virtue of (2) leads to the very same expression, where we must substitute f for $f_j$, i.e., the relation

$$\|f_k - f\|_{W^l_{up}(g_1)} \leqslant \varepsilon \qquad (k > N)$$

obtains for any $g_1$, and so for g. Here f belongs (in addition to $f_k$) to $W^l_{up}(g)$. In this way the completeness of $W^l_{up}(g)$, in particular, $W^{lj}_{x_j p}(g)$ is proving, but then it is obvious that $W^l_{up}(g)$ is also complete.

Let us now prove the completeness of $B^r_{up\theta}(g)$ $(1 \leqslant \theta \leqslant \infty,\ B^r_{up\infty} = H^r_{up})$. We can show (cf remark at the end of the book to section 4.3.6) that functions of the classes $B^{r,\ldots,r}(g) = B^{r,\ldots,r}_{up\theta}(g)$ can be extended from $g_1 \subset \tilde{g}_1 \subset g$ to R with preservation of the norm (with respect to g), i.e., for each $f \in B^{r,\ldots,r}(g)$ its extension $\bar{f}(\bar{f} = f$ on $g_1)$ can be specified, such that

$$\|\bar{f}\|_{B^{r,\ldots,r}(R)} \leqslant c\|f\|_{B^{r,\ldots,r}(g)}. \tag{4}$$

But further, it will be shown (5.6.2) that

$$B^{r,\ldots,r}(R) = B^r(R). \tag{5}$$

Therefore

$$
\begin{gathered}
0 \leftarrow \|f_k - f_j\|_{B^r(g)} \gg \|f_k - f_j\|_{B^{r,\dots,r}(g_1)} \gg \\
\gg \|\bar f_k - \bar f_j\|_{B^{r,\dots,r}(R)} \gg \|\bar f_k - \bar f_j\|_{B^r(R)} \gg |f_k^{(s)} - f_j^{(s)}|_{L_p(g_1)} \\
(s = (s_1, \dots, s_m, 0, \dots, 0),\ |s| \leqslant \bar r,\quad r = \bar r + \alpha,
\end{gathered} \tag{6}
$$

$\bar r$ is an integer, $0 < \alpha \leqslant 1$).

The first inequality is trivial ($B^r$ — $B^{r,\dots,r}$); the second is valid by virtue of the above-noted theorem on extension; here $\bar f_j$ and $\bar f_k$ are functions extending, by this theorem, functions $f_k$ and $f_j$, respectively, from the set $g_1$; the third inequality follows from (5); the fourth will be proven later (6.2(8)). Notice that in the case $H^r_{x_1 p}(g)$ ($m = 1$) the inequality between the first and the last terms in (6) follows immediately from (4.4.3(8)) without bringing in the theorem on extension.

Obviously, it follows from (6) that when $\rho = \bar r$

$$
|f^{\rho}_{ku} - f^{\rho}_{ju}|_{L_p(\bar g_1)} \to 0 \qquad (k, j \to \infty,\ u \in R_m), \tag{7}
$$

where $f_{ku}$ is a derivative of $f_k$ of order $\rho$ in the direction $u$, whatever be the $g_1 \subset \bar g_1 \subset g$.

Now, inspecting the class $H^r_{up}(g)$, and for simplicity asserting that $0 < \alpha < 1$, and assigning $\varepsilon > 0$, we get ($g_1 \subset \bar g_1 \subset g_u$)

$$
\begin{gathered}
\|f_k - f_j\|_{L_p(g)} + \\
+ \frac{|(f^{\rho}_{ku}(x+u) - f^{\rho}_{ju}(x+u)) - (f^{\rho}_{ku}(x) - f^{\rho}_{ju}(x))|_{L_p(g_1)}}{|u|^{\alpha}} < \\
< \|f_k - f_j\|_{H^r_{up}(g)} < \varepsilon \quad (k, j > N,\ u \in R_m),
\end{gathered} \tag{8}
$$

where N is sufficiently large. If k is fixed and $j \to \infty$, then at the limit the first term in (8) by virtue of (7) is converted into the same expression, where $f_j$ must be replaced with f. In this expression let us replace $g_1$ with

$g_u$, which obviously is legitimate. Taking the upperbound of the resulting expression with respect to $u$, we get

$$\|f_k - f\|_{H^r_{up}(g)} \leqslant \varepsilon \quad (k > N),$$

and since it follows from the fact that $f_k \in H^r_{up}(g)$, $f \in H^r_{up}(g)$, then the completeness of $H^r_{up}(g)$ provided $\alpha < 1$ stands proven. When $\alpha = 1$, the first difference*of (8) must be replaced by the second, reasoning analogously.

But for the class $B^r_{up\theta}(g)(1 \leq \theta < \infty)$, for any $\varepsilon > 0$ $(g_1 \subset \bar{g}_1 \subset g_{ku})$

$$\|f_\mu - f_\nu\|_{L_p(g)} + \left(\int\limits_{\lambda < |u| < \varkappa} |u|^{-m-\theta\alpha} \|\Delta^k_u (f^\rho_\mu - f^\rho_\nu)\|^\theta_{L_p(g_1)} du\right)^{1/\theta} \leqslant \varepsilon$$

$(\mu, \nu > N,\ u \in R_m,\ \rho = \bar{r},\ k \geqslant 2,$ provided $\alpha < 1 \quad k \geqslant 1)$, (9)

where $0 < \lambda < \varkappa$ are arbitrary numbers.

The passage to the limit as $\nu \to \infty$ reaches the same inequality where $f_\nu$ must be replaced by $f^\rho_\nu$. This follows from the fact that here we can employ the Lebesgue on the limit under the integral sign. The issue is that the u-dependent norm under the sign of the integral in (9) boundedly approaches the same number, where $f$ appears instead of $f_\nu$ (cf (6) and (7)). In the resulting inequality valid for any indicated $\lambda$ and $\varkappa$, we can obviously set $\lambda = 0$ and $\varkappa = \infty$ and replace $g_1$ with $g_{ku}$, which entails

$$\|f_\mu - f\|_{B^r_{up\theta}} \leqslant \varepsilon, \quad \mu > N.$$

Moreover, from the fact that $f_\mu \in B^r_{up\theta}(g)$ follows $f \in B^r_{up\theta}(g)$. The completeness of $B^r_{up\theta}(g)$ is proven.

## 4.8. Estimate of A Derivative by the Difference Relation

Theorem (inverting inequality 4.4.4(2)). Suppose function $f(x) = f(x_1, y)$ is given on an open set $g$, is locally summable on it, and satisfies the inequality

$$\int\limits_{g_h} \left|\frac{\Delta_{x_1, h} f(x)}{h}\right|^p dx \leqslant M \quad (1 < p < \infty), \tag{1}$$

where M does not depend on h.

*) $f^\rho_{ku} - f^\rho_{j\hat{u}}$

Then on $g$ there exists the derivative $\partial f/\partial x_1$ exhibiting the property

$$\int_g \left|\frac{\partial f}{\partial x_1}\right|^p dx \leq M. \tag{2}$$

Proof. Let us assign two open cubes $\Delta \subset \Delta_1 \subset \bar{\Delta}_1 \subset g$ with faces parallel to the coordinate axes and strictly embedded one in the other. We have

$$\frac{\Delta_{x_1, h} f_\varepsilon(x)}{h} = \left(\frac{\Delta_{x_1, h} f(x)}{h}\right)_\varepsilon$$

$((\cdot)$ — $\varepsilon$-averaging),

Therefore from (1) and 1.4(7) it follows that for sufficiently small h and $\varepsilon$

$$\int_\Delta \left|\frac{\Delta_{x_1, h} f_\varepsilon(x)}{h}\right|^p dx \leq \int_{\Delta_1} \left|\frac{\Delta_{x_1, h} f(x)}{h}\right|^p dx \leq M.$$

Passing to the limit as h --- $\varepsilon$, we get

$$\int_\Delta \left|\frac{\partial f_\varepsilon}{\partial x_1}\right|^p dx \leq M. \tag{3}$$

We have

$$f_\varepsilon(x_1', y) - f_\varepsilon(x_1, y) = \int_{x_1}^{x_1'} \frac{\partial f_\varepsilon}{\partial x_1}(t, y)\,dt, \tag{4}$$

where we assume that $y = (\xi_2, \ldots, \xi_n)$ runs through the orthogonal parallelepiped

$$\Delta_* = \{x_j < \xi_j < x_j + h_j;\ j = 2, \ldots, n\}$$

and

$$[x_1, x_1'] \times \Delta_* \subset \Delta.$$

Integrating (4) with respect to $y \in \Delta_*$, we get

$$\int_{\Delta_*} [f_\varepsilon(x_1', y) - f_\varepsilon(x_1, y)]\,dy = \int_\Delta dy \int_{x_1}^{x_1} \frac{\partial f_\varepsilon}{\partial x_1}(t, y)\,dt. \tag{5}$$

From (3) it follows that there exists a sequence of numbers $\varepsilon_k' \to 0$ and the function $\psi \in L_p(\Delta)$ such that $\partial f_{\varepsilon'_k}/\partial x_1 \to \psi$ weakly in the $L_p(\Delta)$-sense (cf 1.3.11). On the other hand, from the fact that $\|f_{\varepsilon'_k} - f\|_{L_p(\Delta)} \to 0$, we can separate the subsequence $\{\varepsilon_k\}$ from the sequence $\{\varepsilon'_k\}$ such that

$$\int_{\Delta_0} f_{\varepsilon_k}(x_1, y)\,dy \to \int_{\Delta_0} f(x_1, y)\,dy, \quad \varepsilon_k \to 0,$$

for all $x_1$ on the same set $\mathscr{E} \subset [a, b] = \operatorname{np}_{x_1}\Delta$ (projection of $\Delta$ on the $x_1$ axis) of measure (linear) b - a. In this case, if in (5) we set $\varepsilon = \varepsilon_k$, then at the limit as $\varepsilon_k \to 0$ for any $x_1$, $x_1' \in \mathscr{E}$ we get

$$\int_{\Delta_0} [f(x_1', y) - f(x_1, y)]\,dy = \int_{\Delta_0} dy \int_{x_1}^{x_1'} \psi(t, y)\,dt.$$

If we decompose the two parts of this inequality into $h_1, \ldots, h_n$ and pass to the limit as $h_1 \to 0$, then $h_2 \to 0$ and so on, then we get for almost all $y = (x_2, \ldots, x_n)$ and

$$x_1 \in \mathscr{E}, \quad x_1' \in \mathscr{E} \quad ((x_1, y), (x_1', y) \in \Delta):$$

$$f(x_1', y) - f(x_1, y) = \int_{x_1}^{x_1'} \psi(t, y)\,dt. \tag{6}$$

In fact, this equality is valid for almost all admissible $y$ and almost all admissible $x_1$, $x_1' \in [a, b]$, since its right side is continuous with respect to $x_1$, $x_1'$. It indicates the existence on $\Delta$ of a (generalized) partial derivative $\partial f/\partial x_1 = \psi \in L_p(\Delta)$ and by virtue of the arbitrariness of $\Delta$, it also indicates the existence of $\partial f/\partial x_1 \in L_p(\Omega)$, whatever the open $\Omega \subset \bar{\Omega} \subset g$.

Since we now already know that the integrand function in (1) tends almost everywhere on $\Omega$ to $|\partial f/\partial x_1|^p$ then (Fatou theorem 1.3.10)

$$\int_{\Omega} \left|\frac{\partial f}{\partial x_1}\right|^p dx \leqslant \sup_h \int_{\Omega} \left|\frac{\Delta_{x_1, h} f}{h}\right|^p dx \leqslant M.$$

and by virtue of the arbitrariness of $\Omega \subset \bar{\Omega} \subset g$, (2) is valid.

4.8.1. Theorem 4.8 when $p = \infty$ and $n = 1$ is a familiar theorem from theory of functions of a real variable: if function f satisfies on the interval (a, b) the Lipshits condition with constants M, then it has almost everywhere on (a, b) a derivative satisfying the inequality $|f'(x)| \leq M$ (cf P. S. Aleksandrov and A. N. Kolmogorov [1]).

4.8.2. Theorem 4.8 when $p = 1$, $n = 1$ changes into the following: if for a function f locally summable on (a, b) the inequality

$$\int_a^{b-h} |f(x+h) - f(x)|\, dx \leq Mh \quad (0 < h < b-a),$$

is satisfied, then it is equivalent to some function that we will again designate by f, with bounded variation on (a, b) and

$$\operatorname*{Var}_{(a,b)} f \leq M.$$

In fact, reasoning as in the beginning of the proof of theorem 4.8, we get

$$\int_\Delta |f'_\varepsilon|\, dx \leq M,$$

where $\Delta$ is an arbitrary integral such that $\Delta \subset \Delta_1 \subset \bar{\Delta}_1 \subset (a, b)$. Therefore

$$\operatorname*{Var}_{(a,b)} f_\varepsilon = \int_a^b |f'_\varepsilon|\, dx \leq M. \tag{1}$$

Since $f \in L(\Delta_1)$, then $\int_{\Delta_1} |f_\varepsilon - f|\, dx \to 0$ and by virtue of the arbitrariness of $\Delta_1 \subset \bar{\Delta}_1 \subset (a, b)$, there exists the sequence $\varepsilon_k \to 0$ such that

$$f_{\varepsilon_k}(x) \to f(x) \tag{2}$$

almost everywhere on (a, b). But by the Helly theorem (cf I. P. Natanson [1]), from condition (1) and the fact that (2) is satisfied even if at only one point of the interval (a, b), it follows that there exists a subsequence $f_{\varepsilon'_k}$ close the $\{\varepsilon'_k\}$ of sequence $\{\varepsilon_k\}$ such that $f_{\varepsilon'_k}$ tends everywhere on (a, b) to some function $\psi$ bounded on (a, b) and

$$\operatorname*{Var}_{(a,b)} \psi \leq M.$$

But then $\psi$ and f are equivalent on (a, b).

## CHAPTER V DIRECT AND INVERSE THEOREMS OF THE THEORY OF APPROXIMATIONS. EQUIVALENT NORMS

### 5.1. Introduction

Everywhere in this paragraph we will assume that $u = (x_1, \ldots, x_m)$ $R_m$, $y = (x_{m+1}, \ldots, x_n)$ and we will consider the cylindrical measurable set $\mathcal{E} = R_m \times \mathcal{E}'$ of points $x = (u, y) = (x_1, \ldots, x_n)$ where $u \in R_m$, $y \in \mathcal{E}'$. We will let $R_m$ also stand for the subspace $R_n$ of points $(u, 0) = (x_1, \ldots, x_m, 0, \ldots, 0)$. When $m = n$, $\mathcal{E} = R_n$, the case $m = 0$ would be of little interest.

This chapter will be devoted to studying approximations of functions from the H, W, and B classes (of Chapter IV) given on the indicated cylindrical set $\mathcal{E}$. Functions of classes $H_p$ and $W_p$ will be approximated by integral functions of the exponential type (with respect to $u$) in the metric $L_p$, while periodic functions of the classes $H_p^*$ and $W_p^*$ will be approximated by trigonometric polynomials (in $u$) in the metric $L_p^*$.

The direct theorems of the theory of approximation(Jackson type) will be proven for the classes H and W, showing that the numbers r or systems of numbers $(r_1, \ldots, r_m)$ determining the class also define the order of approximation of the functions belonging to it.

We will also prove the inverse theorems of the theory of approximations (Bernshteyn type), showing that the order of approximation of a given function f by means of functions of a finite system for trignometric polynomials frequently completely defines the class H (but not W) to which function f belongs. In several cases of analytic interest, necessary and sufficient conditions will be obtained in the language of orders of approximation for the membership of function f to a given H-class. The concept of the best approximation, which can be placed to P. L. Chebyshev considered as an important artifice in the expression of these theorems.

Classes $B^r_{p\theta}$ will also be examined from this point of view. The functions belonging to them are also completely characterized by the behavior of their best approximations in terms of integral functions of the exponential type or (in the periodic case) by trigonometric polynomials. Namely, for the function to belong to a given B class it is necessary and sufficient that a certain series composed of its best approximations converges. We will see that the definition of classes $B^r_{p\theta}$ in the language of the best approximation naturally is extended to the case $\theta = \infty$ and leads to the equivalency: $B^{(r)}_{p\infty} = H^{(r)}_p$.

In the chapter, based on periodic approximations, we will obtain many different equivalent definitions of norms in the H and B classes. The actual fact of equivalency will be reduced to certain inequalities, in particular, inequalities between partial derivatives of the same function.

Let us consider the functions $g_\nu(x) = g_\nu(u, y)$, $\nu = (\nu_1, \ldots, \nu_m)$, defined on $\mathcal{E} = R_m \times \mathcal{E}'$, where the functions are for almost all $y \in \mathcal{E}'$ integral and exponential type $\nu$ in the variables $u = (x_1, \ldots, x_m)$. The correction of all such functions $g_\nu \in L_p(\mathcal{E})$ for a given $\nu$ forms the subspace $\mathfrak{M}_{\nu p}(\mathcal{E}) \subset L_p(\mathcal{E})$ (cf 3.5).

Let the function $f \in L_p(g)(1 \leq p \leq \infty)$ be given. The quantity

$$E_\nu(f) = E_\nu(f)_{L_p(\mathcal{E})} = \inf_{g_\nu \in \mathfrak{M}_{\nu p}(\mathcal{E})} \|f - g_\nu\|_{L_p(\mathcal{E})} = \inf_{g_\nu} \|f - g_\nu\|_{L_p(\mathcal{E})}. \tag{1}$$

will be called the best approximation of f by means of functions $g_\nu \in \mathfrak{M}_{\nu p}(\mathcal{E})$, where $\nu = (\nu_1, \ldots, \nu_m)$ is a given system of numbers. When m = n, the lower bound of (1) is reached for some (best) function. Actually, from (1) it follows that the sequence of functions $g_{\nu s} \in \mathfrak{M}_{\nu p}(R_n)(s = 1, 2, \ldots)$ exists for which the inequalit es

$$\|f - g_{\nu s}\|_{L_p(R_n)} \leqslant E_\nu(f)_{L_p(R_n)} + \varepsilon_s = d + \varepsilon_s \quad (\varepsilon_s \to 0).$$

are satisfied. From this sequence, we can by 3.3.6 separate a subsequence, which we will again denote by $\{g_{\nu s}\}$ such that it uniformly converges to some function $g \in \mathfrak{M}_{\nu p}(R_n)$ on any bounded domain $g \subset R_n$. But then

$$\|f-g_\nu\|_{L_p(g)} = \lim_{s\to\infty}\|f-g_{\nu s}\|_{L_p(g)} \leq \lim_{s\to\infty}\|f-g_{\nu s}\|_{L_p(R_n)} = d.$$

Consequently,

$$\|f-g_\nu\|_{L_p(R_n)} = d.$$

Since $\mathfrak{M}_{\nu p}(\mathcal{E})$ is a subspace of the space $L_p(\mathcal{E})$, then providing the condition $1 < p < \infty$ is met, the lower bound of (1) is attained for the unique (best) function $g_\nu \in \mathfrak{M}_{\nu p}(\mathcal{E})$. Sometimes, it is convenient to examine functions that we will designate by $g_{u\nu}(x)(\nu > 0)$. These are functions defined on $\mathcal{E}$ and for almost all $y = (x_{m+1}, \ldots, x_n)$ are integral functions of the exponential type in $u = (x_1, \ldots, x_m)$ of spherical degree .

We will call the quantity

$$E_{u\nu}(f) = E_{u\nu}(f)_{L_p(\mathcal{E})} = \inf_{g_{u\nu}}\|f-g_{u\nu}\|_{L_p(\mathcal{E})}, \tag{2}$$

the best approximation of function $f \in L_p(\mathcal{E})$ by means of functions $g_{u\nu}$ (for given $\nu > 0$) where the lower bound is extended over all $g_{u\nu} \in L_p(\mathcal{E})$ for given $\nu$.

A particular case of these concepts is the quantity

$$E_{x_j\nu}(f)_{L_p(\mathcal{E})} = \inf_{g_{x_j\nu}}\|f-g_{x_j\nu}\|_{L_p(\mathcal{E})}, \tag{3}$$

where $\mathcal{E} = R_j \times \mathcal{E}^j$ $(j = 1, \ldots, n)$, $R_j$ is the axis of $x_j$ coordinates, and $g_{x_j\nu}$ are functions from $L_p(\mathcal{E})$ of the exponential type $\nu$ in $x_j$.

## 5.2. Theorem on Approximation

5.2.1. Direct theorem on approximation by integral functions of the exponential type. Let $g(\xi)$ be a nonnegative even function of one variable of exponential type 1, satisfying the condition

$$\varkappa_m\int_0^\infty g(\xi)\xi^{m-1}\,d\xi = \int_{R_m} g(|u|)\,du = 1, \tag{1}$$

where $\varkappa_1 = 2$ and $\varkappa_m$ when $m > 1$ is the area of a unit sphere with radius 1 in the m-dimensional space $R_m$ and let $\mathcal{E} = R_m \times \mathcal{E}'$.

The equality

$$(-1)^{l+1}\Delta_h^l\varphi(x) = \sum_{j=0}^{l}(-1)^{j-1}C_l^j\varphi(x+jh) = \sum_{j=1}^{l} d_j\varphi(x+jh) - \varphi(x), \tag{2}$$

is valid for an arbitrary function $\varphi(x)$ defined on $\mathcal{E}$, vector $h \in R_m$, and natural number l, where

$$\sum_{j=1}^{l} d_j = 1. \tag{3}$$

Let us assign the function $f \in L_p(\mathcal{E})$; then for almost all $y \in \mathcal{E}'$ function $f(u, y)$ of u belongs to $L_p(R_m)$ and the function

$$g_\nu(x) = g_\nu(u, y) = \int_{R_m} g(|t|)\{(-1)^{l-1}\Delta^l_{t/\nu}f(x) + f(x)\}\,dt = \int_{R_m} g(|t|)\sum_{j=1}^{l} d_j f\left(u + j\frac{t}{\nu},\ y\right)dt = \int_{R_m} K_\nu(t-u)f(t, y)\,dt, \tag{4}$$

is meaningful where

$$K_\nu(u) = \sum_{j=1}^{l} d_j\left(\frac{\nu}{j}\right)^m g\left(\frac{|u|\,\nu}{j}\right). \tag{5}$$

By (1)

$$g_\nu(x) - f(x) = (-1)^{l-1}\int_{R_m} g(|t|)\Delta^l_{t/\nu}f(x)\,dt. \tag{6}$$

Let us now assume that function f has on $\mathcal{E}$ with respect to m derivatives of order $\rho$ belonging to $L_p(\mathcal{E})$ and $k = 1 - \rho$. Then from (6) it follows that (explanations below)

$$
\begin{aligned}
E_{\nu}(f)_{L_p(\mathcal{E})} &\leq \|f - g_\nu\|_{L_p(\mathcal{E})} = \left\| \int_{R_m} g(|t|)\, \Delta^l_{t/\nu} f(x)\, dt \right\|_{L_p(\mathcal{E})} \leq \\
&\leq \int_{R_m} g(|t|) \| \Delta^l_{t/\nu} f(x) \|_{L_p(\mathcal{E})} dt \leq \\
&\leq \int_{R_m} g(|t|) \left(\frac{|t|}{\nu}\right)^{\rho} \| \Delta^k_{t/\nu} f^{\rho}_t \|_{L_p(\mathcal{E})} dt \leq \\
&\leq \frac{1}{\nu^{\rho}} \int_{R_m} g(|t|) |t|^{\rho}\, \Omega^k_{R_m}\left(f^{\rho}, \frac{|t|}{\nu}\right) dt \leq \\
&\leq \frac{1}{\nu^{\rho}} \int_{R_m} g(|t|) |t|^{\rho} (1 + |t|)^k\, dt\, \Omega^k_{R_m}\left(f^{\rho}, \frac{1}{\nu}\right)_{L_p(\mathcal{E})} = \\
&= \frac{c}{\nu^{\rho}} \Omega^k_{R_m}\left(f^{\rho}, \frac{1}{\nu}\right)_{L_p(\mathcal{E})} \qquad (\nu > 0),
\end{aligned}
\tag{7}
$$

if the right side is finite.

In particular, it follows from (7) that if $f \in H^r_{up}(\mathcal{E})$, then

$$
E_{\nu}(f) \leq \frac{c_1}{\nu^r}. \tag{8}
$$

We have used generalized Minkowski inequality 1.3.2 and inequality 4.6 (6); $f^{\rho}_t$ is the derivative of f of order $\rho$ in direction t, $\Omega^k_{R_m}(f^{\rho}, \delta)$ is the module of the continuity of f with respect to all derivatives of order $\rho$. Property 4.2(14) was applied to it. Finally, we assert that function g is chosen so that the integral

$$
\int_{-\infty}^{\infty} g(t)\, t^{\rho+k+m-1}\, dt
$$

is finite. We can select a function of the form

$$
\mu \left( \frac{\sin \frac{t}{\lambda}}{t} \right)^{\lambda}. \tag{9}
$$

serve as $g$, where $\lambda \geqslant \rho + k + m + 2$ is an even integer and $\mu$ is a constant for which (1) holds.

Since $g(\xi)$ is an integral function of one variable of exponential type 1, then by (5) function $g_\nu(x)$ is in term an integral function of spherical type $\nu$ with respect to $u = (x_1, \ldots, x_m)$ (cf 3.6.2), belonging to $L_p(\mathcal{E})$.

5.2.1.1. Let us assume that about function f we only know that the continuity module

$$\Omega^k_{R_m}(f^\rho, \delta) < \infty \tag{1}$$

for it is finite for some $\delta > 0$. Then, reasoning as above (from right to left), we can obtain for $1/\nu \leqslant \delta$ the entire chain of relations 5.2.1(7), excluding for the meanwhile the first inequality. The difference $f - g_\nu$ will stand for the formal notation of the function appearing under the integral in the third term in 5.2.1(7). However, if we know that function f is locally integrable in the p-th degree on $\mathcal{E}$ (or even somewhat less: cf below), then it can be concluded that f is integral in the p-th degree on $\mathcal{E}$ with a certain weight, and $g$ given the choice of the suitable kernel $g$ is an integral function of spherical type $\nu$ (integral integrable with the same weight). In fact, from (1) for any $h \in R_m$ with $|h| \leqslant \delta$ it follows that:

$$\|\Delta^l_h f(x)\|_{L_p(\mathcal{E})} \leqslant \delta^\rho \|\Delta^k_h f^\rho_h(x)\|_{L_p(\mathcal{E})} \leqslant \delta^\rho \Omega^k_{R_m}(f^\rho, \delta)$$

and by 4.2.2(5) (replace k with l)

$$\|f(x)(1 + |u|^{-\mu})\|_{L_p(\mathcal{E})} < \infty, \tag{2}$$

where

$$\mu = \frac{m}{p} + l + \varepsilon \qquad (\varepsilon > 0). \tag{3}$$

But then for almost all $y$

$$\|f(u, y)(1 + |u|)^{-\mu}\|_{L_p(\mathcal{E})} < \infty.$$

and by 3.6.2, kernel $g(t)$ of form 5.2.1(9) can be selected so that (taking $\lambda$ sufficiently large) that the function

$$g_\nu(x) = \int_{R_m} K_\nu(t - u) f(t, y)\, dt \tag{4}$$

(cf 5.2.1 (4), (5)) will clearly be of the spherical type for almost all $y$.

Now the first term of formula 5.2.1(7), $E_{uv}(f)_{L_p(\mathcal{E})}$, also becomes meaningful. It can be considered as the best approximation in metric $L_p(\mathcal{E})$ of the function f under consideration by means of integral functions of spherical type $\nu$ (generally not belonging to $L_p(\mathcal{E})$). We have shown that if module (1) is meaningful for the function f locally summable in the p-th degree, then it makes sense to approach it in the matrix $L_p(\mathcal{E})$ by fractions of the spherical type $\nu$ with respect to u.

In fact (cf 4.2.2(4)), instead of local summability of $|f|^p$ (when $p = \infty$, local boundedness and measurability of f), it is sufficient to assume the existence of $\|f\|_{L_p(v \times \mathcal{E}')}$, where $v = \{|u| < \delta(1 + m)\}$.

5.2.2. Other approximation estimates. Below are presented other estimates based on formula 5.2.1(6). If f has generalized derivatives in $u = (u_1, \ldots, u_m)$ up to order inclusively, then from 5.2.1(6) it follows that in any case the equality

$$g_\nu^{(s)} - f^{(s)} = (-1)^{l-1} \int_{R_m} g(|t|)\, \Delta_{t/\nu}^{l} f^{(s)}(x)\, dt \tag{1}$$

and the equality

$$\|g_\nu^{(s)} - f^{(s)}\|_{L_p(\mathcal{E})} < \int_{R_m} g(|t|)\, \|\Delta_{t/\nu}^{l} f^{(s)}(x)\|_{L_p(\mathcal{E})}\, dt. \tag{2}$$

obtain formally for any integral nonnegative vector $s = (s_1, \ldots, s_m, 0, \ldots, 0)$. If for any s with $|s| \le \rho$, the integral in the right side of (2) is finite, then already the nonformal equality (1) and inequality (2) hold.

We will use as well the inequality

$$\|\Delta_h^{l}\varphi(x)\|_{L_p(\mathcal{E})} < c\,\|\Delta_h^{l'}\varphi(x)\|_{L_p(\mathcal{E})} \qquad (0 < l' < l), \tag{3}$$

where $c = 2^{l-l'}$, $h \in R_m$.

Then (from explanation below) when $k = l - \rho$, $|s| \le \rho$

$$\begin{aligned}
&\| g_v^{(s)} - f^{(s)} \|_{L_p(\mathfrak{g})} < \\
&< \int_{R_m} g(|t|) \left(\frac{|t|}{v}\right)^{\rho-|s|} \left\| \Delta_{t/v}^{k+|s|} \frac{\partial^{\rho-|s|}}{\partial t^{\rho-|s|}} f^{(s)}(x) \right\|_{L_p(\mathfrak{g})} dt < \\
&< \frac{1}{v^{\rho-|s|}} \int_{R_m} g(|t|)\, |t|^{\rho-|s|} \sum \| \Delta_{t/v}^{k+|s|} f^{(\rho)} \|_{L_p(\mathfrak{g})} < \\
&< \frac{1}{v^{\rho-|s|}} \int_{R_m} g(|t|)\, |t|^{\rho-|s|} \sum \Omega_{R_m}^{k+|s|} \left(f^{(\rho)}, \frac{|t|}{v}\right) dt < \\
&< \frac{1}{v^{\rho-|s|}} \int_{R_m} g(|t|)\, |t|^{\rho-|s|} (1 + |t|^{k+|s|})\, dt \sum \Omega_{R_m}^{k} \left(f^{(\rho)}, \frac{1}{v}\right) < \\
&< \frac{1}{v^{\rho-|s|}} \int_0^\infty g(t)(1+t)^{\rho+k+m-1}\, dt \sum \Omega_{R_m}^{k} \left(f^{(\rho)}, \frac{1}{v}\right) < \\
&< \frac{c}{v^{\rho-|s|}} \sum_{|s|=\rho} \Omega_{R_m}^{k} \left(f^{(s)}, \frac{1}{v}\right) \quad (v > 0).
\end{aligned} \tag{4}$$

The first inequality is obtained on the basis of 4.6(6); here $(\partial^{\rho-|s|})/(\partial t^{\rho-|s|})$ denotes the derivative of order $\rho-|s|$ in direction t. The second inequality is derived by virtue of the fact that this derivative is a linear combination of ordinary derivatives (in coordinate direction) of the same order with bounded coefficients not dependent on x; here the some extended over all the derivatives $f^{(\rho)}$ of order $\rho$. The third inequality stems from the definition 4.2(13). The fourth, by 4.2(14), when $\rho = 0$ (where necessary $f^{(\rho)}$ can be replaced with f) and the last equality, we must assume that g is selected so that 5.2.1(1) is satisfied.

Let us note further that any derivative $g^{(\rho)}$ with respect to $u \in R_m$ of order $\rho$ can be written (cf 5.2.1(4)) as

$$g_v^{(\rho)}(x) = \int_{R_m} g(|t|) \sum_{l=1}^{l} d_l f_u^{(\rho)}\left(u + \frac{lt}{v}, y\right) dt,$$

from whence

$$\begin{aligned}
\| \Delta_h^k g_v^{(\rho)}(x) \|_{L_p(\mathfrak{g})} &= \int_{R_m} g(|t|) \sum_{l=1}^{l} |d_l| \left\| \Delta_h^k f^{(\rho)}\left(x + \frac{lt}{v}\right) \right\|_{L_p(\mathfrak{g})} dt < \\
&< c \int_{R_m} g(|t|)\, dt\, \omega_{R_m}^{(k)}(|h|, f^{(\rho)})_{L_p(\mathfrak{g})} \leq c_1 \omega_{R_m}^{(k)}(|h|, f^{(\rho)})_{L_p(\mathfrak{g})}.
\end{aligned}$$

or

$$\Omega^k_{R_m}(g_\nu^{(\rho)}, \delta) \leq c\Omega^k_{R_m}(f^{(\rho)}, \delta), \tag{5}$$

where c does not depend on $\nu$ and f.

This shows that the differential properties of f are transferred on $g_\nu$ uniformly relative to $\nu$.

5.2.3. Let us turn again to kernel 5.2.1(5), which we are interested in this time when m = 1. We will, thus, assume that $\mathfrak{G} = R_1 \times \mathfrak{G}' \subset R_n$.

Let us suppose

$$K_{\nu,l}(u) = \sum_{l=1}^{l} d_l \frac{\nu}{l} g\left(\frac{u\nu}{l}\right) \quad (l = \rho + k, \quad -\infty < u < \infty). \tag{1}$$

By 5.2.1(7)

$$\left\| f - \int K_{\nu,l}(t - x_1) f(t, y)\, dt \right\|_{L_p(\mathfrak{G})} \leq \frac{b_l}{\nu^\rho} \omega^k_{x_1}\left(f^{(\rho)}_{x_1}, \frac{1}{\nu}\right)_{L_p(\mathfrak{G})} \tag{2}$$

on the assumption, of course, that the right side of (2) is meaningful. Naturally, we will assume as in 5.2.1 that the integral positive even function g(t) of one variable of exponential type 1 is chosen so that conditions 5.2.1(1) are satisfied when m = 1, ensuring estimate (2). We stress that from these, and the case, it follows that

$$\int_{-\infty}^{\infty} K_{\nu,l}(t)\, dt = 1, \tag{3}$$

$$\int_{-\infty}^{\infty} |K_{\nu,l}(t)|\, dt \leq c_l < \infty \qquad (\nu > 0), \tag{4}$$

where $c_1$ does not depend on $\nu > 0$.

Now let $\mathfrak{G} = R_m \times \mathfrak{G}' \subset R_n$, g(x) is a function which for almost all $y \in \mathfrak{G}'$ is an integral function of exponential type $\nu = (\nu_1, \ldots, \nu_m)$ with respect to $(x_1, \ldots, x_m)$; we will as always denote it by

$$g_\nu = g_{\nu_1, \ldots, \nu_m}(x_1, \ldots, x_m).$$

This definition is meaningful when $\nu_i (i = 1, \ldots, m)$ are positive finite numbers. Let us extend this definition to the case whence certain $\nu_i$ (not all) are equal to $\infty$. Specifically, if $\nu_i = \infty \ (i \leq m)$, we will assume that with respect to the variable $x_i$ function $g$ need not necessarily be integral. For example, $g_{\nu_1, \nu_2, \infty, \ldots, \infty}$, where $\nu_1$ and $\nu_2$ are finite denotes that this function for almost all $(x_3, \ldots, x_n)$ in the sense of the $(n - 2)$-dimensional measure is with respect to $x_1$ and $x_2$ and integral function of exponential type, respectively, $\nu_1$ and $\nu_2$.

5.2.4. Theorem. Let $\rho_j$, $k_j (j = 1, \ldots, m)$ be natural numbers, $l_j = \rho_j + k_j$, $f(x) = f(u, y) \in L_p(\mathscr{G})$, and that the system of functions

$$
\begin{gathered}
g_{\nu_1, \infty, \ldots, \infty}(x) = \int_{-\infty}^{\infty} K_{\nu_1, l_1}(u) f(x_1 + u, x_2, \ldots, x_m; y)\, du, \\
g_{\nu_1, \nu_2, \infty, \ldots, \infty}(x) = \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{\nu_1, l_1}(u_1) K_{\nu_2, l_2}(u_2) \times \\
\times f(x_1 + u_1, x_2 + u_2, x_3, \ldots, x_m; y)\, du_1\, du_2, \qquad (1) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
g_{\nu_1, \ldots, \nu_m}(x) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} K_{\nu_1, l_1}(u_1) \ldots \\
\ldots K_{\nu_m, l_m}(u_m) f(x_1 + u_1, \ldots, x_m + u_m; y)\, du_1, \ldots, du_m, \\
\text{где } \nu_j > 0.
\end{gathered}
$$

be given where $\nu_j > 0$.

Then each of functions $g_{\nu_1, \ldots, \nu_l, \infty, \ldots, \infty}$ (obviously) belong to $L_p(\mathscr{E})$ and is an integral function of exponential type $\nu_1, \ldots, \nu_l$, respectively, for $x_1, \ldots, x_l$ $(1 \leq l \leq m)$. Moreover, the inequalities

$$\|f - g_{\nu_1, \infty, \ldots, \infty}\|_{L_{p_1}(\mathscr{E})} < \frac{c\omega_{x_1}^{k_1}\left(f_{x_1}^{(\rho_1)}, \frac{1}{\nu_1}\right)_{L_{p_1}(\mathscr{E})}}{\nu_1^{\rho_1}},$$

$$\|g_{\nu_1, \infty, \ldots, \infty} - g_{\nu_1, \nu_2, \infty, \ldots, \infty}\|_{L_{p_2}(\mathscr{E})} < \frac{c\omega_{x_2}^{k_2}\left(f_{x_2}^{(\rho_2)}, \frac{1}{\nu_2}\right)_{L_{p_2}(\mathscr{E})}}{\nu_2^{\rho_2}}, \tag{2}$$

$$\|g_{\nu_1, \ldots, \nu_{m-1}, \infty} - g_{\nu_1, \ldots, \nu_m}\|_{L_{p_m}(\mathscr{E})} < \frac{c\omega_{x_m}^{k_m}\left(f_{x_m}^{(\rho_m)}, \frac{1}{\nu_m}\right)_{L_{p_m}(\mathscr{E})}}{\nu_m^{\rho_m}},$$

$$\|g_\nu\|_{L_{p_j}(\mathscr{E})} < c\|f\|_{L_{p_j}(\mathscr{E})}, \tag{3}$$

$$\omega_{x_j}^{k_j}\left(g_{\nu x_j}^{(\rho_j)}, \delta\right)_{L_{p_j}(\mathscr{E})} < c\omega_{x_j}^{k_j}\left(f_{x_j}^{(\rho_j)}, \delta\right)_{L_{p_j}(\mathscr{E})} \quad (j = 1, \ldots, m). \tag{4}$$

are satisfied.

From (1), in particular, when $p = p_1 = \ldots = p_m$ obviously it follows that:

$$\|f - g_\nu\|_{L_p(\mathscr{E})} < c\sum_{j=1}^{m} \frac{\omega_{x_j}^{k_j}\left(f_{x_j}^{(\rho_j)}, \frac{1}{\nu_j}\right)_{L_p(\mathscr{E})}}{\nu_j^{\rho_j}}. \tag{5}$$

Proof. Let us present the proof of the theorem for the case $m = 3$; it is analogous for $m > 3$.

The first inequality in (2) is obtained on the basis of 5.2.3(2):

$$\|f - g_{\nu_1, \infty, \ldots, \infty}\|_{L_p(\mathscr{E})} < \frac{b_{l_1}}{\nu_1^{\rho_1}} \omega_{x_1}^{(k)}\left(f_{x_1}^{(\rho_1)}, \frac{1}{\nu_1}\right)_{L_p(\mathscr{E})}.$$

The second inequality in (2) is derived by means of the following manipulations:

$$\|g_{\nu_1,\infty,\infty}-g_{\nu_1,\nu_2,\infty}\|_{L_{p_1}(\mathfrak{X})}=$$
$$=\left\|\int_{-\infty}^{\infty}K_{\nu_1,l_1}(u_1)\,h_1(x_1+u_1,\ x_2,\ x_3;\ y)\,du_1\right\|_{L_{p_1}(\mathfrak{X})},\tag{6}$$

where

$$h_1(x_1,\ x_2,\ x_3;\ y)=$$
$$=f(x_1,\ x_2,\ x_3;\ y)-\int_{-\infty}^{\infty}K_{\nu_2,l_2}(u_2)\,f(x_1,\ x_2+u_2,\ x_3;\ y)\,du_2$$

and by 5.2.3(2),

$$\|h_1\|_{L_{p_1}(\mathfrak{X})}\leqslant\frac{b_{l_2}\omega_{x_2}^{k_2}\left(f_{x_2}^{(\rho_2)},\ \frac{1}{\nu_2}\right)}{\nu_2^{\rho_2}}.$$

Then, applying to (6) the generalized Minkowski inequality, we get (cf further 5.2.3(4))

$$\|g_{\nu_1,\infty,\infty}-g_{\nu_1,\nu_2,\infty}\|_{L_{p_1}(\mathfrak{X})}\leqslant$$
$$\leqslant\int_{-\infty}^{\infty}|K_{\nu_1,l_1}(u_1)|\,\|h_1(x_1+u_1,\ x_2,\ x_3;\ y)\|_{L_{p_1}(\mathfrak{X})}\,du_1=$$
$$=\|h_1\|_{L_{p_1}(\mathfrak{X})}\int_{-\infty}^{\infty}|K_{\nu_1,l_1}(u_1)|\,du_1\leqslant\frac{c_{l_1}b_{l_2}\omega_{x_2}^{k_2}\left(f_{x_2}^{(\rho_2)},\ \frac{1}{\nu_2}\right)}{\nu_2^{\rho_2}}.$$

Finally, the third inequality in (2) is obtained by means of the considerations:

$$(g_{\nu_1,\nu_2,\infty}-g_{\nu_1,\nu_2,\nu_3})(x_1,\ x_2,\ x_3;\ y)=$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_{\nu_1,l_1}(u_1)\,K_{\nu_2,l_2}(u_2)\,h_2(x_1+u_1,\ x_2+u_2,\ x_3;\ y)\,du_1\,du_2,$$

where

$$h_2(x_1,\ x_2,\ x_3;\ y)=$$
$$=f(x_1,\ x_2,\ x_3;\ y)-\int_{-\infty}^{\infty}K_{\nu_3,l_3}(u)\,f(x_1,\ x_2,\ x_3+u;\ y)\,du.$$

Therefore, using the generalised Minkowski inequality and relations 5.2.3(2) and (4), we get

$$\|g_{\nu_1,\nu_2,\infty} - g_{\nu_1,\nu_2,\nu_3}\|_{L_{p_3}(\mathcal{Y})} \leqslant \frac{c_{l_1}c_{l_2}b_{l_3}\omega_{x_3}^{k_3}\left(f_{x_3}^{(\rho_3)}, \frac{1}{\nu_3}\right)}{\nu_3^{\rho_3}}.$$

Inequalities (2) stand proven. Inequality (3) (when m = 3) is quickly derived if we apply the generalized Minkowski inequality

$$\|g_{\nu_1,\nu_2,\nu_3}\|_{L_{p_i}(\mathcal{Y})} \leqslant$$
$$\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K_{\nu_1,l_1}(u_1)K_{\nu_2,l_2}(u_2)K_{\nu_3,l_3}(u_3)|\,du_1\,du_2\,du_3\|f\|_{L_{p_i}(\mathcal{Y})} \leqslant$$
$$\leqslant c_{l_1}c_{l_2}c_{l_3}\|f\|_{L_{p_i}(\mathcal{Y})} \qquad (i = 1, 2, 3).$$

to the integral appearing in right side of the last equality (1). Finally, if we differentiate the last equality (1) $\rho_1$ times with respect to $x_1$ and use the operation of the $k_1$-th difference with respect to $x_1$, then by the Minkowski inequality we get

$$\left\|\Delta_{x_1,h}^{k_1}\frac{\partial^{\rho_1}g_\nu}{\partial x_1^{\rho_1}}\right\|_{L_{p_1}(\mathcal{Y})} \leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\nu_1,l_1}(u_1)K_{\nu_2,l_2}(u_2)K_{\nu_3,l_3}(u_3)\times$$
$$\times\|\Delta_{x_1,h}^{k_1}f_{x_1}^{(\rho_1)}(x_1+u_1, x_2+u_2, x_3+u_3, y)\|_{L_p(\mathcal{Y})}\,du_1\,du_2\,du_3 \leqslant$$
$$\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K_{\nu_1,l_1}K_{\nu_2,l_2}K_{\nu_3,l_3}|\,du_1\,du_2\,du_3\,\omega_{x_1}^{(k_1)}(|h|, f_{x_1}^{(\rho_1)}),$$

from whence follows (4). When i = 2, 3 the proof is analogous.

## 5.3. Periodic Classes

Theorems in section 5.2 are preserved with certain modifications in the proof if in their formulations the functions considered there are assumed to be periodic with period 2 , and the approximating integral functions g are replaced by the trigonometric polynomial T . As always, in this case we must replace the norms $\|\cdot\|_{L_p(\mathcal{E})}$ ($\mathcal{E} = R_m \times \mathcal{E}_1 \subset R_n$) by the norms $\|\cdot\|_{L_p(\mathcal{E}_*)}$ ($\mathcal{E}_* = \Delta^{(m)} \times \mathcal{E}_1$), where $\Delta^{(m)} = \{0 \leqslant x_j \leqslant 2\pi\ ;\ j = 1, \ldots, m\}$. The first of the simplest direct theorems of approximation were obtained in the periodic case. Namely, Jackson showed that a periodic function, with period $2\pi$, of a single variable that has a continuous derivative of the order r can be approximated by trigonometric polynomials $T_n(x)$ (n = 1, 2, ...) such that the deviation (in a uniform metric C) satisfies the inequality

$$|f(x) - T_n(x)| \leq c_r \frac{\omega_s\left(f^{(r)}, \frac{1}{n}\right)}{n^r} \qquad (n = 1, 2, \ldots),$$

where $\omega_s(f^{(r)}, \delta)$ is the continuity module of the function $f^{(r)}$. The method of approximation of periodic function with trigonometric polynomials, which will be considered below, is a modernised Jackson method. In the simplest cases (cf further 5.3.1(6), (8) $l = 1, \sigma = 2$, and $n = 1$) it coincides with Jackson's method. On the other hand, it is an analog of the above considered method 5.2.1(4) of the approximation with integral functions of exponential type.

5.3.1. The first two equalities 5.2.1(4) when $m = 1$, $-\infty$ $x < \infty$ can further be written as

$$\begin{aligned} g_\nu(x, y) &= \int_{-\infty}^{\infty} \nu g(\nu t) \left\{(-1)^{l+1} \Delta_{x,t}^l f(x, y) + f(x, y)\right\} dt = \\ &= \int_{-\infty}^{\infty} \nu g(\nu t) \sum_{k=1}^{l} d_k f(x + kt, y) dt, \end{aligned} \tag{1}$$

where

$$d_k = (-1)^{k-1} C_l^k \qquad (k = 1, \ldots, l)$$

(2)

and

$$q_\nu(t) = \nu g(\nu t)$$

is an integral nonnegative functions of exponential type , satisfying (cf 5.2.1(1), (2)) the following condition:

$$\int_{-\infty}^{\infty} q_\nu(t)\, dt = 1. \tag{3}$$

Let us introduce into consideration, by analogy, trigonometric polynomials $\tau_\nu(t)$ ($\nu = 0, 1, 2, \ldots$) of order not higher than $\nu$, exhibiting the following properties:

$$\int_0^{2\pi} \tau_\nu(t)\, dt = 1, \tag{4}$$

$$\int_0^{2\pi} |\tau_\nu(t)|\, dt \leq c, \qquad (\nu = 1, 2, \ldots), \tag{5}$$

where c is a constant independent of $\nu$.

Obviously,

$$\tau_0(t) = \frac{2}{\pi}.$$

when $\nu > 0$, polynomials $\tau_\nu(t)$ are defined nonuniquely.

These polynomials can be obtained, for example (cf 2.2.2(2)) by means of the formula

$$d_\nu(t) = \frac{1}{a_\nu}\left(\frac{\sin\frac{\lambda t}{2}}{\sin\frac{t}{2}}\right)^{2\sigma}, \tag{6}$$

where $\sigma > 0$ is an integral number not dependent on $\lambda$ and $\lambda$ is the natural number such that

$$2(\lambda - 1)\sigma \leqslant \nu < 2\lambda\sigma; \tag{7}$$

the constant $a_\nu$ been selected so as to satisfy equality (4). In an example (6) polynomial $\tau_\nu(t)$ are nonnegative, therefore property (5) automatically follows property (4).

Let us define by analogy with (1) the function

$$T_\nu(x, y) = \int_0^{2\pi} \tau_\nu(t)\left\{(-1)^{l+1}\Delta_{x,t}^l f(x, y) + f(x, y)\right\} dt = \\ = \int_0^{2\pi} \tau_\nu(t) \sum_{k=1}^{l} d_k f(x + kt, y)\, dt, \tag{8}$$

where this time $f(x) = f(x, y)$ is a function defined on $\mathcal{E} = R_1 \times \mathcal{E}' \subset R_n$ $(x \in R_1, y \in \mathcal{E}')$ with period $2\pi$ with respect to x and integrable in the p-th degree on $\mathcal{E}_* = [0, 2\pi] \times \mathcal{E}'$.

Let us note that

$$T_0(x, y) = \frac{2}{\pi}\int_0^{2\pi} \sum_{k=1}^{l} d_k f(x + kt, y)\, dt = \\ = \int_0^{2\pi} f(u, y)\, du \sum_{k=1}^{l} (-1)^{k+1} C_l^k = \int_0^{2\pi} f(u, y)\, du = T_0(y). \tag{8'}$$

Thus, for fixed $y$, function $T_0(x, y)$ is a constant (a function of $y$), and the mean value of $f(x, y)$ with period 2 .

By virtue of the periodicity of $f$, we can write further:

$$
\begin{aligned}
T_\nu(x, y) &= \sum_{k=1}^{l} \frac{d_k}{k} \int_0^{2k\pi} \tau_\nu\left(\frac{u}{k}\right) f(x+u, y)\, du = \\
&= \sum_{k=1}^{l} \frac{d_k}{k} \sum_{s=0}^{k-1} \int_{2s\pi}^{2(s+1)\pi} \tau_\nu\left(\frac{u}{k}\right) f(x+u, y)\, du = \\
&= \sum_{k=1}^{l} \frac{d_k}{k} \sum_{s=0}^{k-1} \int_0^{2\pi} \tau_\nu\left(\frac{t+2s\pi}{k}\right) f(x+t, y)\, dt = \\
&= \int_0^{2\pi} K_\nu(t) f(x+t, y)\, dt,
\end{aligned}
\tag{9}
$$

where

$$
K_\nu(t) = \sum_{k=1}^{l} \frac{d_k}{k} \sum_{s=0}^{k-1} \tau_\nu\left(\frac{t+2s\pi}{k}\right). \tag{10}
$$

Let us show that function $K_\nu(t)$ is a trigonometric polynomial of order not higher than $\nu$, from whence it follows that $T_\nu(x, y)$ with respect to $x$ (for almost all $y$) is also a trigonometric polynomial of order not higher than $\nu$.

Actually, the trigonometric polynomial $\tau_\nu$ can be written as a certain linear combination

$$
\tau_\nu(t) = \sum_{-\nu}^{\nu} a_\lambda e^{i\lambda t} \qquad (\bar{a}_\lambda = a_{-\lambda})
$$

with constant coefficients $a_\lambda$ .

But

$$\sum_{s=0}^{k-1} e^{i\lambda\frac{t+2s\pi}{k}} = e^{\frac{i\lambda t}{k}} \sum_{s=0}^{k-1} e^{i\frac{\lambda 2s\pi}{k}} =$$

$$= \begin{cases} ke^{i\frac{\lambda t}{k}} = ke^{i\mu t} & \text{when } \lambda/k = \mu \text{ integral} \\ 0 & \text{when } \lambda/k = \mu \text{ nonintegral} \end{cases}$$

and, therefore, the sum

$$\sum_{s=0}^{k-1} \tau_\nu\left(\frac{t+2s\pi}{k}\right) = \sum_{-\nu}^{\nu} a_\lambda \sum_{s=0}^{k-1} e^{i\lambda\frac{t+2s\pi}{k}}$$

is a trigonometric polynomial of order $\nu$. But then $K_\nu$ is also a trigonometric polynomial of order $\nu$.

From (8) it follows that

$$T_\nu - f = (-1)^{l+1} \int_0^{2\pi} \tau_\nu(t)\, \Delta^l_{x,t} f(x, y)\, dt, \tag{11}$$

from whence by employing the generalized Minkowski inequality, we get the fundamental inequality

$$\|T_\nu - f\|_{L_p(\mathfrak{E}_*)} \leq \int_0^{2\pi} |\tau_\nu(t)|\, \|\Delta^l_{x,t} f(x, y)\|_{L_p(\mathfrak{E}_*)}\, dt \qquad (\nu = 0, 1, \ldots). \tag{12}$$

The following theorem, reducing to an inequality analogous to 5.2.1(7), obtains.

5.3.2. Theorem. Let $1 \leq p \leq \infty$ and $\mathfrak{E} = R_1 \times \mathfrak{E}_1' \subset R_n$, and function $f = f(x, y)$ $(x \in R_1, y \in \mathfrak{E}_1)$ be defined on $\mathfrak{E}$, have the period $2\pi$ with respect to x for almost all $y \in \mathfrak{E}'$ and belong to class $L_p(\mathfrak{E}_*)$, $\mathfrak{E}_* = [0, 2\pi] \times \mathfrak{E}_*'$.

Moreover, let $f$ have on $\mathscr{E}$ the generalised derivative $f_x^{(\rho)} = \partial^\rho f/\partial x^\rho$ of order $\rho$ ($f_x^{(0)} = f$). Finally, let even nonnegative trigonometric polynomials $\tau_\nu(t)$ of order $\nu$ satisfy, along with condition 5.3.1(4), also the auxiliary condition

$$\int_0^\pi \tau_\nu(t)\, t^\rho\, dt < \frac{a_\rho}{(\nu+1)^\rho}, \tag{1}$$

where constant $a_\rho$ does not depend on $\nu = 0, 1, 2, \ldots$ (this polynomial can be obtained by formula 5.3.1(6) with the appropriate selection of $\sigma$ and $\lambda$.)

Then function $T_\nu(x, y)$ defined by equality 5.3.1(8) (trigonometric polynomial of order $\nu$ with respect to x) approaches $f$ in the e matrix $L_p(\mathscr{E}_*)$ with the following estimate:

$$\|f - T_\nu\|_{L_p(\mathscr{E}_*)} < b_\rho \frac{\omega^{*k}_{x, L_p(\mathscr{E})}\left(f_x^{(\rho)}, \frac{\pi}{\nu+1}\right)}{(\nu+1)^\rho}, \quad (\nu = 0, 1, \ldots) \tag{2}$$

where $b_\rho$ is a constant dependent on $\rho$.

Proof. we already know that trigonometric polynomial d (t) defined by relations 5.3.1(6) and 5.3.1 (7) satisfy conditions 5.3.1 (4). Let us show that they, provided $\nu \geqslant 1$, also satisfy inequality (1) for some constant $a_{p+1}$ on the assumption that $2\sigma - \rho \geqslant 3$. By this we will establish the existence of polynomial satisfying the conditions of the theorem. In fact,

$$\int_0^\pi d_\nu(t)\, t^\rho\, dt < \frac{c_1}{a_\nu} \int_0^\pi \left(\frac{\sin\frac{\lambda t}{2}}{t}\right)^{2\sigma} t^\rho\, dt < \frac{c_2}{\lambda^\rho} \int_0^\infty \frac{(\sin u)^{2\sigma}}{u^{2\sigma-\rho}}\, du <$$
$$< \frac{c_3}{\lambda^\rho} < \frac{a_\rho}{(\nu+1)^\rho} \qquad (\nu = 1, 2, \ldots),$$

where the last inequality follows from 5.3.1(7).

We note that the inequality

$$\left\|\Delta^{\rho+k}_{x,t} f(x, y)\right\|_{L^*_p(\mathscr{E})} < |t|^\rho \left\|\Delta^k_{x,t} f_x^{(\rho)}(x, y)\right\|_{L^*_p(\mathscr{E})} < |t|^\rho \omega^k_*(|t|), \tag{3}$$

obtains, where

$$\omega_*^k(\delta) = \omega_{*\, x,\, L_p(\mathfrak{T})}^{k}(f,\, \delta).$$

Let us note further that the inequality

$$\omega_*^k(t) \leqslant c\,(\nu+1)^k\, t^k \omega_*^k\left(\frac{\pi}{\nu+1}\right) \quad \left(\frac{\pi}{\nu} < t\right). \tag{4}$$

is valid, which is proven analogously to the proof for inequality 4.2(8).

Let us use inequality 5.3.1(12) when $l = \rho + k$, taking (3) and (4) into account:

$$\begin{aligned}
\|f - \tau_\nu\|_{L_p^*(\mathfrak{T})} &\leqslant \int_{-\pi}^{\pi} \tau_\nu(t) \left\| \Delta_{x,\,t}^{\rho+k} f(x,\, y) \right\|_{L_p^*(\mathfrak{T})} dt \leqslant \\
&\leqslant \int_{-\pi}^{\pi} \tau_\nu(t)\, |t|^\rho\, \omega_*^k(|t|)\, dt = 2 \int_0^{\pi} \tau_\nu(t)\, t^\rho \omega_*^k(t)\, dt = \\
&= 2 \int_0^{\frac{\pi}{\nu+1}} \tau_\nu(t)\, t^\rho \omega_*^k(t)\, dt + 2 \int_{\frac{\pi}{\nu+1}}^{\pi} \tau_\nu(t)\, t^\rho \omega_*^k(t)\, dt \leqslant \\
&\leqslant 2\omega_*^k\left(\frac{\pi}{\nu+1}\right) \left[ \left(\frac{\pi}{\nu+1}\right)^\rho + \frac{2}{\pi}\,(\nu+1)^k \int_{\frac{\pi}{\nu+1}}^{\pi} \tau_\nu(t)\, t^{\rho+k}\, dt \right] \leqslant \\
&\leqslant 2\omega_*^k\left(\frac{\pi}{\nu+1}\right) \left[ \left(\frac{\pi}{\nu+1}\right)^\rho + \frac{2}{\pi}\, \frac{a_{\rho+k}}{(\nu+1)^\rho} \right] = \frac{b_\rho}{(\nu+1)^\rho}\, \omega_*^k\left(\frac{\pi}{\nu+1}\right),
\end{aligned}$$

where

$$b_\rho = 2\left(\pi^\rho + \frac{2}{\pi}\, a_{\rho+1}\right).$$

Thus the theorem is proven.

Note 1. Equality 5.3.1(11) is satisfied for the trigonometric polynomial $T_\nu$ under consideration, therefore

$$T_\nu^{(\rho)}(x,\, y) = f^{(\rho)}(x,\, y) + (-1)^{l+1} \int_0^{2\pi} \tau_\nu(t)\, \Delta_{x,\,t}^{l} f_x^{(\rho)}(x,\, y)\, dt,$$

and we get an equality analogous to the equality 5.2.2(1). Arguing as in 5.2.2 when $l = \rho + k$, it is easy to obtain an inequality analogous to 5.2.2(5):

$$\omega^{k}_{x, L_p(\mathfrak{T})}(T^{(\alpha)}_{\nu}, \delta) \leq c\,\omega^{k}_{x, L_p(\mathfrak{T})}(f^{(\alpha)}, \delta). \qquad (5)$$

where constant c does not depend on the series of the standing multiplier.

Note 2. If periodic function $f(x, y)$ is such that its mean for the period equals zero, i.e.,

$$\int_0^{2\pi} f(u, y)\, du = 0,$$

then $T_0 = 0$(of 5.3.1(8)), therefore inequality (2) when $\nu = 0$ reduces to the following inequality:

$$\|f\|_{L_p(\mathfrak{T}_x)} \leq b_p \omega^{k}_{x, L_p(\mathfrak{T})}(f^{(\alpha)}_x, \pi). \qquad (6)$$

5.3.3. Just as in 5.2.3, we can define (analogous to $g_{\nu_1, \dots \nu_m}(x)$) functions $T_{\nu_1, \dots \nu_m}(x_1, \dots, x_n)$ given on the measurable set $\mathcal{E} = R_m \times \mathcal{E}_1 \subset R_n$, which are trigonometric polynomials for almost all $y = (x_{m+1}, \dots, x_n) \in \mathcal{E}'$ with respect to variable $x_1, \dots, x_m$, respectively, of orders $\nu_1, \dots, \nu_m$. We will, as in 5.2.3, assume that individual $\nu_k (k = 1, \dots, m)$ can equal $\infty$.

Let $r = (r_1, \dots, r_m) > 0$. Let us define even nonnegative trigonometric polynomials $\tau_{\nu, r_j}(t)$ of order satisfying the conditions of theorem 5.3.2, respectively, for $\rho = r_1, \dots, \rho = r_m$. For these, and thus, the following conditions are satisfied:

$$\int_0^{2\pi} \tau_{\nu, r_j}(t)\, dt = 1, \qquad (1)$$

$$\int_0^{2\pi} \tau_{\nu, r_j}(t)\, dt < \frac{a_{r_j}}{(\nu + 1)^{r_j}} \quad (j = 1, \dots, m;\ \nu = 1, 2, \dots). \qquad (2)$$

Let us, further, define trigonometric polynomial $K_{\nu, r_j}$ (kernels) of orders $\nu$ by the formulas (cf 5.3.1), where $l_j \geqslant r + k$.

Let us further assume

$$K_{\nu, r_j}^{\ (k)}(t) = \sum_{q=1}^{l_j} \frac{d_q}{q} \sum_{s=0}^{q-1} \tau_{\nu, r_j}\left(\frac{t+2s\pi}{q}\right) \qquad (k = 1, \ldots, m)$$

(см. 5.3.1), где $l_j \geqslant r + k$.

Полагаем далее

$$T_{\nu_1, \infty, \ldots, \infty}(x) = \int_0^{2\pi} K_{\nu_1 r_1}(u) f(x_1 + u, x_2, \ldots, x_m; y)\, du,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$T_{\nu_1, \ldots, \nu_m}(x) =$$

$$= \int_0^{2\pi} \ldots \int_0^{2\pi} K_{\nu_1 r_1}(u_1) \ldots K_{\nu_m r_m}(u_m) f(x_1 + u_1, \ldots, x_m + u_m; y)\, du_1 \ldots du_m.$$

For the indicated family $T_{\nu_1, \ldots, \nu_m}$ of functions f, we can on analogy formulate and improve a theorem (generalization of Jackson's theorem) analogous to theorem 5.2.4.

In particular, from it it follows that if $f \in H^r_{* \, up}(\mathcal{E})$, then

$$E_{x_1 \nu}^{*}(f)_p < \frac{c \| f \|_{x_1, p}^{h^*_r}}{(\nu + 1)^r}, \tag{3}$$

where c does not depend on the series of the standing multiplier.

## 5.4. Inverse Theorems of the Theory of Approximations

In this section we will elucidate a scheme by which inverse theorems of theory of approximation can be obtained that indicate to which class a function belongs if its approximation estimates are known.

The general theorem whose basis is the inverse theorem of the theory of approximation (for trigonometric polynomials and integral functions of the exponential type), originating with S. N. Bernshteyn*) is to be proven.

5.4.1. Theorem. Let $R_n$ be an n-dimensional space of points $\mathbf{x} = (\mathbf{u}, \mathbf{y})$, $\mathbf{u} = (x_1, \ldots, x_m)$, $\mathbf{y} = (x_{m+1}, \ldots, x_n)$ and $R_m = (\mathbf{u}, 0)$ be its m-dimensional subspace $(1 \leq m \leq n)$. Further, let $r > 0$, k be a natural number, $1 \leqslant p \leqslant \infty$, and $\mathfrak{M}_\nu$ be linear, dependent on parameter $\nu \geq 1$, sets of functions defined on

*) S. N. Bernshteyn [1], pages 11- 104.

the open space $g \subset R_n$, where

$$\mathfrak{M}_\nu \subset \mathfrak{M}_{\nu'} \quad (\nu < \nu'). \tag{1}$$

Let us assume that each function $\tau_\nu \in \mathfrak{M}_\nu$ exhibits the following property: $\tau_\nu$ has on g derivatives with respect to u of orders less than r + k and that the inequalities

$$\|\tau_\nu^{(s)}\|_{L_p(g)} \leqslant c\nu^{|s|}\|\tau_\nu\|_{L_p(g)}, \quad s=(s_1, \ldots, s_m, 0, \ldots, 0), \quad |s|<r+k, \tag{2}$$

obtain, where constant c does not depend on $\nu$.

Let, moreover, there exists for given function $f \in L_p(g)$ a family of functions $\tau_\nu \in \mathfrak{M}_\nu$ dependent on $\nu$, such that

$$\|f-\tau_\nu\|_{L_p(g)} \leqslant \frac{K}{\nu^r} \quad (\nu \geqslant 1), \tag{3}$$

where K does not depend on $\nu$.

Then $f \in H^r_{up}(g)$ (cf 4.3.3) and the inequality

$$\|f^{(\rho)}\|_{L_p(g)} \leqslant A\left(\|f\|_{L_p(g)}+K\right) \tag{4}$$

are satisfied for all derivatives $f^{(\rho)}$ of f of order $\rho < r$ and

$$\|f\|_{H^r_{up}(g)} \leqslant A\left(\|f\|_{L_p(g)}+K\right), \tag{5}$$

where A does not depend on the series of the standing multiplier.

Note 1. Functions $\tau_\nu$ can even be considered periodic with respect to $x_1$, with period $2\pi$, defined on $g = \mathcal{E} = R_1 \times g'$, and then in (1) - (5) $L_p(g)$ and $H^r_{up}(g)$ must be replaced by $L_p{}^*(\mathcal{E})$ and $H^r_{up}{}^*(\mathcal{E})$.

Note 2. It can be assumed that runs through the values $\nu = \nu(s)$, dependent on s = 0, 1, ..., and satisfying the conditions:

1)
$$\nu(s)\geqslant 1,$$
2)
$$\nu(s)\to\infty\quad(s\to\infty),$$
$$\frac{\nu(s+1)}{\nu(s)}\leqslant\Lambda<\infty\quad(s=0,\ 1,\ \ldots),$$
3)

where $\Lambda$ does not depend on s. In particular, it can be assumed that $\nu(s) = a^s$, $a > 1$.

Actually, let

$$\|f-\tau_{\nu(s)}\|_{L_p(g)}\leqslant\frac{K}{\nu(s)^r}\qquad(s=0,\ 1,\ \ldots)$$

and $\nu_0 = \min \nu(s)$, $s = 0, 1, \ldots$

If $1 \leq \nu \leq \nu_0$, then we assume $\tau_\nu = 0$ and then $\|f - \tau_\nu\| = \|f\| \leq (\|f\|\nu_0^r)/\nu^r$, but if $\nu > \nu_0$, then we select s such that

$$\nu(s)\leqslant\nu<\nu(s+1).$$

Since $\tau_{\nu(s)} \subset \mathfrak{M}_{\nu(s)} \subset \mathfrak{M}_\nu$, then we can assume $\tau_{\nu(s)} = \tau_\nu$ and therefore,

$$\|f-\tau_\nu\|_{L_p(g)}\leqslant\frac{K}{\nu(s)^r}\leqslant\frac{K}{\nu^r}\left(\frac{\nu(s+1)}{\nu(s)}\right)^r\leqslant\frac{K\Lambda^r}{\nu^r},\qquad\nu\geqslant 1.$$

Thus,

$$\|f-\tau_\nu\|\leqslant\frac{K_1}{\nu^r},\quad\nu\geqslant 1,$$

where

$$K_1=\|f\|\nu_0^r+K\Lambda,$$

and inequality (3) is satisfied for all $\nu \geqslant 1$, as required by the theorem. The conclusion of the theorem (cf (4) and (5)) does not change when K is replaced by $K_1$, because $\|f\| + K_1 \leq \|f\| + K$.

Proof of theorem 5.4.1. By (3) function f can be represented as the series

$$f=\sum_0^\infty Q_j, \tag{6}$$

where

$$Q_0=\tau_1=\tau_{2^0},\quad Q_j=\tau_{2^j}-\tau_{2^{j-1}}\quad(j=1,\ 2,\ \ldots),$$

convergent in the $L_p(g)$-sense. Here ($\|\cdot\|_{L_p(g)} = \|\cdot\|$),

$$\|Q_0\| = \|\tau_1\| < K + \|f\|,$$
$$|Q_j| < |\tau_{s^j} - f| + |f - \tau_{s^{j-1}}| <$$
$$< \frac{K}{2^{jr}} + \frac{K}{2^{(j-1)r}} = \frac{c_1 K}{2^{jr}} \quad (j = 1, 2, \ldots). \tag{7}$$

Let us take any derivative of f of order $\rho$ mixed or unmixed:

$$f^{(\rho)} = \sum_0^\infty Q_j^{(\rho)}, \ |\rho| = \rho. \tag{8}$$

Since the sets $\mathfrak{M}_{2^j}$ are linear and $\mathfrak{M}_{2^{j-1}} \subset \mathfrak{M}_{2^j}$ ($j = 1, 2, \ldots$), then $Q_j \in \mathfrak{M}_{2^j}$ and based on estimate (2) the inequalities

$$|Q_0^{(\rho)}| < c\|Q_0\| < c_1(\|f\| + K),$$
$$|Q_j^{(\rho)}| < c2^{\rho j}\|Q_j\| < \frac{c_1 K}{2^{j(r-\rho)}}, \tag{9}$$

hold, showing that the formal memberwise differentiation (8) of series (6) when $\rho < r$ is legitimate and series (8) converges in the $L_p(g)$-sense to $f^{(\rho)}$ (cf lemma 4.4.7). Here $f \in W^{\rho}_{up}(g)$ and inequalities (4) are satisfied.

Let us assign the vector $h = (h_1, \ldots, h_m, 0, \ldots, 0) \in R_m$ and choose a natural N such that

$$\frac{1}{2^{N+1}} < |h| < \frac{1}{2^N}, \quad |h|^2 = \sum_{j=1}^m h_j^2. \tag{10}$$

Let us consider the k-th difference of the function $f^{(\rho)}$ corresponding to the shift h. Considering the equality $\Delta_h \varphi(x) = |h| \int_0^1 \frac{\partial \varphi}{\partial h}(x + th)\, dt$,

we get

$$\Delta_h^k \overset{\vee}{f^{(\rho)}(x)} =$$
$$= \sum_0^N |h|^k \int_0^1 \ldots \int_0^1 \frac{\partial^k}{\partial h^k} Q_l^{(\rho)}(x + h(u_1 + \ldots + u_k))\,du_1 \ldots du_k +$$
$$+ \sum_{N+1}^{\infty} \Delta_h^k Q_l^{(\rho)}(x), \tag{11}$$

where $x \in g_{kh}$. Obviously,

$$|\Delta_h^k f^{(\rho)}(x)|_{L_p(g_{kh})} \leq |h|^k \sum_0^N \left|\frac{\partial^k}{\partial h^k} Q_l^{(\rho)}\right|_{L_p(g_{kh})} +$$
$$+ 2^k \sum_{N+1}^{\infty} |Q_l^{(\rho)}|_{L_p(g)} = I_1 + I_2. \tag{12}$$

Considering that the derivative $\partial^k / \partial h^k$ is a finite linear combination of ordinary derivatives with respect to coordinates $x_1, \ldots, x_m$ and taking the inequalities (2), (9), and (10) into account, we get

$$I_1 \leq |h|^k c_4(\|f\| + K) \sum_0^N 2^{[k-(r-\rho)]l} \leq$$
$$\leq c_5 |h|^k (\|f\| + K)\, 2^{[k-(r-\rho)]N} \leq c_6(\|f\| + K)|h|^{r-\rho}. \tag{13}$$

It is important to know that we have assumed that $k > r - \rho$. If we had held that $k = r - \rho$, then the sum

$$\sum_0^N 2^{[k-(r-\rho)]l} = N + 1$$

would not be of order $2^{[k-(r-\rho)]N} = 1$.

Further

$$I_2 \leq c_7 K \sum_{N+1}^{\infty} \frac{1}{2^{(r-\rho)l}} \leq c_8 K |h|^{r-\rho}. \tag{14}$$

From (12) - (14) and (4), when $\rho = 0$, follows (5).

5.4.2. Theorem. (inverse of 5.2.1(8)*). Let $r > 0$, $1 \leq p \leq \infty$, $1 \leq m \leq n$, $\mathcal{E} = R_m \times \mathcal{E}'$, and $f \in L_p(\mathcal{E})$.

If for the best approximation of f in metric $L_p(\mathcal{E})$ by means of integral functions of exponential spherical type $\nu$ the inequality

$$E_{u\nu}(f)_{L_p(\mathcal{E})} \leq \frac{K}{\nu^r} \quad (\nu \geq 1), \tag{1}$$

is satisfied, where K does not depend on $\nu$ ( $\nu$ can run through the values $\nu = \nu(s)$ satisfying the conditions of note 2 in 5.4.1, $s = 0, 1, \ldots$), then $f \in H_p^r(\mathcal{E})$, and

$$\|f\|_{H_p^r(\mathcal{E})} \leq A(\|f\|_{L_p(\mathcal{E})} + K), \tag{2}$$

$$\|f^{(\rho)}\|_{L_p(\mathcal{E})} \leq A(\|f\|_{L_p(\mathcal{E})} + K), \quad (|\rho| = 0, 1, \ldots, \bar{r}) \tag{3}$$

where A does not depend on the series of the standing multiplier.

Proof. From the condition there follows the existence of a family of functions $g_\nu(u, y)$ ($u \in R_m$, $y \in \mathcal{E}'$) of exponential spherical type $\nu$ with respect to $u$ (for almost all $y \in \mathcal{E}'$) such that

$$\|f - g_\nu\|_{L_p(\mathcal{E})} \leq 2E_\nu(f)_{L_p(\mathcal{E})} \leq \frac{2K}{\nu^r}.$$

But then the confirmation of the theorem directly stems from theorem 5.4.1 if we consider that $g_\nu$ are also functions of exponential type $\nu$ with respect to each of the variable $x_1, \ldots, x_m$ and therefore the inequality (cf 3.2.2(9))

$$\|g_\nu^{(k)}\|_{L_p(\mathcal{E})} \leq \nu^{|k|}\|g_\nu\|_{L_p(\mathcal{E})},$$

is satisfied for them, whatever the derivative of order $k = (k_1, \ldots, k_m, 0, \ldots, 0)$. The case when $\nu = \nu(s)$ runs through values described in 2 in 5.4.1, converges, according to this same note, to the case of a continuously varying .

*) $m = 1$, $p = \infty$ -- S. N. Bernshteyn [1], pages 421-432; $m = n = 1$, $1 \leq p \leq \infty$ -- N. I. Akiyezer [1]; $m = 1$, $1 \leq p \leq \infty$ -- S. M. Nikol'skiy [3].

5.4.3. **Theorem** (inverse of 5.3.3(3)*). Let $r > 0$, $1 \leq p \leq \infty$, $\mathfrak{E} = R_1 \times \mathfrak{E}' \subset R_n$, and function $f(\mathbf{x}) = f(x_1, \mathbf{y})$ $(x_1 \in R_1, \mathbf{y} \in \mathfrak{E}')$ with respect to the variable $x_1$ (for almost all $\mathbf{y} \in \mathfrak{E}'$) is periodically with period $2\pi$ and belongs to $L_p^*(\mathfrak{E})$.

If for the best approximation f in metric $L_p^*(\mathfrak{E})$ by means of functions $T_\nu(x_1, \mathbf{y})$, which are (for almost all $\mathbf{y} \in \mathfrak{E}'$) trigonometric polynomials of order $\nu$, the inequality

$$E_{x_1\nu^*}(f)_{L_p(\mathfrak{E})} \leqslant \frac{K}{(\nu+1)^r} \qquad (\nu = 0, 1, \ldots), \tag{1}$$

is satisfied, then $f \in H^r_{x_1 p^*}$ /illegible, text page 242 and text page 243/

.................

*) Cf note to 5.4 at the end of the book.

Using the Abel transformation, we get

$$\chi_n^{(l)}(u) = \sum_{k=0}^{n-2} (k+1)\Delta^2\left(1-\frac{k}{n}\right)^l F_k(u) + \\ + \frac{1}{n^{l-1}} F_{n-1}(u) + \frac{n-1}{n^l}(2^l-2) F_{n-2}(u),$$

where $F_k(u)$ are Fejer kernels (cf 2.2.2(1)), and $\Delta^2\mu_k = \mu_k - 2\mu_{k+1} + \mu_{k+2}$.

It is essential to note that $F_k(u) \geqslant 0$ and $\Delta^2(1 - k/n)^i \geqslant 0$, by means of which $\chi_n^{(i)}(u) \geqslant 0$ and $1/\pi\int_0^2 |\chi_n^{(j)}(u)|\, du = 1$ ($i = 1, 2, \ldots$). Applying to (6) the generalized Minkowski inequality, we get

$$\|\Psi_n^{(l)}\|_{L_p(\sigma_R)} \leqslant 2|\lambda_l| n^l R^{-l} \|\Phi_n\|_{L_p(\sigma_R)} \frac{1}{\pi}\int_0^{2\pi} |\chi_n^{(l)}(u)|\, du \leqslant \\ \leqslant c_R n^l \|\Phi_n\|_{L_p(\sigma_R)}$$

from whence follow (3) and (4). Inequality (5) stems from the fact that $\Phi_n(\rho, \theta)$ is a trigonometric polynomial with respect to $\theta$ of order n.

## 5.5. Direct and Inverse Theorems on the Best Approximations. Equivalent H-Norms

In this section the above-proven direct and inverse theorems on best approximations are compared. We will see that functions of classes H are completely characterised by the behavior of their best approximations. As everywhere in this chapter, $\mathcal{E} = R_m \times \mathcal{E}' \subset R_n$.

The best approximation of a function f measurable on $\mathcal{E}$ by means of integral functions of the exponential spherical type $\nu$ with respect to u, by 5.2.1(7), satisfies the inequality

$$E_\nu(f) = E_{u\nu}(f)_{L_p(\mathcal{E})} \leqslant \frac{c}{\nu^\rho}\,\Omega^k_{R_m}\left(f^\rho, \frac{1}{\nu}\right) \leqslant \frac{c}{\nu^\rho}\sum_{|s|=\rho}\Omega^k_{R_m}\left(f^{(s)}, \frac{1}{\nu}\right), \tag{1}$$

if, of course, its right side is meaningful. Thus,

$$E_\nu(f) = o(\nu^{-\rho}) \quad (\nu \to \infty) \tag{2}$$

for $f \in W^{\rho}_{up}(\mathcal{E})$ when p is finite ($1 \leq p < \infty$) and when $p = \infty$, if derivatives $f^{(s)}(|s| = \rho)$ are uniformly continuous on $\mathcal{E}$ in direction $R_m$ which means that for any $\varepsilon$ we can find a $\delta > 0$ such that

$$|f^{(s)}(x+h) - f^{(s)}(x)| < \varepsilon, \quad (|h| < \delta, \ h \in R_m).$$

As shown by the example (5.5.5) given above, estimate (1) when $\rho > 0$ is not becoming inverted, i.e., from the fact that for $f \in L_p(\mathcal{E})$ (2) is satisfied it does not follow that $f \in W^{\rho}_{up}(\mathcal{E})$.

But in the case $\rho = 0$, it does invert. Namely, the following two theorems (Weierstrass) hold.

5.5.1. Theorem. Let $1 \leq p < \infty$. For the function $f \in L_p(\mathcal{E})$, it is necessary and sufficient that there exists a family of functions $g_\nu \in L_p(\mathcal{E})$ that are integrable and of exponential spherical type $\nu$ with respect to $u$ such that

$$\|f - g_\nu\|_{L_p(\mathcal{E})} \to 0 \quad (\nu \to \infty). \tag{1}$$

The necessity follows from 5.5(1) when $\rho = 0$, and the sufficiency is trivial.

5.5.2. Theorem 4*). For a function f to be bounded and uniformly continuous on $\mathcal{E}$ in the direction $R_m$, it is necessary and sufficient that there exist a family of functions $g_\nu$ that are integral and of the exponential spherical type $\nu$ with respect to $u$ bounded in a set on $\mathcal{E}$, such that

$$\lim_{\nu \to \infty} g_\nu(x) = f(x) \tag{1}$$

uniformly on $\mathcal{E}$.

Proof. Again the necessity follows from 5.5.(1) when $\rho = 0$. Let us prove the sufficiency. Since $g_\nu$ are bounded and since the uniform convergence of (1) obtains, then f is bounded and their exists a constant $\lambda$ such that for all $\nu$ and $x \in \mathcal{E}$

$$|g_\nu(x)| \leq \lambda.$$

Therefore for $h \in R_m$ $$|g_\nu(x+h) - g_\nu(x)| \leq |h| \sup_x \left|\frac{\partial}{\partial h} g_\nu(x)\right| \leq \leq |h| \nu \sup_x |g_\nu(x)| \leq \lambda\nu|h|.$$

*) S. N. Bernshteyn [2], page 371, when n = 1.

i.e., $g_\nu$ (for given $\nu$) are uniformly continuous on $\mathcal{E}$ in direction $R_m$, and because f is also uniformly continuous on $\mathcal{E}$ in direction $R_m$.

5.5.3. Let us consider the norms

$$\|f\|_{H^r_{mp}(\mathcal{E})} = {}^j\|f\| = |f| + {}^j\|f\|_h, \quad |\cdot| = |\cdot|_{L_p(\mathcal{E})},$$

and the classes ${}^jH$ and ${}^jh$ corresponding to them, where

$${}^j\|f\|_h = {}^jM_f' \quad (j = 1, 2, 3, 4)$$

are the smallest constants M for which, respectively, inequalities given below (cf 4.3.3) are satisfied:

$$\Omega^k_{R_m}(f^{(s)}, \delta)_{L_p(\mathcal{E})} \leqslant M\delta^{r-\rho} \quad (|s| = \rho), \tag{1}$$

$$\Omega^k_{R_m}(f^\rho, \delta)_{L_p(\mathcal{E})} \leqslant M\delta^{r-\rho}, \tag{2}$$

$$|\Delta^k_h f^{(s)}(x)|_{L_p(\mathcal{E})} \leqslant M|h|^{r-\rho} \quad (|s| = \rho), \tag{3}$$

$$|\Delta^k_h f^\rho_h(x)|_{L_p(\mathcal{E})} \leqslant M|h|^{r-\rho} \tag{4}$$

($\rho \geq 0$, $k > r - \rho > 0$) and $h \in R_m$. We further introduce the norm (of classes)

$$\|f\|_h = \sup_{\nu > 0} \nu^r E_\nu(f), \tag{5}$$

where $E(f) = E_u(f)$ is the best approximation of function f in matrix $L_p(\mathcal{E})$ by integral functions of spherical $\nu$ with respect to u. Here $\nu$ can also run through the values $\nu(s) = a^s$, $a > 1$ ($s = 0, 1, 2, \ldots$).

Moreover,

$${}^6\|f\|_H = \sup_{s=0,1,\ldots} a^{sr} |Q_{a^s}|, \tag{6}$$

where it is assumed that function f is representable in the form of the series

$$f=\sum_{s=0}^{\infty} Q_{a^s}(x), \tag{7}$$

convergent in it in metric $L_p(\mathcal{E})$, whose terms are integral functions of spherical type $a^s$ with respect to m, where norm (6) is finite. Let us note that the norm of f does not explicitly appear in (6).

When j = 1, 2, 3, 4, we can further examine modified constants ${}^jM_f$, which we will denote by ${}^jM_f$ -- these are the smallest constants in the corresponding inequalities (1) - (4), when $\delta \leq \eta$ or $|h| < \eta$, where $\eta$ is a given arbitrary positive number. The corresponding classes will be symbolized by ${}^jH$ and ${}^jh'$ and the norms by

$${}^j\|f\|_{h'} = \|f\| + {}^j\|f\|_{h'}.$$

Our aim will be proved that all the classes ${}^jH$ and ${}^1H'$ (but in general, not ${}^jh$ and ${}^1h'$) are equivalent to each other; here each of them can be taken with any independent system of admissible parameters k, $\rho$, $\eta$, and a. Incidentally, the constants of the corresponding embeddings depend on these parameters (along with r, n, and m).

The foregoing lays the foundation for employing, in the following treatment, the single notation $\|f\|_{H_p^r(\mathcal{E})}$ for all norms ${}^j\|\cdot\|$ and ${}^j\|\cdot\|_{H'}$, omitting the j and the stroke; as for the norms ${}^j\|\cdot\|_h$ and ${}^j\|\cdot\|_{h'}$, then this notation generally speaking is essential to them. In passing we will obtain certain embeddings for the classes h that are interesting in themselves.

It directly follows from the definiticn of the continuity modulos appearing in (1) and (2) that the equivalency

$${}^1H \rightleftarrows {}^3H, \quad {}^3H \rightleftarrows {}^4H, \tag{8}$$

holds, if the classes compared are taken over the same pairs k, $\rho$. This in fact does hold if in (8) we replace H with h, H' or with h' (upon comparison with the identical ). Below it will be shown that ${}^1H \rightleftarrows {}^2H \rightleftarrows {}^1H \rightleftarrows {}^2H'$ and that here these classes can be taken independently with any admissible k, $\rho$, and also with any $\eta > 0$. Then by (8) /illegible text pages 248 and 249/

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and we have proven that

$$^{1}H \to {}^{4}H \to {}^{1}H,$$

i.e.

$$^{1}H = {}^{4}H.$$

The results obtained, in particular, contain the following theorem.

5.5.4. Theorem. For a function f defined on $\mathcal{E} = R_m \times \mathcal{E}'$ to belong to one of the classes ${}^{j}H_p^r(\mathcal{E})$ (j = 1, 2, 3, and 4) or ${}^{j}H'^{r}_p(\mathcal{E})$ (j = 1, 2, 3, and 4), it is necessary and sufficient that its best approximation by means of integral functions of the exponential and spherical type with respect to $\nu$ satisfies the inequality

$$E_{u\nu}(f)_{L_p(\mathcal{E})} < \frac{c}{\nu^r},$$

where c does not depend on $\nu > 0$ or $\nu = a^s$ ($s = 0, 1, \ldots; \nu > 0$, $a > 1$).

5.5.5. Example 1. It is well known that if a real-valued function f(x) with period 2 belongs to $L_2^*$, then it can be expanded in the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} (a_k \cos kx + b_k \sin kx), \tag{1}$$

converging to it in the sense of $L_2^* = L_2(0, 2\pi)$, where

$$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\} = \frac{1}{\pi} \int_0^{2\pi} f(t) \left\{\begin{matrix} \cos kt \\ \sin kt \end{matrix}\right\} dt \qquad (k = 0, 1, \ldots). \tag{2}$$

Here

$$\frac{1}{\pi} \int_0^{2\pi} f^2\, dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2). \tag{3}$$

In contrast, if a series of any real number $a_k$ and $b_k$ appearing in the right side of three converge, then series (1) converges in the $L_2^*$-sense to some function $f \in L_2^*$ and equalities (2) hold.

As a consequence of familiar orthogonal properties of trigonometric functions, the square of the best approximation by means of trigonometric polynomials of order n - 1 (in the $L^*_2$-sense) of function $f \in L^*_2$, defined by series (1), is

$$E_n(f)_{L_2^*} = \min_{\gamma_k, \delta_k} \int_0^{2\pi} \left[ f(x) - \frac{\gamma_0}{2} - \sum_1^{n-1} (\gamma_k \cos kx + \delta_k \sin kx) \right]^2 dx = \int_0^{2\pi} \left[ f(x) - \frac{a_0}{2} - \sum_1^{n-1} (a_k \cos kx + b_k \sin kx) \right]^2 dx = \int_0^{2\pi} \left[ \sum_n^{\infty} (a_k \cos kx + b_k \sin kx) \right]^2 dx = \pi \sum_n^{\infty} (a_k^2 + b_k^2). \tag{4}$$

If function f belongs to $W^1_{2*}$, i.e., is absolutely convergent and its (existing almost everywhere) derivative $f' \in L^*_2$, then its Fourier coefficients $a_k$ and $b_k$ can (by integrating by parts) be represented as

$$a_k = -\frac{\beta_k}{k}, \quad b_k = \frac{\alpha_k}{k} \quad (k = 1, 2), \tag{5}$$

where $\alpha_k$ and $\beta_k$ are Fourier coefficients of the derivative f' for which the series

$$\sum_1^{\infty} \alpha_k^2 + \beta_k^2 = \sum_1^{\infty} k^2 (a_k^2 + b_k^2) \tag{6}$$

converges. Conversely, function f belongs to $W^1_{2*}$ if it is representable as series (1) (convergent in $W^1_{2*}$ in the $L^*_2$-sense), where

$$\sum_1^{\infty} k^2 (a_k^2 + b_k^2) < \infty.$$

The best approximation of the function $f \in W^1_{2*}$ by means of trigonometric polynomials of (n - 1)-th order is subjected to the inequality

$$E_{n-1}(f)^2_{L_2^*} = \pi \sum_n^{\infty} \frac{\alpha_k^2 + \beta_k^2}{k^2} < \frac{\pi}{n^2} \sum_n^{\infty} (\alpha_k^2 + \beta_k^2) = o(n^{-2}) \quad n \to \infty, \tag{7}$$

which agrees with the general theory (the periodic analog of formula 5.5(1)).

In order to see that, conversely, the membership of f in class $W_{*2}^{1}$ does not stem from (7), let us examine the function

$$\varphi(x) = \sum_{1}^{\infty} \frac{\cos kx}{k^{3/2}\sqrt{\ln k}}.$$

Obviously,

$$E_n(f)^2_{L_2} = \pi \sum_{n}^{\infty} \frac{1}{k^3 \ln k} < \pi \frac{1}{\ln n} \sum_{n}^{\infty} \frac{1}{k^3} = o(n^{-2}) \qquad (n \to \infty).$$

On the other hand, $f \notin W_{*2}^{1}$, since the series

$$\sum_{1}^{\infty} \frac{1}{k \ln k},$$

corresponding to series (6) diverges.

Example 2. The function with period 2

$$f(x) = \sum_{1}^{\infty} \frac{\sin kx}{k^2 \ln k},$$

is obviously continuous and has the best approximation by means of trigonometric polynomials of (n - 1)-th order in metric C (or $L_\infty$) satisfying the inequality

$$E_{n-1}(f)_C < \sum_{n}^{\infty} \frac{1}{k^2 \ln k} < \frac{c}{n \ln n} = \frac{o(1)}{n} \qquad (n \to \infty).$$

At the same time the memberwise differentiated series

$$f'(x) = \sum_{2}^{\infty} \frac{\cos kx}{k \ln k} \tag{8}$$

by virtue of the monotonic diminishing to zero of its coefficients in the formerly converges on $[\varepsilon, 2\pi - \varepsilon]$ for any $\varepsilon > 0$ (cf Zigmund[1],2.6). Thus, its sum is continuous on the integral $(0, 2\pi)$ and is equal to the derivative $f'(x)$. Here series (8) is a Fourier series for $f'(x)$, since

$$\sum \frac{1}{k^2 \ln^2 k} < \infty.$$

In this case $f'$ is discontinuous at the point $x = 0$, because if $f'$ were continuous everywhere, then its n-th Fejer sum at $x = 0$ would tend to $f'(0)$.

Even so, the Fejer sum as the arithmetic mean of the first n + 1 Fejer sums at x = 0 tends to $\infty$ together with the sums.

5.5.6. Anisotropic case. We will begin from the estimate

$$\|f - g_\nu\|_{L_p(\mathscr{E})} \leqslant c \sum_{j=1}^{m} \frac{\omega_{x_j}^{k_j}\left(f_{x_j}^{(\rho_j)}, \frac{1}{\nu_j}\right)_{L_p(\mathscr{E})}}{\nu_j^{\rho_j}} \tag{1}$$
$$(\mathscr{E} = R_m \times \mathscr{E}', \ \nu_j > 0).$$

proven in 5.2.4(5). From it, for the best approximation $f \in W_{up}^r(\mathscr{E})$ by means of integral functions $g_\nu$ of exponential type $\nu = (\nu_1, \ldots, \nu_m)$ with respect to $u = (x_1, \ldots, x_m)$ follows the inequality

$$E_\nu(f)_{L_p(\mathscr{E})} = \sum_{j=1}^{m} \frac{o(1)}{\nu_j^{r_j}} \qquad (\nu_j \to 0) \tag{2}$$

provided $1 \leqslant p < \infty$ or $p = \infty$, if the partial derivatives $f_{x_j}^{(r_j)}$ are correspondingly uniformly continuous on $\mathscr{E}$ in the direction $x_j$.

If $f \in H_p^r(\mathscr{E})$, then from (1) it follows that

$$E_\nu(f)_{L_p(\mathscr{E})} \leqslant \|f - g_\nu\|_{L_p(\mathscr{E})} \leqslant c \|f\|_{h_p^r(\mathscr{E})} \sum_1^m \frac{1}{\nu_j^{r_j}}. \tag{3}$$

In particular, if in this inequality we replace $\nu_j$ accordingly by $\nu^{1/r_j}$ $(\nu > 0)$, then we get (omitting $L_p(\mathscr{E})$)

$$\nu E_{\nu^{1/r_1}, \ldots, \nu^{1/r_m}}(f) \leqslant c_1 \|f\|_{h_p^r(\mathscr{E})} \quad (\nu > 0). \tag{4}$$

Let us assume $a > 1$ and introduce the norms

$$^j\|\cdot\|_H = \|\cdot\| + {}^j\|\cdot\|_h \quad (j = 1, 2, 3), \ \|\cdot\| = \|\cdot\|_{L_p(\mathscr{E})}, \tag{5}$$

where

$$ {}^{1}\|f\|_h = \|f\|_{h_p^r(\mathcal{E})}, \qquad (6) $$

$$ {}^{2}\|f\|_h = \sup_{\nu > 0} \nu E_{\nu^{1/r_1}, \ldots, \nu^{1/r_m}}(f), \qquad (7) $$

$$ {}^{3}\|f\|_h = \sup_{s=0,1,\ldots} a^s E_{a^{s/r_1}, \ldots, a^{s/r_m}}(f), \qquad (8) $$

Additionally, let us suppose

$$ {}^{4}\|f\|_H = \sup_{s=0,1,\ldots} a^s \| Q_s \|, \qquad (9) $$

where the last norm (not explicitly containing $\|f\|$ ) must be understood in the sense that f is representable in the form of the series

$$ f = \sum_0^\infty Q_s, \qquad (10) $$

convergent to it in the metric $L_p(\mathcal{E})$, be terms of whose functions are integral and of type $a^{s/r_j}$ with respect to $x_j$ ($j = 1, \ldots, m$) such that norm (9) is finite.

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Therefore $f \in H^r_p(\mathcal{E}) = {}^1H$ and

$$^1\|f\|_H \ll {}^3\|f\|_H.$$

We have proven that

$$^1H \to {}^2H \to {}^3H \to {}^1H,$$

i.e., these classes are equivalent.

The results contain, in particular, the following theorem.

5.5.7. Theorem*) . For a function $f \in H^r_p(\mathcal{E})$, it is necessary and sufficient that the inequalities

$$E_\nu(f) \leqslant c \sum_1^m \frac{1}{\nu_j^{r_j}} \qquad (\nu_j > 0). \tag{1}$$

be satisfied.

Inequality (1) must follow from 5.5.6(3). Conversely, if it is satisfied for any independent $\nu_j > 0$, then still more so for $\nu_j$ of the form $\nu_j = \nu^{1/r_j}$ ($j = 1, \ldots, m$), and then the upper bound of 5.5.6(7) is finite.

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*) S. N. Bernshteyn [2], pages 421-426, $p = \infty$; S. M. Nikol'skiy [1], $1 \leq p < \infty$.

## 5.6. Definition of B-Classes by Means of Best Approximations. Equivalent Norm

Let $\mathscr{E} = R_m \times \mathscr{E}' \subset R_n$, $r > 0$, $k$ and $\rho$ be admissible integers (satisfying the inequalities $\rho \geqslant 0$, $k > r - \rho > 0$), $1 \leq p \leq \infty$, $1 \leq \theta < \infty$, $a > 1$, and the function $f$ is assumed measurable on $\mathscr{E}$.

The principal goal will be prove that the norms

$$ {}^{j}\|f\|_{B^r_{up\theta}(\mathscr{E})} = {}^{j}\|f\|_B = \|f\| + {}^{j}\|f\|_b \qquad (j = 1, \ldots, 5), $$

where $\|\cdot\| = \|\cdot\|_{L_p(\mathscr{E})}$,

$$ {}^{1}\|f\|_b = \sum_{|s|=\rho} \left( \int_0^\infty t^{-1-\theta(r-\rho)} \Omega^k (f^{(s)}, t)^\theta \, dt \right)^{1/\theta}, \tag{1} $$

$$ {}^{2}\|f\|_b = \left( \int_0^\infty t^{-1-\theta(r-\rho)} \Omega^k (f^{\rho}, t)^\theta \, dt \right)^{1/\theta}, \tag{2} $$

$$ {}^{3}\|f\|_b = \sum_{|s|=\rho} \left( \int_{R_m} |u|^{-m-\theta(r-\rho)} \|\Delta_u^k f^{(s)}(x)\|^\theta_{L_p(\mathscr{E})} \, du \right)^{1/\theta}; \tag{3} $$

$$ {}^{4}\|f\|_b = \left( \int_{R_m} |u|^{-m-\theta(r-\rho)} \|\Delta_u^k f_u^{\rho}(x)\|^\theta_{L_p(\mathscr{E})} \, du \right)^{1/\theta}, \tag{4} $$

$$ {}^{5}\|f\|_b = \left\{ \sum_{l=0}^\infty a^{lr\theta} E^\theta_{ua^l}(f)_{\mathscr{E}L_p} \right\}^{1/\theta}, \tag{5} $$

are equivalent; in addition, they are equivalent to the norm (not explicitly containing $\|f\|$)

$$ {}^{6}\|f\|_B = \left\{ \sum_{l=0}^\infty a^{lr\theta} \|Q_{ua^l}\|^\theta_{L_p(\mathscr{E})} \right\}^{1/\theta} \qquad (a > 1), \tag{6} $$

which must be understood in the sense that $f$ can be represented as the series

$$ f = \sum_{l=0}^\infty Q_{ua^l}(x), \tag{7} $$

convergent in it in the $L_p(\mathscr{E})$-sense, and the terms of this series are integral and of the exponential spherical type $a^l$ with respect to $u \in R_m$, such that the norm ${}^{6}\|f\|_B$ is finite.

Here $f^{(s)}$ denotes an arbitrary derivative of f of order $s = (s_1, \ldots, s_m)$, $|s| = \rho$, with respect to the variables $u_1, \ldots, u_m$, and $f_u$ is a derivative in the direction $u \in R_m$ of order $\rho$,

$$\Omega^k(f^{(s)}, \delta) = \Omega^k_{R_m}(f^{(s)}, \delta)_{L_p(\mathfrak{F})} = \sup_{|u|<\delta} \|\Delta^k_u f^{(s)}(x)\|_{L_p(\mathfrak{F})},$$

$$\Omega^k(f^{\rho}, \delta) = \Omega^k_{R_m}(f^{\rho}, \delta)_{L_p(\mathfrak{F})} = \sup_{\substack{|u|=1 \\ u\in R_m}} \sup_{|t|<\delta} \|\Delta^k_{tu} f^{\rho}_u(x)\|_{L_p(\mathfrak{F})}.$$

We further introduce the norms

$$^j\|f\|_{B'} = \|f\| + {}^j\|f\|_{b'} \qquad (j = 1,\ 2,\ 3,\ 4).$$

These are the same norms as, respectively, $^j\|\ \ \|_B$ and $^j\|\ \ \|_b$, but integration in them by definition is performed with respect to $t \in [0, \eta]$ or with respect to u with $|u| < \eta$.

It will be proven that these norms are equivalent to the preceding (with strokes) but with constants dependent on $\eta$. We must remember that each of the classes listed depends further on the admissible pair $(k, \rho)$. It will be shown that any two of these classes corresponding to different pairs are also equivalent (with constant dependent on these pairs).

Let us note that the equivalency of norm (5) with one of the remaining norms for the classes $B^r_{up\theta}(\mathfrak{F})$ corresponds to confirmation of theorem 5.5.4, which yields in terms of best approximations the necessary and sufficient conditions for the function f belongs to the class $H^r_{up}(\mathfrak{F})$. From (5) it follows that $B^r_{up\infty}(\mathfrak{F}) = H^r_{up}(\mathfrak{F})$.

The classes corresponding to these norms are multiples of the series, which we will denote by $^jB$ and $^jb$ $(j = 1, \ldots, 6)$ and $^jB'$ and $^jb'$ $(j = 1, \ldots, 4)$.

It must be born in mind that of themselves seminorms $^jb$ and $^1b'$, generally speaking, are not equivalent, while their sums with $\|f\| = \|f\|_{L_p(\mathfrak{F})}$ are equivalent, i.e., the norms $^jB$ and $^1B'$.

Below we will prove several embeddings, from which will follow the confirmation of equivalence stated above. These embeddings are of interest in themselves. Several of them are valid not only for admissible pairs k, , i.e., those satisfying inequalities $k > r - \rho > 0$.

We have thus far for the same, but not necessarily admissible, pair of natural k, $\rho$

$$ {}^1b \to {}^1b' \to {}^3b' \to {}^4b'. \tag{8}$$

The first and second embeddings are obvious, and the third follows from the relations

$$\left|\Delta^k_u f^\rho_u(x)\right| = \left|\Delta^k_u \sum_{|s|=\rho} f^{(s)}\left(\frac{u}{|u|}\right)^s\right| \le \sum_{|s|=\rho}\left|\Delta^k_u f^{(s)}\right|.$$

Similarly, also for the same, and not necessarily admissible, pair k, $\rho$:

$$ {}^1b \to {}^1b' \to {}^2b' \to {}^4b'. \tag{9}$$

Now let $f \in {}^4B'$ for several, not necessarily admissible k, $\rho$ pair.

For each $\nu > 0$ there exists an integral function $g_\nu$ of spherical type with respect to $u \in R_m$, such that (5.2.1(6))

$$g_\nu - f = (-1)^{l-1}\int_{R_m} g(|u|)\,\Delta^{k+\rho}_{u/\nu} f(x)\,du, \tag{10}$$

and then

$$E_{e^j}(f) \le \|g_{e^j} - f\| = \left\|\int_{R_m} g(|u|)\,\Delta^{k+\rho}_{e^{-j}u} f(x)\,du\right\| =$$
$$= c\left\|\int_0^\infty \int_{|\xi|=1} g(t)\,\Delta^{k+\rho}_{e^{-j}t\xi} f(x)\,t^{m-1}\,d\xi\,dt\right\|.$$

Therefore (explanations given below)

$$
\begin{aligned}
{}^5\|f\|_b &= \left\{ \sum_{j=0}^{\infty} a^{jr\theta} E^{\theta}_{a^j}(f) \right\}^{1/\theta} \leqslant a^r \left\{ \int_{-1}^{\infty} a^{jr\theta} E^{\theta}_{a^j}(f)\, dj \right\}^{1/\theta} \ll \\
&\ll \left\{ \int_{-1}^{\infty} a^{jr\theta} \left| \int_0^{\infty} \int_{|\xi|=1} g(t)\, \Delta^{k+\rho}_{a^{-j}t\xi} f(x)\, t^{m-1}\, d\xi\, dt \right|^{\theta} dj \right\}^{1/\theta} \leqslant \\
&\leqslant \int_0^{\infty} t^{m-1} g(t) \left\{ \int_{-1}^{\infty} a^{jr\theta} \left| \int_{|\xi|=1} \left| \Delta^{k+\rho}_{a^{-j}t\xi} f(x) \right| d\xi \right|^{\theta} dj \right\}^{1/\theta} dt \ll \\
&\ll \int_0^{\infty} t^{m-1+r} g(t) \left\{ \int_0^{at} \int_{|\xi|=1} v^{-r\theta-1} \left| \Delta^{k+\rho}_{v\xi} f(x) \right|^{\theta} d\xi\, dv \right\}^{1/\theta} dt \ll \\
&\ll \int_0^{\infty} t^{m-1+r} g(t)\, dt \left\{ \int_{|u|<\eta} |u|^{-m-(r-\rho)\theta} \left| \Delta^k_u f^{\rho} \right|^{\theta} du + \right. \\
&\left. + \int_{\eta}^{\infty} v^{-r\theta-1}\, dv \|f\|^{\theta} \right\}^{1/\theta} \ll {}^4\|f\|_{b'} + \eta^{-r} \|f\| \ll {}^4\|f\|_{B'}. \qquad \ldots
\end{aligned}
\tag{11}
$$

The generalized Minkowski inequality was applied to the fourth relation (inequality): first the norm $\|\cdot\|$ with respect to x is brought under the sign of the integral with respect to j, and then the norm with respect to j -- under the sign of integral with respect to t. In the fifth relation, j and the integral was replaced with v by means of the substitution $a^{-j}t = v$.

If $\eta = \infty$, then

$$
{}^5\|f\|_b \ll {}^4\|f\|_{b'} \tag{12}
$$

i.e.,

$$
{}^4B' \to {}^5B, \tag{13}
$$

$$
{}^4b \to {}^5b. \tag{14}
$$

In the following we use only embedding (13), but embedding (14) is of interest for its own sake.

Now let $f \in {}^5B$. We will let $g_{a^l}$ stand for a function that is integral and of spherical degree $a^l$ with respect to u such that

$$
|f - g_{a^l}| \leqslant 2E_{a^l}(f) \qquad (l = 0, 1, \ldots),
$$

and set

$$
Q_{a^0} = g_{a^0}, \quad Q_{a^l} = g_{a^l} - g_{a^{l-1}} \qquad (l = 1, 2, \ldots).
$$

Then in the $L_p(\xi)$-sense

$$
f = \sum_{l=0}^{\infty} Q_{a^l},
$$

because from the finiteness of ${}^5\|\cdot\|_b$ it follows that $E_{a^l}(f) \to 0$ $(l \to \infty)$. Further

$$\|Q_{a^0}\| \ll \|f\| + 2E_{a^0}(f),$$
$$|Q_a^l| \ll |g_a^l - f| + |f - g_a^{l-1}| \ll 4E_a^{l-1}(f),$$

therefore $E_a^l(f)$ does not increase with increasing l. Therefore

$$^{\theta}\|f\|_b \ll \left\{ (\|f\| + 2E_{a^0}(f))^{\theta} + \sum_{l=1}^{\infty} a^{lr\theta} E_{a^{l-1}}(f)^{\theta} \right\}^{1/\theta} \ll$$
$$\ll \|f\|_1 + \left\{ \sum_{l=0}^{\infty} a^{lr\theta} E_{a^l}(f)^{\theta} \right\}^{1/\theta} = {}^{\theta}\|f\|_b,$$

and we have proven that

$$^{\theta}B \to {}^{\theta}B. \tag{15}$$

Now let $f \in {}^{\theta}B$ and let f be representable as (7). We will assign arbitrary admissible natural $k, \rho$. For any $u \in R_m$, integral vector $s =$

$(s_1, \ldots, s_m, 0, \ldots, 0)$ with $|s| = \rho$ and natural N

$$\Delta_u^k f^{(s)}(x) = \sum_{l=0}^{N-1} \Delta_u^k Q_{a^l}^{(s)}(x) + \sum_{l=N}^{\infty} \Delta_u^k Q_{a^l}^{(s)}(x),$$
$$|\Delta_u^k f^{\rho}| \ll |u|^k \sum_{l=0}^{N} a^{l(\rho+k)} |Q_a^l| + 2^k \sum_{l=N}^{\infty} a^{l\rho} |Q_a^l|.$$

From whence

$$\Omega^k(f^{(s)}, a^{-N}) = \sup_{|u| < a^{-N},\, u \in R_m} |\Delta_u^k f^{(s)}(x)| \ll$$
$$\ll a^{-Nk} \sum_{l=0}^{N} a^{l(\rho+k)} |Q_a^l| + \sum_{l=N}^{\infty} a^{l\rho} |Q_a^l|.$$

Let us estimate ${}^1\|f\|_{b'}$. We have

$$\int_0^1 t^{-1-\theta(r-\rho)} \Omega^k(f^{(s)}, t)^{\theta}\, dt =$$
$$= \ln a \int_0^{\infty} a^{\theta(r-\rho)n} \Omega^k(f^{(s)}, a^{-n})^{\theta}\, dn =$$
$$= \ln a \sum_{N=0}^{\infty} \int_N^{N+1} a^{\theta(r-\rho)n} \Omega^k(f^{(s)}, a^{-n})^{\theta}\, dn \ll \tag{16}$$
$$\ll \sum_{N=0}^{\infty} a^{\theta(r-\rho)N} \Omega^k(f^{(s)}, a^{-N})^{\theta} = J_1 + J_2,$$

where (explanations given below)

$$J_1 = \sum_{N=0}^{\infty} a^{\theta(r-\rho-k)N}\left(\sum_{l=0}^{N} a^{l(\rho+k)} \|Q_a{}^l\|\right)^{\theta} \ll \sum_{l=0}^{\infty} a^{r\theta l}\|Q_a{}^l\|^{\theta}, \tag{17}$$

$$J_2 = \sum_{N=0}^{\infty} a^{\theta(r-\rho)N}\left(\sum_{l=N} a^{l\rho} \|Q_a{}^l\|\right)^{\theta} \ll \sum_{l=0}^{\infty} a^{lr\theta}\|Q_a{}^l\|^{\theta}. \tag{18}$$

Inequalities $\ll$ are justified thusly. If $a > 1$, $0 < \delta < \beta$, and $b_l \geqslant 0$ ($l = 0, 1, \ldots$), then

$$\begin{aligned}\sum_{N=0}^{\infty} a^{-\theta\beta N}\left(\sum_{l=0}^{N} b_l\right)^{\theta} &= \sum_{N=0}^{\infty} a^{-\theta\beta N}\left(\sum_{l=0}^{N} a^{(\beta-\delta)l}a^{(\delta-\beta)l}b_l\right)^{\theta} \ll \\ &\ll \sum_{N=0}^{\infty} a^{-\theta\delta N}\sum_{l=0}^{N} a^{(\delta-\beta)\theta l}b_l^{\theta} = \sum_{l=0}^{\infty} a^{(\delta-\beta)\theta l}b_l^{\theta}\sum_{N=l}^{\infty} a^{-\theta\delta N} \ll \\ &\ll \sum_{l=0}^{\infty} a^{-\beta\theta l}b_l^{\theta},\end{aligned} \tag{19}$$

$$\begin{aligned}\sum_{N=0}^{\infty} a^{\theta\beta N}\left(\sum_{l=N}^{\infty} b_l\right)^{\theta} &= \sum_{N=0}^{\infty} a^{\theta\beta s}\left(\sum_{l=N}^{\infty} a^{(\delta-\beta)l}a^{(\beta-\delta)l}b_l\right)^{\theta} \ll \\ &\ll \sum_{N=0}^{\infty} a^{\theta\beta N}\left(\sum_{l=N}^{\infty} a^{(\delta-\beta)\theta' l}\right)^{\theta/\theta'}\left(\sum_{l=N}^{\infty} a^{(\beta-\delta)\theta l}b_l^{\theta}\right) \ll \\ &\ll \sum_{N=0}^{\infty} a^{\theta\delta N}\sum_{l=N}^{\infty} a^{(\beta-\delta)\theta l}b_l^{\theta} = \sum_{l=0}^{\infty} a^{(\beta-\delta)\theta l}b_l^{\theta}\sum_{N=1}^{l} a^{\theta\delta N} \ll \\ &\ll \sum_{l=0}^{\infty} a^{\beta\theta l}b_l^{\theta};\end{aligned} \tag{20}$$

where $A \ll B$ must be understood in the sense of $A \leqslant cB$, where $c$ is a constant dependent on and $\delta$, but not on $b_l$.

Inequality (17) is derived from (19) if we set $\beta = k - r + \rho\ (>0)$ and $b_l = a^{l(\rho + k)}\|Q_a{}^l\|$, but inequality (18) is derived from (20) we set

$$\beta = r - \rho, \quad b_l = a^{l\rho}\|Q_a{}^l\|.$$

The use of these two inequalities requires the assumption of the admissibility of the pair k, $\rho$ , i.e. that the conditions $k > r - \rho > 0$ be satisfied.

We have proven that

$$\left(\int_0^\eta t^{-\theta(r-\rho)-1}\Omega^k(f^{(s)}, t)^\theta\, dt\right)^{1/\theta} \ll {}^\theta\|f\|_B. \tag{21}$$

Further, assuming $1/\theta + 1/\theta' = 1$, we get

$$\|f\| \ll \sum_0^\infty |Q_s^l| = \sum_0^\infty a^{-lr}a^{lr}|Q_s^l| \ll$$
$$\ll \left(\sum_0^\infty a^{-lr\theta'}\right)^{1/\theta'} {}^\theta\|f\|_B = c\,{}^\theta\|f\|_B, \tag{22}$$

therefore from (21) and (22) it follows that (for any admissible pair k, )

$${}^\theta B \to {}^1B'. \tag{23}$$

Finally, by using (7) it follows that ($|s| = \rho < r$)

$$\|f^{(s)}\| \ll \sum_{l=0}^\infty a^{l\rho}|Q_s^l| = \sum_{l=0}^\infty a^{-l(r-\rho)}a^{lr}|Q_s^l| \ll$$
$$\ll \left(\sum_0^\infty a^{\theta lr}|Q_s^l|^\theta\right)^{1/\theta} = {}^\theta\|f\|_B. \tag{24}$$

and since the function $t^{-\theta(r-\rho)-1}$ is integrable on $(1,\infty)$ and ( $|s| = \rho$)

$$\Omega^k(f^{(s)}, t) \ll \|f^{(s)}\|,$$

then

$$\int_\eta^\infty t^{-\theta(r-\rho)-1}\Omega^k(f^{(s)}, t)\, dt \ll {}^\theta\|f\|_B,$$

from whence obtains the embedding

$${}^\theta B \to {}^1B, \tag{25}$$

which is stronger than (23) and develop for any admissible k, $\rho$ pair.

Now let k, $\rho$ be an admissible pair. Combining (8), (9), (13), (15), and (25), we get

$$\begin{tikzcd}[column sep=small, row sep=small] & {}^1b' \arrow[r] & {}^3b' \arrow[dr] & \\ {}^1b \arrow[ur] \arrow[dr] & & & {}^4b', \\ & {}^1b' \arrow[r] & {}^2b' \arrow[ur] & \end{tikzcd} \quad {}^4B' \to {}^5B \to {}^6B \to {}^1B.$$

Since here b can be replaced everywhere with B (because this signifies merely that the corresponding inequality remains unchanged if to both of its parts we add $\|f\|$), then

$$\begin{tikzcd}[column sep=small, row sep=small] & & {}^3B' \arrow[dr] & & & & \\ {}^1B \arrow[r] & {}^1B' \arrow[ur] \arrow[dr] & & {}^4B' \arrow[r] & {}^5B \arrow[r] & {}^6B \arrow[r] & {}^1B. \\ & & {}^2B' \arrow[ur] & & & & \end{tikzcd}$$

On the other hand, it is obvious that (chains (8) and (9) are valid if the strokes everywhere in them are omitted)

$$\begin{tikzcd}[column sep=small, row sep=small] & {}^3B \arrow[dr] & & & \\ {}^1B \arrow[ur] \arrow[dr] & & {}^4B \arrow[r] & {}^4B' \arrow[r] & {}^1B. \\ & {}^2B \arrow[ur] & & & \end{tikzcd}$$

This shows that all classes appearing in both chains are equivalent. We again achieve the equivalency of these classes for another admissible pair k', $\rho'$ and since class ${}^5B$, just as ${}^6B$ is independent of(admissible ) k, $\rho$ pairs, then obviously all the indicated classes (${}^jB$(j = 1, ..., 6), ${}^jB'$(j = 1, ..., 4) are equivalent to each other independently of by which k, $\rho$ or parameter $\eta > 0$ they are defined. Of course, the embedding constants emerging here depend generally speaking on k, $\rho$, $\eta$ , and a. Let us note further that the classes ${}^5B$ and ${}^6B$ remain equivalent given the variation $a > 1$. This follows from the fact that, for example, they are equivalent,(but with constants dependent on a) to the classes ${}^1B$ not dependent on a.

Note. Let $f \in {}^1B$. Let us define for f functions $g_\nu$ by means of equality (10). It is easy to see that $g_\nu$ is obtained from f by means of the linear operation $g_\nu = A_\nu(f)$ (cf 5.2.1(4)). From the chain of inequalities (11) that we must read starting with the third term and from the subsequent estimates (cf (12)) follows the inequality

$$\left(\sum_{i=0}^{\infty} a^{i\theta} |g_{a^i} - f|^\theta\right)^{1/\theta} \ll \|f\| \ll \|f\|.$$

Therefore, if we set

$$Q_{a^0} = g_{a^0}, \quad Q_{a^i} = g_{a^i} - g_{a^{i-1}} \qquad (i = 1, 2, \ldots)$$

and consider that

$$|Q_{a^l}| \leqslant |g_{a^l} - f| + |f - g_{a^{l-1}}|,$$

then it is easy to obtain the inequality

$$c_1\|f\|_B \leqslant \left(\sum_{l=0}^{\infty} a^{lr\theta}\|Q_{a^l}\|^{\theta}\right)^{1/\theta} \leqslant c_2\|f\|_B.$$

This line of reasoning was advanced in order to emphasize that if we introduce a norm of the form from 6 $\|\cdot\|_B$ for functions $f \in B^r_{p\theta}(\mathscr{E})$, then we can always assume that here functions $Q_s$ are obtained from f by means of wholly determined linear operations (5.2.1(4)). It is important to note still further that for a given $r_0 > 0$ for all $r < r_0$ these operations for each s can be taken as the same.

5.6.1. Anisotropic case. Let us assign a function $f \in B^r_{p\theta}(\mathscr{E})$ where

$$p = (p_1, \dots, p_m), \quad \theta = (\theta_1, \dots, \theta_m), \quad r = (r_1, \dots, r_m),$$
$$1 \leqslant m \leqslant n, \quad 1 \leqslant p_j, \ \theta_j \leqslant \infty, \quad r_j > 0, \quad a > 1, \quad \mathscr{E} = R_m \times \mathscr{E}'.$$

Let us define for it a family of functions $g_{\nu_1,\dots,\nu_m}$ that are integral and of exponential type $\nu_j$ with respect to $x_j$ by formulas 5.2.4(1), where $0 \leqslant \nu_j \leqslant \infty$ and let us introduce the constant $a > 0$. We will show that there exist inequalities generalising inquality 5.2.4(2) for the case of finite $\theta$:

$$\left(\sum_{s=0}^{\infty} a^{\theta_1 s}\|f - g_{a^{s/r_1},\infty,\dots,\infty}\|^{\theta_1}_{L_{p_1}(\mathscr{E})}\right)^{1/\theta_1} \leqslant c\|f\|_{b^r_{p\theta}(\mathscr{E})},$$

$$\left(\sum_{s=0}^{\infty} a^{\theta_2 s}\|g_{a^{s/r_1},\infty,\dots,\infty} - g_{a^{s/r_1},a^{s/r_2},\infty,\dots,\infty}\|^{\theta_2}_{L_{p_2}(\mathscr{E})}\right)^{1/\theta_2} \leqslant$$
$$\leqslant c\|f\|_{b^r_{p\theta}(\mathscr{E})}, \tag{1}$$

$$\left(\sum_{s=0}^{\infty} a^{\theta_m s}_m\|g_{a^{s/r_1},\dots,a^{s/r_{m-1}},\infty} - g_{a^{s/r_1},\dots,a^{s/r_m}}\|^{\theta_m}_{L_{p_m}(\mathscr{E})}\right)^{1/\theta_m} \leqslant$$
$$\leqslant c\|f\|_{b^r_{p\theta}(\mathscr{E})}.$$

When $\theta_j = \infty$, the corresponding j-th inequality is of the form

$$a^s \| g_{a^{s/r_1}, \ldots, a^{s/r_{j-1}}, \infty, \ldots, \infty} - g_{a^{s/r_1}, \ldots, a^{s/r_j}, \infty, \ldots, \infty} \|_{L_{p_j}(\mathscr{G})} \leqslant$$
$$\leqslant c \| f \|_{h_p^r(\mathscr{G})}.$$

It follows directly from 5.2.4(2). However $\theta_j$ is finite, and ($r_j - \rho_j > 0$, $\rho_j \geqslant 0$, cf 5.2.4(2); 5.6)

$$\sum_{s=0}^{\infty} a^{\theta_j \rho_j} \| g_{a^{s/r_1}, \ldots, a^{s/r_{j-1}}, \infty, \ldots, \infty} - g_{a^{s/r_1}, \ldots, a^{s/r_j}, \infty, \ldots, \infty} \|_{L_{p_j}(\mathscr{G})}^{\theta_j} \leqslant$$
$$\leqslant \sum_{s=0}^{\infty} a^{\theta_j \frac{r_j - \rho_j}{r_j} s} \omega_{x_j}^{k_j} \left( f_{x_j}^{(\rho_j)}, a^{-s/r_j} \right)_{L_{p_j}(\mathscr{G})}^{\theta_j} \ll$$
$$\ll \int_0^1 t^{-(r_j - \rho_j)\theta_j - 1} \omega_{x_j}^{k_j} \left( f_{x_j}^{(\rho_j)}, t \right)^{\theta_j} dt \leqslant \| f \|_{b_{p\theta}^r(\mathscr{G})}^{\theta_j}.$$

and we have proven (1).

Now let $p = p_1 = \ldots = p_n$, $\theta = \theta_1 = \ldots = \theta_m$. Let us introduce the norm ($\|f\| = \|f\|_{L_p(\mathscr{G})}$):

$$^j\|f\|_B = \|f\| + {}^j\|f\|_b \quad (j = 1, 2, 3). \tag{2}$$

We assume that $^1b = b_{p\theta}^r(\mathscr{G})$, i.e., this is already the familiar class

and

$$^2\|f\|_b = \left( \sum_{s=0}^{\infty} a^{\theta s} E_{a^{s/r_1}, \ldots, a^{s/r_m}} (f)_{L_p(\mathscr{G})}^{\theta} \right)^{1/\theta} \tag{3}$$

$$^3\|f\|_B = \left( \sum_{s=0}^{\infty} a^{\theta s} \| Q_s \|^{\theta} \right)^{1/\theta}, \tag{4}$$

where it is assumed that f is representable as the series

$$f = \sum_{s=0}^{\infty} Q_s, \tag{5}$$

convergent in the $L_p(\mathscr{E})$-sense, and whose terms $Q_s$ are functions that are integral and of the type $a^{s/r_j}$, respectively, in $x_j (j = 1, \ldots, m)$.

Norms (2)(of class B), but not ${}^j\|\cdot\|_b$, are equivalent.

Actually, let $f \in {}^1B = B^r_{p\theta}(\mathscr{E})$

$$^2\|f\|_b \leqslant \left(\sum_{s=0}^{\infty} a^{\theta s}\|f - g_{a^{s/r_1}, \ldots, a^{s/r_m}}\|^{\theta}_{L_p(\mathscr{E})}\right)^{1/\theta} \ll {}^1\|f\|_B \tag{6}$$

(the middle part of (6) does not exceed the sum of the left sides of inequalities (1) given equal $p_j$ and equal $\theta_j$).

On the other hand,

$$^2\|f\|_B = \|f\|_{L_p(\mathscr{E})} + \left(\sum_0^{\infty} a^{\theta s} E_{x_j a^{s/r_j}}(f)^{\theta}\right)^{1/\theta} \gg \|f\|_{B^{r_j}_{x_j p\theta}(\mathscr{E})}, \tag{7}$$

where the second quality is valid by virtue of the equivalency of the norms corresponding to seminorms 5.6(1) and 5.6(5).

From (6) and (7) it follows that ${}^1B = {}^2B$.

Let us proceed to the proof of the equivalency ${}^1B = {}^3B$. Suppose $f \in {}^1B$. Let us define for f a family of integral functions $g_s = g_{a^{s/r_1}, \ldots, a^{s/r_m}}$ $(a > 1, s = 0, 1, \ldots)$ for which (6) obtains:

$$\left(\sum_{s=0}^{\infty} a^{\theta s}\|f - g_s\|^{\theta}\right)^{1/\theta} \ll {}^1\|f\|_B. \tag{8}$$

Hence, in particular, it follows that

$$\|f - g_0\| \ll {}^1\|f\|_B \quad \text{and} \quad \|g_0\| \ll {}^1\|f\|_B.$$

Let

$$Q_0 = g_0, \quad Q_s = g_s - g_{s-1} \quad (s = 1, 2, \ldots). \tag{9}$$

It follows from the convergence of the series appearing in (8) that the function is representable in the form of series (5) convergent in it in the $L_p(\mathscr{E})$-sense.

Further,

$$ {}^{3}\|f\|_B = \left(\sum_{s=0}^{\infty} a^{\theta s}\|Q_s\|^{\theta}\right)^{1/\theta} \leqslant \|Q_0\| + \left(\sum_{s=1}^{\infty} a^{\theta s}\|g_s - f\|^{\theta}\right)^{1/\theta} + $$
$$ + \left(\sum_{s=1}^{\infty} a^{\theta s}\|g_{s-1} - f\|^{\theta}\right)^{1/\theta} \leqslant 3\|f\|_{1_B}. $$

Finally, if $f \in {}^3B$, then f is representable as series (5) with finite norm (4). But $Q_s$ is for each j integral and the type $a^{s/r_j}$ with respect to $x_j$, therefore $f \in B^{r}_{x_j p}(\mathscr{E})$ (cf 5.6(6), replace $a^r$ with a, and set m = 1, $R_m = R_{x_j}$) and

$$ \|f\|_{B^{r_j}_{x_j p}} \ll {}^3\|f\|_B \qquad (j = 1, \ldots, m). $$

Thus, $f \in B^r_p(\mathscr{E})$ and

$$ {}^1\|f\| = \|f\|_{B^r_p(\mathscr{E})} \ll {}^3\|f\|_B. $$

We have proven that ${}^1B = {}^3B$.

In conclusion let us emphasize that the norms of classes ${}^1B = B^r_p(\mathscr{E})$ are expressed in (4.3.4) by means of norms $B^{r_j}_{x_j p}(\mathscr{E})$ (j = 1, ..., m) which can be conceived in any equivalent norms described in 5.6 (when m = 1, $R_m = R_{x_j}$).

We observed that everywhere here we have assumed that $\theta_j$ and $\theta$ can be equal to infinity, therefore, in particular, it has been proven that ${}^3H = {}^4H$ obtained in the notations of section 5.5.6.

5.6.2. Let us show the equivalency of the classes

$$ B^{r,\ldots,r}_{p\theta}(\mathscr{E}) - B^r_{p\theta}(\mathscr{E}) \qquad (1 \leqslant \theta \leqslant \infty). \tag{1} $$

We denote first of these by B, and the second by B'. Let us choose a number a such that $a^{1/r} \geqslant \sqrt{m}$, then $\sqrt{m}\, a^{s/r} \leqslant a^{s+1/r}$ (s = 0, 1, ..., m). We observed that the integral function $Q_{a^{s/r}, \ldots, a^{s/r}}$ of type $a^{s/r}$ with respect to each variable $x_j$ (j = 1, ..., m) is of the same time spherical of the type $\sqrt{m}\, a^{s/r}$

with respect to $u$, and so more so of the spherical type $a^{(s+1)/r}$ with respect to $u$:

$$Q_{a^{s/r},\ldots,a^{s/r}} = Q_{ua^{(s+1)/r}}.$$

Now let $f \in B$. Then

$$f = \sum_{s=0}^{\infty} Q_{a^{s/r},\ldots,a^{s/r}} = \sum_{s=0}^{\infty} Q_{ua^{(s+1)/r}}$$

and

$$\|f\|_B = \left(\sum_{s=0}^{\infty} a^{\theta s} \| Q_{ua^{(s+r)/r}} \|_p^\theta\right)^{1/\theta} =$$

$$= \frac{1}{a}\left(\sum_{s=0}^{\infty} a^{\theta(s+1)} \| Q_{ua^{(s+1)/r}} \|_p^\theta\right)^{1/\theta} \leqslant \frac{1}{a}\left(\sum_{s=0}^{\infty} a^{\theta s} \| Q_{ua^{s/r}} \|_p^\theta\right)^{1/\theta} =$$

$$= \frac{1}{a}\|f\|_{B'},$$

where we set $Q_{ua^0} = Q_{u1} \equiv 0$.

And thus, it has been proven that if the function $f \in B$, then it is represented as the series

$$f \in \sum_{0}^{\infty} Q_{ua^{s/r}}$$

of integral functions of spherical type $a^{s/r}$ with respect to $u$ such that

$$\|f\|_{B'} \ll \|f\|_B,$$

i.e., it is proven that $B \rightarrow B'$. The inverse embedding is trivial, and we have proven (1).

5.6.3. Theorem*). Let $f \in B^r_{p\theta}(\mathscr{E})$ and $l = (l_1, \ldots, l_m)$ be an integral on negative that is nonnegative vector ($l_j \geqslant 0$) such that

$$\varkappa = 1 - \sum_{j=1}^{m} \frac{l_j}{r_j} > 0. \qquad (1)$$

Then there exists the derivative

$$f^{(l)} \in B^{\varkappa r}_{p\theta}(\mathscr{E}) \qquad (2)$$

and

$$\|f^{(l)}\|_{B^{\varkappa r}_{p\theta}(\mathscr{E})} \leqslant c\|f\|_{B^r_{p\theta}(\mathscr{E})}. \qquad (3)$$

*) cf note at end of book to sections 5.6.2 - 5.6.3.

where c does not depend on f.

The theorem ceases to be valid when $\rho = \varkappa r$ is replaced by $\rho + \varepsilon$, where $\varepsilon > 0$(cf 7.5). Additionally, generally speaking it is invalid for $\varkappa = 0$ (cf note to 5.6.3).

Proof. By the condition of the theorem

$$f = \sum_{s=0}^{\infty} Q_{a^{s/r_1}, \dots, a^{s/r_m}} = \sum_{0}^{\infty} Q_s \qquad (a > 1),$$

where the terms of the series are integral functions of the exponential type $a^{s/r_j}$ with respect to $x_j$ ($j = 1, \dots, m$), where

$$\|f\|_B = \left(\sum_{0}^{\infty} a^{\theta s} \| Q_s \|^\theta\right)^{1/\theta}$$

$$\left(B = B^r_{p\theta}(\mathscr{E}),\ \|\cdot\| = \|\cdot\|_{L_p(\mathscr{E})},\ a > 1\right).$$

We have for the present, formally,

$$f^{(k)} = \sum_{0}^{\infty} Q_s^{(k)}, \tag{4}$$

where k is any of the vectors $(l_1, 0, \dots, 0)$, $(l_1, l_2, 0, \dots, 0)$, ..., $l = (l_1, \dots, l_n)$. Let us note that

$$|Q_s^{(k)}| < a^{s \sum_{1}^{m} \frac{l_j}{r_j}} \| Q_s \| = a^{s(1-\varkappa)} \| Q_s \|.$$

Therefore

$$\left(\sum_{0}^{\infty} a^{s\varkappa\theta} |Q_s^{(k)}|^\theta\right)^{1/\theta} < \left(\sum_{0}^{\infty} a^{s\theta} \| Q_s \|^\theta\right)^{1/\theta} = \|f\|_B. \tag{5}$$

From (5) it follows that series (4) converges in the $L_p$-sense, therefore member-wise differentiation in (4) (in the generalized sense) is legitimate based on lemma 4.4.7.

Let us note that $Q_s^{(l)}$ just as $Q_s$, is an integral function of the type $a^{s/r_j}$ with respect to $x_j$ ($j = 1, \dots, m$). If we set $a^{\varkappa} = b$ ($b > 1$) then equality (5) for $k = 1$ will be written as

$$\left(\sum_{0}^{\infty} b^{s\theta} |Q_s^{(l)}|^{\theta}\right)^{1/\theta} < \|f\|_b,$$

where $Q_s^{(l)}$ is an integral function of the type $b^{s/r_j \varkappa}$ with respect to $x_j$. In this case $f^{(l)} \in B_{p\theta}^{\varkappa r}(\xi)$ and inequality (3) is satisfied.

Let us make also the following addition. Let us assume that we wished to differentiate the derivative $f^{(l)}$ mentioned in the theorem another $l' = (l'_1, \ldots, l'_m)$ "times". This is possible by this theorem, if the quantity

$$\varkappa' = 1 - \sum_{j=1}^{m} \frac{l'_j}{r_j \varkappa} > 0.$$

Hence

$$\varkappa\varkappa' = \varkappa - \sum_{j=1}^{m} \frac{l'_j}{r_j} = 1 - \sum_{j=1}^{m} \frac{l_j + l'_j}{r_j} = \varkappa_* > 0.$$

But the quantity $\varkappa_*$ in term is the constant $\varkappa$ appearing in our theorem if in it $l$ is replaced by $l + l'$.

In this sense the theorem is transitive character.

5.6.4. **Example.** Below is presented an example showing that seminorms $^3b$ and $^3b'$, generally speaking, are not equivalent (cf 5.6(3), (4)). Let us confine ourselves to the 1-dimensional case

$$m = 1, \quad r = 1 - \frac{1}{p} < 1, \quad \rho = 0, \quad k = 1, \quad \theta = p.$$

Let $f_N(x)$ be an even function, equal to

$$f_N(x) = \begin{cases} \frac{x}{N}, & 0 \leqslant x \leqslant N, \\ 1, & N < x. \end{cases}$$

Then

$$^3\|f_N\|_b^p =$$

$$= 2\int_0^{\infty} dh \int_{-\infty}^{\infty} \left|\frac{f_N(x+h) - f_N(x)}{h}\right|^p dx \geqslant 2\int_0^N dx \int_0^{N-x} \frac{dh}{N^p} = N^{2-p},$$

$$\frac{1}{2}\,^3\|f_N\|_{b'}^p = \int_0^1 dh \left\{\int_0^{\infty} + \int_{-\infty}^{-h} + \int_{-h}^{0}\right\} dx = J_1 + J_2 + J_3 = O(N^{1-p}),$$

because

$$J_2 - J_1 \leqslant \int_0^{N-1} dx \int_0^1 \frac{dh}{N^p} + \int_{N-1}^{N} dx \left\{ \int_0^{N-x} \frac{dh}{N^p} + \int_{N-x}^{1} \left| \frac{1-\frac{x}{N}}{h} \right|^p dh \right\} = O(N^{1-p}),$$

$$J_3 = \int_0^1 dh \int_{-h}^{0} \left| \frac{\frac{x+h}{N} + \frac{x}{N}}{h} \right|^p dx = O(N^{1-p}).$$

From this it is clear that it does not exist a constant c such that for all $N > 0$ the inequality $^3\|f_N\|_b \leq c^3 \|f_N\|_{b'}$ is satisfied.

5.6.5. Translationwise continuity. Theorem. When $h \to 0$

$$\|f(x+h) - f(x)\|_W \to 0 \quad (f \in W = W_p^l(R_n),\ 1 \leqslant p < \infty,\ l \geqslant 0). \tag{1}$$

$$\|f(x+h) - f(x)\|_B \to 0 \quad (f \in B = B_p^r(R_n),\ 1 \leqslant p, \theta < \infty,\ r \geqslant 0). \tag{2}$$

The confirmation of (1) when $p = \infty$ does not obtain, just as (2) when $\theta = \infty$ ($B^r_{p\infty} = H^r_p$, cf. further 7.4.1); when $p = \infty$, $1 \leq \theta < \infty$ (2) remains valid.

Proof. In the case $l = 0$ ($W^0_p = L_p(R_n)$), property (1) is a well-known fact (invalid, however, when $p = \infty$). The general case actually reduces to it because $\|f\|_W$ is the sum of norms f and $\partial^{l}f/\partial x_j^{l_j}$ in $L_p(R_n)$ $(j = 1, \ldots, n)$.

The representation

$$f = \sum_0^\infty Q_s$$

$$\|f\|_B = \left\{ \sum_0^\infty 2^{s\theta} \|Q_s\|_p^\theta \right\}^{1/\theta},$$

where $Q_s$ are integral functions of the type $2^{s/r_j}$ with respect to $x_j$ obtains for the function $f \in B$. Therefore

$$\|f(x+h)-f(x)\|_B \leqslant \left\{\sum_0^{N-1} 2^{s\theta}\|Q_s(x+h)-Q_s(x)\|_p^\theta\right\}^{1/\theta} +$$

$$+2\left\{\sum_N^\infty 2^{s\theta}\|Q_s\|_p^\theta\right\}^{1/\theta} < \varepsilon+\varepsilon = 2\varepsilon,$$

if we take N sufficiently large and then choose a sufficiently small .

Note. We can replace p in (1) and (2) with $p = (p_1, \ldots, p_n)(1 \leqslant p_j < \infty)$, because these relations are valid, in particular, for the classes $W^{l_j}_{x_j, p_j}(R_n)$ and $B^{r_j}_{x_j, p_j}(R_n)$, $j = 1, \ldots, n$.

5.6.6. Under the condition that $1 \leqslant \theta$, $p < \infty$, and $g \subset R_n$ is an open set, $g^N = g(R_n - V_N)$, where $V_N$ is a sphere with center at the zero point and of radius N, and $f \in B^r_{p\theta(g)} = B(g)$,

$$\|f\|_{B(g^N)} \to 0 \qquad (N \to \infty). \tag{1}$$

holds. This is evident from the definition of the norm $\|\cdot\|_B$, for example, in the form 4.3.4(2) ($\rho = 0$, $k \geqslant 2$):

$$\|f\|_{b^r_{p\theta}(g^N)} = \left(\int_0^\infty t^{-1-\theta r}\Omega^k_{R_n}(f, t)^\theta_{L_p(g^N)}\right)^{1/\theta} \to 0 \qquad (N \to \infty).$$

In fact $f \in L_p(g)$, therefore $\Omega^k_{R_n}(f, t)_{L_p}(g^N)$ is finite for any t and tends to zero, monotonically diminishing as $N \to \infty$, and we can use the Lebesgue theorem on the limit passage under the sign of the integral.

# CHAPTER VI THEOREMS OF EMBEDDING OF DIFFERENT METRICS AND MEASURES

## 6.1. Introduction

Begin by setting up the S.L. Sobolev embedding theorem [3] with latter supplements due to V. I. Kondrashov [1] and V. P. Il'yin [2]*). As applied to space $R_n$ and to its coordinate subspace $R_m (1 \leq m \leq n)$, this theorem reads:

If a function $f \in W_p^l(R_n)$ and

$$0 \leqslant \rho = l - \frac{n}{p} + \frac{m}{p'}, \quad 1 < p < p' < \infty, \tag{1}$$

then**)

$$W_p^l(R_n) \to W_{p'}^{[\rho]}(R_m), \tag{2}$$

where $[\rho]$ is the integral part of $\rho$. This means that the trace of the function $f|_{R_m} = \varphi$ belonging to the class $W_{p'}^{[\rho]}(R_m)$ exists, and that the inequality

$$\|\varphi\|_{W_{p'}^{[\rho]}(R_m)} \leqslant c \|f\|_{W_p^l(R_n)}, \tag{3}$$

is met where c does not depend on f**).

This concept of the trace of f will be explained in the following, but for the present we will state that in any case if f is continuous on $R_n$, then its trace on $R_m$ is the name given to the function $\varphi = f|_{R_m}$ induced by the function f on $R_m$.

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*) Cf note to 6.1 and the book.

**) The general S. L. Sobolev theorem can be written in the form of formula (2), where we must replace $R_n$ and $R_m$ by $g$ and $\Delta_m = R_m g$ and assume that $g$ is a star-shaped domain relative to some n-dimensional sphere.

In particular, when $m = n$, from (2) follows the "pure" embedding of different metrics:

$$W_p^l(R_n) \to W_{p'}^{[\rho]}(R_n), \tag{4}$$

asserting that if $f \in W_p^l(R_n)$, then $f \in W_{p'}^{[\rho]}(R_n)$ and

$$\|f\|_{W_{p'}^{[\rho]}(R_n)} \leqslant c\|f\|_{W_p^l(R_n)} \tag{5}$$

provided the condition that (1) is satisfied (when $m = n$).

The S. L. Sobolev embedding theorems will be proven in Chapter IX.

But in this chapter we will set out to discuss these questions for the classes $B_{p\theta}^r(R_n)$, in particular, when $\theta = \infty$, the classes $H_p^r(R_n)$. Incidentally, from the theorem obtained in this chapter, in particular, there will follow the above-formulated theorems for the case when $\rho > 0$ is nonintegral, and then, as we will see, they are valid under more sweeping conditions: $1 \leq p \leq p' \leq \infty$.

Let us present even at this stage the characteristic theorem of the embedding of different metrics, which in particular will be obtained in this chapter:

$$B_{p\theta}^r(R_n) \to B_{p'\theta}^{\rho}(R_n), \tag{6}$$

if

$$1 \leqslant p < p' \leqslant \infty, \quad 1 \leqslant \theta \leqslant \infty, \quad \rho = r - n\left(\frac{1}{p} - \frac{1}{p'}\right) > 0. \tag{7}$$

Thus, if function f belongs to the left class of (6), then it belongs also to the right class and, moreover, the inequality

$$\|f\|_{B_{p'}^{\rho}(R_n)} \tag{8}$$

is satisfied where c does not depend on f.

The characteristic (direct) theorem of embedding of different measures which will be proven in this chapter, is written thusly:

$$B_{p\theta}^r(R_n) \to B_{p\theta}^{\rho}(R_m), \tag{9}$$

where

$$1 \leqslant p, \ \theta \leqslant \infty, \quad 1 \leqslant m < n, \quad \rho = r - \frac{n-m}{p} > 0. \tag{10}$$

It asserts that provided the conditions (10), if a function f of class $B^r_{p\theta}(R_n)$ is given on $R_n$, then it has the trace $\varphi$ on $R_m$ belonging to the class $B^\rho_{p\theta}(R_m)$ and the inequality

$$\|\varphi\|_{B^\rho_{p\theta}(R_m)} \leq c\|f\|_{B^r_{p\theta}(R_n)}, \tag{11}$$

is satisfied, where c does not depend on f.

Inequality (11) is important for applications; it indicates a certain (stable) dependence of the norms of traces of functions f on the norms of f.

Theorems of embedding of different measures for the classes $B^r_{p\theta}$ are characterized by the fact that they are wholly invertible. Let us present by way of example the theorem that is the inverse of theorem (9). It is described thusly:

$$B^\rho_{p\theta}(R_m) \to B^r_{p\theta}(R_n) \tag{12}$$

(provided condition (10)) and reads: to each function $\varphi$ defined on $R_m$ and belonging to the class $B^\rho_{p\theta}(R_m)$ there can be brought in correspondence its extension on $R_n$ -- the function $f \in B^r_{p\theta}(R_n)$ — such that $f|_{R_m} = \varphi$ and

$$\|f\|_{B^r_{p\theta}(R_n)} \leq c\|\varphi\|_{B^\rho_{p\theta}(R_m)}, \tag{13}$$

where c does not depend on $\varphi$.

More general theorems of embeddings of different measures that the reader can find in this chapter are correspondingly also wholly invertible. This indicates, in particular, the unimprovability of these theorems. As far as theorems of embedding of different metrics are concerned, they also are unimprovable (in the terms in which they are stated); this is proven in the next chapter. There the reader can find out about certain interesting so-called transitive properties of embedding theorems.

We will commence this chapter by establishing the simplest relationships between the classes W, H, and B expressible by means of embeddings.

Here we note only the following relationships:

$$H^{r+\varepsilon}_p \to W^r_p \to H^r_p \qquad (\varepsilon > 0,\ r = 0,\ 1,\ \ldots), \tag{14}$$

the second of which is already known to us.

From (14), (6), and (9) follows ($\rho = 1 - n/p + m/p' > 0$ is nonintegral):

$$W_p^l(R_n) \to H_p^l(R_n) \to H_{p'}^{l-n\left(\frac{1}{p}-\frac{1}{p'}\right)}(R_n) \to H_{p'}^{\rho}(R_m) \to W_{p'}^{[\rho]}(R_m), \tag{15}$$

i.e., (5).

Theorems of embedding of different metrics and measures, just as the inverse theorems of embedding of different metrics, were proven for classes $H_p^r(R_n)$ by S. M. Nikol'skiy [3] using methods of approximation by integral functions of exponential type. They were generalized by O. V. Besov [2, 3] for the classes $B_{p\theta}^r(R_n)$ ($H_p^r = B_{p\infty}^r$) he introduced. O. V. Besov also founded his approach on the method of approximation with integral functions of exponential type. Certain embedding theorems of different metrics for one-dimensional classes $H_p^r$ were found by Hardy and Littlewood [1]. The theorem of embedding of different measures was also proven for more general classes $H_p^r(R_n)$ ($p_j$, generally speaking, are different) by S. M. Nikol'skiy [10] by the methods of approximation. Then it was generalized for the classes $B_{p\theta}^r$ by V. P. Il'yin and V. A. Solonnikov [1, 2], but then by different methods.

Below everywhere in our proof we will operate with methods of approximation, including our examination of this theorem for the general classes $B_{p\theta}^r$*).

## 6.2. Relationships Between Classes B, H, and W

We will consider the functions of these classes on a cylindrical measurable space $\mathscr{E} = \mathscr{E}_m \times \mathscr{E}'$ ($1 \leq m \leq n$, $u = (x_1, \ldots, x_m)$, $w = (x_{m+1}, \ldots, x_n)$).

We will suppose for sake of brevity that

$$B_{up\theta}^r(\mathscr{E}) = B_{p\theta}^r, \quad H_{up}^r(\mathscr{E}) = H_p^r, \quad W_{up}^r(\mathscr{E}) = W_p^r,$$
$$\|f\|_{L_p(\mathscr{E})} = \|f\|, \quad r > 0, \quad 1 \leqslant \theta \leqslant \infty.$$

The following embeddings ($r = \bar{r} + \alpha$, $0 < \alpha \leq 1$, $\bar{r}$ is an integer) obtain:

$$B_{p1}^r \to B_{p\theta}^r \to B_{p\theta'}^r \to B_{p\infty}^r = H_p^r \qquad (1 \leqslant \theta < \theta' \leqslant \infty), \tag{1}$$

$$W_p^r \to H_p^r \qquad (r = 1, 2, \ldots), \tag{2}$$

*) Cf T. I. Amanov [3].

$$H_p^{r+\varepsilon} \to B_{p\theta}^r \to H_p^r \qquad (\varepsilon > 0), \tag{3}$$

$$H_p^r \to W_p^\rho \qquad (\rho = 0, 1, \ldots, \bar{r}), \tag{4}$$

$$B_{p\theta}^{r+\varepsilon} \to B_{p\theta}^r \qquad (\varepsilon > 0). \tag{5}$$

Embeddings (1) show that classes $B_{p\theta}^r$ expand with increment in $\theta$. The proof of (1) directly follows from the fact that (cf 5.6(6), (7)) function $f \in B_{p\theta}^r$ can be defined as the sum of the series

$$f = \sum_{s=0}^{\infty} Q_s \tag{6}$$

convergent in it in the $L_p$-sense, the members of whose function $Q_s$ are integral and of the spherical type $a^s$ $(a > 1)$ with respect to $x$ such that

$$\|f\|_{B_{p\theta}^r} = \left(\sum_{s=0}^{\infty} a^{sr\theta} \|Q_s\|^\theta\right)^{1/\theta} \qquad (a > 1). \tag{7}$$

In fact, the right side of (7) diminishes with increment in $\theta$ (cf 3.3.3). It is also clear from the chain (1) that for fixed r and p, the "worst" class is class $H_p^r$ and the "best" is $B_{p1}^r$.

Embedding (3), from which it follows that

$$B_{p\theta'}^{r+\varepsilon} \to B_{p\theta}^r \to B_{p\theta''}^{r-\varepsilon}$$

for any $1 \leq \theta', \theta, \theta'' \leq \infty$, however small the $\varepsilon > 0$, show that the class $B_{p\theta}^r$ depends more strongly on r than on $\theta$. The second embedding in (3) was already proved in (1). Suppose $f \in H_p^{r+\varepsilon}$; then

$$\|f\|_{H_p^{r+\varepsilon}} = \sup_s a^{s(r+\varepsilon)} \|Q_s\| = M < \infty,$$

therefore

$$\|f\|_{B_{p\theta}^r} < \left\{\sum_{s=0}^{\infty}\left(a^{sr} \frac{M}{a^{s(r+\varepsilon)}}\right)^\theta\right\}^{1/\theta} < cM,$$

where c does not depend on M, from whence follows the first embedding of (3). Embedding (5) follows from the fact that the right side of (7) increases together with r. We must bear in mind that for a given $r_0 > 0$, functions $Q_s$

for all $r < r_0$ can be assumed to be the same (cf note at end of section 5.6).

Embedding (2) follows from the inequalities ($h \in R_n$):

$$|\Delta_h^k f^{(\bar r)}(x)| \leq 2^{k-1} |\Delta_h f^{(\bar r)}(x)| \leq 2^{k-1} |h| \left| \frac{\partial}{\partial h} f^{(\bar r)}(x) \right| \leq$$
$$\leq |h| \sum |f^{(r)}| \quad (\bar r = r - 1),$$

where the sum is extended over all derivatives $f^{(r)}$ from f of order r.

From (1) and (4) it follows that

$$B_{p\theta}^r \to H_p^r \to W_p^\rho \quad (\rho = 0, 1, \ldots, \bar r). \tag{8}$$

For the anisotropic classes

$$B_{\varkappa p\theta}^r(\mathscr{E}) = B_{p\theta}^r, \quad H_{\varkappa p}^r(\mathscr{E}) = H_{\varkappa p}^r, \quad W_{\varkappa p}^r(\mathscr{E}) = W_p^r$$

the following embedding ($p = (p_1, \ldots, p_n)$) obtain:

$$B_{p1}^r \to B_{p\theta}^r \to B_{p\theta'}^r \to B_{p\infty}^r = H_p^r, \quad 1 \leq \theta < \theta' < \infty, \tag{9}$$

$$W_p^\rho \to H_p^\rho \quad (\rho \text{ is an integral vector}) \tag{10}$$

$$H_p^{r+\varepsilon} \to B_{p\theta}^r \to H_p^r \quad (\varepsilon > 0, \text{ i.e., } \varepsilon_j > 0) \tag{11}$$

$$B_{p\theta}^{r+\varepsilon} \to B_{p\theta}^r \quad (r > 0), \tag{12}$$

$$B_{p\theta}^r \to H_p^r \to W_p^\rho \quad (\rho < r, \rho \text{ is an integral vector}) \tag{13}$$

They are analogous to embeddings (1) - (5) and (8) and directly follow from them. If $p = p_1 = \ldots = p_n$, then p can everywhere be replaced by p.

## 6.3. Embedding of Different Metrics

$$B_{p\theta}^r(R_n) \to B_{p'\theta}^{r'}(R_n), \tag{1}$$

obtains*) if the following conditions are met:

*) [*) on following page]

$$1 \leqslant p < p' \leqslant \infty, \quad 1 \leqslant \theta \leqslant \infty, \tag{2}$$

$$\varkappa = 1 - \left(\frac{1}{p} - \frac{1}{p'}\right) \sum_{1}^{n} \frac{1}{r_j} > 0, \tag{3}$$

$$r' = \varkappa r. \tag{4}$$

( We assume that $r > 0$.)

In particular, if we consider that $B_p^r(R_n) = B_p^{r,\ldots,r}(R_n)$(cf 5.6.2),

$$B_{p\theta}^r(R_n) \to B_{p'\theta}^{r'}(R_n) \tag{1'}$$

obtains provided the conditions

$$1 \leqslant p < p' \leqslant \infty, \tag{2'}$$

$$\varkappa = 1 - \left(\frac{1}{p} - \frac{1}{p'}\right) \frac{n}{r} > 0, \tag{3'}$$

$$r' = \varkappa r. \tag{4'}$$

For example, when $p' = \infty$ and

$$r' = r - n\left(\frac{1}{p} - \frac{1}{\infty}\right) = r - \frac{n}{p} > 0,$$

$$B_{p\theta}^r(R_n) \to B_{\infty\theta}^{r'}(R_n) \to H_{\infty}^{r'}(R_n)$$

and, therefore, if the function $f \in B_{p\theta}^r(R_n)$, then it is continuous and bounded on $R_n$ together with its partial derivatives of order less than $r'$. Additionally, if, for example, $r' = \rho + \alpha$, is an integer, and $0 < \alpha < 1$, then the derivative $f^{(\rho)}$ of order $\rho$ satisfies on $R_n$ the Lipshits condition of degree $\alpha$.

Let us prove (1). Suppose $B$ and $B'$, respectively, stand for the first and second classes (1) and $\|\cdot\|_p = \|\cdot\|_{L_p(R_m)}$. Let us assign the function $f \in B$ and the number $a > 1$. It can be represented as the series

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*) S. M. Nikol'skiy [3], case $H_p^r(R_n) = B_{p\,\infty}^r(R_n)$; O. V. Besov [2, 3], case $1 \leqslant \theta < \infty$ ; Hardy and Littlewood [1] for certain one-dimensional classes $H_p^r$.

$$f = \sum_{s=0}^{\infty} Q_s, \tag{5}$$

whose terms $Q_s$ are integral functions of type $a^{s/r_j}$ with respect to $x_j$ $(j = 1, \ldots, n)$, and where

$$\|f\|_B = \left(\sum_0^{\infty} a^{\theta s}\|Q_s\|_p^{\theta}\right)^{1/\theta} < \infty \qquad (a > 1). \tag{6}$$

The inequality of different metrics (3.3.5(1))

$$\|Q_s\|_{p'} < 2^n a^{(1-\varkappa)s}\|Q_s\|_p,$$

is satisfied for the functions $Q_s$, therefore

$$\left(\sum_0^{\infty} a^{\theta\varkappa s}\|Q_s\|_{p'}^{\theta}\right)^{1/\theta} < \left(\sum_0^{\infty} a^{\theta s}\|Q_s\|_p^{\theta}\right)^{1/\theta} = \|f\|_B.$$

But if we set $a^{\varkappa} = b\ (b > 1)$, then we get the inequality

$$\left(\sum_0^{\infty} b^{\theta s}(\|Q_s\|_{p'})^{\theta}\right)^{1/\theta} < \|f\|_B, \tag{7}$$

where $Q_s$ are integral functions of the type $b^{s/r'_j}$ with respect to $x_j$. From this it follows that series (5) converges in the metric $L_{p'}$ and here to f, because it already converges to f in the metric $L_p$ (cf (1.3.7)). Moreover, from (7) it follows that $f \in B'$ and the left side of (7) is $\|f\|_{B'}$. We have proven that

$$\|f\|_{B'} < \|f\|_B,$$

and embedding (1) stands proven.

In this case conditions (2') - (4') are equivalent to the following:

$$r, r' > 0, \quad 1 \leq p < p' \leq \infty, \quad r - \frac{n}{p} = r' - \frac{n}{p'}.$$

The quantity r - n/p appears in them, which must be invariant in order to insure embedding. In the general case cf 7.1 on this issue.

Let us note that $R_n$ cannot be replaced by $R_n \times \mathscr{E}'$ in (1), since in this case there would be no inequality similar to (7). In fact, this never occurs, and can be easily seen in examples.

## 6.4. Place of Function

The function f belonging to a given class $B(R_n)$ and $W(R_n)$ is defined on $R_n$ only with accuracy up to the set of n-dimensional measure zero or, as we will additionally state, with an accuracy up to equivalence relative to $R_n$ or in the $R_n$-sense. Therefore the trace of function f

$$f|_{R_m} = \varphi = \varphi(x_1, \ldots, x_m) \tag{1}$$

for any subspace $R_m \subset R_n (m < n)$ is not meaningful, if it is understood literally.

Below we give the definition of the trace of function f on $R_m$ leading to the unique function $\varphi$ with an accuracy up to equivalence relative to $R_m$.

We will denote each point $x \in R_n$ as the pair $x = (u, v)$, where $u = (x_1, \ldots, x_m)$, $v = (x_{m+1}, \ldots, x_n)$, and let $R_n(v)$ be the m-dimensional subspace of points $(u, v)$ where $v$ is fixed, and let $u$ runs through all possible values. In particular, let $R_m(0) = R_m$.

Suppose $f(x)$ is a function measurable on $R_n$.

We will state that the function

$$\varphi = \varphi(u) = f|_{R_m} \tag{2}$$

is the trace of f on $R_m$ if f can be modified on a set of m-dimensional measure zero such that after this, for a certain p, $1 \leq p \leq \infty$, the following properties will be satisfied:

1)

2)

3)

where $\delta$ is sufficiently small.

Let us show that the trace of f on $R_m$ defined in this way is unique with an accuracy up to equivalence in the $R_m$-sense.

Actually, assume that we will be able to find the two modifications $f_1$ and $f_2$ of function f on the set of n-dimensional measure zero and such numbers $p_1$ and $p_2$ ($1 \leq p_1 \leq p_2 \leq \infty$) that for $f_1$, $\varphi_1$, $p_1$, and $f_2$, $\varphi_2$, $p_2$, relations 1) - 3) are individually fulfilled, and suppose $g \subset R_m$ is not arbitrarily bounded open set. Then

$$\|\varphi_2(u)-\varphi_1(u)\|_{L_{p_1}(g)} \leqslant \|\varphi_1(u)-f_1(u, w)\|_{L_{p_1}(g)} + \\ + \|f_1(u, w)-f_2(u, w)\|_{L_{p_1}(g)} + c\|f_2(u, w)-\varphi_2(u)\|_{L_{p_1}(g)}, \qquad (3)$$

where c is a constant dependent on the measure of g. Functions $f_1$ and $f_2$ are equivalent in the $R_n$-sense, therefore $\int\int_{R_n} |f_1 - f_2|^{p_1}\, du\, dw = 0$

and by Fubini's theorem, for almost all w

$$\int_{R_m(w)} |f_1 - f_2|^{p_1}\, du = 0.$$

But from the set of points w for which this equality holds, we can always select this sequence $w_1$, $w_2$, ... with $|w_k| \to 0$. The right side of (3), when w runs through this sequence, tends to zero, but then the left side equals zero, and since $g \subset R_m$ arbitrarily, then $\varphi_1 = \varphi_2$ on $R_m$.

It is not difficult to see that if function f not only is measurable on $R_n$, but also is continuous in the n-dimensional neighborhood $R_m$, then its trace $\varphi$ coincides with the trace of f on $R_m$ with accuracy up to equivalence in the $R_m$-sense in the ordinary meaning of this word, Denote further that if for the two measurable functions $f_1$ and $f_2$, for some p the above-described operation of removal of trace (2), which we will further denote as:

$$\varphi = A(f) = f|_{R_m}, \qquad (4)$$

is possible, then it is possible also for any linear combination

$$c_1 f_1 + c_2 f_2,$$

where $c_1$ and $c_2$ are arbitrary real numbers and where the equality

$$A(c_1 f_1 + c_2 f_2) = c_1 A(f_1) + c_2 A(f_2).$$

obtains.

Thus, the set of all measurable functions f for which operation (4) is possible for some p is linear and (4) is the linear operation (operator) defined in it. As will be clear from the following, functions of classes $B^r_{p\theta}$ and $W^r_p$, with the corresponding values of parameters p and r, have traces on $R_m$ in the above-indicated sense.

Suppose the domain $g \subset R_n$ and $g' \subset \bar{g}$ such that, in particular, g' can the boundary of g. Further assume that the class of function $\mathfrak{M}$ is defined on g'. Let us assign function f on g and assume that on g' it has the trace:

$$\varphi = f|_{g'},$$

belonging to $\mathfrak{M}$. Then we will not only write: $\varphi = f|_{g'} \in \mathfrak{M}$, but also $f \in \mathfrak{M}$.

## 6.5. Embeddings of Different Measures

There obtains*)

$$B^r_{p\theta}(R_n) \to B^{r'}_{p\theta}(R_m) \tag{1}$$

given the conditions

$$0 \leqslant m < n, \quad 1 \leqslant p, \ \theta \leqslant \infty, \tag{2}$$

$$\varkappa = 1 - \frac{1}{p} \sum_{j=m+1}^{n} \frac{1}{r_j} > 0, \tag{3}$$

$$r' = (r'_1, \ldots, r'_m), \quad r'_j = \varkappa r_j. \tag{4}$$

Here $R_m$ stands for the m-dimensional subspace of points

$u = (x_1, \ldots, x_m)$, $y = (x_{m+1}, \ldots, x_n)$, where $y$ is fixed. Let B and B',

*) S. M. Nikol'skiy [3], case $H^r_p = B^r_{p\infty}$; O. V. Besov [2, 3], case $1 \leq \theta < \infty$.

respectively, be the first and second classes in (1) and $\|\cdot\|^m = \|\cdot\|_{L_p(R_m)}$.
Embedding (1) states that any function $f \in B$ has the trace

$$f|_{R_m} = \varphi \in B',$$

and that the inequality*)

$$\|\varphi\|_{B'} \leqslant c\|f\|_B,$$

is satisfied by where c does not depend on f.

For the case when $m = 0$, it is assumed that

$$B^{r'}_{p\theta}(R_0) = B^{r'}_{\infty,\theta}(R_n).$$

Thus, in this case we are talking about embedding in different metrics (from p to $p' = \infty$), and it has already been proven in 6.2.

If we consider that $B^r_p(R_n) = B^{r,\ldots,r}_p(R_n)$, and $B^r_p(R_m) = B^{r,\ldots,r}_p(R_m)$ then from (1), in particular, it follows that

$$B^r_{p\theta}(R_n) \to B^{r'}_{p\theta}(R_m) \tag{1'}$$

providing the conditions

$$0 \leqslant m < n, \quad 1 \leqslant p, \ \theta \leqslant \infty, \tag{2'}$$

$$\varkappa = 1 - \frac{n-m}{rp} > 0, \tag{3'}$$

$$r' = r\varkappa = r - \frac{n-m}{p}. \tag{4'}$$

Let us now turn to the proof when $1 \leqslant m < n$. We represent the function $f \in B^r_{p\theta}(R_n)$ as the series

$$f = \sum_0^\infty Q_s, \tag{5}$$

which are integral functions of the type $a^{s/r_j}$ $(a > 1)$ with respect to $x_j$ $(j = 1, \ldots, n)$ with the norm

$$\|f\|_B = \left(\sum_0^\infty a^{s\theta}(\|Q_s\|^n)^\theta\right)^{1/\theta}. \tag{6}$$

---

*) The more exact inequality $\|f\|_{b'} \leqslant c\|f\|_b$ obtains given certain reservations (cf 7.2(10) and (11)).

Let us use estimate (3,4.2(1)

$$\|Q_s\|^m \leqslant 2^{n-m} a^{s(1-\varkappa)} \|Q_s\|^r,$$

for $Q_s$, from whence

$$\left(\sum_0^\infty a^{\theta\varkappa s}(\|Q_s\|^m)^\theta\right)^{1/\theta} \ll \left(\sum_0^\infty a^{s\theta}(\|Q^s\|^n)^\theta\right)^{1/\theta} = \|f\|_B.$$

Setting $a^{\varkappa} = b$ $(b > 1)$, we get

$$\left(\sum_0^\infty b^{\theta s}(\|Q_s\|^m)^\theta\right)^{1/\theta} \ll \|f\|_B \quad \left(Q_s = Q_{b^s/r'_1, \ldots, b^s/r'_m}\right). \tag{7}$$

This inequality, in particular, shows that series (5) converges for any fixed $y$ and the $L_p(R_m)$-sense with respect to $u = (x_1, \ldots, x_m)$ to some function $f_1(x) = f_1(u, y) \in L_p(R_m)$. But then $f_1 = f$ almost everywhere in the sense of the n-dimensional measure (cf 1.3.9).

By virtue of inequality (7), $f_1(u, y) \in B'$ for any $y$

$$\|f_1(u, y)\|_{B'} \ll \|f\|_B. \tag{8}$$

The constant in this inequality does not depend on $y$.

If it will be proven that $f_1(u, y)$ is the trace of $f$ on $R_m$ for any $y$, then together with inequality (7) this leads to the required embedding (1). Since (cf 6.1(14))

then

$$B = B^r_{p\theta}(R_n) \to H^r_p(R_n), \tag{9}$$

$$\|Q_s\|^r \ll a^{-s}.$$

The increment in $Q_s(x)$ in turn is an integral function of the type $a^{s/r_j}$ with respect to $x_j$ ($j = 1, \ldots, n$), therefore based on 3.4.2(1), 3.2.2(7), and 4.4.4(2)

$$|\Delta_{x_j h} Q_s|^m \leq 2^{n-m} a^{s(1-\varkappa)} |\Delta_{x_j h} Q_s|^{\prime} \leq 2^{n-m} a^{s\left(1-\varkappa+\frac{1}{r_j}\right)} |Q_s|^{\prime} |h|.$$

The inequality

$$|\Delta_{x_j h} f_1|^m \leq \sigma'_\mu + \sigma''_\mu,$$

is valid, where $\quad \sigma'_\mu < \sum_0^{\mu-1} |\Delta_{x_j h} Q_s|^m, \quad \sigma''_\mu = \sum_\mu^\infty |\Delta_{x_j h} Q_s|^m.$

Let us assign the number h with $|h| < 1$ and choose an integral $\mu$ such that

$$a^{-\mu/r_j} < |h| \leq a^{-(\mu-1)/r_j}. \tag{10}$$

Then (cf (9))

$$\sigma'_\mu \leq 2^{n-m} \sum_0^{\mu-1} a^{s\left(1-\varkappa+\frac{1}{r_j}\right)} \| Q_s \|^{\prime} |h| \ll |h| \sum_0^{\mu-1} \frac{1}{a^{s\delta}}, \tag{11}$$

where

$$\delta = \varkappa - \frac{1}{r_j}.$$

If $\delta < 0$, in other words, if $r'_j = r_j \varkappa < 1$, then

$$\sigma'_\mu \ll |h| |a^{-\mu\delta}| \ll |h|^{r'_j},$$

and if $\delta > 0$, i.e., if $1 < r'_j$, then

$$\sigma'_\mu \ll |h|.$$

When $\delta = 0$, i.e., $r'_j = 1$

$$\sigma'_\mu \ll |h| \mu \ll |h| |\ln |h||.$$

On the other hand,

$$\sigma''_\mu \leqslant 2\sum_{\mu}^{\infty}\|Q_s\|^m \ll \sum_{\mu}^{\infty} a^{s(1-\varkappa)}\|Q_s\|^m \ll$$

$$\ll \sum_{\mu}^{\infty}\frac{1}{a^{\varkappa s}} \ll a^{-\varkappa\mu} \ll |h|^{r'_i}. \tag{12}$$

From the estimate obtained it obviously follows that*)

$$\|\Delta_{x_j h} f_1(x)\|^m = \begin{cases} O(|h|^{r'_i}), & 0<r'_i<1, \\ O(|h||\ln|h||), & r'_i=1, \\ O(|h|), & r'_i>1. \end{cases} \tag{13}$$

The right sides of (13) tend to zero together with h, therefore $f_1(u, y)$ has the trace $f(u, y)$ for any $y$.

Let us emphasize that the desired function $f(x) = f(u, y)$ was known to an accuracy up to a set of n-dimensional measure zero, therefore it was not meaningful to consider it as a function of $u$ for fixed $y$. The method of obtaining the trace of function $f \in B$ was given above. This requires that f be expanded in series (5) with finite norm (6) and that $y$ be fixed in its term $Q_s$. Then the resulting series of functions of $u$ converges in the $L_p(R_m)$-sense namely, to the trace $f_1(u, y)$ of function f.

Ordinarily, in inequalities of the type (13), it amounts to the same to write f instead of $f_1$, understanding this in the sense that f can be modified on a set of n-dimensional measure such that after this (13) will obtain for $y$ and in this case with a constant independent $y$.

6.5.1. Note. Embedding 6.5(1) remains valid for the same condition 6.5(2)-(4) if in it $R_n$ and $R_m$ are replaced, respectively, with the measurable cylindrical sets $\mathcal{E}_n = R_n \times \mathcal{E}'$ and $\mathcal{E}_m = R_m \times \mathcal{E}'$, where as before, $R_m \subset R_n$ and $z = (x, w)$, $x \in R_n$, and $w \in \mathcal{E}'$. In fact, an inequality corresponding to 6.5(1) where constant c does not depend not only on $y$, but also does not depend on $w$ is valid for almost any $w$ given a finite p. Let us raise both its sides

*) L. D. Kudryavtsev [2], part 1.

to the power p, integrate with respect to w and then raise the result to the power 1/p. We finally get the necessary inequality. When $p = \infty$, this statement is trivial.

6.5.2. Inequalities 6.5(13) are of interest in themselves. They indicate for functions of the class $H_p^r(R_n)$ the average order of the trend of their traces. This order is unimprovable (cf 7.6).

It is not difficult to show that the inequality

$$\|\Delta_{x_j h}^k f_1(x)\|^m = O_j(|h|) \qquad (r'_j = 1,\ k > 1) \tag{1}$$

obtains (even without ln), supplementing the second inequality 6.5(13).

Since

$$W_p^r(R_n) \to H_p^r(R_n) \qquad (\mathbf{r} \text{ -- integral vector}),$$

then estimate 6.5(13) are applicable also to $W_p^r(R_n)$*). In this case as well they are improvable in the sense that the powers of $|h|$ shown in their right sides cannot be replaced by larger values. However, for each individual function $f \in W_p^r(R_n)$ the following estimates obtain:

$$\|\Delta_{x_j h} f(x)\|^m = \begin{cases} o\left(|h|^{r'_j}\right)(|h| \to 0), & 0 < r'_j < 1, \\ o(|h|\,|\ln|h||)(|h| \to 0), & r'_j = 1, \\ O(|h|)(|h| < 1), & r'_j > 1, \end{cases} \tag{2}$$

improving, provided $\overline{r'_j} \leq 1$, the estimates 6.5(13).

In fact (cf 5.6.1(9) and 5.2.4(5)), in this case

$$\|Q_s\|^m \leq |g_{a^{s/r_1}, \ldots, a^{s/r_n}} - f| + \\ + |f - g_{a^{(s-1)/r_1}, \ldots, a^{(s-1)/r_n}}| = o(a^{-s}) \qquad (s \to \infty),$$

and then inequalities 6.5(11), (12) are replaced by these:

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*) These estimates for the class $W_p^l$ $(l = 1, 2, \ldots)$ were obtained directly by V. I. Kondrashov [1].

$$\sigma'_\mu \ll o(|h|^{r'_l}) \quad (r'_l<1),\ \sigma'_\mu \ll$$
$$\ll o(|h||\ln|h||) \quad (r'_l=1),\ \sigma''_\mu \ll o(|h|^{r'_l})$$

In fact the estimate $O(|h|)$ cannot be improved. This is easily verified for the example $f = g(x)g(y)$, where $g(x) \in L_p(R_1)$ is an integral function of type 1 such that $g'(0) \neq 0$ and $R_m = R_1$.

## 6.6. Simplest Inverse Theorem of Embedding of Different Measures

Let $1 \leq m < n$ and $R_m$ be a coordinate subspace of $R_n$. For definiteness, we will assert that it consists of the points $(u, 0) = (x_1, \ldots, x_m, 0, \ldots, 0)$. In section 6.5 the following theorem is proven:

$$B^{r}_{p\theta}(R_n) \to B^{r'}_{p\theta}(R_m) \tag{1}$$

provided the condition that

$$r' = r\varkappa = r - \frac{n-m}{p} > 0. \tag{2}$$

Below it will be proven that a theorem that is its complete inverse exists*):

$$B^{r'}_{p\theta}(R_m) \to B^{r}_{p\theta}(R_n) \tag{3}$$

provided condition (2). For explanation, cf 6.1(12).

Let B' and B stand, respectively, for the first and second classes in (1). Let us assign the arbitrary function $\varphi \in B'$. It can be represented as the series

$$\varphi(u) = \sum_{s=0}^{\infty} Q_{a^{s/r}},$$

convergent in it in the $L_p(R_m)$-sense, where $Q_{a^{s/r}}$ are integral functions of spherical type $a^{s/r} (a > 1)$ and

*) S. M. Nikol'skiy [5], case $H^r_p = B^r_p$ ; O. V. Besov [2, 3], case $1 \leq \theta < \infty$.

$$\|\varphi\|_{B'} = \left\{\sum_0^\infty a^{\varkappa s\theta}(\|Q_s\|^m)^\theta\right\}^{1/\theta}, \quad \|\cdot\|^m = \|\cdot\|_{L_p(R_m)}. \tag{4}$$

Suppose

$$F_\nu(t) = \left(\frac{\sin\frac{\nu t}{2}}{\frac{\nu t}{2}}\right)^2 \quad (\nu > 0).$$

This is an integral function of one variable t of type $\nu$ such that

$$\|F_\nu\|_{L_p(R_1)} = \left(\int\left(\frac{\sin\frac{\nu t}{2}}{\frac{\nu t}{2}}\right)^{2p} d\left(\frac{\nu t}{2}\right)\right)^{1/p}\left(\frac{2}{\nu}\right)^{1/p} = \frac{c_p}{\nu^{1/p}} \quad (\nu > 0).$$

Let us introduce a new function of $x \in R_n$ defined by the series

$$f(x) = \sum_0^\infty q_{a^{s/r}}(x), \tag{5}$$

where

$$q_{a^{s/r}}(x) = Q_{a^{s/r}}(u)\prod_{m+1}^n F_{a^{s/r}}(x_j). \tag{6}$$

Obviously (cf (2)),

$$|q_{a^{s/r}}|^n = |Q_{a^{s/r}}|^m a^{-(1-\varkappa)s} c_p^{n-m}.$$

In this case

$$\left(\sum_0^\infty a^{s\theta}(|q_{a^{s/r}}|^n)^\theta\right)^{1/\theta} = c_p^{n-m}\left(\sum_0^\infty a^{\varkappa s\theta}(|Q_{a^{s/r}}|^m)^\theta\right)^{1/\theta} = c_p^{n-m}\|\varphi\|_{B'}. \tag{7}$$

Since functions $q_{a^{s/r}}$ are integral of the (exponential) type $a^{s/r}$ with respect to each variable $x_1, \ldots, x_n$, then by 5.6.1(4), (5) the left side of (7) is the norm of f in the sense of $B^{r,\ldots,r}_{p\theta}(R_n)$. But $B^{r,\ldots,r}_{p\theta}(R_n)$ --- $B^{r}_{p\theta}(R_n) = B$, and we will prove that $f \in B, \|f\|_B \ll \|\varphi\|_{B'}$.

The function f is defined by series (5) convergent in it in the sense $L_p(R_n)$. But the series for any $y = (x_{m+1}, \ldots, x_n)$ converted also in the sense of $L_p(R_m)$ to some function $f_1(x)$, which can differ from f(x) only by a set of n-dimensional measure zero (1.3.9). Obviously,

$$f_1(u, 0) = \varphi(u)$$

in the $R_m$ sense, i.e., for almost all u and the sense of the m-dimensional measure. Further, considering that $F_\nu(0) = 1$, $F_\nu(t)$ are bounded with respect to $\nu$ and t and that $\sum_0^\infty \|Q_{a^{s/r}}\|^n < \infty$, we get

$$\|f_1(u, y) - f_1(u, 0)\|^m \leqslant \sum_\nu^\infty \left| \prod_{m+1}^n F_{a^{s/r}}(x_j) - 1 \right| \|Q_{a^{s/r}}\|^m \to 0$$
$$(y = (x_{m+1}, \ldots, x_n) \to 0).$$

This reasoning shows that $\varphi$ is the trace of f (6.3). In this way the statement (3) is previously completely proven.

Let us note that the class $B = B^r_{p\theta}(R_n)$ is a Banach space. If

$$r' = r - \frac{n-m}{p} > 0,$$

then, by (1), the operation of obtaining the trace

$$Af = f|_{R_m} = \varphi \tag{8}$$

holds for the functions $f \in B$ on $R_m \subset R_n (1 \leq m < n)$. This operation is linear; moreover, by (1), it boundedly maps B into B', where owing to the invertibility of embedding (1), it does this already not into B', but on B'. Above we proved that in turn B' can map on some portion of B by means of some bounded linear operator. This latter is not unique, because an infinite set of such operator can be specified.

In the language of functional analysis the linear bounded operator A mapping Banach space B onto Banach space B' is called continuously invertible*).

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*) /*) on following page7

In the last sections we will prove an embedding more general than 6.5(1), in which we will speak about the boundary property not only of the function f itself on the subspace $R_m \subset R_n$, but also of some of its partial derivatives. We will then completely invert this theorem.

## 6.7. General Theorem of Embedding of Different Measures

Theorem**). Suppose $f \in B^r_{p\theta}(R_n)$, $0 \leq m < n$, and for some vector $\lambda = (\lambda_{m+1}, \ldots, \lambda_n)$ with nonnegative integral coordinates inequalities

$$\rho_i^{(\lambda)} = r_i\left(1 - \sum_{j=m+1}^{n} \frac{\lambda_j}{r_j} - \frac{1}{p} \sum_{j=m+1}^{n} \frac{1}{r_j}\right) > 0 \tag{1}$$

are satisfied. Further, let

$$\psi = \frac{\partial^{\lambda_{m+1}+\cdots+\lambda_n} f}{\partial x_{m+1}^{\lambda_{m+1}} \cdots \partial x_n^{\lambda_n}},$$

and let $R_m$ denote the m-dimensional subspace $R_n$ obtained when vector $y = (x_{m+1}, \ldots, x_n)$ and $\rho(\lambda) = (\rho_1^{(\lambda)}, \ldots, \rho_n^{(\lambda)})$ are specified.

Then

$$\psi|_{R_m} = \varphi \in B^{\rho(\lambda)}_{p\theta}(R_m)$$

and the inequality

$$\|\varphi\|_{B^{\rho(\lambda)}_{p\theta}(R_m)} \leq c\|f\|_{B^r_{p\theta}(R_n)}$$

with constant c not dependent on f and y is satisfied.

Proof. From (1) it follows that

$$\sum_{j=m+1}^{n} \frac{\lambda_j}{r_j} < 1,$$

*) F. Hausdorf [1], addition.

**) S. M. Nikol'skiy [5], case $H^r_p - B^r_{p\infty}$; O. V. Besov [2, 3], case $1 \leq \theta \leq \infty$.

Therefore based on theorem 5.6.3

$$\psi \in B^{r'}_{p\theta}(R_n),$$

$$r' = (r'_1, \ldots, r'_n), \quad r'_i = r_i\left(1 - \sum_{k=m+1}^{n} \frac{\lambda_k}{r_k}\right), \quad i = 1, \ldots, n,$$

and the inequality

$$\|\psi\|_{B^{r'}_{p\theta}(R_n)} \ll \|f\|_{B^{r}_{p\theta}(R_n)}.$$

is satisfied.

In order to see to which class the trace $\psi$ on $R_m$ belongs, let us employ embedding theorem 6.5(1). It is applicable because

$$\varkappa = 1 - \frac{1}{p}\sum_{m=1}^{n}\frac{1}{r'_l} = \frac{1 - \sum\limits_{m+1}^{n}\frac{\lambda_j}{r_j} - \frac{1}{p}\sum\limits_{m+1}^{n}\frac{1}{r_j}}{1 - \sum\limits_{m+1}^{n}\frac{\lambda_j}{r_j}} > 0,$$

and thus we have confirmation of the theorem.

## 6.8. General Inverse Embedding Theorem

Theorem*). Let there be given the vector $r = (r_1, \ldots, r_n)$ 0 and all possible vectors

$$\lambda = (\lambda_{m+1}, \ldots, \lambda_n) \tag{1}$$

with nonnegative integral coordinates for which vectors $\rho(\lambda) = (\rho_1^{(\lambda)}, \ldots, \rho_n^{(\lambda)})$ defined by formula 6.7(1) are positive.

Suppose, in addition, that the function

$$\varphi_{(\lambda)}(x_1, \ldots, x_m) \in B^{\rho(\lambda)}_{p\theta}(R_m). \tag{2}$$

*) S. M. Nikol'skiy [5], case $H_p^r = B^r_{p\infty}$; O. V. Besov [2, 3], case $1 \leq \theta < \infty$.

is brought into correspondence with each vector $(\lambda)$. Then we can construct on $R_n$ function $f \in B^r_{p\theta}(R_n)$ such that

$$\|f\|_{B^r_{p\theta}(R_n)} < c \sum_{(\lambda)} |\varphi_{(\lambda)}|_{B^{\rho(\lambda)}_{p\theta}(R_m)}, \tag{3}$$

where c does not depend on $\varphi(\lambda)$, the sum is extended over all possible indicated vectors $\lambda$, and $\varphi_{(\lambda)}$ are traces of partial derivatives of function f:

$$\left.\frac{\partial^{\lambda_{m+1}+\dots+\lambda_n} f}{\partial x_{m+1}^{\lambda_{m+1}} \dots \partial x_n^{\lambda_n}}\right|_{R_m} = \varphi_{(\lambda)}. \tag{4}$$

Proof. Let

$$r_i^{(\lambda)} = r_i\left(1 - \sum_{m+1}^{n} \frac{\lambda_j}{r_j}\right) \qquad (i = 1, \dots, n), \tag{5}$$

$$\varkappa^{(\lambda)} = 1 - \frac{1}{p} \sum_{m+1}^{n} \frac{1}{r_j^{(\lambda)}}. \tag{6}$$

Then it is obvious

$$\rho_i^{(\lambda)} = r_i^{(\lambda)} \varkappa^{(\lambda)}. \tag{7}$$

Let $\varphi_{(\lambda)} \in B^{\rho(\lambda)}_{p\theta}(R_m) = B^{(\lambda)}$. Then

$$\varphi_{(\lambda)} = \sum_{0}^{\infty} Q_{s(\lambda)},$$

where $Q_{s(\lambda)}$ are integral functions of the type $2^{s/ri(\lambda)}$ with respect to $x_i$ $(i = 1, \dots, m)$ and

$$|\varphi_{(\lambda)}|_{B^{(\lambda)}} = \left(\sum_{0}^{\infty} 2^{\theta s \varkappa(\lambda)} (|Q_s|^*)^\theta\right)^{1/\theta}, \tag{8}$$
$$(\|\cdot\|^* = \|\cdot\|_{L_p(R_m)}).$$

Let us introduce trigonometric polynomials $T_\nu(x)$ $(\nu = 0, 1, \dots, l)$ where l denotes the largest of the numbers $\lambda_j$ encountered in the different vectors $\lambda = (\lambda_{m+1}, \dots, \lambda_n)$ considered. Suppose these polynomials exhibit the following properties: the function

$$\Phi_\nu(x) = \frac{T_\nu(x)}{x^3}$$

is integral and, moreover,

$$\Phi_\nu^{(\nu)}(0) = \frac{d^\nu}{dx^\nu}\Phi_\nu(x)\Big|_{x=0} = 1, \quad \Phi_\nu^{(k)}(0) = 0 \tag{9}$$
$$(k = 0, \ldots, \nu-1, \nu+1, \ldots, l).$$

We would not be concerned about the magnitude of the power of the trigonometric polynomial, because the conditions indicated above do not uniquely define it. We will assume that we have chosen wholly determined polynomials that are of power $\mu(\nu)$. Then $\phi_\nu(x)$ is an integral function of the type $\mu(\nu)$ and $\phi_\nu\left(\frac{k}{\mu(\nu)}x\right)$ is an integral function of the type k. Obviously, further

$$\left\|\Phi_\nu\left(\frac{k}{\mu(\nu)}x\right)\right\|_{L_p(R_1)} = \left(\int_{-\infty}^{\infty}\left|\Phi_\nu\left(\frac{k}{\mu(\nu)}x\right)\right|^p dx\right)^{1/p} =$$
$$= \left(\frac{\mu(\nu)}{k}\right)^{1/p}\left(\int_{-\infty}^{\infty}|\Phi_\nu(u)|^p du\right)^{1/p} = \frac{A_\nu}{k^{1/p}}, \tag{10}$$

where $A_\nu$ depends only on $\nu$.

Let us define functions $f_{(\lambda)}(x_1, \ldots, x_n)$ corresponding to different $\lambda$ vectors by means of the series

$$f_{(\lambda)}(x_1, \ldots, x_n) =$$
$$= \sum_{s=0}^{\infty} Q_{s(\lambda)} \prod_{j=m+1}^{n}\left(\frac{\mu(\lambda_j)}{2^{s/r_j(\lambda)}}\right)^{\lambda_j}\Phi_{\lambda_j}\left(2^{s/r_j(\lambda)}\frac{x_j}{\mu(\lambda_j)}\right) = \sum_{s=0}^{\infty} R_{s(\lambda)}, \tag{11}$$

where, obviously, $R_{s(\lambda)}$ are integral functions of the type $2^{s/r_i(\lambda)}$ with respect to $x_i$ $(i = 1, \ldots, n)$.

Considering (5)-(8) and (10), we have

$$\|R_{s(\lambda)}\|^{m} \leqslant \frac{\|Q_{s(\lambda)}\|^{m}}{2^{s\left[\sum\limits_{m+1}^{n} \frac{\lambda_j}{r_j(\lambda)} + \frac{1}{p}\sum\limits_{m+1}^{n} \frac{1}{r_j(\lambda)}\right]}} = \frac{\|Q_{s(\lambda)}\|^{m} 2^{s\kappa(\lambda)}}{2^{s\frac{r_i(\lambda)}{r_i}}},$$

or

$$2^{s\frac{r_i(\lambda)}{r_i}} \|R_{s(\lambda)}\|^{p} \leqslant \|Q_{s(\lambda)}\|^{m} 2^{s\kappa(\lambda)}.$$

Therefore by (8), considering further that

$$\frac{r_i(\lambda)}{r_i} = 1 - \sum_{m+1}^{n} \frac{\lambda_j}{r_j}$$

does not in fact depend on i, we get the inequality

$$\|f_{(\lambda)}\|_{B^{r}_{p\theta}(R_n)} \leqslant \|\varphi_{(\lambda)}\|_{B^{(\lambda)}}.$$

Let us note further that by virtue of the properties of the function $\Phi_\nu$ , the equality

$$\left.\frac{\partial^{\lambda_{m+1}+\cdots+\lambda_n} f_{(\lambda)}}{\partial x_{m+1}^{\lambda_{m+1}} \ldots \partial x_n^{\lambda_n}}\right|_{R_m} = \varphi_{(\lambda)}(x_1, \ldots, x_m). \tag{12}$$

is satisfied for the function $f_{(\lambda)}$, if the vector $\lambda$ is admissible, i.e., satisfies conditions 6.7(1).

In fact, if series (11) is formally differentiated memberwise with respect to $x_{m+1}, \ldots, x_n$, respectively, $\lambda_{m+1}, \ldots, \lambda_n$ times, then we get

$$\frac{\partial^{\lambda_{m+1}+\cdots+\lambda_n}}{\partial x_{m+1}^{\lambda_{m+1}} \ldots \partial x_n^{\lambda_n}} f = \\ = \sum_{s=0}^{\infty} Q_{s(\lambda)} \prod_{j=m+1}^{n} \Phi_{\nu_j}^{(\lambda_j)}\left(2^{s/r_j(\lambda)} \frac{x_j}{\mu_j(\lambda)}\right) = \sum_{s=0}^{\infty} \mu_{s(\lambda)}. \tag{13}$$

From the estimates derived for $R_{s(\lambda)}$ it follows that at any stage of differentiation series convergent in the $L_p(R_n)$-sense are obtained, therefore equality (13) actually obtained in the sense of the convergence of $L_p(R_n)$ (cf lemma 4.4.7). Further, by virtue of the boundedness of $\phi_{\lambda_j}$, the derivatives $\phi_{\lambda_j}^{(\lambda_1)}$ are also bounded, therefore

$$\left|\sum_0^{\infty}(\mu_{s(\lambda)} - Q_{s(\lambda)})\right|_{L_p(R_m)} <$$
$$< \sum_0^N \| Q_{s(\lambda)} \|_{L_p(R_m)} \left| \Phi_{\lambda_j}^{(\lambda_j)}\left(2^{s/r_j(\lambda)} \frac{x_j}{\mu_j(\lambda)}\right) - 1 \right| +$$
$$+ 2c \sum_{N+1}^{\infty} \| Q_{s(\lambda)} \|_{L_p(R_m)}. \tag{14}$$

Let us now assign $\varepsilon > 0$ and choose N sufficiently large so that the second term in the right side of (14) is smaller than $\varepsilon$, and now we select (cf (9)) $\delta$ to be sufficiently small that for $|x_j| < \delta$ ($j = m + 1, \ldots, n$) with first term is small than $\varepsilon$.

If however $\{\lambda'\}$ is another admissible vector $(\lambda'_{m+1}, \ldots, \lambda'_n)$, then by similar arguments we get

$$\left.\frac{\partial^{\lambda'_{m+1} + \ldots + \lambda'_n} f_{(\lambda)}}{\partial x_{n+1}^{\lambda'_{n+1}} \ldots \partial x_n^{\lambda'_n}}\right|_{R_m} = 0.$$

In this case the function $f = \sum_{\lambda} f_{\lambda}$,

where summation is extended over all possible admissible vectors $\lambda$ and satisfies all requirements of the theorem.

## 6.9. Generalization of the Theorem of Embedding of Different Metrics

Below is given the generalization of embedding theorem 6.3(1) for the case of classes $B^r_{p\theta}(R_n)$.

Theorem*). Suppose that for the numbers considered below the inequality ($r_j > 0$)

$$1 < p_j < p' < \infty, \tag{1}$$

$$\varkappa' = 1 - \sum_{j=1}^{n}\left(\frac{1}{p_j} - \frac{1}{p'}\right)\frac{1}{r_j} > 0, \tag{2}$$

$$\varkappa_i = 1 - \sum_{j=1}^{n}\left(\frac{1}{p_j} - \frac{1}{p_i}\right)\frac{1}{r_j} > 0 \quad (i = 1, \ldots, n) \tag{3}$$

are satisfied and that

$$\rho_i = \frac{r_i\varkappa'}{\varkappa_i} \text{ **)}. \tag{4}$$

Further, let $r = (r_1, \ldots, r_n)$, $\rho = (\rho_1, \ldots, \rho_n)$, and $p = (p_1, \ldots, p_n)$. Then the embedding

$$B^r_{p\theta}(R_n) \to B^\rho_{p'\theta}(R_n). \tag{5}$$

obtains.

From (5), when $p = p_1 = \ldots = p_n$ follow 6.3(1), $\rho = r'$. Let us further note that from the fact that $\varkappa' > 0$ it follows that $\varkappa_i > 0$ for all i, since $p' \geqslant p_i$.

Proof. Let us introduce a family of functions $g_\nu = g_{\nu_1,\ldots,\nu_n}$ $(0 < \nu_j \leqslant \infty ;\ j = 1, \ldots, n)$ that are integral and of the exponential type

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*) S. M. Nikol'skiy [10], case $H^r_p = B^r_{p\infty}$; V. P. Il'yin and V. A. Solonnikov [1, 2], case $1 \leq \theta < \infty$ (using the T. I. Amanov approximation theory [3]).

**) In this theorem we can proceed from the condition that all $\rho_j > 0$, since for such an i for which $p_i$ takes on the smallest value, $\varkappa_i > 0$, then also $\varkappa' > 0$, and so do the remaining $\varkappa_i > 0$.

$\nu_j$ with respect to $x_j$, defined by the last equality in 5.2.4(1) when $m = n$.

Let us suppose

$$\nu_k = \nu_k(s) = 2^{s/\rho_k} \qquad (k = 1, \ldots, n;\ s = 0, 1, \ldots, \quad \text{and } s = \infty)$$

and

$$Q_0 = g_{\nu(0)}, \quad Q_s = g_{\nu(s)} - g_{\nu(s-1)} \qquad (s = 1, 2, \ldots). \tag{6}$$

Obviously,

$$Q_s = \sum_{l=1}^{n} Q_s^{(l)} \qquad (s = 1, 2, \ldots), \tag{7}$$

where

$$Q_s^{(l)} = g_{\nu_1(s), \ldots, \nu_l(s), \nu_{l+1}(s-1), \ldots, \nu_n(s-1)} - g_{\nu_1(s), \ldots, \nu_{l-1}(s), \nu_l(s-1), \ldots, \nu_n(s-1)}. \tag{8}$$

We have

$$\|Q_s\|_{p'} \leq \sum_{l=1}^{n} \|Q_s^{(l)}\|_{p'} \qquad (s = 1, 2, \ldots) \qquad \left(\|\cdot\|_{p'} = \|\cdot\|_{L_{p'}(R_n)}\right). \tag{9}$$

Let us apply to each i-th term of this sum inequality of different metrics (3.3.5)

$$\|Q_s^{(l)}\|_{p'} \leq 2^n 2^{s\left(\frac{1}{p_l} - \frac{1}{p'}\right)\sum_1^n \frac{1}{\rho_j}} \|Q_s^{(l)}\|_{p_l} = 2^n 2^{-s\left[\frac{r_l}{\rho_l} - \left(\frac{1}{p_l} - \frac{1}{p'}\right)\sum_{j=1}^{n} \frac{1}{\rho_j}\right]} 2^{s\frac{r_l}{\rho_l}} \|Q_s^{(l)}\|_{p_l} \qquad (l = 1, \ldots, n). \tag{10}$$

Let us now select numbers $\rho_j$ such that the expressions in the brackets equal unity:

$$1 = \frac{r_l}{\rho_l} - \left(\frac{1}{p_l} - \frac{1}{p'}\right)\sum_{l=1}^{n} \frac{1}{\rho_l} \qquad (l = 1, \ldots, n). \tag{11}$$

Dividing all equalities by $r_i$, and replacing i with l and summing up with respect to l, we get

$$\sum_{l=1}^{n}\frac{1}{r_l}=\left(1-\sum_{l=1}^{n}\frac{\frac{1}{p_l}-\frac{1}{p}}{r_l}\right)\sum_{l=1}^{n}\frac{1}{\rho_l}. \tag{12}$$

Cancelling out the sum from (11) and (12), we obtain

$$\rho_l=r_l\frac{\varkappa}{\varkappa_l} \qquad (l=1,\ \dots,\ n). \tag{13}$$

Therefore summation (10) with respect to i brings us (cf (7)) from which it follows that (explanations below)

$$2^{s}\|Q_s\|_p<2^{s}\sum_{l=1}^{n}2^{s\frac{r_l}{\rho_l}}\|Q_s^{(l)}\|_{p_l},$$

from which it follows that (explanations below)

$$\begin{aligned}\left\{\sum_{1}^{\infty}2^{s\theta}\|Q_s\|_p^{\theta}\right\}^{1/\theta}&<\sum_{l=1}^{n}\left\{\sum_{s=1}^{\infty}2^{s\theta\frac{r_l}{\rho_l}}\|Q_s^{(l)}\|_{p_l}^{\theta}\right\}^{1/\theta}<\\&<\sum_{l=1}^{n}\left(\sum_{s=1}^{\infty}2^{s\theta\frac{\alpha_l}{\rho_l}}\omega_{x_l}^{k}\left(f_{x_l}^{r_l},2^{-\frac{s}{\rho_l}}\right)_{p_l}^{\theta}\right)^{1/\theta}<\\&<\sum_{l=1}^{n}\left(\int_{0}^{\infty}2^{s\theta\frac{\alpha_l}{\rho_l}}\omega_{x_l}^{k}\left(f_{x_l}^{r_l},2^{-\frac{s}{\rho_l}}\right)_{p_l}^{\theta}ds\right)^{1/\theta}<\\&<\sum_{l=1}^{n}\left(\int_{0}^{1}\frac{\omega_{x_l}^{k}(f_{x_l}^{r},t)_{p_l}^{\theta}}{t^{1+\theta\alpha_l}}dt\right)^{1/\theta}<\|f\|_{b_{p\theta}^{r}(R_n)}.\end{aligned} \tag{14}$$

The second inequality (14) follows from the fact that if $\nu_1, \dots, \nu_n$ and $\nu_n'$ are arbitrary numbers and $\nu_n \leq \nu_n'$, then by 5.2.4(2)

$$\left| g_{v_1, \ldots, v_n} - g_{v_1, \ldots, v_{n-1}, v_n'} \right|_{p_n} < \left| g_{v_1, \ldots, v_n} - f \right|_{p_n} +$$

$$+ \left| f - g_{v_1, \ldots, v_{n-1}, v_n'} \right|_{p_n} < \frac{2c\omega_{x_n}^k \left( f_{x_n}^{\bar{r}_n}, \frac{1}{v_n} \right)_{p_n}}{v_n^{r_n}} \qquad (r_n - \bar{r}_n = a_n).$$

Further, since $f \in L_{p_i}$, therefore we also have $Q_0 \in L_{p_i}$ (cf integral representation 5.2.4(1)), so more so $Q_0 \in L_{p'}$, since $p_i \leq p'$ and

$$\| Q_0 \|_{p'} < \| f \|_{p_i} < \| f \|_{B^r_{p\theta}(R_n)}. \tag{15}$$

From (14) and (15), in particular, it follows that the series

$$\sum_0^\infty Q_s \tag{16}$$

converges in the $L_{p'}$-sense. It clearly converges in the $L_{p_i}$-sense to f because

$$\left| f - \sum_0^N Q_s \right|_{p_i} = \left| f - g_{v(N)} \right|_{p_i} < \frac{\omega_{x_i}^k \left( f_{x_i}^{\bar{r}_i}, v_i(N)^{-1} \right)_{p_i}}{v_i^{r_i}} <$$

$$< \frac{1}{v_i(N)^{r_i}} \to 0 \qquad (N \to \infty),$$

snnce $B^r_{p\theta} \to H^r_p \to H^{r_i}_{x_i p_i}$.

And thus, series (16) converges to f; inequalities (14) and (15) are valid, $Q_s$ are integral functions of the type $2^{s/r_i}$ with respect to $x_i$ ($i = 1, \ldots, n$), therefore $f \in B^{\rho}_{p'}(R_n)$, and embedding (5) obtains.

6.9.1. Suppose instead of the number p' (cf 6.9) the vector $p' = (p_1', \ldots, p_n')$ is given such that $p_i' \geqslant p_j$ ($i, j = 1, \ldots, n$). We will assume

$$\varkappa_i' = 1 - \sum_{l=1}^n \left( \frac{1}{p_l} - \frac{1}{p_i'} \right) \frac{1}{r_l} > 0 \tag{1}$$

and

$$r_i' = \frac{r_i \varkappa_i'}{\varkappa_i}, \tag{2}$$

where $\mathcal{H}_i$ are defined as in 6.9(3).

Then, it comes by theorem 6.9

$$B^{r}_{p\theta}(R_n) \to B^{r_i'}_{x_i p_i' \theta}(R_n) \quad (i = 1, \ldots, n) \tag{3}$$

and, therefore,

$$B^{r}_{p\theta}(R_n) \to B^{r'}_{p'\theta}(R_n). \tag{4}$$

6.10. Additional Information

6.10.1. Theorems derived in this chapter are automatically transferable to the periodic case. Their formulations remain valid if in them the symbols W, H, and B are replaced, respectively, by W*, H*, and B*.

In presenting the proof for the periodic case, the role of integral functions of the exponential type is now played, of course, by a trigonometric polynomial. It will be central to our exposition that integral functions of the exponential type exhibit following properties: for them the inequalities 1) for derivatives (Bernshteyn type inequalities), 2) inequalities of different metrics, and 3) inequalities of different measures are valid. Trigonometric polynomials exhibit such properties. Additionally, we can for periodic and nonperiodic functions, as we know (compare 5.2.1(6) and 5.3.1(11)), construct analogous methods for their approximation with trigonometric polynomials and, accordingly, with functions of the exponential type. We in fact used these methods in presenting the theory in the nonperiodic phase.

6.10.2. We can indicate the method of obtaining general systems of functions that are not analytic, but such that inequalities very similar to the inequalities discussed above for derivatives are valid for them, as are the inequalities of different metrics and different measures.

Suppose (O. V. Besov)

$$h = (h_1, \ldots, h_n), \quad h_i > 0, \quad y : h = \left(\frac{y_1}{h_1}, \ldots, \frac{y_n}{h_n}\right),$$

$$\varphi_h(x) = \int_{R_n} \prod_{i=1}^{n} \frac{1}{h_i} \chi(y : h) \varphi(x + y)\, dy, \tag{1}$$

where function $\chi(\mathbf{y})$ is infinitely differentiable on $R_n$ and is concentrated (has a carrier) within the first coordinate junction, and

$$\int_{R_n} \chi(y)\,dy = 1. \tag{2}$$

We call function $\varphi_h(x)$ the mean function for $\varphi(\mathbf{x})$ with vector pitch $\mathbf{h} = (h_1, \ldots, h_n)$.

The inequality

$$\|D^{\alpha}\varphi_h(x)\|_{L_q(R_m)} \leqslant c_1 \left(\prod_1^n h_i^{-\alpha_i}\right)\left(\prod_1^n h_i^{-\frac{1}{p}}\right)\left(\prod_1^m h_i^{\frac{1}{q}}\right)\|\varphi\|_{L_p(R_n)}. \tag{3}$$

is valid for mean functions.

Inequality (3) is to some extent*) analogous to the corresponding estimates for integral functions of finite degrees $\nu_i = 1/h_i$, which enables the theorem expounded here to be transferred without essential changes to the case of approximation with mean functions $\varphi_h$ (or with secondary mean functions $\varphi_{hh} = (\varphi_h)_h$), adopting in 5.2.1(5) in place of

$$g(t) = \mu\left(\frac{\sin\frac{t}{\lambda}}{t}\right)^{\lambda}$$

a smooth finite function $\xi(t)$.

In this way, for example, we can arrive at the integral representation (obtained from other considerations) by V. P. Il'yin [6] for the function in terms of its difference. Let us note that only values of the function $\varphi(\mathbf{x} + \mathbf{y})$ for the points $\mathbf{y}$ from the portion of the vicinity of point $\mathbf{y} = 0$ lying in the first coordinate junction participate in construction (1) of mean function $\varphi_h(\mathbf{x})$. Thus we have made it possible to construct the corresponding "local" theory.

*) There is some difference in that $\varphi$ and not $\varphi_h$ appears in the right side of (3) under the sign of the norm. A way out of this predictment can be found in the fact that inequality (3) is used for $\varphi_h$, and then instead of $\varphi_h$, $\varphi_{hh}$ will appear in the left side.

Let us prove inequality (3). It can be obviously asserted that $\alpha = 0$. Using Hölder's inequality for the three functions

$$|\chi|^{1-\varepsilon}, \quad |\varphi(x+y)|^{\frac{q-p}{q}}, \quad |\chi|^{\varepsilon}\,|\varphi(x+y)|^{\frac{p}{q}} \quad (\varepsilon > 0)$$

with the exponents $\lambda_1 = \dfrac{p}{p-1}$, $\lambda_2 = \dfrac{pq}{q-p}$, and $\lambda_3 = q$, we have*)

$$\left|\int_{R_n} \prod_{i=1}^{n} \frac{1}{h_i}\, \chi(y:h)\,\varphi(x+y)\,dy\right| <$$

$$< \left(\prod_{i=1}^{n} \frac{1}{h_i}\right)\left(\int_{R_n} |\chi(y:h)|^{\frac{1-\varepsilon}{p-1}p}\,dy\right)^{1-\frac{1}{p}} \times$$

$$\times \|\varphi\|_{L_p(R_n)}^{1-\frac{p}{q}} \left(\int_{R_n} |\chi(y:h)|^{\varepsilon q}\,|\varphi(x+y)|^{p}\,dy\right)^{\frac{1}{q}};$$

whence

$$\|\varphi_h\|_{L_q(R_m)} < c_1\left(\prod_{i=1}^{n} h_i\right)^{-\frac{1}{p}} \|\varphi\|_{L_p(R_n)}^{1-\frac{p}{q}} \times$$

$$\times \operatorname*{supvrai}_{x_{m+1},\ldots,x_n} \left\{\int_{R_n} |\chi(y:h)|^{\varepsilon q} \int_{R_m} |\varphi(x+y)|^{p}\,dx_1\ldots dx_n\,dy\right\}^{1/q} <$$

$$< c\left(\prod_{1}^{n} h_i^{-\frac{1}{p}}\right)\left(\prod_{1}^{m} h_i^{\frac{1}{q}}\right)\|\varphi\|_{L_p(R_n)}.$$

6.10.3. It is useful to bear in mind the following lemma.

**Lemma.** Suppose on $R_n = R_m \times R_{n-m}$ $(x = (u, v),\ u \in R_m,\ v \in R_{n-m}$, two functions $f \in L_p(R_n)(1 \leq p \leq \infty)$ and $f_*$ be given, along with the sequence of functions $f_k(k = 1, 2, \ldots)$ continuous on $R_n$, such that the following properties are satisfied:

1) $\|f_k - f\|_{L_p(R_n)} \to 0 \quad (k \to \infty);$

2) $\|f_k(u, w) - f_*(u, w)\|_{L_p(R_m)} \to 0 \quad (k \to \infty).$

---

*) Relations $1 \leq \lambda_i < \infty$ $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} = 1$ are fulfilled for the exponents $\lambda_1$, $\lambda_2$, and $\lambda_3$.

uniformly relative to $w$ ($|w| < a$);

3) $$\|f_k(u, w) - f_k(u, w')\|_{L_p(R_m)} \to 0$$
$$(|w - w'| \to 0, \ |w|, \ |w'| < a)$$

uniformly relative to $k = 1, 2, \ldots$

Then $f_*$ for any fixed $w(|w| < a)$ is the trace of function $f$ on the corresponding m-dimensional subspace $R_m(w)$.

Proof. From properties 1) and 2) it follows, by lemma 1.3.9, that $f$ and $f_*$ are equivalent on $R_n$: $f = f_*$ almost everywhere on $R_n$. Further, for the indicated $w$ and $w'$

$$\|f_*(u, w) - f_*(u, w')\|_{L_p(R_m)} \leqslant \|f_*(u, w) - f_k(u, w)\|_{L_p(R_m)} +$$
$$+ \|f_k(u, w) - f_k(u, w')\|_{L_p(R_m)} + \|f_k(u, w') - f_*(u, w')\|_{L_p(R_m)} <$$
$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \ (k > k_0, \ |w - w'| < \delta)$$

for sufficiently small $\delta$ and large enough $k_0$. This is possible owing to properties 1), 2) and 3).

In order to be clear as to the significance of this lemma, let us turn to the theorem of embedding of different measures, for simplicity confine ourselves to the isotropic case. Employing this lemma we can easily conclude that it is sufficient to prove the theorem on traces only for continuous or even infinitely differentiable functions, of the corresponding class, as it will automatically be valid for all functions of this class. Let us explain this reasoning.

Suppose $B = B^r_{p\theta}(R_n)$, $B' = B_{p\theta}(R_m(w))$ $(\rho = r - \frac{n - m}{p} > 0, \ 1 \leq m < n)$ and $\mathfrak{M} \subset B$ is a set of continuous functions dence in B (in the metric B)*). Further let the inequalities

$$\|f\|_{B'} \leqslant c\|f\|_B, \tag{1}$$

$$\|f(u, w) - f(u, w')\|_{L_p(R_m)} \leqslant \|f\|_B \, \lambda(|w - w'|),$$
$$(\lambda(\delta) \to 0, \ \delta \to 0), \tag{2}$$

*) In this reasoning B can be replaced with $W^l_p(R_n)$ $(l = 1, 2, \ldots)$.

be proved, where c does not depend on w and indicated f, just as function $\lambda(\delta)$ does not depend on f and w and w'. Then these inequalities with the same constant c and function $\lambda(\delta)$ obtained for all $f \in B$. In fact, let $f_k$ ($k = 1, 2, \ldots$) and $\|f_k - f\|_B \to 0$ ($k \to \infty$). Then

$$\|f_k\|_{B'} \leqslant c\|f_k\|, \tag{3}$$

$$\|f_k(u, w) - f_k(u, w')\|_{L_p(R_m)} \leqslant K\lambda(|w - w'|), \tag{4}$$

where the constant K does not depend on k. From (3) it further follows that

$$\|f_k - f_l\|_{B'} \leqslant c\|f_k - f_l\|_B,$$

from (1) owing to the completeness of B' for any w there exists the function $f_*(x) = f_*(u, w)$ such that

$$\begin{gathered} \|f_k - f_*\|_{L_p(R_m)} \leqslant \|f_k - f_*\|_{B'} \leqslant c\|f_k - f_*\|_B \to 0, \quad k \to \infty, \\ \|f_*(u, w) - f_*(u, w')\|_{L_p(R_m)} \leqslant K\lambda(|w - w'|), \\ \|f_*\|_{B'} \leqslant c\|f\|_B. \end{gathered} \tag{5}$$

Thus, conditions 1) - 3) of the lemma are satisfied for f, $f_*$, and $f_k$, and therefore, $f_*$ for any w is the trace of f on $R_m(w)$. By this we have proven inequalities (1) and (2) for arbitrary function $f \in B$ (we must bear in mind that constant K in (4) can for sufficiently large k and l be taken as little differing from $\|f\|_B$ as desired).

This argumentation can be pursued for the case of the inverse theorem of embedding. Suppose $\mathfrak{M}' \subset B'$ is a set of continuous functions, dence in B', and let to each continuous function $\varphi \in \mathfrak{M}'$ defined on $R_m = R_m(0)$ there be brought into correspondence the continuous function $A\varphi = f(x) \in B$, defined on $R_n$, such that the trace f on $R_m$ is $\varphi$, and the inequality*)

$$\|f\|_B \leqslant c\|\varphi\|_{B'}, \tag{6}$$

is satisfied, where c does not depend on $\varphi \in \mathfrak{M}'$. Let us assign the arbitrary function $\varphi \in B'$, and let $\varphi_k \in \mathfrak{M}'$, $\|\varphi - \varphi_k\|_{B'} \to 0$ ($k \to \infty$), $A\varphi_k = f_k$, then

$$\|f_k - f_l\|_B \leqslant c\|\varphi_k - \varphi_l\|_{B'} \to 0 \quad (k, l \to \infty)$$

*) [*) on following page]

and there exists $f \in B$ ($B$ is complete) such that $\|f - f_k\|_B \to 0$. Obviously, inequality (6) (with the same constant c) is satisfied for functions $\varphi$ and f.

Let us note that for finite $\theta$ the set $\mathfrak{M}$ of integral functions $f \in L_p(R_n)(1 \leq p \leq \infty)$ of exponential spherical types (all) is compacted in any $B = B^r_{p\theta}(R_n)$ (in metric B). Actually, it is compacted in any $B = B^r_{p\theta}(R_n)$ $(1 \leq \theta < \infty)$ because if $f \in B$, then (cf 6.2(6))

$$f = \sum_0^\infty Q_s, \quad \|f\|_B = \left(\sum_0^\infty a^{sr\theta}\|Q_s\|^\theta\right)^{1/\theta}$$

and

$$\|f - f_k\|_B = \left(\sum_{k+1}^\infty a^{sr\theta}\|Q_s\|^\theta\right)^{1/\theta} \to 0 \qquad (k \to \infty),$$

where

$$f_k = \sum_0^k Q_s \in \mathfrak{M}.$$

When $1 \leq p < \infty$, $\mathfrak{M}$ is also compact in $W^l_p(R_n)(l = 0, 1, 2, \ldots)$, which follows from estimates 5.2.2(4).

Of course, from the foregoing it follows that a set of all infinitely differentiable functions of the class $B^r_{p\theta}(R_n)(1 \leq \theta < \infty)$ or $W^l_p(R_n)$ is compact in the corresponding class, because it includes the set of functions of exponential types belonging to $L_p(R_n)$.

*) Here again B can be replaced with $W^l_p(R_n)(l = 1, 2, \ldots)$. The corresponding theorem on extension from $R_m$ to $R_n$ is proven in 9.5.2.

## CHAPTER VII TRANSITIVITY AND UNIMPROVABILITY OF EMBEDDING THEOREMS. COMPACTNESS

### 7.1. Transitive Properties of Embedding Theorems*)

Let us assign systems of numbers

$$r=(r_1, \ldots, r_n)>0, \quad p=(p_1, \ldots, p_n) \quad (1 \leqslant p_i \leqslant \infty) \tag{1}$$

and the numbers p' and p" satisfying the inequalities

$$p_i \leqslant p' < p'' \leqslant \infty. \tag{2}$$

If the condition

$$\rho'_i = \frac{r_i \varkappa}{\varkappa_i}, \tag{3}$$

$$\varkappa = 1 - \sum_{l=1}^{n} \frac{\frac{1}{p_l} - \frac{1}{p'}}{r_l} > 0, \tag{4}$$

$$\varkappa_i = 1 - \sum_{l=1}^{n} \frac{\frac{1}{p_l} - \frac{1}{p_i}}{r_l} > 0 \quad (i = 1, \ldots, n), \tag{5}$$

are satisfied, then embedding theorem (6.8) $B^{r}_{p\theta}(R_n) \to B^{\rho'}_{p'\theta}(R_n)$,

obtains, that permits passing from system of numbers (1) to system of numbers

$$\rho' = (\rho'_1, \ldots, \rho'_n), \; p'.$$

*) S. M. Nikol'skiy [3, 10].

We must bear in mind that $\mathcal{H} \leq \mathcal{H}_1$, also because inequality (5) is a consequence of inequality (4).

But now the class $B_{p'\theta}(R_n)$ can be taken as the starting class and given the existence of the inequality

$$\varkappa' = 1 - \left(\frac{1}{p'} - \frac{1}{p''}\right) \sum_1^n \frac{1}{\rho'_k} > 0$$

we can conclude that the further embedding of classes

$$B^{(\rho')}_{p'\theta}(R_n) \to B^{(\rho'')}_{p''\theta}(R_n),$$

obtains, where

$$\rho'' = (\rho''_1, \ldots, \rho''_n) = \varkappa'\rho'.$$

Thus, we have transformed the system $(r, p)$ into the system $(\rho', p')$, which in turn was converted into the system $(\rho'', p'')$. We must remember that $\rho'$ is defined by means of $r$, $p$ and $p'$, and $\rho''$ — terms of $\rho'$, $p'$, and $p''$. It is remarkable that these transformations are transitive in character: the passage from the first system to the second, and then from the second to the third can be replaced by the single passage from the first system to the third.

In fact,

$$\rho''_k = \frac{r_k \varkappa \varkappa'}{\varkappa_k} \qquad (k = 1, \ldots, n),$$

where it is assumed that

$$\varkappa, \varkappa', \varkappa_k > 0 \qquad (k = 1, \ldots, n). \tag{6}$$

On the other hand, suppose $p_k \leq p''$ $(k = 1, \ldots, n)$ and let the inequalities

$$\varkappa'', \varkappa_k > 0 \qquad (k = 1, \ldots, n), \tag{7}$$

obtain, where

$$\varkappa'' = 1 - \sum_{l=1}^n \frac{\frac{1}{p_l} - \frac{1}{p''}}{r_l}.$$

Then the embedding

$$B^r_{p\theta}(R_n) \to B^{(\rho'')}_{p''\theta}(R_n),$$

holds, i.e., the passage from $(r, p)$ directly to $(\rho''_*, p'')$, where

$$\rho''_* = (\rho''_{*1}, \ldots, \rho''_{*n})$$

and

$$\rho''_{*k} = \frac{r_k \varkappa''}{\varkappa_k} \quad (k = 1, \ldots, n).$$

But it is easy to compute that

$$\varkappa'' = \varkappa\varkappa', \tag{8}$$

therefore

$$\rho''_* = \rho''.$$

Moreover, by virtue of the inequality $p' < p''$ it is obvious that $\varkappa' > \varkappa''$, i.e., $\varkappa' > 0$. But then as a consequence of (8) $\varkappa > 0$ and the transitivity stands proven.

The transitivity of the theorems of embedding of different measures

$$B^r_{p\theta}(R_n) \to B^{r'}_{p\theta}(R_{m_1}) \to B^{r''}_{p\theta}(R_{m_2})$$
$$(1 \leqslant m_2 < m_1 < n),$$

where

$$r'_i = r_i \varkappa' \quad (i = 1, \ldots, m_1),$$

$$\varkappa' = 1 - \frac{1}{p} \sum_{m_1+1}^{n} \frac{1}{r_j} > 0,$$

$$r''_i = r'_i \varkappa'' \quad (i = 1, \ldots, m),$$

$$\varkappa'' = 1 - \frac{1}{p} \sum_{m_2+1}^{n} \frac{1}{r'_j},$$

follows from the easily verified equality

$$\varkappa = 1 - \frac{1}{p} \sum_{m_2+1}^{n} \frac{1}{r_j} = \varkappa'\varkappa''.$$

## 7.2. Inequalities With Parameter $\varepsilon$. Multiplicative Inequalities

Let us assign the function $f(x) \in H^r_p(R_n) = H^r_p$ and the positive vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Let us assume $F(x) = f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) = f_\varepsilon(x)$. Obviously $(k_j > r_j - \rho_j > 0)$,

$$\frac{\left\|\Delta_h^{k_j} F_{x_j}^{(\rho_j)}\right\|_p}{h^{r_j-\rho_j}} = \frac{\varepsilon_j^{r_j}\left\|\Delta_{x_j\varepsilon_j h}^{k_j} f_{x_j}^{(\rho_j)}(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)\right\|_p}{(\varepsilon_j h)^{r_j-\rho_j}} =$$

$$= \frac{\varepsilon_j^{r_j}\varepsilon^{-\frac{1}{p}}\left\|\Delta_{x_j\varepsilon_j h}^{k} f_{x_j}^{(\rho_j)}(x)\right\|_p}{(\varepsilon_j h)^{r_j-\rho_j}} \qquad (\varepsilon^{\alpha} = \varepsilon_1^{\alpha} \ldots \varepsilon_n^{\alpha}).$$

Taking the upper bound of both parts of the inequality in h, we get

$$\|f_\varepsilon(x)\|_{h_{x_j p}^{r_j}} = \varepsilon_j^{r_j}\varepsilon^{-\frac{1}{p}}\|f\|_{h_{x_j p}^{r_j}}, \tag{1}$$

whatever the $\varepsilon > 0$.

Let us further consider the seminorm $b_{x_j p}^{r_j} = b_{x_j p\theta}^{r_j}(R_n)$, $1 \leq \theta \leq \infty$;

$$\|f\|_{b_{x_j p}^{r_j}} = \left(\int_0^\infty t^{1-\theta(r_j-\rho_j)}\Omega_{x_j}^{k_j}\left(f_{x_j}^{(\rho_j)}, t\right)_{L_p(R_n)}^{\theta} dt\right)^{1/\theta}. \tag{2}$$

The function $f_\varepsilon$ can also appear in it and a change of variables can be made in the integral under the sign $\Omega$. As a result, we get an equality analogous to (1):

$$\|f_\varepsilon(x)\|_{b_{x_j p}^{r_j}} = \varepsilon_j^{r_j}\varepsilon^{-\frac{1}{p}}\|f\|_{b_{x_j p}^{r_j}}. \tag{3}$$

Thus it is valid for any $\theta$ $(1 \leq \theta \leq \infty)$.

Obviously, further,

$$\|f_\varepsilon\|_p = \varepsilon^{-\frac{1}{p}}\|f\|_p, \tag{4}$$

therefore

$$\|f_\varepsilon\|_{B^r_p} = \varepsilon^{-\frac{1}{p}}\left\{\|f\|_p + \sum_{j=1}^{n} \varepsilon_j^{r_j}\|f\|_{b^{r_j}_{x_j,p}}\right\}. \tag{5}$$

Assuming $f_\varepsilon(x) = f(\varepsilon x)$ for functions f belonging to isotropic classes, where now $\varepsilon$ is a positive scalar, and reasoning as above, we get

$$\|f(\varepsilon x)\|_p = \varepsilon^{-\frac{n}{p}}\|f\|_p, \quad \|f(\varepsilon x)\|_{h^r_p} = \varepsilon^{r-\frac{n}{p}}\|f\|_{h^r_p}, \tag{6}$$
$$\|f(\varepsilon x)\|_{b^r_p} = \varepsilon^{r-\frac{n}{p}}\|f\|_{b^r_p}.$$

Let us present an example of the application of formulas (3)-(6).

The inequality

$$\|f\|_{B^\rho_p} < c\left(\|f\|_p + \|f\|_{b^r_p}\right), \tag{7}$$

is associated with the embedding $B^r_p \longrightarrow B_p(0 < \rho < r)$, and from this inequality by (6) follows the inequality

$$\|f\|_{B^\rho_p} < c\left(\varepsilon^{-\rho}\|f\|_p + \varepsilon^{r-\rho}\|f\|_{b^r_p}\right) \tag{7'}$$

with arbitrary parameter $\varepsilon$.

Conversely, from (7') when $\varepsilon = 1$ follows (7). Inequality (7') is used in applications when it is desired that a certain term of its right side be sufficiently small. Minimizing the right side of (7') with respect to $\varepsilon$, we get the inequality

$$\|f\|_{b^\rho_p} < c\left[\left(\frac{r-\rho}{\rho}\right)^{\rho/r} + \left(\frac{\rho}{r-\rho}\right)^{1-\rho/r}\right]\|f\|^{1-\frac{\rho}{r}}\left(\|f\|_{b^r_p}\right)^{\rho/r}, \tag{7''}$$

which is called a multiplicative inequality. Conversely, from (7") obviously follow (7').

Let us also consider the inequalities

$$\|f\|_{B_p^\rho(R_m)} \leq c\left(\|f\|_{L_p(R_n)} + \|f\|_{b_p^r(R_n)}\right) \tag{8}$$
$$\left(1 \leq m < n,\ \rho = r - \frac{n-m}{p} > 0\right),$$
$$\|f\|_{B_{p'}^{r'}(R_n)} \leq c\left(\|f\|_{L_p(R_n)} + \|f\|_{b_p^r(R_n)}\right),$$
$$r' = r - \left(\frac{1}{p} - \frac{1}{p'}\right)n > 0, \tag{9}$$

associated with embeddings of different measures and metrics, where $R_m$ is a subspace of points $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$ with arbitrary fixed $y = (x_{m+1}, \ldots, x_n)$ and where c does not depend on f and $\varphi$. If f is replaced by $f_\varepsilon$ in these inequalities, and then is removed from the norms by means of (6), then we get, respectively,

$$\|f\|_{b_p^\rho(R_m)} \leq c\left(\varepsilon^{-r}\|f\|_{L_p(R_n)} + \|f\|_{b_p^r(R_n)}\right),$$
$$\|f\|_{b_{p'}^{r'}(R_n)} \leq c\left(\varepsilon^{-r}\|f\|_{L_p(R_n)} + \|f\|_{b_p^r(R_n)}\right).$$

Passing to the limit as $\varepsilon \longrightarrow \infty$, we get the inequalities

$$\|f\|_{b_p^\rho(R_m)} \leq c\|f\|_{b_p^r(R_n)}, \tag{10}$$
$$\|f\|_{b_{p'}^{r'}(R_n)} \leq c\|f\|_{b_p^r(R_n)}, \tag{11}$$

refining inequalities (8) and (9), for the same constant c appears in them, but they no longer contains the term $\|f\|_{L_p(R_n)}$, which was finite. However, if $\|f\|_{L_p(R_n)} = 0$, then inequalities (10) and (11) generally speaking are valid. Thus, when $r - \rho \geq 1$, the polynomial

$$P_l(x) = \sum_{|k| \leq l} a_k x^k,$$

where $l = \bar{r}$, if r is a noninteger, and $l = \bar{r} + 1$ if r is an integer, the right side of inequality (10) approaches zero, while at the same time its left side in general does not equal zero. When $r - \rho < 1$, inequalities (10) can be satisfied without the norm being finite (cf note 7.2).

We can in the spirit of formulas (10)-(11) attain a refinement of the theorem on estimating mixed derivatives. For example, the inequality (cf 9.2.2)

$$\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right| \leqslant c\left(\left|\frac{\partial^2 u}{\partial x_1^2}\right| + \left|\frac{\partial^2 u}{\partial x_2^2}\right| + \|u\|\right).$$

obtains for $W_p^2(1 < p < \infty)$, whence

$$\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right| \leqslant c\left(\left|\frac{\partial^2 u}{\partial x_1^2}\right| + \left|\frac{\partial^2 u}{\partial x_2^2}\right| + \varepsilon^{-2}\|u\|\right).$$

and after passage to the limit as $\varepsilon \to 0$, we get the inequality

$$\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right| \leqslant c\left(\left|\frac{\partial^2 u}{\partial x_1^2}\right| + \left|\frac{\partial^2 u}{\partial x_2^2}\right|\right)$$

which is valid providing the conditions $\|u\| < \infty$

Such refinements do not always obtain. For example, in inequality (7) the first term of its right side cannot be dropped, as evident from the inequality (7') equivalent to it. If the first term were absent in the latter, then after passage to limit as $\varepsilon \to 0$ we will obtain the result at the left side equal zero, which is possible only if f were a polynomial.

Let us further consider an example applying to the anisotropic case.

In the inequality of different measures

$$\|f\|_{b^{\rho_j}_{x_j,p}(R_m)} \leqslant c\left(\|f\|_{L_p} + \|f\|_{b^r_p(R_n)}\right), \tag{12}$$

$$\varkappa = 1 - \frac{1}{p}\sum_{m+1}^{n}\frac{1}{r_j} > 0, \quad \rho_j = \varkappa r_j$$
$$(j = 1, \ldots, m)$$

(13)

the first term of the right side is superfluous. In fact, taking for convenience j = 1 and substituting $f_\varepsilon$ in (12) based on (3), we get

$$\varepsilon_1^{\rho_1}(\varepsilon_1 \ldots \varepsilon_m)^{-1/p}\|f\|_{b^{\rho_1}_{x_1,p}(R_m)} \leqslant$$
$$\leqslant c(\varepsilon_1 \ldots \varepsilon_n)^{-1/p}\left\{\|f\|_{L_p(R_n)} + \sum_{j=1}^{n}\varepsilon_j^{r_j}\|f\|_{b^{r_j}_{x_j,p}(R_n)}\right\}.$$

Let us cancel out $(\varepsilon \ldots \varepsilon_m)^{-1/p}$ and pass to the limit as $\varepsilon_j \to 0$ only when $j = 2, \ldots, m$, and let us set $\varepsilon_j = \varepsilon_1^{r_1/r_j}$ when $j = m+1, \ldots, n$. Then

$$\|f\|_{b^{\rho_1}_{x_1 p}(R_m)} \leqslant c\left\{\varepsilon_1^{-r_1}\|f\|_{L_p(R_n)} + \sum_{j=m+1}^{n}\|f\|_{b^{r_j}_{x_j p}(R_n)}\right\}.$$

The passage to the limit as $\varepsilon_1 \to 0$ leads to inequality (12), but no longer without the first term in the right side for $j = 1$. But this can be done for any $j = 1, \ldots, m$. Summing up with respect to j, we get (if $\|f\|_{L_p} < \infty$) the inequality

$$\|f\|_{b^{\rho}_{p}(R_m)} \leqslant c_1\|f\|_{b^{r}_{p}(R_n)},$$

revising the corresponding theorem on embedding of different measures.

## 7.3. Extremal Functions in $H^r_p$. Unimprovability of Embedding Theorems

Let us write $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ 0, if for all $\varepsilon_j \geq 0$ and even if one of the components $\varepsilon_j > 0$. We will call function f the extremal function in the class $H^r_p$ if it belongs to $H^r_p$ but does not belong to $H^{r+\varepsilon}_p$, whatever the vector $\varepsilon > 0$.

We will consider the class $H^r_p(R_n)$, where $r = (r_1, \ldots, r_n) > 0$, $p = (p_1, \ldots, p_n)$, $1 \leq p_j \leq \infty$, and $j = 1, \ldots, n$. As always, if $p = p_1 = \ldots = p_n$, then in place of the vector p we will talk about the number p and instead of $H^r_p$, write $H^r_p$. Let us impose the condition

$$\varkappa_j = \varkappa_j(p) = 1 - \sum_{l=1}^{n}\left(\frac{1}{p_l} - \frac{1}{p_j}\right)\frac{1}{r_l} > 0 \qquad (j = 1, \ldots, n). \tag{1}$$

on vector p. In particular,

$$\varkappa_j(p) = 1 \qquad (j = 1, \ldots, n),$$

and in the case of the classes $H_p^r(R_n)$ condition (1) is automatically satisfied.

Let us note that

$$\sum_{l=1}^{n} \frac{\varkappa_l}{r_l} = \sum_{l} \frac{1}{r_l} + \sum_{l} \sum_{l} \left(\frac{1}{p_l} - \frac{1}{p_j}\right) \frac{1}{r_l r_j} = \sum_{l} \frac{1}{r_l}.$$

Suppose

$$F(t) = \left(\frac{\sin \frac{t}{2}}{\frac{t}{2}}\right)^2 \tag{2}$$

and

$$\psi(x) = \psi_{p,r}(a, x) = \sum_{s=0}^{\infty} \frac{\prod_{j=1}^{n} F\left(a^{\frac{s\varkappa_j}{r_j}} x_j\right)}{a^{s\left(1 - \sum_{l=1}^{n} \frac{1}{p_l r_l}\right)}} \tag{3}$$

$$(a > 1,\ \varkappa_j = \varkappa_j(p)).$$

In particular,

$$\psi_{p,r}(a, x) = \sum_{s=0}^{\infty} \frac{\prod_{j=1}^{n} F\left(a^{\frac{s}{r_j}} x_j\right)}{a^{s\left(1 - \frac{1}{p} \sum_{l=1}^{n} \frac{1}{r_l}\right)}}. \tag{4}$$

Let us show that $\psi_{p,r}(a, x) \in H_p^r(R_n)$. In fact, suppose $Q_s$ is the $s$-th term of series (3). Since $F$ is an integral function of the type 1 of a single variable, then $Q_s$ is an integral function of the type $\nu_j(s) = a^{s\varkappa_j/r_j}$ with respect to $x_j$ and here

$$\|Q_s\|_{L_{p_j}(R_n)} \sim a^{-s\varkappa_j} \qquad (s = 0, 1, \ldots), \tag{5}$$

because

$$1-\sum_1^n \frac{1}{p_i r_i}+\frac{1}{p_i}\sum_1^n \frac{\varkappa_i}{r_i}=1-\sum_1^n\left(\frac{1}{p_l}-\frac{1}{p_i}\right)\frac{1}{r_l}=\varkappa_i. \tag{6}$$

Consequently*),

$$\nu_i(s)^{r_i}\|Q_s\|_{L_{p_i}(R_n)}=\left(a^{s\frac{\varkappa_i}{r_i}}\right)^{r_i}\|Q_s\|_{L_{p_i}(R_n)}\sim 1\quad (s=0,1,\ldots). \tag{7}$$

Thus, the left side of (7) is bounded for $\nu_j(s)$ running through an ascending progression. This shows (cf 5.5.3(6)) that $\psi\in H^{r_i}_{x_i p_i}(R_n)$ for any $i = 1, \ldots, n$, i.e., $\psi\in H^r_p(R_n)$.

But it will be proven below (cf 7.4) that in any case, for sufficiently large $a > 1$ function $\psi_{p,r}$ not only belongs to $H^r_p(R_n)$, but is extremal in this class, though for the present we will draw several conclusions that follow from this.

Let us assign the number $p' \geq p_j (j = 1, \ldots, n)$, which in particular can be equal to $\infty$, such that

$$\varkappa=1-\sum_1^n\left(\frac{1}{p_l}-\frac{1}{p'}\right)\frac{1}{r_l}>0$$

(then automatically $\varkappa_j > 0$, $j = 1, \ldots, n$), and let us define, as in the theorem of embedding of different metrics the numbers

$$\rho_i=\frac{r_i\varkappa}{\varkappa_i}\qquad (i=1,\ldots,n).$$

If we set

$$b=a^{\varkappa},\quad a^{s\frac{\varkappa_j}{r_j}}=b^{s/\rho_j}\qquad (i=1,\ldots,n), \tag{8}$$

then we get

*) By definition $a_s \sim b_s$ $(s\in e)$ if there exists positive constant $c_1$ and $c_2$ not dependent on $s \in e$, such that $c_1 a_s \leq b_s \leq c_2 a_s (s\in e)$.

$$\psi(x) = \psi_{p',\rho}(b, x) = \sum_{s=0}^{\infty} \frac{F\left(b^{s\frac{1}{\rho_l}} x_l\right)}{b^{s\left(1-\frac{1}{p'}\sum\frac{1}{\rho_l}\right)}}. \tag{9}$$

In fact,

$$\varkappa\left(1-\frac{1}{p'}\sum_l\frac{1}{\rho_l}\right) = \varkappa - \frac{1}{p'}\sum_l\frac{\varkappa_l}{r_l} = 1 - \sum_l\left(\frac{1}{p_l}-\frac{1}{p'}\right)\frac{1}{r_l} - \\ - \frac{1}{p'}\sum_l\frac{1}{r_l} = 1 - \sum\frac{1}{p_l r_l}.$$

Equalities (3) and (9) point to the fact that beside $\psi$ is at the same time functions $\psi_{p,r}(a, x)$ and $\psi_{p',p}(b, x)$, where b and a are associated by equality (8). But if a is sufficiently large, then $\psi_{p,r} \in H_p^r(R_n)$ and $\psi_{p',\rho} \in H_{p'}(R_n)$, which is in agreement with the embedding theorem. But $\psi_{p',\rho}$ is the extremal function in the class $H_{p'}(R_n)$. It does not belong to any such class $H_{p'}^{\rho+\varepsilon}(R_n)$, where $\varepsilon > 0$. This shows that the embedding $H_p^r(R_n) \longrightarrow H_{p'}^{\rho+\varepsilon}(R_n)$ $(\varepsilon > 0)$ is invalid. But then the embedding $B_{p\theta}^r(R_n) \to B_{p'}^{\rho+\varepsilon}(R_n)$ is also invalid, because if we assume that it is valid, then we would have

$$H_p^{r+\frac{1}{2}\varepsilon}(R_n) \to B_{p\theta}^r(R_n) \to B_{p\theta}^{\rho+\varepsilon}(R_n) \to H_p^{\rho+\varepsilon}(R_n),$$

which, as we have proven, is impossible. We will now proceed from the function $\psi_{p,r}(a, x)$ (cf (3)), and assume

$$\varkappa = 1 - \frac{1}{p}\sum_{m+1}^{n}\frac{1}{r_l} > 0 \qquad (1 \leqslant m < n). \tag{10}$$

We will assert that the vector $\rho = (\rho_1, \ldots, \rho_m)$, here already m-dimensional, is defined, as in the theorem of embedding of different measures, by the equalities

$$\rho_j = r_j\varkappa \qquad (j = 1, \ldots, m),$$

and we will assume $x = (u, y)$, $u = (x_1, \ldots, x_m)$, $y = (x_{m+1}, \ldots, x_n)$, and $\psi(x) = \psi(u, y)$. Let $R_m$ stand for the coordinate subspace of points $(u, 0)$. The trace of $\psi$ on $R_m$ is the function $(F(0) = 1)$

$$\psi(u, 0) = \sum_{s=0}^{\infty} \frac{\prod_{j=1}^{m} F\left(a^{s/r_j} x_j\right)}{a^{s\left(1-\frac{1}{p}\sum_{1}^{n}\frac{1}{r_i}\right)}} = \sum_{s=0}^{\infty} \frac{\prod_{j=1}^{m} F\left(b^{s/\rho_j} x_j\right)}{b^{s\left(1-\frac{1}{p}\sum_{1}^{m}\frac{1}{\rho_i}\right)}} = \psi_{p,\rho}(u) \qquad (b = a^{x}), \tag{11}$$

because

$$x\left(1-\frac{1}{p}\sum_{1}^{m}\frac{1}{\rho_i}\right) = x - \frac{1}{p}\sum_{1}^{m}\frac{1}{r_i} = 1 - \frac{1}{p}\sum_{1}^{n}\frac{1}{r_i}.$$

From (11) we see that the trace of $\psi_{p,r}(x)$ on $R_m$ is $\psi_{p,\rho}(u)$. Here $\psi_{p,r}(x) \in H_p^r(R_n)$, and $\psi_{p,\rho}(u) \in H_p^{\rho}(R_m)$, which is in agreement with the theorem of embedding of different measures. But $\psi_p^{\rho}$ is an extremal function in $H_p(R_m)$ and does not belong to $H_p^{\rho+\varepsilon}(R_m)(\varepsilon > 0)$. Therefore the embedding $H_p^r(R_n) \to H_p^{\rho+\varepsilon}(R_m)$ is invalid. Reasoning as above, we arrive at the conclusion that the embedding $B_{p\theta}^r(R_n) \to B_{p\theta}^{\rho+\varepsilon}(R_m)$ is also invalid. By this we have proven that the theorem of embedding of different metrics in this sense is unimprovable. Nevertheless, improvement is possible in terms of more general classes. For example, A. S. Dzhafarov [1] obtained refinements of embedding theorems for the classes $H_p^r$ by considering the more general classes $H_p^{r,s}(H_p^{r,0} = H_p^r)$ of functions f, which, for example, provided $n = 1$, $r < 1$, are defined thusly: $f \in H_p^{r,s}$, if $f \in L_p$, and

$$\|f(x+h) - f(x)\| \leqslant M|h|^r \left|\ln\frac{1}{|h|}\right|^s.$$

We have seen that the conclusion of the impossibility of this embedding leads to the proof of impossibility of the inequality accompanying it. Even though, it remains unclear whether there does exist in the class $B^r_{p\theta}(R_n)$ a function not belonging to $B_{p\theta}(R_m)$. It would be shown in 7.6 that such a function does exist.

## 7.4. More on Extremal Functions in $H^r_p$

Let us proceed to the proof that $\psi = \psi_{pr}(a, x)$ (7.3(3)) given sufficiently largo a is an extremal function in $H^r_p(R_n)$.

Let us note that function $F(t)$ exhibits the following properties: for each natural $l$ we can indicate such numbers $c$ and $\delta$, dependent on $l$, that

1) the derivative $F^{(l)}(t)$ preserves its sign at $(0, \delta)$;
2) at $(0, \delta)$ the following inequality is satisfied:

$$|F^{(l)}(t)| \geq ct. \tag{1}$$

The first property stems from the analyticity of $F$. The second stems from the fact that

$$F(t) = a_0 + a_2 t^2 + a_4 t^4 + \ldots,$$

where $a_i \neq 0$ for any $i = 0, 1, \ldots$

Suppose

$$\gamma = 1 - \sum_{1}^{n} \frac{1}{p_i r_i} \tag{2}$$

and note that

$$\frac{1}{p_i} \sum_{j=1}^{n} \frac{x_j}{r_j} - x_i = \frac{1}{p_i} \sum_{i} \frac{1}{r_i} - \left(1 - \sum_{i} \left(\frac{1}{p_i} - \frac{1}{p_i}\right) \frac{1}{r_i}\right) = -\gamma. \tag{3}$$

Let us assign the sequence

$$h = h_\mu = \frac{\delta}{2} a^{-\mu \frac{x_i}{r_i}} \quad (\mu = 0, 1, \ldots), \tag{4}$$

where $\delta$ is the number specified above, selected for $l = \bar{r}_1 + 2(r_1 = \bar{r}_1 + \alpha$, $\bar{r}_1$ is an integer, $0 < \alpha \leq 1)$.

Our function can be written as

$$\psi = \sum_{0}^{\infty} Q_s, \quad Q_s = a^{-s\gamma} \prod_{j=1}^{n} F\left(a^{s\frac{\varkappa_j}{r_j}} x_j\right).$$

Our goal is an estimate from below of the norm $\Delta^2_{x_1 h} \psi^{(\bar{r}_1)}_{x_1}(x)$ in the metric $L_{p_1}(R_n)$, where $\psi^{(r_1)}_{x_1}$ denotes the derivative of $\psi$ with respect to $x_1$ of order $\bar{r}_1$.

We have

$$\Delta^2_{x_1 h} \psi^{(\bar{r}_1)}_{x_1} = \sum_{s \le \mu} \Delta^2_{x_1 h} Q^{(\bar{r}_1)}_{s x_1} + \sum_{s > \mu} = s(h) + \sigma(h), \tag{5}$$

$$\|\sigma(h)\|_{L_{p_1}(R_n)} \le 4 \sum_{s > \mu} \|Q^{(\bar{r}_1)}_{s x_1}\|_{L_{p_1}(R_n)} \le 4 \sum_{s > \mu} a^{-s\varkappa_1\left(1 - \frac{\bar{r}_1}{r_1}\right)} =$$

$$= 4a^{-(\mu+1)\frac{\varkappa_1}{r_1}\alpha} \sum_{0}^{\infty} a^{-s\frac{\varkappa_1}{r_1}\alpha} = \left(\frac{2}{\delta} h\right)^{\alpha} \frac{4}{a^{\frac{\varkappa_1}{r_1}\alpha}\left(1 - a^{-\frac{\varkappa_1}{r_1}\alpha}\right)}. \tag{6}$$

We usedthe estimate 7.3(5) in the second inequality, while in the last equality the substitution $h = h_\mu$ by formula (4) was made. We computed the constant for $h^\alpha$ not in vain -- here it is clear that it approaches zero as $a \to \infty$.

On the other hand (explanations below)

$$\begin{aligned}
&\|s(h)\|_{L_{p_1}(R_n)} = \\
&\quad = \Bigg\| \sum_{s \le \mu} a^{-s\gamma} \left(h a^{s\frac{\varkappa_1}{r_1}}\right)^2 a^{s\bar{r}_1 \frac{\varkappa_1}{r_1}} F^{(\bar{r}_1+2)}_{x_1}\left(a^{s\frac{\varkappa_1}{r_1}}(x_1 + \theta h)\right) \times \\
&\quad \times \prod_{j=2}^{n} F\left(a^{s\frac{\varkappa_j}{r_j}} x_j\right) \Bigg\|_{L_{p_1}(R_n)} \ge \|\cdot\|_{L_{p_1}[(0,h)\times R_{n-1}]} \ge \\
&\quad \ge \left(h a^{\mu\frac{\varkappa_1}{r_1}}\right)^2 a^{-\mu\left(\gamma - \bar{r}_1 \frac{\varkappa_1}{r_1}\right)} \times \\
&\times \Bigg\| F^{(\bar{r}_1+2)}_{x_1}\left(a^{\mu\frac{\varkappa_1}{r_1}}(x_1 + \theta h)\right) \prod_{j=2}^{n} F\left(a^{\mu\frac{\varkappa_j}{r_j}} x_j\right) \Bigg\|_{L_{p_1}[(0,h)\times R_{n-1}]} \ge \\
&\ge c_1 \left(\frac{\delta}{2}\right)^2 a^{-\mu\left(\gamma - \varkappa_1 + \frac{\alpha\varkappa_1}{r_1}\right)} \left(\int_0^h \left| c a^{\mu\frac{\varkappa_1}{r_1}} x_1 \right|^{p_1} dx_1\right)^{1/p_1} a^{-\mu\frac{1}{p_1}\sum_2^n \frac{\varkappa_1}{r_j}} = \\
&= c_2 \left(\frac{\delta}{2}\right)^2 \left(h a^{\mu\frac{\varkappa_1}{r_1}}\right)^{1+\frac{1}{p_1}} \left(h a^{\mu\frac{\varkappa_1}{r_1}}\right)^{-\alpha} h^{\alpha} = c_2 \left(\frac{\delta}{2}\right)^{3-\alpha+\frac{1}{p_1}} h^{\alpha}.
\end{aligned} \tag{7}$$

In the second relation (inequality), the domain of integration of $R_n$ is replaced by its portion $(0, h) \times R_{n-1}$, consisting of points $x$, where $0 < x_1 < h$, $-\infty < x_j < \infty$ for $j = 2, \ldots, n$. In this case when $s \leq \mu$, by virtue of (4) $a^{s\varkappa_1/r_1}(x_1 + \theta h) \leq a^{\mu\varkappa/r_1} 2h \leq \delta$ also because functions $F_{x_1}^{(r_1+2)}(a^{s\varkappa_1/r_1}(x_1 + \theta h))$ retain their sign and, since further $F \geqslant 0$, then the norm to which we have arrived can only decrease, if one term corresponding to $s = \mu$ remains in the sum. This explains the passage from the third term to the fourth. The passage from the fourth term to the fifth is executed by (4) and inequalities (1) when $l = \bar{r}_1 + 2$; for integration with respect to $R_{n-1}$, we must consider that

$$\left(\int |F(Nx)|^p\,dx\right)^{1/p} = \frac{1}{N^{1/p}}\left(\int |F(u)|^p\,du\right)^{1/p} = c_1 N^{-1/p}.$$

The passage from the fifth term to the sixth is based on application of (3). Finally, we apply (4) to the last inequality. It is essential to observe that the constants $c$, $c_1$, and $c_2$ in (7) did not depend not only on $h$ and $\mu$, but neither on $a$. On the other hand, as has already denoted above, the constant for $h^\alpha$ in inequality (6) can be made as small as desired, given a sufficiently large $a$. Consequently, from (5), (6), and (7) it follows that for a sufficiently large $a$ the inequality

$$\|\Delta^2_{x,h}\psi^{r_1}_{x_1}\|_{L_p(R_n)} \geqslant \|s(h)\|_{L_p(R_n)} = \|\sigma(h)\|_{L_p(R_n)} \geqslant c(a)\,h^\alpha, \tag{8}$$

is satisfied, where $h$ runs through the sequence (4) diminishing to zero. But then function $\psi$ cannot belong to the class $H_{p_1}^{r_1+\varepsilon}(R_n)$ $(\varepsilon > 0)$. In fact, let us assume that $\psi \in H_{p_1}^{r_1+\varepsilon}(R_n)$ and let $0 < \eta < \min\{\varepsilon, 1\}$. Then also $\psi \in H_{p_1}^{r_1+\eta}(R_n)$ and here $r_1 + \eta - \bar{r}_1 = \alpha + \eta < 2 = k$, therefore the inequality

$$\|\Delta^2_{x_1 h}\psi^{r_1}_{x_1}\|_{L_{p_1}(R_n)} \leqslant M|h|^{\alpha+\eta}$$

must be satisfied for all $h$, which contradicts (8). Similarly, it is proven that $\psi \notin H_{p_i}^{r_i+\varepsilon}(R_n)$ for any $i = 1, \ldots, n$, if $\varepsilon > 0$.

We have proven that the function $\psi_{p,r}(a, x)$ given sufficiently large $a$, does not belong to any such class

$$H_p^{r+\varepsilon}(R_n), \qquad \text{where } \varepsilon > 0.$$

7.4.1. For the function $\psi$ that is extremal in $H_p^r(R_n) = H(r > 0)$, the $\psi$ norm $\| \psi(x + h) - \psi(x)\|_H$ does not tend to zero as $h \to 0$. In fact, let $r_1 > 0$ and $r_1 = \bar{r}_1 + \alpha$ ($\bar{r}_1$ is an integer, $0 < \alpha < 1$). By 7.4(8), for real $h > 0$ running through some sequence ($\|\cdot\|_{L_p(R_n)} = \|\cdot\|_p$) converging to zero:

$$\|\Delta_h \psi\|_H \geq \sup_{k>0} \frac{\|\Delta_{x_1 k} \Delta_{x_1 h} \psi_{x_1}^{\bar{r}_1}\|_p}{k^\alpha} \geq \frac{\|\Delta_{x_1 h}^2 \psi_{x_1}^{\bar{r}_1}\|_p}{h^\alpha} \geq m > 0. \tag{1}$$

When $\alpha = 1$, the second difference with respect to $k$ (instead of the first) would figure in (1), which will lead to the need to prove inequality 7.4(8) for the third difference (instead of the second). This is done analogously.

## 7.5. Unimprovability of Inequalities for Mixed Derivatives

The inequality

$$\|f^{(l)}\|_{B_{p\theta}^{\rho}(R_n)} \leqslant c \|f\|_{B_{p\theta}^{r}(R_n)} \tag{1}$$

was proven in 5.6.3, provided the condition that

$$\rho = \varkappa r, \quad \varkappa = 1 - \sum_1^n \frac{l_k}{r_k} > 0. \tag{2}$$

It ceases to be valid if $\rho$ in it is replaced with $\rho + \varepsilon$ ($\varepsilon > 0$). This can also be proven by considering the extremal function

$$\psi = \psi_{p,r} = \sum_{s=0}^{\infty} \frac{\prod_{j=1}^{n} F(a^{s/r_j} x_j)}{a^{s\left(1 - \frac{1}{p} \sum \frac{1}{r_j}\right)}} \qquad (a > 1).$$

its derivative

$$\psi^{(l)}(x) = \sum_s \frac{\prod_{j=1}^{n} F^{(l_j)}(b^{s/p_j} x_j)}{b^{s\left(1-\frac{1}{p}\sum\frac{1}{p_l}\right)}} \qquad (a^{x} = b)$$

even though not a particular case of the families of extremal functions we have considered, nevertheless is extremal in the class $H_p(R_n)$, and this is proven quite analogously to the procedure in 7.4 where we had to assert $p = p_1 = \ldots = p_n$. The fact that now different functions $F^{(l_j)}$ appears under the sign $\prod$ is not significant.

This proves our assertion for H-classes*), but now also for B-classes.

## 7.6. Another Proof of the Unimprovability of Embedding Theorems

Let us consider a problem relating to the general theory of functional spaces. Let $E_1$ and $E_2$ be Banach (i.e., linear normed complete) spaces. The following are valid:

Theorem 1**). If a linear bounded operator A mutually uniquely maps $E_1$ onto $E_2$, then the operator $A^{-1}$ inverse to it, which is obviously linear and maps $E_2$ onto $E_1$, is in turn bounded.

Let Banach spaces $E_1$ and $E_2$ have the nonempty intersection $E_1E_2$. We will write for the elements $x \in E_1E_2$ the norm

$$\|x\|_{E_1E_2} = \|x\|_{E_1} + \|x\|_{E_2}. \tag{1}$$

$E_1E_2$ with it is a normed space.

Theorem 2. If $E_1E_2$ is a complete space, i.e., a Banach space, and if the constants $c > 0$ such that $\|x\|_{E_2} \leqslant c\|x\|_{E_1}$

for all $x \in E_1E_2$ do not exist, then there does exist in $E_1$ an element not belonging to $E_2$.

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*) S. M. Nikol'skiy [2], the case when $p = \infty$.
**)Cf book by Hausdorff [1], Addendum.

Proof. Actually, let us assume this is not so, i.e., $E_1 \subset E_2$. Each element $x$ of Banach space $E_1E_2$ can be assumed to be mapped (uniquely) in $x$, but still belonging to $E_1$. This operation is linear and bounded:

$$\|x\|_{E_1} \leqslant \|x\|_{E_1} + \|x\|_{E_2} = \|x\|_{E_1E_2},$$

and maps $E_1E_2$ onto $E_1$. But then, based on theorem 1, the constant $c$ must exist such that

$$\|x\|_{E_1} + \|x\|_{E_2} \leqslant c\|x\|_{E_1}$$

or

$$\|x\|_{E_2} \leqslant c\|x\|_{E_1}, \qquad x \in E_1E_2,$$

and we have reached a contradiction with the condition for the theorem.

Use of theorem 2 requires that the completeness of $E_1E_2$ be collaborated.

If $E_1 = B_p^r(R_n)$ and $E_2 = B_{p'}^{\rho}(R_n)$ (here $B_p = B_{p\theta}$), then the completeness of $E_1E_2$ does obtain, because in this case from the fact that $\|f_k - f_l\|_{E_1E_2} \to 0$, $k, l \to \infty$, owing to the completeness of $E_1$ and $E_2$, there follows the existence $f \in E_1$ and $F \in E_2$ such that $\|f - f_k\|_{E_1} \to 0$, $\|F - f_k\|_{E_2} \to 0$, $(k \to \infty)$, but then also

$$\|f - f_k\|_{L_{p_i}(R_n)} \to 0, \quad \|F - f_k\|_{L_{p'}(R_n)} \to 0 \qquad (i = 1, \ldots, n).$$

Hence (cf 1.3.9) $f = F$ almost everywhere, and we have proven the existence $f \in E_1E_2$, such that

$$\|f - f_k\|_{E_1E_2} \to 0.$$

Let $r = (r_1, \ldots, r_n) > 0$, $p = (p_1, \ldots, p_n)$, $1 \leq p_j \leq \infty$, $\varkappa_j = \varkappa_j(r, p) > 0$ (cf 7.3(1)).

Let us introduce the function $F(t)$ of one variable, finite and infinitely differentiable.

Its norms in the metric $B_p^{\rho}(R_1)$ $(0 < \rho \leq 1)$ are positive, otherwise it would be zero.

Let us construct a family of functions (cf 7.4(2),

$$= 1 - \sum_{j=1}^{n} \frac{1}{p_{f_j}} \Big)$$

$$\Phi_N = \Phi_{N,p,r}(x) = \frac{1}{N^{\nu}} \prod_{j=1}^{n} F\left(N^{\frac{x_j}{r_j}} x_j\right), \tag{2}$$

dependent on parameter N $>0$.

Based on formulas 7.2(1) and (6)

$$\|\Phi_{N,p,r}\|_{p_i} \sim N^{-\frac{1}{p_i}\sum_{j=1}^{n}\frac{x_j}{r_j} - \nu} = N^{-x_i} \qquad (N>0), \tag{3}$$

$$\|\Phi_{N,p,r}\|_{b^{r_i}_{x_i p_i}} \sim N^{\frac{r_i x_i}{r_i}} N^{-x_i} = N^0 = 1 \qquad (i = 1, \ldots, n)$$

$$\left(b^{r_i}_{x_i p_i} = b^{r_i}_{x_i p_i \theta}\right).$$

Let us direct our attention at a specific i and assign the numbers $p^* \geqslant p_i$ and $r^* \geqslant r_j$, where even one of these inequalities is rigorous. Let us compute for comparison the norms

$$\|\Phi_{N,p,r}\|_{p_*} \sim N^{-\frac{1}{p_*}\sum_{1}^{n}\frac{x_j}{r_j} - \nu} = N^{-(x_j - \varepsilon)},$$

$$\|\Phi_{N,p,r}\|_{b^{r^*}_{x_i p_*}} \sim N^{-(x_j-\varepsilon)} N^{\frac{r^* x_i}{r_i}} = N^{\left(\frac{r^*}{r_i} - 1\right) x_j + \varepsilon}$$

$$(\varepsilon > 0, \ N > 0).$$

It is essential to know that here $\varepsilon$ is a positive number, therefore

$$\|\Phi_{N,p,r}\|_{b^{r^*}_{x_i p_*}} \to \infty \qquad (N \to \infty). \tag{4}$$

And thus, to each pair of vectors p, r satisfying the above indicated conditions we have brought it into correspondence the family of functions $\Phi$ (N, p, r), whose norms

$$\|\Phi(N, p, r)\|_{B_p^r(R_n)} \leq c < \infty \qquad (N > 0)$$

are bounded, or at the same time for any i property (4) is satisfied if and only if $r^* \geq r_i$, $p^* \geq p_i$ and one of these inequalities is rigorous.

We will call the family $\Phi$ (N, p, r) the boundary family of functions in the class $B_p^r(R_n)$.

Let us show that (proven in 6.9) the embedding

$$B_{p\theta}^r(R_n) \to B_{p'\theta}^\rho(R_n)$$

given the conditions $1 \leq p_j \leq p' \leq \infty$, $\varkappa > 0$ (therefore also $\varkappa_j > 0$, cf 7.1(4), (5)),

$$\rho_j = \frac{r_j \varkappa}{\varkappa_j} \qquad (j = 1, \ldots, n) \tag{5}$$

ceases to be valid if in it even one of the components $\rho_j$ or the number p' is increased, or both are increased. In fact, if we take $N_1 = N^\varkappa$, then (explanations below)

$$\begin{aligned}\Phi_{N,p,r} &= \frac{1}{N^\gamma}\prod_{j=1}^n F\left(N^{\frac{\varkappa_j}{r_j}} x_j\right) = \\ &= \frac{1}{N_1^{\gamma_1}}\prod_{j=1}^n F\left(N_1^{1/\rho_j} x_j\right) = \Phi_{N_1,\rho,\theta} \qquad (N_1 = N^\varkappa).\end{aligned} \tag{6}$$

Here

$$\gamma_1 = 1 - \frac{1}{p'}\sum_1^n \frac{1}{\rho_i} = \gamma(\rho, p'),$$

because

$$\begin{aligned}\varkappa\gamma_1 = \varkappa - \frac{1}{p'}\sum\frac{\varkappa_j}{r_j} = 1 - \sum\frac{1}{p_i r_i} + \frac{1}{p'}\sum\frac{1}{r_i} - \frac{1}{p'}\sum\frac{1}{r_i} - \\ - \frac{1}{p'}\sum\frac{1}{r_i} + \frac{1}{p'}\sum_j\sum_i\left(\frac{1}{p_i} - \frac{1}{p_j}\right)\frac{1}{r_i r_j} = 1 - \sum\frac{1}{p_i r_i} = \gamma.\end{aligned}$$

further

$$\varkappa_j(\rho, p') = 1 - \sum\left(\frac{1}{p'} - \frac{1}{p'}\right)\frac{1}{r_l} = 1.$$

This proves equality (6).

Thus, the family of functions $\Phi_N$ is simultaneously a boundary family in the classes $B_p^r$ and $B_{p'}^{\rho}$ and the norm $\Phi_N$ in the metrics of these classes are uniformly bounded with respect to N. However, the norms $\Phi_N$ in the metric $B_{p'+\eta}^{\rho+\varepsilon}$ are not bounded. But then constant c not dependent on N and such that

$$\| \Phi_N \|_{B_{p'+\eta}^{\rho+\varepsilon}} \leqslant c \| \Phi_N \|_{B_p^r},$$

does not exist, and we have proven our assertion.

By virtue of theorem 2, in this case it follows that for any $\varepsilon \geqslant 0$, $\eta \geqslant 0$, where one of the inequalities is rigorous, there exists in the class $B_p^r(R_n)$ a function not belonging to $B_{p'+\eta}^{\rho+\varepsilon}(R_n)$. In particular, a function not belonging to $B_{p+\eta}^{r+\varepsilon}(R_n)$ exists in the class $B_p^r(R_n)$.

The embedding (proven in 6.5)

$$B_p^r(R_n) \to B_p^{\rho}(R_m),$$
$$1 \leqslant m < n, \quad \rho_j = \varkappa r_j, \quad j = 1, \ldots, m,$$
$$\varkappa = 1 - \frac{1}{p}\sum_{m+1}^{n}\frac{1}{r_l}, \tag{7}$$

ceases to be valid if in it we replace $\rho$ and p with $\rho^* \geqslant \rho$, $p^* \geqslant p$, where one of the inequalities is rigorous.

Actually, there exists the function $\varphi \in B_p^{\rho}(R_m)$, but not belonging to $B_{p+\eta}^{\rho+\varepsilon}(R_m)$. Based on the theorem on extension, $\varphi$ can be extended from $R_m$ to $R_n$ such that the extended function $f \in B_p^r(R_n)$. Since $f|_{R_m} = \varphi$, then f is an example of the function $f \in B_p^r(R_n)$ whose trace on $R_m$ does not belong to $B_{p+\eta}^{\rho+\varepsilon}(R_m)$.

Incidentally, from the foregoing it follows that the theorem on extension

$$B_p^\rho(R_m) \to B_p^r(R_n)$$

can also not been improved in terms of the classes considered. However, this does not signify that this theorem cannot be improved in other terms. For example, it will be shown in Chapter IX that given the same relationship between r and $\rho$, and between n and m, the mutually inverse embeddings

$$B_p^\rho(R_m) \rightleftarrows L_p^r(R_n),$$

hold, where the class $L_p^r$ when $p \neq 2$ is not equivalent to $B_p^r(R_n)$.

Let us further assign the family (boundary in $B_p^r(R_n)$)

$$\Phi_N = \Phi_{N,p,r} = \frac{1}{N^\gamma} \prod_{j=1}^{n} F(N^{1/r_j} x_j).$$

In this case

$$\gamma = 1 - \frac{1}{p} \sum_{1}^{n} \frac{1}{r_j} > 0, \quad \varkappa_j(p, r) = 1$$
$$(j = 1, \ldots, n).$$

Let us further assume that F(t), in addition to being finite and infinitely differentiable, as the Taylor expansion

$$F(t) = 1 + a_1 t + \ldots + a_{l+1} t^{l+1} + R_{l+1}$$

with coefficients not equal to zero appearing at the odd positions or even. Then, as we can easily see, we can specify a positive number $\delta$ and constant B such that

$$|F^{(k)}(t)| > Bt \qquad (k = 0, 1, \ldots, l, \ |t| < \delta).$$

Let $R_m$ be a subspace of points $(x_1, \ldots, x_m, 0, \ldots, 0) = (\mathbf{u}, 0)$, $\mathbf{u} = (x_1, \ldots, x_m)$. Then

$$\Phi_N(u, 0) = \frac{1}{N_1^{\gamma_1}} \prod_{j=1}^{m} F\left(N_1^{\frac{1}{\rho_j}} x_j\right) = \Phi_{N_1, p, \rho} \quad (N_1 = N^\varkappa),$$

where

$$\gamma_1 = \gamma_1(p, \rho) = 1 - \frac{1}{p} \sum_{1}^{m} \frac{1}{\rho_j}$$

(considering that $\nu = \varkappa\nu_1$).

Let $h > 0$ and $i = m+1, \ldots, n$. Let us consider the increment

$$\Delta_{x_i h}\Phi_N(u,0) = \frac{1}{N_1^{\nu_i}} \prod_{j=1}^{m} F\left(N_1^{\frac{1}{\rho_j}} x_j\right)\left[F\left(N_1^{\frac{1}{\rho_i}} h\right) - F(0)\right] =$$

$$= \Phi_{N_1,p,\rho}(u)\left[F\left(N_1^{\frac{1}{\rho_i}} h\right) - F(0)\right].$$

Function F does not identically equal to zero, therefore we can find such a $\delta > 0$ that $|F(\delta) - F(0)| = K > 0$. We will consider the values of h and $N_1$ associated by the equality $\delta = N_1^{1/\rho_i} h$. By virtue of the first estimate (3), we have (in our case $\varkappa_i = 1$)

$$\|\Delta_{x_i h}\Phi_N(u,0)\|_{L_p(R_m)} = \|\Phi_{N_1,p,\rho}\|_{L_p(R_m)} K \gg \frac{1}{N} \gg |h|^{\rho_i}.$$

This estimate from below shows that the first inequality (6.4) (13) ($\rho_i = r'_i$) derived earlier is reached and in this case not only for the class $H_p^r(R_n)$, but also for $B_{p\theta}^r(R)$.

## 7.7. Theorems on Compactness

**Theorem.** Let there be assigned a sequence of functions $\{f_l\}$ exhibiting one of the following properties:

a)
$$\|f_l\|_{L_p(R_n)} \leqslant M, \quad \|f_l\|_{W_p^r(R_n)} \leqslant N \tag{1}$$

(r is an integral vector)

b)
$$\|f_l\|_{L_p(R_n)} \leqslant M, \quad \|f_l\|_{B_{p\theta}^r(R_n)} \leqslant N \tag{2}$$

$(1 \leqslant p, \theta \leqslant \infty)$.

Then we can separate the subsequence $\{f_{l_k}\}$ and such a function f satisfying*), respectively, conditions (1) and (2) that whatever the numbers $r'_j$ for which $0 < r'_j < r_j$ ($j = 1, \ldots, n$),

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*) on following page.

$$\|f_{l_k} - f\|_{W_p^{r'}(g)} \to 0 \qquad (l_k \to \infty) \tag{3}$$

obtains for any bounded domain $g \subset R_n$.

The proof of this theorem will be based on the following lemma from functional analysis.

Lemma. Suppose that the same linear set of elements x is normed by two norms $\|\cdot\|$ and $\|\cdot\|_*$, where the normed spaces E and $E_*$ obtained are complete and $\|x\|_* \leqslant c\|x\|$, where the constant c does not depend on x.

Let there be assigned in E a bounded set F and a sequence of operator $A_n(x)$ ($n = 1, 2, \ldots$) mapping E onto $E_*$ defined by the equalities

$$A_n(x) = x - U_n(x)$$

and satisfying the conditions:

1) operator $y = U_n(x)$ ($x \in E$, $y \in E_*$) are wholly continuous (the linearity of $U_n$ is not required);

2) $\sup\limits_{x \in F} \|A_n(x)\| = \eta_n \to 0 \quad (n \to \infty)$.

Then the set F is compact in $E_*$.

Proof. Let us assign an arbitrary sequence of elements $x_1, x_2, \ldots$ belonging to F. It is bounded and owing to property 1), from it the subsequence $x_1^{(1)}, x_2^{(1)}, \ldots$ for which $U_1(x_k^{(1)})$ ($k = 1, 2, \ldots$) converges in $E_*$ can be separated. In turn, from this sequence we can separate the subsequence $x_1^{(2)}, x_2^{(2)} \ldots$ for which $U_2(x_k^{(2)})$ ($k = 1, 2, \ldots$) converges in $E_*$. Continuing this process without limit, and taking the diagonal sequence $z_1 = x_1^{(1)}$, $z_2 = x_2^{(2)}, \ldots$ we find that $U_n(z_k)$ converges in $E_*$ as $k \to \infty$ and for any n. Let us now assign $\varepsilon > 0$. By condition (2), for some $n = N$ the inequality

$$\|A_N(x)\|_* \leqslant c\|A_N(x)\| < \varepsilon$$

is satisfied for all $x \in F$. If p and q exceed a sufficiently large number, then

$$\|z_p - z_q\|_* \leqslant \|A_N(z_p)\|_* + \|U_N(z_p) - U_N(z_q)\|_* + \|A_N(z_q)\|_* < 3\varepsilon,$$

*) Function f satisfies (1) or (2) with the same constant N if we understand thennorm in the same sense; in the case b) the proof will be given below for the variants of the norm $\|\cdot\|_B$, $\rho = \bar{r}$ (cf 5.6).

and the compactness of F in $E_*$ is proven.

Proof of theorem. Let $K = M + N$. Let us now first consider the case b) when $\theta = \infty$, i.e., the case of class $H_p^r = H_p^r(R_n)$.

Let $\mathfrak{M}$ be a set of all functions f for which any quality (2) (when $\theta = \infty$) is satisfied. The expansion (5.5.3(6), (7))

$$f = \sum_{s=0}^{\infty} Q_s,$$

obtains for each of these, where

$$Q_s = Q_{a^{s/r_1}, \dots, a^{s/r_n}} \qquad (a > 1)$$

are integral functions of exponential type $a^{s/r_j}$, respectively, with respect to $x_j (j = 1, \dots, n)$ and

$$\sup_s a^s \| Q_s \|_p = \| f \|_{H_p^r} \leqslant cK.$$

Let us assign a number $\gamma$ satisfying the inequality $0 < \gamma < 1$ and set

$$T_m(f) = T_m = \sum_0^{m-1} Q_s, \quad a^\gamma = b.$$

Then $(\| \cdot \|_p = \| \cdot \|_{L_p(R_n)})$

$$\| f - T_m \|_{H_p^{\gamma r}} = \sup_{s \geqslant m} b^s \left| Q_{b^{\frac{s}{\gamma r_1}}, \dots, b^{\frac{s}{\gamma r_n}}} \right|_p =$$

$$= \sup_{s \geqslant m} a^{\gamma s} \| Q_s \|_p \leqslant \frac{1}{a^{(1-\gamma) m}} \sup_s a^s \| Q_s \|_p \leqslant \frac{cK}{a^{(1-\gamma) m}}.$$

Moreover,

$$\| f \|_{H_p^{\gamma r}} \leqslant c \| f \|_{H_p^r} \leqslant cK$$

(cf 6.2(3)).

We will consider the function f from the space

$$E = H_p^{\gamma r} = H_p^{\gamma r}(R_n)$$

also as elements of the space

$$E_* = H_p^{\gamma r}(g) \qquad (g \subset R_n),$$

where, obviously,

$$\| f \|_{E_*} \leqslant \| f \|_E.$$

We have $$f = T_m(f) + (f - T_m(f)),$$

where for f

$$\| f - T_m(f) \|_E \leqslant \frac{cK}{a^{(1-\nu)m}} \to 0 \qquad (m \to \infty).$$

Further

$$\| T_m(f) \|_p \leqslant \| T_m(f) \|_{H_p^{\nu r}} \leqslant \| f \|_E + \frac{cK}{a^{(1-\nu)m}}.$$

Therefore, the image of any sphere E in the transformation $T_m$ is a set of functions $T_m(f)$ of exponential type $a^{m/r_j}$ with respect to $x_j$ bounded in the sense $L_p = L_p(R_n)$. In this case, this set is compact on any bounded set $g \subset R_n$ in the sense of the metric $c^l(g)$ (cf 3.3.6*)), and therefore for any natural l it is also compact in the sense $E_* = H_p^r(g)$. We have proven that $T_m(f)$ is a wholly continuous operator ( generally speaking, nonlinear).

As a consequence of the above proven lemma, $\mathfrak{M}$ is a set compact in $H_p^{\nu r}(g)$. Since this argumentation applies to any $\nu$ with $0 < \nu < 1$, then $\mathfrak{M}$ is compact in the $H_p^{\nu r}$-sense for any specified $\nu$. Let us take a specific sequence of numbers $\{\nu_k\}$ monotone-approaching 1, and let us specify an arbitrary sequence of functions $\{f_l\}$ from $F(\subset \mathfrak{M})$. By virtue of the proven completeness of $H_p^{\nu_1 r}$ as well (cf 4.7), from it we can separate a subsequence $\{f_{l_k^1}\}$ convergent in the metric $H_p^{\nu_1 r}$ to some function $f \in H_p^{\nu_1 r}$. In turn, from the resulting subsequence we can separate a subsequence $\{f_{l_k^2}\}$ convergent in the metric $H_p^{\nu_2 r}$ to the function $f \in H_p^{\nu_2 r}$, which is obviously the same. Continuing this process without limit and taking the diagonal sequence that we denote by $\{f_{l_k}\}$, we find that $f_{l_k} \to f$ in the sense of the metric $H_p^{\nu_s r}$, whatever the s, but then by (6.2(3)), this is true also in the sense of the metric $H_p^{r'}$, where $r_j' < r_j (j = 1, \ldots, n)$.

*)From the boundedness (in the $L_p$-sense) of functions $T_m(f_k)(k = 1, 2, \ldots)$ follows the boundedness of their derivatives of any given order. The application of 3.3.6 not only to functions, but also to their derivatives up to order l inclusively and the diagonal process leads to compactness not only in the $c(g)$-sense, but also in the $c^l(g)$-sense.

We have proven (3) in the case b) for $\theta = \infty$; the remaining case a) and b) when $1 \leq \theta < \infty$ reduce to the same case, because $W_p^r$, $B_{p\theta}^r \to H_p^r$. But it still remains for us to prove a more subtle fact, that the limit function f belongs, specifically, to $W_p^r$ and $B_{p\theta}^r$ and that the inequalities hold, respectively

$$\|f\|_{w_p^r}, \quad \|f\|_{b_{p\theta}^r} \leq N.$$

Inequality $\|f\| \leq M$ follows from (1), (2), and (3).

As always, we will assert that $r_j = \bar{r}_j + \alpha_j$ where $\bar{r}_j$ is an integer and $0 < \alpha_j \leq 1$. Let $f_{x_j}^{\bar{r}_j}$ stand for the partial derivative of f of order $\bar{r}_j$ with respect to $x_j (\bar{r}_j < r'_j < r_j)$.

Then (6.2(3))

$$\left|f_{l_k} - f\right|_p, \quad \left|f_{l_k x_j}^{r'_j} - f_{x_j}^{r'_j}\right|_p \leq c\left|f_{l_k} - f\right|_{H_p^{r'}} \to 0 \quad (l_k \to \infty). \tag{4}$$

In the case b) the functions $f_{l_k}$ are subject to inequality (2), therefore

$$\left(\int_{-\infty}^{\infty} |u|^{-1-\theta\alpha_j} \left|\Delta_{x_j u}^2 f_{l_k x_j}^{r'_j}(x)\right|_p^\theta du\right)^{1/\theta} = m_j^{(k)} \quad (1 \leq \theta < \infty), \tag{5}$$

$$\left|\Delta_{x_j u}^2 f_{l_k x_j}^{r'_j}(x)\right|_p \leq m_j^{(k)} |u|^{\alpha_j} \quad (\theta = \infty),$$

where

$$\sum_{j=1}^{n} m_j^{(k)} \leq N \quad (j = 1, \ldots, n;\ k = 1, 2, \ldots).$$

Passing (5) to the limit as $k \to \infty$, based on (4) we get

$$m_j = \left(\int_{-\infty}^{\infty} |u|^{-1-\theta\alpha_j} \left|\Delta_{x_j u}^2 f_{x_j}^{r'_j}(x)\right|_p^\theta du\right)^{1/\theta} \leq \varlimsup_{k \to \infty} m_j^{(k)} \quad (1 \leq \theta < \infty),$$

$$\left|\Delta_{x_j u}^2 f_{l_k x_j}^{r'_j}(x)\right|_p \leq \varlimsup m_j^{(k)} |u|^{\alpha_j} \quad (\theta = \infty),$$

therefore ($f \in L_p$) $f \in B^r_{p\theta}$

$$\|f\|_{b^r_{p\theta}} = \sum_{j=1}^{n} m_j \leq N.$$

In the case a) the functions $f_{l_k}$ are subject to inequalities

$$\left\| \frac{\Delta_{x_j u} f'_{l_k x_j}}{u} \right\|_p \leq \| f'_{l_k x_j} \|_p \leq m_j^{(k)}, \tag{6}$$

where

$$\sum_{j=1}^{n} m_j^{(k)} \leq N.$$

Passing to the limit in (6) as $k \to \infty$, we get

$$m_j = \left\| \frac{\Delta_{x_j u} f'_{x_j}}{u} \right\|_p \leq \varliminf_{k\to\infty} m_j^{(k)},$$

and since further $f \in L_p$, then (cf 4.8) $f \in W^r_p$ and

$$\|f\|_{w^r_p} = \sum_{j=1}^{n} m_j \leq N.$$

Note. In the theorem proven $W^r_p$ and $B^r_{p\theta}$ can be replaced, respectively, by $W^r_p$ and $B^r_{p\theta}$, and then in (3) we can replace r' with r', $0 < r' < r$. The case $W^r_p$ and similar cases that can be proven on analogy find application in the theory of variational methods. It is very essential to application that the inequality of type (1) entails the same inequality for the limit function with the same constant. In the theorem, the classes involved can be roplaced by the corresponding periodic classes.

7.7.1. Theorem. In order that the set $\mathfrak{M}$ are functions $f \in L_p = L_p(g)$ where $g \subset R_n$ is an arbitrary domain, be compact, it is necessary and sufficient that it be: 1) bounded in $L_p$, and 2) equicontinuous translation-wise in $L_p$:

$$\Lambda(\delta) = \sup_{f \in \mathfrak{M}} \omega(\delta, f)_p \to 0 \qquad (\delta \to 0),$$
$$\omega(\delta, f)_p = \sup_{|h| < \delta} \|f(x+h) - f(x)\|_p \qquad (f = 0 \text{ на } R_n - g),$$

3) and that the functions $f \in \mathfrak{M}$ diminish uniformly with respect to the norm in $L_p$ at infinity

$$\sup_{f \in \mathfrak{M}} \|f\|_{L_p(|x| > N,\ x \in g)} \to 0 \qquad (N \to \infty).$$

This theorem was proven in the book by S. L. Sobolev [4], Chapter I, section 4.3. Property 3), obviously, drops out for a bounded domain g. When $p = \infty$, the theorem generally ceases to be valid. In this case the translationwise norm of an individual function in general will not tend to zero as $h \to 0$.

7.7.2. Theorem. For the set $\mathfrak{M}$ of functions $f \in W \in W_p^l(R_n), (1 \leq p < \infty,\ l \geq 0)$ bounded in $L_p = L_p(R_n)$ to be compacted W, it is necessary and sufficient that $\mathfrak{M}$ be equicontinuous translationwise:

$$\Lambda(\delta) = \sup_{f \in \mathfrak{M}} \sup_{|h| < \delta} \|f(x+h) - f(x)\|_W \to 0 \qquad (\delta \to 0) \tag{1}$$

and that the functions $f \in \mathfrak{M}$ uniformly diminish normwise at infinity:

$$\sup_{f \in \mathfrak{M}} \|f\|_{L_p(|x| > N)} \to 0 \qquad (N \to \infty). \tag{2}$$

In this formulation W can be replaced by $B = B_{p\theta}^r(R_n)(1 \leq p,\ \theta < \infty,\ r \geq 0)$.

Proof. We will consider the space W, but W can everywhere be replaced by B. But $\mathfrak{M}$ be compact in W. Then it is compact also in $L_p$, because (cf 7.7.1) satisfies property (2). By the general compactness criterion (Hausdorff [1]), for a given $\varepsilon > 0$, we can specify a finite system of functions $f_j$ ($f = 1, \ldots, N$) such that for any function $f \in \mathfrak{M}$ we can find a j (dependent on f) for which

$$\|f - f_j\|_W < \varepsilon.$$

We can also specify $\delta$ and N such that the inequality (cf 5.6.5)

$$\|f(x+h) - f(x)\|_W < \varepsilon, \quad \|f\|_{L_p(|x| > N)} < \varepsilon, \quad |h| < \delta$$

will be satisfied for all $f_j$ ($J = 1, \ldots, N$). But then for any $f \in \mathfrak{M}$, given suitable j,

$$\|f(x+h) - f(x)\|_W \leq \|f(x+h) - f_j(x+h)\|_W + \\ + \|f_j(x+h) - f_j(x)\|_W + \|f_j(x) - f(x)\|_W < 3\varepsilon \qquad (|h| < \delta),$$

if $\delta$ is sufficiently small, and must be proven in (1). The necessity of the conditions in the theorem is proven.

Suppose, conversely, that $\mathfrak{M}$ is a set bounded in $L_p$ and satisfying conditions (1) and (2). Then based on 7.7.1, it is compacted in $L_p(\|\cdot\|_W \geqslant \|\cdot\|_{L_p})$. Let us introduce a new concept -- the module of continuity of $f \in W$:

$$\omega(t) = \omega(f, t) = \sup_{|h|<t} \|f(x+h) - f(x)\|_W.$$

It satisfies the conditions

$$\begin{gathered} 0 \leqslant \omega(\delta_2) - \omega(\delta_1) \leqslant \omega(\delta_2 - \delta_1) \qquad (0 < \delta_1 < \delta_2), \\ \omega(l\delta) \leqslant (l+1)\,\omega(\delta) \qquad (l, \delta > 0). \end{gathered} \tag{3}$$

This is proven precisely as for the module of continuity of f in $L_p$(cf 4.2). From (3) it follows that for the function $\Lambda(\delta)$(cf (1)), the inequality

$$\Lambda(l\delta) \leqslant (l+1)\,\Lambda(\delta) \qquad (l, \delta > 0). \tag{4}$$

is also satisfied. Let us further introduce a function of one variable

$$K_k(t) = a_k\left(\frac{\sin kt}{t}\right)^{\lambda} \qquad (k > 1)$$

that is integral and of the exponential type $k\lambda$, where $\lambda > n + 1$ is an even natural number and the constant $a_k$ defined from the equality

$$1 = \int_{R_n} K_k(|u|)\,du = a_k \varkappa_n \int_0^\infty \left(\frac{\sin kt}{t}\right)^{\lambda} t^{n-1}\,dt =$$

$$= k^{\lambda-n} a_k \varkappa_n \int_0^\infty \left(\frac{\sin t}{t}\right)^{\lambda} t^{n-1}\,dt = ck^{\lambda-n} a_k$$

($\varkappa_n$ is the area of a unit field in $R_n$, and c does not depend on k and $a_k$). Hence it follows that

$$a_k = O(k^{n-\lambda}) \qquad (k > 1).$$

Let us suppose

$$U_k f = \int K_k(|u|)\,f(x+u)\,du,$$

from whence $$\| U_k f \|_p \leqslant \| K_k \|_L \| f \|_p.$$

Для $f \in \mathfrak{M}$

$$f - U_k f = \int K_k(|u|)[f(x) - f(x+u)]\,du, \qquad (5)$$

For $f \in \mathfrak{M}$

therefore

$$\begin{aligned} \| f - U_k f \|_W &\leqslant \int K_k(|u|) \| f(x) - f(x+u) \|_{x,W}\,du \leqslant \\ &\leqslant \int K_k(|u|)\Lambda(|u|)\,du \leqslant \int\limits_{|u|<\delta} K_k(|u|)\Lambda(|u|)\,du + \\ &+ \int\limits_{|u|>\delta} K_k(|u|)\Lambda\left(\frac{|u|}{\delta}\delta\right)du \leqslant \\ &\leqslant \Lambda(\delta) + \Lambda(\delta)\int\limits_{|u|>\delta} K_k(|u|)\left(1 + \frac{|u|}{\delta}\right)du < \\ &< \varepsilon + \varepsilon = 2\varepsilon \qquad (k > k_0), \end{aligned} \qquad (6)$$

where $k_0$ is sufficiently large, because by (1) we can specify such a $\delta$ that $\Lambda(\delta) < \varepsilon$ and consequently $\delta$ -- the second member of the penultimate term in (6) -- can be made also smaller than $\varepsilon$ for sufficiently large k:

$$\begin{aligned} \int\limits_{|u|>\delta} K_k(|u|)\left(1 + \frac{|u|}{\delta}\right)du &\ll \\ &\ll k^{n-\lambda}\int\limits_{\delta}^{\infty}\left(\frac{\sin kt}{t}\right)^{\lambda}\left(1 + \frac{t}{\delta}\right)t^{n-1}\,dt \ll \\ &\ll k^{n-\lambda}\int\limits_{\delta}^{\infty}\left(1 + \frac{t}{\delta}\right)t^{n-\lambda-1}\,dt = c_\delta k^{n-\lambda} \to 0 \qquad (k \to \infty). \end{aligned}$$

We have proven that

$$\sup_{f \in \mathfrak{M}} \| f - U_k f \|_W \to 0 \qquad (k \to \infty). \qquad (7)$$

Now let a sequence of functions $f_l \in \mathfrak{M}$ be given. If it is compact in $L_p$, therefore from it we can separate a subsequence that we will again denote by $\{f_l\}$ convergent to some function $f \in L_p$. For any fixed k (cf (5))

$$U_k f_l \to U_k f \qquad (l \to \infty)$$

in $L_p$, but then also in W, because for fixed k the functions $U_k f_l$ ($l = 1, 2, \ldots$) are integral and the exponential spherical type $k\lambda$ (cf 3.6.2 and lemma 7.7.3 below).

By (7), for any $\varepsilon > 0$ we can select a k such that

$$\|f_l - U_k f_l\|_W < \varepsilon \qquad (\text{for all } l = 1, 2, \ldots)$$

Consequently, the sequence $\{f_l\}$ exhibits the property that for any $\varepsilon > 0$ we can specify a k such that

$$f_l = U_k f_l + (f_l - U_k f_l),$$

where the first term converges as $l \to \infty$ in the W-sense, and the second, with respect to the norm W, does not exceed $\varepsilon$ for any $l = 1, 2, \ldots$ But then by virtue of the completeness of W

$$f_l \to f \qquad (l \to \infty)$$

in W. The theorem is proven.

7.7.3. Lemma. The inequality

$$|g_\nu|_W \leq \left(1 + \sum_1^n \nu_i^{r_i}\right)|g_\nu|_{L_p}, \tag{1}$$

$$|g_\nu|_B < c\left(1 + \sum_1^n \nu_i^{r_i}\right)|g_\nu|_{L_p}, \tag{2}$$

is obtained in the notation of theorem 7.7.2, where c is a constant not dependent on the series of the standing multiplier and $g$ is an integral function of exponential type $\nu = (\nu_1, \ldots, \nu_n) \geq 0$. In (2) B can be replaced with $H = H_p^r(R_n)$.

Thus, if the sequence $g^l$ of integral functions of the same type tends to some function $g_\nu$ (cf 3.5) in the $L_p$-sense, then it also does so in the sense of W, H, and B.

Proof. Inequality (1) borrows directly from the definition of W and Bernshteyn's inequality 3.2.2(9). The function $g = g_\nu$ is integral and the type $\nu_j$ with respect to $x_j$, and consequently, is also of the type $2^s > 1 + \nu_j$, where s is the smallest natural number for which inequality is satisfied. Let us set $g_{2^0} = g_2 = \ldots = g_{2^{s-1}} = 0, \quad g_{2^s} = g,$

then (cf 5.6.6(6)

$$g = g_{2^0} + \sum_1^s (g_{2^j} - g_{2^{j-1}}),$$

$$\|g\|_{B^{r_j}_{x_j p\theta}} = 2^{sr_j}\|g\|_{L_p} \leqslant 2^{r_j}(1+\nu_j)^{r_j}\|g\|_p \leqslant c(1+\nu_j^{r_j})\|g\|_p$$

$$(j = 1, \ldots, m),$$

from whence it follows (2). Obviously, in these considerations we can replace B with H.

7.7.4. Theorem 7.7.2 remains valid and is proven precisely just as when W in it is replaced by $H = H_p^r(R_n)$ $(r \geqslant 0,\ 1 \leq p < \infty)$, but it is presupposed that for each function $f \in \mathfrak{M}$ the reaction

$$\|f(x+h) - f(x)\|_H \to 0 \qquad (|h| \to 0) \tag{1}$$

obtains (which in general does not hold).

In the case $p = \infty$, it is valid.

7.7.5. Theorem. Let there be given a set $\mathfrak{M} \subset H = H_\infty^r(R_n)$ $(r \geqslant 0)$ of functions f, each of which belongs further to the class $\tilde{C} = \tilde{C}(R_n)$ of functions continuous on $R_n$ and with a finite limit at the point $x = \infty$. Then for each function $f \in \mathfrak{M}$, 7.7.4(1) obviously holds. Let, moreover, $\mathfrak{M}$ be bounded in C.

For $\mathfrak{M}$ to be compact in H, it is necessary and sufficient that the conditions

$$\Lambda(\delta) = \sup_{f \in \mathfrak{M}} \sup_{|h| < \delta} \|f(x+h) - f(x)\|_H \to 0 \qquad (\delta \to 0)$$

be satisfied and that for any $\varepsilon > 0$ we can also find an $N > 0$ such that

$$|f(x)-f(x')|<\varepsilon, \qquad (1)$$

whatever the x and x' satisfying inequalities $|x|$ , $|x'| > N$ for all $f \in \mathfrak{M}$.

The proof of this theorem is also exactly the same as the proof of 7.7.2, if we take note of the fact that the following assertion holds here: For the set $\mathfrak{M} \subset \tilde{C}$ of functions to be compact in $\tilde{C}$, it is necessary and sufficient that it be: 1) bounded, 2) equicontinuous (on $R_n$), and that 3) for any $\varepsilon > 0$ a N be found such that property (1) holds.

This latter assertion can be easily obtained by starting from Arzela's theorem: satisfying for $\mathfrak{M}$ conditions 1) and 2) for an arbitrary sphere $|x| < N$ is necessary and sufficient for the compactness of $\mathfrak{M}$ on this sphere.

## CHAPTER VIII INTEGRAL REPRESENTATIONS AND ISOMORPHISM OF ISOTROPIC CLASSES

### 8.1. Bessel-Macdonald Kernels

The Fourier transform of the function $(1 + |x|^2)^{-r/2}$ for sufficiently large $r > 0$ can be obtained effectively; since it is a function of $|x|$, then to it the familiar formula*)

$$\widehat{(1+|x|^2)^{-r/2}} = \frac{1}{(2\pi)^{n/2}} \int \frac{e^{ix\xi}\, d\xi}{(1+|\xi|^2)^{r/2}} = \frac{1}{|x|^{\frac{n-r}{2}}} \int_0^\infty \frac{\rho^{n/2}}{(1+\rho^2)^{r/2}} I_{\frac{n-2}{2}}(|x|\rho)\, d\rho,$$

where $I_\mu$ is the Bessel function of order $\mu$, is applied.

This integral (Hankel type) is computed, for example, in the book by Titchmarsh**), where we must take $\mu + 1 = r/2$, $\nu + 1 = n/2$, which yields

$$\widehat{(1+|x|^2)^{-r/2}} = \frac{1}{2^{\frac{r-2}{2}}\Gamma\left(\frac{r}{2}\right)} \frac{K_{\frac{n-r}{2}}(|x|)}{|x|^{\frac{n-r}{2}}} = G_r(|x|). \tag{1}$$

$$K_\nu(z) = K_{-\nu}(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \xi^{-\nu-1} e^{-\xi-\frac{z^2}{4\xi}}\, d\xi. \tag{2}$$

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*) Bochner [1], theorem 5.6, page 263.

**) Titchmarsh [1], 7.11.6, page 264, see further Watson [1], section 13.6(2), page 476 and N. Ya. Sonin [1].

Function K  (z) is called the Macdonald function of order $\nu$ or the modified Bessel function of order  .

Asymptotic estimates are familiar or the kernel K (x) is a function of the single variable x. Here we will give them without proof, referring to the book by Watson [1] (below these references will be denoted by the letter B). The following asymptotic equalities hold:

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left(1 + O\left(\frac{1}{x}\right)\right) \qquad (1 < x)$$

(B 7.2.3(1), page 226),

$$K_0(x) = \ln\frac{1}{x} + O(1) \qquad (0 < x < 1)$$

(B 3.7.1 (14), page 95)

$$K_n(x)\frac{1}{2} = \frac{(n-1)!}{\left(\frac{1}{2}x\right)^n} + O\left(\frac{1}{x^{n-1}}\right) \quad (0 < x < 1,\ n \neq 0 \text{ is an integer})$$

(B 3.7.1(15), page 95)

$$K_\nu(x) = \frac{\pi}{2\sin|\nu|\pi\Gamma(-|\nu|+1)}\left(\frac{1}{2}x\right)^{-|\nu|} + O(x^{-|\nu|})$$

($x \to 0$, $\nu$ is a noninteger)

(B 3.7(6), page 92; 3.1 (8), page 51)

For our purposes, it would be quite sufficient to bear in mind that from these estimates it follows that

$$|K_\nu(x)| \leq \frac{ce^{-x}}{x^{1/2}} \qquad (1 < x), \tag{3}$$

$$|K_0(x)| \leq c\left(\ln\frac{1}{x} + 1\right) \qquad (0 < x < 1),$$

$$|K_\nu(x)| \leq \frac{c}{x^{|\nu|}}$$

$(0 < x < 1,\ \nu \neq 0$ is any number),

where c depends on $\nu$, but not on x.

Incidentally, inequalities (3) can easily be obtained directly, by estimating the integral

$$\Phi(\nu, x) = \int_0^\infty \xi^{-\nu-1} e^{-\xi - \frac{x^2}{4\xi}} d\xi. \tag{4}$$

Parameter $\nu = \lambda + i\mu$ can be assumed complex in integral (4). If we consider that

$$|\xi^{-\nu}| = |\xi^{-\lambda}|, \tag{5}$$

then estimates (3) remain valid when $\nu$ in them is replaced by $\lambda$ and for complex $\nu$. Let us note that the integral has only two singularities $\xi = \infty$ and $\xi = 0$, and the integrand is continuous with respect to $(\xi, x, \nu)(\xi>0)$ for any real x and complex $\nu$; moreover, the integral uniformly converges relative to the indicated $x, \nu$ in a fairly small neighborhood of any indicated point $x_0$, $\nu_0$. This shows that $\phi(\nu, x)$ is continuous relative to $\nu$, x. These facts also obtain for the integral formally differentiated with respect to $\nu$. This shows that the function $\phi(\nu, x)$ has the derivative $\partial/\partial\nu\ \phi(\nu, x)$ with respect to $\nu$ and this derivative is continuous with respect to $(\nu, x)$. Thus, $\phi(\nu, x)$ is analytic with respect to $\nu$.

In equality (1) its left side, if it is considered as a generalized function, is meaningful for any complex r. The right side, expressible by means of integral (2), also is meaningful as an ordinary function of (r, x), whatever be the complex number r(Re r > 0) and points $x \in R_n$, $x \neq 0$. Additionally, $G_r(|x|)$ is continuous relative to the indicated (r, x), just as its derivative in r. Thus, it is analytic in r.

It follows from estimates (3) and equality (1) that

$$|G_r(|x|)| < c_r \begin{cases} \dfrac{e^{-|x|}}{|x|^{\frac{n-r+1}{2}}} & (|x| > 1,\ n,\ r \text{ are any numbers}), \\ \ln\dfrac{1}{|x|} + 1 & (|x| < 1,\ n - r = 0), \\ \dfrac{1}{|x|^{n-r}} & (|x| < 1,\ n - r > 0), \\ 1 & (|x| < 1,\ n - r < 0), \end{cases} \tag{6}$$

where $c_r > 0$ is a continuous function of r.

We took r as real in the inequalities. They are valid also by virtue of (5) if in their left members we take r as complex, but in their right substitute everywhere $\lambda$ for $r = \lambda + i\mu$.

It is easy to see from (6) that $G_r(|x|) \in L(R_n) = L$. From the foregoing it follows that equality (1) is actually valid for any complex r if $\operatorname{Re} r = \lambda > 0$. Actually, let $\varphi \in S$, then the function

$$\widetilde{((1+|x|^2)^{-r/2}, \varphi)} = ((1+|x|^2)^{-r/2}, \tilde{\varphi}) = \psi(r)$$

is, as easily verified, an analytic function in r. On the other hand, using estimates (6) it can be directly established that function $G_r(|x|)$ with respect to module does not exceed the summable function*) relative to r and satisfying the inequality $|r - r_0| < \delta$ $(\lambda_0 > \delta > 0)$, and since $G_r(|x|)\varphi(x)$ is continuous from (r, x), $x \neq 0$ and $\varphi$ is bounded, then by the Weierstrass characteristic, the function

$$\psi_1(r) = (G_r(|x|), \varphi(x)) = \int G_r(|x|)\varphi(x)\,dx$$

is a finite continuous function in $r(\lambda > 0)$. By means of estimates (3) and (6), an analogous fact**) is established for the derivative

$$\frac{d}{dr}\psi_1(r) = \left(\frac{d}{dr} G_r(|x|), \varphi(x)\right).$$

This shows that $\psi_1(r)$ is analytic for $\lambda > 0$. Moreover, it is equal to $\psi(r)$ for sufficiently large real r, therefore, also for any complex r with $\lambda > 0$, whatever be the $\varphi \in S$. This is entailed by equality (1). Let us show that the following estimates

$$|D^s G_r(|x|)| \leqslant c \begin{cases} \dfrac{e^{-|x|}}{|x|^{\frac{n-r+1}{2}}} & (|x| > 1,\ n,\ r,\ s - \text{любые}), ^{1} \\ \ln\dfrac{1}{|x|} + 1 & (|x| < 1,\ n - r + |s| = 0,\ |s| - \text{четное}) ^{2} \\ \dfrac{1}{|x|^{n-r+|s|}} & (|x| < 1,\ n - r + |s| > 0 \text{ и } n - r + |s| = 0,\ \text{а } |s| - \text{нечетное}), ^{3} \\ 1 & (|x| < 1,\ n - r + |s| < 0), \end{cases} \tag{7}$$

LEGEND for (7):

1. s is any number
2. s is an even number
3. s is an odd number

* and ** [* and ** on following page]

where c (continuously) depends on n, r, and s, but does not depend on $\mathbf{x}$, obtain for derivatives of $G_r(|\mathbf{x}|)$ of order $\mathbf{s} = (s_1, \ldots, s_n)$.

Notice that it is easily verified by induction that

$$D^s e^{-\frac{|x|^2}{4\xi}} = e^{-\frac{|x|^2}{4\xi}} \sum \frac{A_{k,l} x^k}{\xi^l} \quad (2l - |k| \leqslant |s|;\ |k| \leqslant l \leqslant |s|), \tag{8}$$

where $D^s$ is the operator of differentiation of order $\mathbf{s} = (s_1, \ldots, s_n)$, $\mathbf{x}^k = x_1^{k_1} \ldots x_n^{k_n}$, $\mathbf{k} = (k_1, \ldots, k_n)$ are integral nonnegative vectors, $A_{k,l}$ are constants, and the sum is extended over the pairs $\mathbf{k}$, l satisfying the inequality indicated in the brackets.

Therefore

$$\begin{gathered}|D^s G_r(|x|)| \ll \left| D^s \int_0^\infty \xi^{\frac{n-r}{2}-1} e^{-\xi - \frac{|x|^2}{4\xi}} d\xi \right| \ll \\ \ll \sum \left| x^k \int_0^\infty \xi^{-\frac{n-r+2l}{2}-1} e^{-\xi - \frac{|x|^2}{4\xi}} d\xi \right| \ll \sum |x^k G_{r-2l}(|x|)|,\end{gathered} \tag{9}$$

where the sums are extended over the pairs $\mathbf{k}$, l specified in (8).

If $|\mathbf{x}| > 1$, then by virtue of the first assumption (6)

$$|D^s G_r(|x|)| \ll \sum \frac{|x^k| e^{-|x|}}{|x|^{\frac{n-(r-2l)+1}{2}}} \ll \sum \frac{e^{-|x|}}{|x|^{\frac{n-r+1}{2}+l-|k|}} \ll \frac{e^{-|x|}}{|x|^{\frac{n-r+1}{2}}},$$

because $l - |\mathbf{k}| \geqslant 0$. We have proven the first inequality in (7).

Now suppose $|\mathbf{x}| \leqslant 1$. If, additionally, $n - r + 2l > 0$, then by virtue of the third estimates (6)

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*) Constants $c_r$ in inequality (6) are bounded for the specified r.

**) The analogous anisotropic case is examined in detail in 9.4.

$$|x^k G_{r-2l}(|x|)| \ll \frac{|x^k|}{|x|^{n-r+2l}} =$$

$$= \frac{1}{|x|^{n-r+2l-|k|}} \ll \frac{1}{|x|^{n-r+|s|}}, \tag{10}$$

because $2l - |k| \leq |s|$.

Further, if $n - r + 2l < 0$, then by the fourth estimate (6)

$$|x^k G_{r-2l}(|x|)| \ll |x^k| \ll 1. \tag{11}$$

If however for some $l$ (one) $n - r + 2l = 0$, then by the second estimates (6)

$$|x^k G_{r-2l}(|x|)| \ll |x^k| \ln \frac{1}{|x|}. \tag{12}$$

Further, if $n - r + |s| > 0$, the right member of (10) is larger than the right sides of (11) and (12) ($|k| \geqslant 0$). We have proven the third estimate for $n - r + |s| \geqslant 0$. If however $n - r + |s| = 0$ and $|s|$ is an odd number, then there is no natural $l$ which $n - r + 2l = 0$ and in this case estimate (12) does not emerge, while estimates (10) and (11) yield 1. By this means, the third estimate (7) is completely proven. If however $n - r + |s| = 0$ ( $|s|$ is an even number), then estimate (12) also arises. By this we have proven the second inequality in (7).

Finally, if $n - r + |s| < 0$, then the right sides of (10) and (11) and when $|k|$ 0 are estimated by unity. It remains only to explore the case (12) when $k = 0$, but it is not possible, because from $n - r + |s| < 0 = n - r + 2l$ follows inequality $|s| < 2l$, which contradicts the fact that in addition to this inequality $2l - |k| \leq |s|$, i.e., $2l \leq |s|$, must be satisfied when $|k| = 0$. Thus we have proven the last inequality in (7).

From inequality (7) it is easily seen that $G_r(|x|)$ for any $r > 0$ and any natural $n$ belong to $L(R_n) = L$, therefore for the functions $f \in L_p(R_n) = L_p (1 \leq p \leq \infty)$ the convolution

$$F(x) = \frac{1}{(2\pi)^{n/2}} \int G_r(|x-u|) f(u) du = \widetilde{(1+|u|^2)^{-r/2} \tilde{f}} = I_r f. \tag{13}$$

is meaningful. Here, obviously, $F \in L_p$. In fact, function F exhibits, as we will see, considerably better properties.

## 8.2. Isomorphism of the Classes $W_p^l$

We will state that Banach spaces $E_1$ and $E_2$ are isomorphic if there exists a linear operator A mapping $E_1$ onto $E_2$ mutually uniquely, and two positive constants $c_1$ and $c_2$ not dependent on $x \in E_1$, such that

$$c_1 \|x\|_{E_1} \leqslant \|A(x)\|_{E_2} \leqslant c_2 \|x\|_{E_1} \tag{1}$$

for all $x \in E_1$.

We will state about operator $A^{-1}$ that it executes the isomorphism $E_1$ and $E_2$:

$$A(E_1) = E_2. \tag{2}$$

Then the inverse operator $A^{-1}$ obviously does exist, is linear, and in turn executes the isomorphism

$$A^{-1}(E_2) = E_1.$$

We will prove that the operation $I_l$ for natural $l$ executes the isomorphism

$$I_l(L_p) = W_p^l \tag{3}$$

$$(1 < p < \infty;\ W_p^l = W_p^l(R_n),\ L_p = W_p^0,\ l = 0, 1, \ldots).$$

Suppose $F \in W_p^l$. Then

$$\widehat{(iu_j)^l \tilde{F}} = \frac{\partial^l F}{\partial x_j^l} \in L_p \qquad (j = 1, \ldots, n),$$

and by virtue of the fact that $(i^3 \operatorname{sign} u_j)^l$ is a Marcinkievicz multiplier (cf 1.5.5, example 1, and 1.5.4.1),

$$\|\widehat{|u_j|^l \tilde{F}}\|_p \leqslant c_1 \left\| \frac{\partial^l F}{\partial x_j^l} \right\|_p.$$

Therefore, considering further that $F \in L_p$, we get

$$\left\| \widehat{\left(1 + \sum_{j=1}^n |u_j|^l\right) \tilde{F}} \right\|_p \leqslant c_2 \|F\|_{W_p^l}.$$

But function $$(1+|u|^2)^{l/2}\left(1+\sum_1^n|u_j|\right)^{-l}$$

is a Marcinkiewicz multiplier (1.5.5, example 7), therefore

and $$f=\widetilde{(1+|u|^2)^{l/2}\tilde{F}}\in L_p$$

$$\|f\|_p\leqslant c_3\|F\|_{W_p^l}. \tag{4}$$

Now suppose $f \in L_p$; then $F = \mathfrak{G}_1 f = (1+|\lambda|^2)^{-1/2}\tilde{f}$ and, by (1.5(10)),

$$\widetilde{F^{(k)}}=(i\lambda)^k(1+|\lambda|^2)^{-\frac{l}{2}}\tilde{f}.$$

But when $|k| = 1$, the function

$$(i\lambda)^k(1+|\lambda|^2)^{-l/2}$$

is a Marcinkiewicz multiplier (cf 1.5.5, example 5). Therefore

$$\|F^{(k)}\|_p\leqslant c_1\|f\|_p. \tag{5}$$

But also (8.1(13)) $\|F\|_n \leqslant c_5\|f\|_p$, therefore $F \in W_p^1$ and

$$\|F\|_{W_p^l}\leqslant c\|f\|_p.$$

We have proved that the operation $I_1$ executes isomorphism (3).

In the following it will be shown that it can serve as an artifice for defining an executing isomorphism of other classes of differentiable functions.

## 8.3. Properties of Bessel-Macdonald Kernels

Below it is proven for the Bessel-Macdonald kernel $G_r(|x|)$ when $r > 0$ that the estimate ($s$ is a natural number, $-\infty < h < \infty$)

$$\Lambda=\int\left|\Delta^2_{hx_j}\frac{\partial^s G_r(|x|)}{\partial x_j^s}\right|dx\leqslant M_r|h|^\alpha \tag{1}$$

$(j=1,\ldots,n;\ s=\bar{r},\ r=\bar{r}+\alpha,$ $\bar{r}$ is an integer, $0<\alpha\leq 1)$

obtains.

Since $G_r(|x|) \in L = L(R_n)$ when $r > 0$ then from (1) it follows (cf definition of the classes $H_p^r$ and 5.6.2), that

$$G_r(|x|) \in H_1^r = H_1^r(R_n) \tag{2}$$

and

$$\|G_r(|x|)\|_{H_1^r} = \|G_r(|x|)\|_L + M_r, \tag{3}$$

where $M_r$ is the least constant for which inequalities (1) are satisfied.

Let us set $u = (u_j, u^j)$, $u^j = (u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n)$,

$$g^{(s)}(x) = \frac{\partial^s G_r(|x|)}{\partial x_j^s}, \quad \Delta_h^2 \varphi(t) = \varphi(t+h) - 2\varphi(t) + \varphi(t-h).$$

We will employ the four estimates 8.1(7)(we will denote them by, respectively, 1), 2), 3), and 4)).

By 1) - 3)

$$\Lambda \leqslant 4 \int |g^{(s)}(u)|\, du \ll$$
$$\ll \int_{|u|<1} \left(\ln \frac{1}{|u|} + \frac{1}{|u|^{n-r+s}}\right) du + \int_{|u|>1} e^{-\frac{|u|}{2}}\, du \leqslant c < \infty,$$

because $n - (r - s) = n - \alpha < n$. Therefore, for $|h| \geqslant 1$

$$\Lambda \leqslant c \leqslant c|h|^{\alpha}. \tag{4}$$

We will proceed to the case $|h| < 1$. For definiteness we will assume that $0 < h < 1$. We have

$$\Lambda = \Lambda_1 + \Lambda_2, \tag{5}$$

where $\Lambda_1$ is the same as the $\Lambda$, but now taken not over the entire space, but over the sphere $|u| < 4h$. Then

$$\Lambda_1 \leq 4 \int\limits_{|u|<2h} |g^{(s)}(u)|\,du \ll$$

$$\ll \int\limits_{|u|<2h} \frac{du}{|u|^{n+s-r}} \ll \int\limits_0^{2h} \rho^{\alpha-1}\,d\rho \ll h^{\alpha} \qquad (6)$$

by virtue of estimate (3).

However, there remains the case

$$n - r + s = 0, \quad s \text{ is an even number.}$$

Since $0 < r - s \leq 1$, then this can obtain if and only if $n = 1$, $s = r - 1$ is an even number, i.e., $\alpha = r - s = 1$.

The required estimate then is obtained thusly (the integrals are one-dimensional):

$$\Lambda_1 \leq \int\limits_{|u|<4h} |g^{(s)}(u+h) - g^{(s)}(u)|\,du +$$

$$+ \int\limits_{|u|<4h} |g^{(s)}(u) - g^{(s)}(u-h)|\,du \ll$$

$$\ll 2 \int\limits_{|u|<5h} |g^{(s)}(u+h) - g^{(s)}(u)|\,du =$$

$$= 2\int\limits_{-5h}^{-h} |g^{(s)}(u+h) - g^{(s)}(u)|\,du + 2\int\limits_{-h}^{0} + 2\int\limits_{0}^{5h} = \Lambda_1^{(1)} + \Lambda_1^{(2)} + \Lambda_1^{(3)},$$

where

$$\Lambda_1^{(3)} = 2\int\limits_0^{5h} du \left| \int\limits_u^{u+h} g^{(s+1)}(t)\,dt \right| \ll \int\limits_0^{5h} du \int\limits_u^{u+h} \frac{dt}{t} =$$

$$= \int\limits_0^{5h} \ln\left(1 + \frac{h}{u}\right) du = h \int\limits_0^{5} \ln(1+t)\,dt \ll h,$$

and analogously

$$\Lambda_1^{(1)} \ll h,$$

and (considering that $G_r(|u|)$ is an even function, and because when $s$ $r-1$ is even, function $g^{(s)}(u)$ is also even)

$$\Lambda_1^{(\cdot)} = 2\int_{-h}^{0} |g^{(s)}(u+h) - g^{(s)}(u)|\,du =$$

$$= 2\int_{-h}^{0} |g^{(s)}(u+h) - g^{(s)}(-u)|\,du =$$

$$= 2\int_{0}^{h} |g^{(s)}(u) - g^{(s)}(h-u)|\,du = 2\int_{0}^{h} du \left| \int_{h-u}^{u} g^{(s+1)}(t)\,dt \right| \ll$$

$$\ll \int_{0}^{h} du \left| \int_{h-u}^{u} \frac{dt}{t} \right| = 2\int_{0}^{h} \left| \ln\frac{u}{h-u} \right| du = 2h\int_{0}^{1} \left| \ln\frac{t}{1-t} \right| dt \ll h.$$

By this way we fully proved (6). Let us proceed to the estimate

$$\Lambda_2 = \int_{|u|>4h} |\Delta^2_{h,x_j} g^{(s)}|\,du =$$

$$= \int_{|u|>4h} \left| \int_0^h \int_0^h \frac{\partial^2 g^{(s)}}{\partial x_j^2}(u_j + v + t,\, u^j)\,dv\,dt \right| du =$$

$$= \int_{\substack{|u|>4h \\ u_j>0}} + \int_{\substack{|u|>4h \\ -2h<u_j<0}} + \int_{\substack{|u|>4h \\ h_j<-2h}} = \Lambda_2^{(1)} + \Lambda_2^{(2)} + \Lambda_2^{(3)},$$

where by virtue of 3) (considering that $n + s - r + 2 = n - \alpha + 2 \geq n + 1 > 0$)

$$\Lambda_2^{(1)},\ \Lambda_2^{(3)} \ll h^2 \int_{\substack{|u|<2h \\ u_j>0}} \frac{du}{|u|^{n+s-r+2}} \ll h^2 \int_{2h}^{\infty} \rho^{\alpha-3}\,d\rho \ll h^{\alpha}, \tag{7}$$

and noting that for $|u| > 4h$, $-2h < u_j < 0$, $|u^j| \geq |u| - |u_j| \geq 4h - 2h = 2h$, we get

$$\Lambda_2^{(2)} \ll h^3 \int_{|u^j|>2h} \frac{du^j}{|u^j|^{n+s-r+2}} \ll h^3 \int_{2h}^{\infty} \frac{\rho^{n-2}\,d\rho}{\rho^{n+s-r+2}} \ll$$

$$\ll h^3 \int_{2h}^{\infty} \rho^{\alpha-4}\,d\rho \ll h^{\alpha}. \tag{8}$$

From (4), (6), (7), and (8) follows (1).

## 8.4. Estimate of Best Approximation for $I_r f$

Let the function $f \in L_p = L_p(R_n)$, $r > 0$ and (8.1(13))

$$F = I_r f = \frac{1}{(2\pi)^{n/2}} \int G_r(|u|) f(x-u)\, du. \qquad (1)$$

Further, let $\omega_\nu \in L$, $\lambda_\nu \in L_p$ be arbitrary functions of the exponential spherical type $\nu$. Thus, $\omega_\nu \in S\mathfrak{M}_\nu, \lambda_\nu \in S\mathfrak{M}_\nu$. We set

$$F(x) - \Omega_\nu(x) = \frac{1}{(2\pi)^{n/2}} \int [G_r(u) - \omega_\nu(u)]\, [f(x-u) - \lambda_\nu(x-u)]\, du.$$

Obviously, $\Omega_\nu \in S\mathfrak{M}_{\nu p}$ (cf 3.6.2) and

$$\| F - \Omega_\nu \|_p \leqslant \frac{1}{(2\pi)^{n/2}} \| G_r(|x|) - \omega_\nu(x) \|_L \| f - \lambda_\nu \|_p.$$

Therefore, considering that the function $G_r(|x|) \in H_1^r$(cf 8.3) and that, consequently, its best approximation in the metric L by means of the integral functions of spherical degree $\nu$ or of the order $Q(\nu^{-r})$(cf 5.5.4), we will have

$$E_\nu(F)_p \leqslant \frac{1}{(2\pi)^{n/2}} E_\nu(G_r(|x|))_L\, E_\nu(f)_p = \frac{b_r}{\nu^r} E_\nu(f)_p, \qquad (2)$$

where $E_\nu(\varphi)_p$, $E_\nu(\varphi)_L$ denote the best approximations of $\varphi$ by means of integral functions of the spherical type $\nu$, respectively, in the metrics $L_p$ and L, where the constant $b_r$ does not depend on the series of the standing multiplier.

Now again let $f \in L_p$ and, additionally, let the Fourier transformation $\tilde{f}$ (usually a generalized function) be equal to zero on the sphere $v_\nu$ with its center at the origin of coordinates, with radius $\nu$ (cf 3.2.6(5)):

$$\tilde{f} = 0 \quad \text{on v} \quad v_\nu. \qquad (3)$$

Then (cf 3.2.6(6)), if $0 < \lambda < \nu$, then the convolution of any function $\omega_\lambda \in S\mathfrak{M}_{\nu l}$ with f equals zero:

$$\omega_\lambda * f = \frac{1}{(2\pi)^{n/2}} \int \omega_\lambda(u) f(x-u)\, du = 0,$$

therefore

$$F(x) = I_r f = \frac{1}{(2\pi)^{n/2}} \int [G_r(|u|) - \omega_\lambda(u)] f(x-u)\, du$$

and

$$\| F \|_p \leq \frac{1}{(2\pi)^{n/2}} \| G_r - \omega_\lambda \|_L \| f \|_p.$$

But then, taking the lower bound with respect to $\omega_\lambda$, we get the inequality

$$\| F \|_p \leq \frac{1}{(2\pi)^{n/2}} E_\lambda (G_r)_L \| f \|_p = \frac{b_r \| f \|_p}{\lambda^r},$$

which is valid for any $\lambda < \nu$, therefore

$$\| I_r f \|_p = \| F \|_p \leq \frac{b_r \| f \|_p}{\nu^r} \qquad (\nu > 0,\ (\tilde{f})_{v_\nu} = 0), \tag{4}$$

where $b_r$ is the constant entering into inequality (2). It does not depend on $\nu > 0$ and on the f considered.

## 8.5. Multiplicator Equal to Unity on a Domain

By definition the generalized function $f \in S'$ is equal to unity on the open set $g \subset R_n$ if for any function $\varphi$ finite in g, the relation

$$(f, \varphi) = 0.$$

obtains. If here f does not only belong to S', but also is a function locally summable on g, then almost everywhere

$$f(x) = 0 \quad \text{on } g.$$

Actually, suppose $\sigma \subset g$ is an arbitrary sphere. There exists (cf 1.4.2) a set of functions $\varphi_N$ finite in $\sigma$ for which the bounded convergence

$$\lim_{N \to \infty} \varphi_N(x) = \text{sign } f(x) \text{ almost everywhere on } \sigma$$

obtains. Therefore, by virtue of the Lebesgue theorem

$$0=(f,\varphi_N)=\int_\sigma f(x)\varphi_N(x)\,dx\to\int_\sigma|f(x)|\,dx\qquad(N\to\infty),$$

i.e., $f(x)=0$ on $\sigma$ almost everywhere, and consequently, also on $g$.

If $f_1, f_2 \in S'$ and $f_1 - f_2 = 0$ on the open set $g$, then we can naturally say that $f_1 = f_2$ on $g$.

8.5.1. Lemma. Suppose $\mu$ is a multiplicator in $L_p(1 \leqslant p \leqslant \infty; \mu \in L$ when $p=\infty$; cf 1.5.1, 1.5.1.1) equal to unity on the open set $g \subset R_n$. Then for $f \in L_p$ and the general for the function $f$ that is regular in the $L_p$-sense,

$$\widetilde{K*f} = \tilde{f} = \tilde{f} \quad \text{on } g\ (K = \hat{\mu}). \tag{1}$$

Proof. For $\varphi \in S$ that has a carrier in $g$, and for the infinitely differentiable finite function $f$

$$(\mu\tilde{f},\varphi)=(\mu,\tilde{f}\varphi)=(1,\tilde{f}\varphi)=(\tilde{f},\varphi). \tag{2}$$

Here we must consider that $\mu$(by the definition of a multiplicator) is an ordinary measurable function by the condition of the lemma, equal to unity on $g$, therefore the second term in (2) is a Lebesgue integral; moreover, by the condition of the lemma $\mu(x)=1$ on $g$, and $\tilde{f}\varphi$ has a carrier in $g$, which proves the second equality.

If $f \in L_p$, then we can find a set of infinitely differentiable finite functions $f_l$ such that $f_l \to f$, $\mu\tilde{f}_l \to \mu\tilde{f}$ weakly. Substituting $f_l$ instead of $f$ in (2) and passing to the limit as $l \to \infty$, we again get (2), but now for $f \in L_p$.

If now $f$ is a function that is regular in the $L_p$-sense, then for $\varphi \in S$ with a carrier in $g$, for sufficiently large $\rho$ we get

$$(\widetilde{K*f},\varphi)=(\overline{I_{-\rho}(K*I_\rho f)},\varphi)=(\widetilde{K*I_\rho f},(1+|\lambda|^2)^{\rho/2}\varphi)=$$
$$=(\widetilde{I_\rho f},(1+|\lambda|^2)^{\rho/2}\varphi)=(\tilde{f},\varphi),$$

i.e. (1).

8.5.2. Lemma. Suppose the multiplicator $\mu=\mu_N=1$ on $\Delta_N=\{|x_j|<N;\ j=1,\dots,n\}$. Then if $N'<N$ and the function $\omega_{N'}\in \mathfrak{M}_{N'p}$(integral and of the exponential type $N'$ with respect to all variables, and belonging to $L_p$), then

$$K * \omega_{N'} = \widehat{\mu \tilde{\omega}_{N'}} = \omega_{N'} \qquad (K = \hat{\mu}). \tag{1}$$

Proof. Suppose $\varepsilon > 0$ and $N' + \varepsilon < N$. Since $\psi_\varepsilon$ is of exponential type $\varepsilon$, $\psi_\varepsilon \omega_{N'} \in \mathcal{M}_{N+\varepsilon, p}$. Moreover, $\psi_\varepsilon \omega_{N'} \in S$, because $\psi_\varepsilon \in S$, and $\omega_{N'}$ together with any of its derivatives is bounded ($\omega_{N'}$ is of polynomial group). Therefore

$$(\mu \widetilde{\psi_\varepsilon \omega_{N'}}, \varphi) = (\mu, \widetilde{\psi_\varepsilon \omega_{N'}} \varphi) = (1, \widetilde{\psi_\varepsilon \omega_{N'}} \varphi) = (\overline{\psi_\varepsilon \omega_{N'}}, \varphi),$$

where the second equality obtains because the carrier $\widetilde{\psi_\varepsilon \omega_{N'}} \varphi$ belongs to $\Delta_N$. Consequently,

$$\widehat{\mu \widetilde{\psi_\varepsilon \omega_{N'}}} = \psi_\varepsilon \omega_{N'}. \tag{2}$$

Passing to the limit in (2) in the weak sense as $\varepsilon \to 0$, we get (1). This follows from 1.5.8(6) for right side of (1). As far as left side is concerned, then we must consider that

$$\| \psi_\varepsilon \omega_{N'} - \omega_{N'} \|_p^p = \int | (\psi_\varepsilon(x) - 1) \omega_{N'}(x) |^p \, dx \to 0 \qquad (\varepsilon \to 0)$$

by the Lebesgue theorem, from whence by virtue of the fact that $\mu$ is a multiplicator, the left side of (2) tends to the left side of (1) not only weakly, but even in the $L_p$-sense.

## 8.6. de la Valleé-Poussin Sums of a Regular Function

In the theory of Fourier integrals, the kernel

$$\frac{\sin Nt}{t} = \int_0^N \cos nt \, dn \tag{1}$$

for integral N corresponds to the trigonometric polynomial

$$D_N^*(t) = \frac{1}{2} + \sum_{n=1}^{N} \cos nt = \frac{\sin\left(N + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}} \qquad (N = 0, 1, \ldots), \tag{2}$$

is called a Dirichlet kernel of order N.

The arithmetic mean

$$v_N^* = v_N^*(t) = \frac{D_{N+1}^* + \dots + D_{2N}^*}{N} =$$
$$= \frac{1}{2} + \sum_0^N \cos kt + \frac{1}{N}\sum_{N+1}^{2N}(2N+1-k)\cos kt =$$
$$= \frac{\cos(N+1)t - \cos(2N+1)t}{4N\sin^2\frac{t}{2}} \qquad (3)$$

is called the de la Valleė-Poussin kernel*). We will state that it is of order N.

Important properties of the de la Valleė-Poussin kernel are as follows:

1*) $*_N$ is an even trigonometric polynomial of order 2N;

2*) The Fourier coefficients $*_N$ with indexes $k = 0, 1, \dots, N$ are equal to unity;

$$3^*)\ \frac{1}{\pi}\int_{-\pi}^{\pi} v_N^*(t)\,dt = 1;$$

$$4^*)\ \frac{1}{\pi}\int_{-\pi}^{\pi} |v_N^*(t)|\,dt = \frac{1}{2N\pi}\int_0^{\pi}\frac{|\cos(N+1)t - \cos(2N+1)t|}{\sin^2\frac{t}{2}}\,dt \leqslant$$
$$\leqslant \frac{\pi}{N}\int_0^{\pi}\frac{\left|\sin\frac{N}{2}t\sin\left(\frac{3N}{2}+1\right)t\right|}{t^2}\,dt \leqslant$$
$$\leqslant \frac{\pi}{N}\int_0^{\pi}\frac{\left|\sin\frac{N}{2}t\sin\frac{3N}{2}t\right|dt}{t^2} + \frac{\pi}{N}\int_0^{\pi}\frac{\left|\sin\frac{N}{2}t\right|}{t}\,dt \leqslant$$
$$\leqslant \pi\int_0^{N\pi}\frac{\left|\sin\frac{u}{2}\sin\frac{3}{2}u\right|}{u^2}\,du + \frac{\pi^2}{2} < A < \infty,$$

where A does not depend on $N \geq 1$.

*) de la Valleė-Poussin [1].

Below we will consider the corresponding analog of the de la Valleé-Poussin kernel for the case of Fourier integral in the n-dimensional case.

Let us begin with considering an ordinary measurable function $g(x)$ bounded on $R = R_n$, such that its Fourier transform $\tilde{g}$ in turn in an ordinary bounded function. Suppose further $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a vector parameter that will vary on the rectangle $\Omega_a = \{a < \lambda_j < 2a;\ j = 1, \ldots, n\}$, where $a > 0$. The equality

$$\overline{\int_{\Omega_a} g(\lambda_1 x_1, \ldots, \lambda_n x_n)\, d\lambda} = \int_{\Omega_a} \overline{g(\lambda_1 x_1, \ldots, \lambda_n x_n)}\, d\lambda. \tag{4}$$

obtains. In fact, if $\varphi \in S$, then

$$\left(\overline{\int_{\Omega_a} g(\lambda_1 x_1, \ldots, \lambda_n x_n) d\lambda}, \varphi\right) = \int_{\Omega_a} d\lambda \int \overline{g(\lambda_1 x_1, \ldots, \lambda_n x_n)}\, \varphi(x)\, dx =$$

$$= \int \left(\int_{\Omega_a} \overline{g(\lambda_1 x_1, \ldots, \lambda_n x_n)}\, d\lambda\right) \varphi(x)\, dx =$$

$$= \left(\int_{\Omega_a} \overline{g(\lambda_1 x_1, \ldots, \lambda_n x_n)}\, d\lambda,\ \varphi(x)\right).$$

All the inequalities here are obvious, and explanation is required only for the fact that $\tilde{g}(\lambda_1 x_1, \ldots, \lambda_n x_n)$ is when $\lambda \in \Omega_a$, an ordinary bounded function. But this follows from the equality

$$\overline{g(\lambda_1 x_1, \ldots, \lambda_n x_n)} = \frac{1}{(2\pi)^{n/2}} \int g(\lambda_1 u_1, \ldots, \lambda_n u_n)\, e^{-ixu}\, du =$$

$$= \frac{1}{\prod_1^n \lambda_j\, (2\pi)^{n/2}} \int g(u)\, e^{-i \sum_{j=1}^{n} \frac{x_j u_j}{\lambda_j}}\, du$$

and the assumptions that $\tilde{g}$ is an ordinary bounded measurable function.

The analog of the de la Valleé-Poussin kernel is defined by means of an equality analogous to (3):

$$V_N(t) = \frac{1}{N^n} \int\limits_{\Omega_N} \prod_{j=1}^{n} \frac{\sin \lambda_j t_j}{t_j} d\lambda =$$

$$= \frac{1}{N^n} \prod_{j=1}^{n} \int\limits_{N}^{2N} \frac{\sin \nu t_j}{t_j} d\nu = \frac{1}{N^n} \prod_{j=1}^{n} \frac{\cos N t_j - \cos 2N t_j}{t_j^2}. \tag{5}$$

The kernel $V_N$ satisfies properties analogous to properties 1*)-4*):

1) $V_N(z)$ is an integral function of the exponential type of degree 2N with respect to each of the variables $z_j$ ($j = 1, \ldots, n$), and is bounded and summable on R;

2) $$\left(\frac{2}{\pi}\right)^{n/2} \tilde{V}_N = \frac{1}{\pi^n} \int V_N(t) e^{-ixt} dt = 1 \quad \text{on} \quad N, \tag{6}$$

$$\Delta_N = \{|x_j| \leqslant N;\ j = 1, \ldots, n\},$$

3) $$\frac{1}{\pi^n} \int V_N(t) dt = 1, \tag{7}$$

4) $$\frac{1}{\pi^n} \int |V_N(t)| dt \leqslant M \qquad (N \geqslant 1). \tag{8}$$

Property 1) is established without difficulty. Property 3) follows from the equality

$$\frac{1}{\pi} \int \frac{\sin \nu t}{t} dt = 1 \qquad (\nu > 0),$$

where the improper Riemann integral converges uniformly relative to $\nu \in [N, 2N]$, owing to which the integration of this integral with respect to parameter can be validly carried out under the sign of the integral

$$\frac{1}{\pi^n} \int V_N(t) dt = \frac{1}{(\pi N)^n} \prod_{j=1}^{n} \int dt_j \int\limits_{N}^{2N} \frac{\sin \nu t_j}{t_j} d\nu =$$

$$= \frac{1}{N^n} \left( \int\limits_{N}^{2N} d\nu \frac{1}{\pi} \int \frac{\sin \nu t}{t} dt \right)^n = 1.$$

Property 4) is obvious:

$$\frac{1}{N}\int_{-\infty}^{\infty}\frac{|\cos Nt-\cos 2Nt|}{t^2}\,dt=\frac{2}{N}\int_{0}^{\infty}\frac{\left|\sin\frac{N}{2}t\sin\frac{3}{2}Nt\right|}{t^2}\,dt=$$
$$=2\int_{0}^{\infty}\frac{\left|\sin\frac{u}{2}\sin\frac{3}{2}u\right|}{u^2}\,du<\infty.$$

Let us consider the function

$$D_\lambda(t)=\prod_{j=1}^{n}\frac{\sin\lambda_j t_j}{t_j},$$

which is an analog of the Dirichlet kernel in the n-dimensional case. Its Fourier transform (cf 1.5.7(10)) is

$$\widetilde{D_\lambda(t)}=\widetilde{\prod_{j=1}^{n}\frac{\sin\lambda_j t_j}{t_j}}=\left(\sqrt{\frac{\pi}{2}}\right)^n(1)_{\Delta_\lambda},$$

where $(1)_{\Delta_\lambda}$ is a function equal to unity on $\Delta_\lambda=\{|x_j|<\lambda_j;\ j=1,\ldots,n\}$ and equal to zero outside of $\Delta_\lambda$. Thus, it is bounded together with its Fourier transform, so equality (4) when $a=N$ can be applied to it, consequently, noting that

$$(1)_{\Delta_\lambda}(x)=\prod_{j=1}^{n}(1)_{\lambda_j}(x_j),$$

where $(1)_{\lambda_j}$ is a function of the single variable $x_j$ equal to unity on the interval $|x_j|<\lambda_j$ and equal to zero for the remaining $x_j$, we get

$$\hat{V}_N=\frac{1}{N^n}\int_{\Omega_N}\widetilde{\prod_{j=1}^{n}\frac{\sin\lambda_j t_j}{t_j}}\,d\lambda=\left(\frac{1}{N}\sqrt{\frac{\pi}{2}}\right)^n\int_{\Omega_N}(1)_{\Delta_\lambda}(x)\,d\lambda=$$
$$=\prod_{j=1}^{n}\frac{1}{N}\sqrt{\frac{\pi}{2}}\int_{N}^{2N}(1)_{\lambda_j}(x_j)\,d\lambda_j=\prod_{j=1}^{n}\mu(x_j), \qquad (9)$$

where

$$\mu(\xi) = \sqrt{\frac{\pi}{2}} \left| \begin{array}{ll} 1 & (|\xi| < N), \\ \frac{1}{N}(2N - \xi) & (N < |\xi| \leqslant 2N), \\ 0 & (2N < |x|). \end{array} \right. \tag{10}$$

We have obtained an effective formula for $\tilde{V}_N$. From it (6) follows directly.

Suppose $f \in L_p (1 \leq p \leq \infty)$. Then

$$\sigma_N(f, x) = \left(\frac{2}{\pi}\right)^{n/2} (V_N \dot{*} f) = \frac{1}{\pi^n} \int V_N(x-u) f(u)\, du \tag{11}$$

is a function belonging to $L_p$, differing only by its constant multiplier from the convolution $V_N * f$. This function is an analog of the periodic de la Vallee'-Poussin sum of order N. Since $V_N \in \mathcal{M}_{2N}$ (integral function of exponential type 2N with respect to all $x_j$ belonging to L), therefore $\sigma_N(f, x) \in \mathcal{M}_{2N,p}$ (cf 3.6.2) for all $f \in L_p$. Moreover, if $\omega_N \in \mathcal{M}_{N,p}$, then the identity

$$\sigma_N(\omega_N, x) = \omega_N(x). \tag{12}$$

obtains.

In fact, $V_N \in L$; therefore, $\tilde{V}_N$ is a multiplicator. Additionally, by virtue of (9) and (10) $\tilde{V}_N = (\pi/2)^n$ on $\Delta_N$; therefore, by lemma 8.5.2,

$$\sigma_{N_0}(\omega_N, x) = \omega_N(x) \quad (N < N_0).$$

(12) follows from this equality as $N_0 \to N$. The validity of the passage to the limit can easily be established by considering the effective formula (5) for $V_N$.

If $f \in L_p$ and $\omega_N \in L_p$ is an integral function of the exponential type N, then by (12) $\sigma_N(f, x) - f(x) = \sigma_N(f - \omega_N, x) + \omega_N(x) - f(x)$, from whence

$$\|\sigma_N(f, x) - f(x)\|_p \leqslant (1 + M) E_N(f)_p, \tag{13}$$

i.e., the approximation of f by means of $\sigma_N(f)$ is of the order of the best approximation of f by means of functions of exponential type N.

If p is finite, then the right side of (13) tends to zero as $N \to \infty$ (cf 5.5.1); whence it follows that

$$\sigma_N(f) \to f \quad (N \to \infty) \qquad \text{weakly} \tag{14}$$

When $p = \infty$, the quantity $E_N(f)$ no longer tends to zero, but property (14) still obtains. In fact, based on 8.3(1)$(0 < \alpha \leq 1)$

$$\int |\Delta^2_{hx_j} G_\alpha(|u|)| du \leqslant M |h|^\alpha.$$

Therefore

$$\left|\Delta^2_{hx_j} \int G_\alpha(|x-u|) f(u) du\right| =$$
$$= \int |\Delta^2_{hx_j} G_\alpha(u) f(x-u) du| \leqslant \|f\|_{L_\infty} M |h|^\alpha \quad (j = 1, \ldots, n),$$
$$\|f\|_{L_\infty} = \operatorname*{supvrai}_{x \in R} |f(x)|.$$

We seek that the function $F(x) = I(f)$ satisfies the condition

$$|\Delta^2_{hx_j} F(x)| \leqslant c |h|^\alpha \qquad (j = 1, \ldots, n),$$

and since it, moreover, is bounded, then it belong to $H^\alpha_\infty(R)$ and, therefore, is uniformly continuous on R, i.e., belongs to C.

But then $\quad E_N(I_\alpha f)_\infty = E_N(F)_\infty \to 0 \qquad (N \to \infty).$

This shows that

$$\sigma_N(F) \to F \qquad \text{weakly}$$

therefore

$$(\sigma_N(f), \varphi) = \left(\frac{2}{\pi}\right)^{n/2} (V_N * f, \varphi) = \left(\frac{2}{\pi}\right)^{n/2} (I_{-\alpha}(V_N * I_\alpha f), \varphi) =$$
$$= (\sigma_N(I_\alpha f), \overset{\bullet}{I}_{-\alpha}\varphi) \to (I_\alpha f, \overset{\bullet}{I}_{-\alpha}\varphi) = (f, \varphi), \qquad N \to \infty, \tag{15}$$

and we have proven (14) also for the case $p = \infty$.

Thus, (14) does obtain for any $f \in L_p$ and any $p$ $(1 \leq p \leq \infty)$. It is important to note that this property is preserved for any function f regular in the $L_p$-sense. In order to be convinced of this, it is sufficient to perform the manipulation described above (15) on f.

Finally, let us note the following inequalities ($f \in L_p$) that are important to us:

$$\| I_r[\sigma_N(f) - \sigma_{2N}(f)] \|_p \leq \gamma_r N^{-r} \| \sigma_N(f) - \sigma_{2N}(f) \|_p, \tag{16}$$
$$\| I_r \sigma_1(f) \|_p \leq \gamma_r \| \sigma_1(f) \|_p, \tag{17}$$

where r is any real number and $\gamma_r$ does not depend on N and f.

When $r > 0$, inequality (17) follows from the fact that operation $I_r$ as a kernel belonging to L(cf 8.1(13) and 1.5.1(5)), and inequality (16) -- from the fact that (cf 8.5.1 and 8.6(6))

$$\widetilde{\sigma_N}(f) - \widetilde{\sigma_{2N}}(f) = 0 \quad \text{on} \quad \Delta_N.$$

But when r is negative, inequality (16) and (17) derive from the inequality which will be shown in the next section, if we consider that

$$\sigma_N(f) - \sigma_{2N}(f) \in \mathfrak{M}_{4N, p}.$$

It will be proven in 8.8 that inequalities (16) and (17) remain in effect for any (generalized) function regular in the $L_p$-sense.

## 8.7. Inequality for the Operation $I_{-r}$ $(r > 0)$ on Functions of the Exponential Type

Let $g = g_\nu \in \mathfrak{M}_{\nu p}(R_n) = \mathfrak{M}_{\nu p}$, i.e., g is a function of the exponential type $\nu$ with respect to each variable $x_j$ belonging to $L_p = L_p(R_n)$. Let us apply to it the operation (cf 1.5.9)

$$I_{-r}g = \overbrace{(1+|x|^2)^{r/2}\tilde{g}}. \tag{1}$$

The main goal of this section is to show that the inequality

$$\|I_{-r}g\|_{L_p(R_n)} \leqslant \varkappa_r (1+\nu)^r \|g\|_{L_p(R_n)} \tag{2}$$
$$(r,\ \nu > 0,\quad 1 \leqslant p \leqslant \infty),$$

where $\mathcal{H}_r$ is a constant not dependent on $\nu$, obtains.

Let us set $\omega(x) = (1 + |x|^2)^{r/2}$ and for any $\nu > 0$ let us introduce the function $\omega_\nu(x)$ with period $2\nu$ (with respect to each variable $x_j$) defined by the equality

$$\omega_\nu(x) = (1 + |x|^2)^{r/2} \quad \{|x_j| \leqslant \nu,\ j = 1, \ldots, n\}. \tag{3}$$

Suppose

$$\omega_\nu(x) = \sum_k c_k^\nu e^{i\frac{\pi}{\nu}kx} \tag{4}$$

is its Fourier series. We show that for $r > r_0$, where $r_0$ is sufficiently large the inequality

$$\sum_k |c_k^\nu| \leqslant \varkappa_r (1+\nu^2)^{r/2}, \tag{5}$$

obtains, where $\mathcal{H}_r$ does not depend on $\nu$, from whence by theorem 3.2.1 follows the validity of the interpolation formula

$$I_{-r}g = \sum c_k^\nu g\left(x + \frac{k\pi}{\nu}\right), \tag{6}$$

from which directly follows inequality (2):

$$\|I_{-r}g\|_p = \sum_k |c_k^\nu| \|g\|_p \leqslant \varkappa_r (1+\nu^2)^{r/2} \|g\|_p \tag{7}$$

and the fact that $I_{-r}g$ is an integral function of the exponential type $\nu$ (cf 3.5).

For small r, the considerations associated with the estimate of the sum $\sum |c_k^\nu|$ become more involved. From the fact that inequality (2) is valid for large r, we derive from general consideration that it is valid with the corresponding constant $\mathcal{K}_r$ and for any $r > 0$.

Let us limit ourselves to considering the two-dimensional case. When $n \geqslant 2$, the argument is more complex, but is analogous.

We have $\left(\sum_k{}' = \sum_{k \neq 0}\right)$

$$\sum |c_{kl}^\nu| = |c_{00}^\nu| + \sum_k{}' |c_{k0}^\nu| + \sum_l{}' |c_{0l}^\nu| + \sum_k{}' \sum_l{}' |c_{kl}^\nu| =$$
$$= J_0 + J_1 + J_2 + J_3,$$

$$J_0 = \frac{1}{\nu^2} \int_{-\nu}^{\nu} \int_{-\nu}^{\nu} (1 + u^2 + v^2)^{r/2} du\, dv \leqslant (1 + 2\nu^2)^{r/2} \leqslant c_1 (1 + \nu^2)^{r/2}.$$

$$J_1 = \frac{1}{\nu^2} \sum_k{}' \left| \int_{-\nu}^{\nu} \int_{-\nu}^{\nu} \omega(u, v) e^{-\frac{ik\pi}{\nu} u} du\, dv \right| \leqslant$$
$$\leqslant \max_{|v| \leqslant \nu} \frac{2}{\nu} \sum_k{}' \left| \int_{-\nu}^{\nu} \omega(u, v) e^{-\frac{ik\pi}{\nu} u} du \right| =$$
$$= \max_v c \sum_k{}' \frac{1}{k} \left| \int_{-\nu}^{\nu} \frac{\partial \omega}{\partial u} e^{-\frac{ik\pi u}{\nu}} du \right| \leqslant$$
$$\leqslant \max_v c_2 \left\{ \sum_k{}' \left| \int_{-\nu}^{\nu} \frac{\partial \omega}{\partial u} e^{-\frac{ik\pi u}{\nu}} du \right|^2 \right\}^{1/2} \leqslant$$
$$\leqslant \max_v c_3 \left\{ \nu \int_{-\nu}^{\nu} \left( \frac{\partial \omega}{\partial u} \right)^2 du \right\}^{1/2} \leqslant$$
$$\leqslant \max_v c_4 \left\{ \nu \int_{0}^{\nu} (1 + u^2 + v^2)^{r-2} u^2 du \right\}^{1/2} \leqslant c_5 (1 + \nu^2)^{r/2} \quad (r \geqslant 2).$$

Here we employ integration by parts and Parseval's inequality in the variable u. Similarly

$$J_2 \leqslant c_5(1+\nu)^{r/2} \quad (r \geqslant 2).$$

Finally, application of integration by parts and Parseval's equality for both variables u and v yields

$$J_3 = \sum_k{}' \sum_l{}' \frac{1}{kl\pi^2} \left| \int_{-\nu}^{\nu} \int_{-\nu}^{\nu} \frac{\partial^2 \omega}{\partial u \, \partial v} e^{-\frac{i\pi}{\nu}(ku+lv)} \, du \, dv \right| \leqslant$$

$$\leqslant c_6 \left\{ \sum{}' \sum{}' \left| \int_{-\nu}^{\nu} \int_{-\nu}^{\nu} \frac{\partial^2 \omega}{\partial u \, \partial v} e^{-\frac{i\pi}{\nu}(ku+lv)} \, du \, dv \right|^2 \right\}^{1/2} \leqslant$$

$$\leqslant c_7 \left\{ \nu^2 \int_0^{\nu} \int_0^{\nu} u^2 v^2 (1+u^2+v^2)^{r-4} \, du \, dv \right\}^{1/2} \leqslant c_8 (1+\nu^2)^{r/2}$$

$$(r \geqslant 4).$$

We have proven (5) when $r \geqslant 4$.

Now let r be an arbitrary positive number and as before $g \in \mathfrak{M}_{\nu p}$. Let us select the natural s such that

$$2^{s-1} < 1+\nu \leqslant 2^s,$$

and let us represent g in the form (cf 8.6(11), (12))

$$g(x) = \sigma_{2^s}(g, x) = \sum_{j=0}^{s} q_j,$$

where

$$q_0 = \sigma_{2^0}(g, x), \quad q_j = \sigma_{2^j}(g, x) - \sigma_{2^{j-1}}(g, x) \quad (j = 1, \ldots, s).$$

Suppose the number $\rho > r$ is sufficiently large that for it inequality (2) is satisfied. Then we have

$$I_{-r} g = \sum_0^s J_{\rho - r} J_{-\rho} q_j$$

and (explanations below)

$$\|I_{-r}g\|_p \ll \sum_{j=0}^{s} \frac{1}{2^{(\rho-r)j}} \|I_{-\rho}q_j\|_p \ll \sum_{j=0}^{s} \frac{1}{2^{(\rho-r)j}} 2^{\rho j} \ll$$
$$\ll \sum_{j=0}^{s} 2^{rj} \ll 2^{sj} \ll (1+\nu)^r. \tag{8}$$

The first relation in this chain obtains on the basis of the already established inequalities 8.6(16), (17) ($\rho - r > 0$). The second relation follows from the fact that $\rho$ is such a number that inequality (2) is valid for it when $r = \rho$.

By this, inequality (2) is proven for any r. Of course, these considerations give a crude constant $\varkappa_r$. But cases are known when its exact (least) value can be obtained. Thus it is considered as an example the case n = 1.

Owing to the evenness of $\omega(t) = (1+t^2)^{r/2}$ $\left(\theta_j = \left(j+\frac{1}{2}\right)\frac{\nu}{k}\right)$

$$c_{-k}^{\nu} = c_k^{\nu} = \frac{1}{\nu}\int_0^{\nu} \omega(t)\cos\frac{k\pi}{\nu}t\,dt =$$
$$= \frac{1}{\nu}\sum_{j=0}^{k-1} \int_{\theta_j-\frac{\nu}{2k}}^{\theta_j+\frac{\nu}{2k}} \omega(t)\cos\frac{k\pi}{\nu}t\,dt =$$
$$= \frac{1}{\nu}\sum_{0}^{k-1}(-1)^{j-1}\int_0^{\frac{\nu}{2k}} [\omega(\theta_j+t)-\omega(\theta_j-t)]\sin\frac{k\pi}{\nu}t\,dt. \tag{9}$$

If $r \geqslant 1$, then computations show that

$$\omega''(t) > 0$$

and therefore the difference appearing in the square brackets under the integral in the right side of (9) monotone-increases with j. Hence it follows that the terms in the sum in the right side of (9) increase in absolute magnitude, successively changing sign, and the sign of $c_k$ coincides with the sign of the last term in the sum corresponding to j = k - 1. By this we have proven that

$$(-1)^k c_k^\nu \geqslant 0 \qquad (k = 0, \pm 1, \pm 2, \ldots; r \geqslant 1). \tag{10}$$

From the evenness of $\omega_\nu(t)$ it follows that

$$\omega_\nu(t) = c_0^\nu + 2 \sum_1^\infty c_k^\nu \cos \frac{k\pi}{\nu} t,$$

therefore by (10) the remarkable equality*) holds.

$$(1 + \nu^2)^{r/2} = \omega_\nu(\nu) = c_0^\nu + 2 \sum_1^\infty (-1)^k c_k^\nu = \sum_{-\infty}^{\infty} |c_k^\nu|.$$

So we have proven that for $r \geqslant 1$ and $n = 1$, we can take $\mathcal{K}_r = 1$ in inequality (7). This constant is unimprovable*) in this form, but we will not dwell here on proving this.

## 8.8. Expansion of a Regular Function in Series by de la Valleé-Poussin Sums

If f is a generalized function regular in the $L_p$-sense, then naturally we assume (cf 1.5.10)

$$\sigma_N(f) = \left(\frac{2}{\pi}\right)^{n/2} (V_N * f) = \left(\frac{2}{\pi}\right)^{n/2} I_{-\rho}(V_N * I_\rho f), \tag{1}$$

where $\rho > 0$ is sufficiently large that $I_\rho f \in L_p$. But $V_N \in L$, therefore $V_N * I_\rho f$ belongs to $L_p$, and by virtue of the fact that the function $V_N$ is of exponential type 2N (cf 3.6.2), $V_N * I_\rho f \in \mathfrak{M}_{2N,p}$. Applying operation $I_-$ (cf 8.7) to this last function does not remove it from the class $\mathfrak{M}_{2N,p}$.

Thus,

$$\sigma_N(f) \in \mathfrak{M}_{2N,p}, \tag{2}$$

whatever be the function f that is regular (in the $L_p$-sense).

Further, for any real

$$I_\lambda \sigma_N(f) = \sigma_N(I_\lambda f) \tag{3}$$

*) P. I. Lizorkin [8].

(cf 1.5.10(5)). Since for the regular function f, $\sigma_N(f) \in \mathfrak{M}_{2N,p}$, then when $r > 0$ obviously inequalities 8.6(16) and (17) hold for it. When $r < 0$, these inequalities also obtain for any regular function f, because for it $\sigma_N(f) - \sigma_{2N}(f) \in L_p$ and $\tilde{\sigma}_N(f) - \tilde{\sigma}_{2N}(f) = 0$ on $\Delta_N$ (cf 8.5.1 and 8.6(6)).

It is convenient for us to associate with each regular function the following series:

$$f = \sigma_{2^0}(f) + \sum_{k=1}^{\infty} [\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f)], \tag{4}$$

weakly convergent, as we will explain, to f. We will further call this series the expansion of the regular function f in sums of the de la Valleė-Poussin type.

For any real r, it is legitimate to apply to it, memberwise, the operation

$$I_r f = I_r \sigma_{2^0}(f) + \sum_{k=1}^{\infty} I_r [\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f)] =$$
$$= \sigma_{2^0}(I_r f) + \sum_{k=1}^{\infty} [\sigma_{2^k}(I_r f) - \sigma_{2^{k-1}}(I_r f)], \tag{5}$$

because if f is a regular function, then $I_r f$ also is, and therefore $I_r f$ can be expanded in the form of its de la Valleė-Poussin series weakly convergent to it -- the second series in (5). The terms of the first and second series are correspondingly equal by virtue of (3).

## 8.9. Representation of Functions of the Classes $B^r_{p\theta}$ in Terms of de la Valleė-Poussin Series. Zero Classes $(1 \leq p \leq \infty)$

We have assumed that $r > 0$, $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$, and $B^r_{p\theta}(R_n) = B^r_{p\theta}$ $(B^r_{p\infty} = H^r_p)$. Let us proceed from the following definition of the class $B^r_{p\theta}$ (5.6(5)): the function f belongs to $B^r_{p\theta}$ if for it the norm

$$\|f\| = \|f\|_{B^r_{p\theta}} = \|f\|_p + \left(\sum_{s=0}^{\infty} 2^{s\theta r} E_{2^s}(f)^\theta_p\right)^{1/\theta}. \tag{1}$$

is finite. Another definition (5.6(6)) equivalent to it is as follows: $f \in B^r_{p\theta}$, if is possible to represent f in the form of the following series, convergent to f in the $L_p$-sense:

$$f = \sum_{s=0}^{\infty} Q_s$$

of functions that are integral and of the exponential type of spherical degree $2^s$ (on $R_n$) such that the norm

$$^{6}\|f\| = \|f\|_{B^r_{p\theta}} = \left(\sum_{s=0}^{\infty} 2^{s\theta r} \|Q_s\|_p^\theta\right)^{1/\theta} \tag{2}$$

is finite.

Let us show that the following is also equivalent to these definitions:

Function $f \in B^r_{p\theta}$ if f is a (generalized) function regular in the $L_p$-sense and if to it the corresponding de la Valleè-Poussin series (convergent to it weakly) corresponding to it

$$f = \sum_{s=0}^{\infty} q_s, \tag{3}$$

$$q_0 = \sigma_{2^0}(f), \quad q_s = \sigma_{2^s}(f) - \sigma_{2^{s-1}}(f) \qquad (s = 1, 2, \ldots), \tag{4}$$

is such that

$$^{7}\|f\| = \|f\|_{B^r_{p\theta}} = \left(\sum_{s=0}^{\infty} 2^{sr\theta} \|q_s\|_p^\theta\right)^{1/\theta} < \infty. \tag{5}$$

In fact, let f be a regular function in the $L_p$-sense for which (3) - (5) hold. Then $q_s$ is an integral function of the exponential type $2^{s+1}$ in all variables, but then of the exponential spherical type $\sqrt{n}2^{s+1}$ (cf 3.2.6) and, consequently, of the type $2^{s+1}$, where we have assumed that l is a natural number such that $2\sqrt{n} \leq 2^l$. Setting $q_s^* = 0$ ($s = 0, 1, \ldots, l$) and $q_{s+1}^* = q_s$ ($s = 0, 1, \ldots$), we find that $f = \sum_{s=0}^{\infty} q_s^*$, $q_s^*$ is of the spherical type $2^s$ and

$$\left(\sum_{s=0}^{\infty} 2^{sr\theta}\|q_s\|_p^{\theta}\right)^{1/\theta} < \infty,$$

i.e., ${}^6\|f\| < \infty$ and $f \in B^r_{p\theta}$.

Now let ${}^5\|f\| < \infty$, then $f \in L_p$ and (cf 8.6(13))

$$\begin{aligned} \|q_s\|_p \leqslant \|\sigma_{2^s}(f) - f\|_p + \|\sigma_{2^{s-1}}(f) - f\|_p \leqslant \\ \leqslant 2ME_{2^{s-1}}(f)_p \quad (s = 1, 2, \ldots), \\ \|q_0\| \leqslant M\|f\|_p. \end{aligned} \tag{6}$$

Therefore ${}^7\|f\| \ll {}^5\|f\|$, and we have proven the equivalence of the norms (5) with the norms (1) and (2).

Note. The equivalency is preserved if the de la Valleė-Poussin sums $\sigma_{2^s}(f)$ are replaced, correspondingly, by the Dirichlet sums $S_{2^s}(f)$ (cf further 8.10), however, given the condition that $1 < p < \infty$. When $p = 1, \infty$, the constant M in the inequality (6) depends on s and thereby is not bounded as $s \to \infty$.

Let us introduce the zero class $B^0_{p\theta}$ of, usually, generalized functions.

By the definition $f \in B^0_{p\theta}$, if f is regular in the $L_p$-sense and if its de la Valleė-Poussin series (3) is such that

$$\|f\|_{B^0_{p\theta}} = \left(\sum_{s=0}^{\infty} \|q_s\|_p^{\theta}\right)^{1/\theta} < \infty. \tag{7}$$

In particular

$$\|f\|_{H^0_p} = \sup_s \|q_s\|_p < \infty. \tag{8}$$

The definition (7) and (8) of zero classes given here have the advantage that they do not depend on the definitions of the corresponding classes for positive r values. But the following definition of $B^0_{p\theta}$ is also possible: this is a class of functions f of the form $I_{-r}\varphi$, where $\varphi \in B^r_{p\theta}$, and $r > 0$ is any number. Let us note further that the apparatus by means of which the original definitions of the H- and B-classes will be given for positive r evidentally is not adapted for immediate generalizations to the case $r \leqslant 0$.

8.9.1. Isomorphism of the classes $B^r_{p\theta}$ for different $r$. Theorem. The operation $I_r (r > 0)$ executes the isomorphism

$$I_r(B^0_{p\theta}) = B^r_{p\theta} \qquad (1 \leqslant p,\ \theta \leqslant \infty,\ B^r_{p\infty} = H^r_p). \tag{1}$$

Equality (1) gives the representation of functions of the class $B^r_{p\theta}$ in term of the convolution of the Bessel-Macdonald kernel $G_r$ with functions of the class $B^0_{p\theta}$ that are, generally speaking, generalized.

Proof. Suppose $f \in B^0_{p\theta}$, then $f$ is a function regular in the $L_p$-sense and expandable in the series

$$f = \sum_{s=0}^{\infty} q_s, \quad q_0 = \sigma_{2^0}(f), \quad q_s = \sigma_{2^s}(f) - \sigma_{2^{s-1}}(f) \quad (s = 1, 2, \ldots), \tag{1}$$

where

$$\|f\|_{B^0_{p\theta}} = \left(\sum_{s=0}^{\infty} \|q_s\|^\theta\right)^{1/\theta} < \infty. \tag{2}$$

But

$$F = I_r f$$

is also a regular function that can be expanded in the series

$$F = \sum_{s=0}^{\infty} Q_s,$$
$$Q_0 = \sigma_{2^0}(F), \quad Q_s = \sigma_{2^s}(F) - \sigma_{2^{s-1}}(F) \qquad (s = 1, 2, \ldots). \tag{3}$$

Here

$$\|Q_s\|_p = \|I_r q_s\|_p \leqslant c \cdot 2^{-rs} \|q_s\|_p.$$

Therefore,

$$\|F\|_{B^r_{p\theta}} = \left(\sum_{s=0}^{\infty} 2^{s\theta r} \|Q_s\|_p^\theta\right)^{1/\theta} \leqslant c \left(\sum_{s=0}^{\infty} \|q_s\|_p^\theta\right)^{1/\theta} = \|f\|_{B^0_{p\theta}}. \tag{4}$$

Conversely, if $F \in B^r_{p\theta}$, then the expansion (3)

$$\|F\|_{B^r_{p\theta}} = \left(\sum_{s=0}^{\infty} 2^{sr\theta} \|Q_s\|_p^\theta\right)^{1/\theta} < \infty,$$

obtains for F, and expansion (1) and (cf 8.6(16))

$$\|q_s\|_p = \|I_{-r} Q_s\|_p \leqslant c \cdot 2^{sr} \|Q_s\| \quad (s = 0, 1, \ldots),$$

for $f = I_{-r}F$, therefore

$$\|f\|_{B^0_{p\theta}} = \left(\sum_{s=0}^{\infty} \|q_s\|_p^\theta\right)^{1/\theta} \leqslant c \|F\|_{B^r_{p\theta}}. \tag{5}$$

The theorem is proven.

8.9.2. Classes $B^r_{p\theta}$ when $r < 0$. The concept of a regular function and its expansion in the de la Valleé-Poussin series yields the possibility of enlarging the classes $B^r_{p\theta}$ to negative r. It is natural to assume that the function $f \in B^r_{p\theta}$, where r is an arbitrary real number if f is regular (in the $L_p$-sense) and if its expansion in the de la Valleé-Poussin series

$$f = \sum_{s=0}^{\infty} q_s = \sigma_{2^0}(f) + \sum_{s=1}^{\infty} [\sigma_{2^s}(f) - \sigma_{2^{s-1}}(f)] \tag{1}$$

is such that

$$\|f\|_{B^r_{p\theta}} = \left(\sum_{s=0}^{\infty} 2^{sr\theta} \|q_s\|_p^\theta\right)^{1/\theta} < \infty. \tag{2}$$

It is easy to see, by reasoning as in the previous section, that for any real r and $r_1$ the operation $I_{r_1-r}$ performs the isomorphism

$$I_{r_1-r}(B^r_{p\theta}) = B^{r_1}_{p\theta} \quad (1 \leqslant p,\ \theta \leqslant \infty;\ B^r_{p\infty} = H^r_p). \tag{3}$$

## 8.10. Series in Dirichlet Sum ($1 < p < \infty$)

If p satisfies the inequalities $1 < p < \infty$, then the above-performed theorem can be developed based on Dirichlet kernels

$$D_N(t) = \prod_{j=1}^{n} \frac{\sin Nt_j}{t_j}, \tag{1}$$

which are analogs of the periodic Dirichlet sum.

The kernels $D_N(z)$ exhibit the following properties:

1) $D_N(z)$ is an integral function of the exponential type N in the each variable $z_j (j = 1, \ldots, n)$ belonging to $L_p$, where $1 < p \leq \infty$

$$2) \quad \left(\frac{2}{\pi}\right)^{n/2} \tilde{D}_N = (1)_{\Delta_N} = \begin{cases} 1 & \text{on } \Delta_N = \{|x_j| < N\}, \\ 0 & \text{outside } \Delta_N \end{cases} \tag{2}$$

(cf 1.5.7(10)).

$$\frac{1}{\pi^n} \int D_N(t)\, dt = 1 \qquad (N > 0). \tag{3}$$

4) The convolution

$$S_N(f, x) = D_N * f = \frac{1}{(2\pi)^n} \int D_N(x - t) f(t)\, dt$$

for $f \in L_p (1 < p < \infty)$ is an integral function of the exponential type N with respect to each variable (cf 3.6.2) belonging to $L_p$(cf 1.5.7(9), (13); 3.6.2; $S_N(f) \in \mathfrak{M}_{Np}$). Here

$$\| D_N * f \|_p \leqslant \varkappa_p \| f \|_p, \tag{4}$$

where $\varkappa_p$ depends only on p. When $p = 1, \infty$ this fact ceases to be valid.

5) If $\omega_N \in \mathfrak{M}_{Np}$, then

$$S_N(\omega_N) = \omega_N. \tag{5}$$

The fact that

$$D_{N_0} * \omega_N = \frac{1}{(2\pi)^n} \int D_{N_0}(x-t)\,\omega_N(t)\,dt = \omega_N(x) \quad (N < N_0), \tag{6}$$

follows from 8.5.2. Further

$$\left|\frac{\sin N_0 t_j}{t_j}\right| \leqslant \varphi(t_j) = \begin{cases} N_1 & |t_j| \leqslant 1, \\ \dfrac{1}{|t_j|} & |t_j| > 1 \end{cases} \qquad (N < N_0 < N_1),$$

therefore

$$|D_{N_0}(x-t)\,\omega_N(t)| \leqslant \varphi(x-t)\,\omega_N(t) \in L(R_n), \quad \varphi(t) = \prod_{j=1}^{n} \varphi(t_j)$$

$$\left(\omega_N \in L_p(R_n), \quad \varphi \in L_q(R_n), \quad \frac{1}{p} + \frac{1}{q} = 1\right).$$

Moreover, $D_{N_0}(x - t) \longrightarrow D_N(x - t)(N_0 \rightarrow N)$ for all $t$. Consequently, by the Lebesgue theorem, in (6) we can replace $N_0$ with $N$.

6) $\widetilde{D_N * f} = \tilde{f}$ on $\Delta_N$ (cf (2) and 8.5.1).

From (4) and (5) it follows that if $f \in L_p(1 < p < \infty)$ and $\omega_N$ is its best approximating function of the class $\mathfrak{M}_{Np}$ in the $L_p$-sense, then $(1<p<\infty)$

$$\begin{aligned} \|f - S_N(f)\|_p &\leqslant \|f - \omega_N\|_p + \|S_N(\omega_N) - f\|_p \leqslant \\ &\leqslant (1 + \varkappa_p) E_N(f)_p \to 0 \qquad (N \to \infty). \end{aligned} \tag{7}$$

In particular, thus

$$S_N(f) \rightarrow f(N \rightarrow \infty) \qquad \text{weakly} \tag{8}$$

Arguing as in the proof of 8.6(14) (cf 8.6(15)), where $V_N$ must be replaced with $D_N$, we get the result that property (8) remains in effect for any function that is regular in the $L_p$-sense.

In this case the function f regular in the $L_p$-sense can be expanded in the series

$$f = S_{2^0}(f) + \sum_{k=1}^{\infty} [S_{2^k}(f) - S_{2^{k-1}}(f)], \tag{9}$$

converging weakly to it (analog of 8.8(4)). Application of the operation $I_\rho$, where $\rho$ is any real number, memberwise to this series is logitimate.

It is important to note that the k-th term of series (9) is an ordinary function of the class $\mathfrak{M}_{2^k p}$, moreover, it is important that

$$\overline{S_{2^k}(f)} - \overline{S_{2^{k-1}}(f)} = 0 \qquad \text{on } \Delta_{2^{k-1}} \quad (k = 1, 2, \ldots)$$

and the inequalities

$$\| I_r [S_N(f) - S_{2N}(f)] \|_p \leqslant \lambda_r N^{-r} \| S_N(f) - S_{2N}(f) \|_p, \tag{10}$$

$$\| I_r S_1(f) \| \leqslant \lambda_r \| S_1(f) \|_p \tag{11}$$

obtain for any real r and any function f regular in the $L_p$-sense.

Arguing as 8.9 - 8.9.2, where it is necessary everywhere to replace $_1(f)$ with $S_1(f)$, we can obtain based on the theorem set forth above

8.10.1. **Theorem.** Suppose $1 < p < \infty$, $1 \leq \theta \leq \infty$, and r is an arbitrary real number. Then $f \in B^r_{p\theta}$ ($B^r_p = H^r_p$) if and only if f is regular in the $L_p$-sense and its series (convergent to it weakly)

$$f = \sum_{k=0}^{\infty} \beta_s, \tag{1}$$

$$\beta_0 = S_{2^0}(f), \quad \beta_s = S_{2^s}(f) - S_{2^{s-1}}(f) \qquad (s = 1, 2, \ldots),$$

is such that

$$\| f \|_{B^r_{p\theta}} = \left( \sum_{s=0}^{\infty} 2^{sr\theta} \| \beta_s \|_p^\theta \right)^{1/\theta} < \infty, \tag{2}$$

$$\| f \|_{H^r_p} = \| f \|_{B^r_{p\infty}} = \sup_s 2^{sr} \| \beta_s \|_p < \infty \tag{3}$$

(cf note to 8.9).

Let us prove the lemma supplementing the results of 1.5.6 (in the same notation).

8.10.2. **Lemma.** Suppose f is a generalized function regular in the $L_p$-sense ($1 < p < \infty$), for which

$$\left\| \left\{ \sum_k \Delta_k(f)^2 \right\}^{1/2} \right\|_p < \infty. \tag{1}$$

then $f \in L_p$.

Proof. Suppose N is a natural number, then

$$\Delta_N = \sum_{|k_j| \le N} \Delta_k = \{|x_j| \le 2^N;\ j = 1, \ldots, N\}$$

and

$$\sum_{|k_j| \le N} \Delta_k(f) = \widehat{(1)_{\Delta_N}} = S_{2^N}(f, x) = F_N, \tag{2}$$

i.e., $F_N$ is a Dirichlet sum of order $2^N$ for the function f. It belongs to $L_p$. We have further (cf 1.5.1.1) for $N < N'$

$$\delta_k(F_{N'} - F_N) = \widehat{(1)_{\Delta_k}((1)_{\Delta_{N'}} - (1)_{\Delta_N})f} = \begin{cases} \Delta_k(f), & \tilde{\Delta}_k \subset \Delta_{N'} - \Delta_N, \\ 0, & \tilde{\Delta}_k \cap (\Delta_{N'} - \Delta_N) = 0, \end{cases}$$

where $\tilde{\Delta}_k$ is the open kernel $\Delta_k$. And therefore, applying equality 1.5.6(1) to $F_N$, we get

$$\| F_{N'} - F_N \|_p \ll \left\| \left\{ \sum_{\Delta_k \subset \Delta_{N'} - \Delta_N} \Delta_k(f)^2 \right\}^{1/2} \right\|_p \to 0 \quad (N, N' \to \infty).$$

But $F_N \to f$ weakly, consequently $F_N \to f$ in the matrix $L_p$ and $f \in L_p$.

The following theorem is analogously proven (notations found in 1.5.6.1).

8.10.3. **Theorem.** If the function f(x) of one variable is a generalized function regular in the $L_p$-sense $(-\infty, \infty)$ $1 < p < \infty$, for which

$$\left\| \left\{ \sum_k \delta_k(f)^2 \right\}^{1/2} \right\|_p < \infty, \tag{1}$$

then it belongs to $L_p$.

8.10.4. Example. Below we present an example of the function $g(x)$ $L_p(-\infty,\infty) = L_p(2 < p < \infty)$ that is integral and the exponential type 1, whose Fourier transform is a generalized function not summable on any interval $(a,b) \subset (-1, +1)$.

Suppose

$$\psi_k(x) = \begin{cases} \alpha_k & \left(2^k - \frac{1}{2} < |x| < 2^k + \frac{1}{2}\right), \\ 0 & \text{(for remaining k)} \end{cases} \qquad (k = 1, 2, \ldots), \tag{1}$$

where the numbers $\alpha_k > 0$ are such that

$$\sum_1^\infty \alpha_k^2 = \infty, \quad \sum_1^\infty \alpha_k^p < \infty \qquad (2 < p < \infty). \tag{2}$$

Let us further set $f_N = \sum_1^N \psi_k$ and

$$f(x) = \sum_1^\infty \psi_k(x). \tag{3}$$

Series (3) obviously converges in the sense of $L_p = L_p(-\infty, \infty)$, and consequently, also in the sense of $S'$, and $f \subset L_p \subset S'$ and $\|f\|_p = (2 \sum |\alpha_k|^p)^{1/p} < \infty$. Suppose further

$$\lambda_k(x) = \alpha_k \sqrt{\frac{2}{\pi}} \int_{2^k - \frac{1}{2}}^{2^k + \frac{1}{2}} \frac{\sin xy}{y} \, dy \qquad (k = 1, 2, \ldots).$$

It is easy to verify (cf 1.5.7(10)) that

$$\lambda_k'(x) = 2\alpha_k \sqrt{\frac{2}{\pi}} \cos 2^k x \frac{\sin \frac{x}{2}}{x} =$$

$$= \alpha_k \sqrt{\frac{2}{\pi}} \left(e^{i2^k x} + e^{-i2^k x}\right) \frac{\sin \frac{x}{2}}{x} = \tilde{\psi}_k(x).$$

Let us further suppose

$$F_N(x) = \sum_1^N \lambda_k(x) = \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \sum_1^N \psi_k(y) \frac{\sin xy}{y}\, dy,$$

$$F(x) = \sum_1^\infty \lambda_k(x).$$

Then

$$|F_N(x)| < \int_{-\infty}^{\infty} f(y) \left|\frac{\sin xy}{y}\right| dy <$$

$$< \|f\|_p \left( \int_{-\infty}^{\infty} \left| \frac{\sin xy}{y} \right|^q dy \right)^{1/q} < \|f\|_p |x|^{1-\frac{1}{q}}, \tag{4}$$

where the constants appearing in these inequalities do not depend on N, $\|f\|_p$, and x. Therefore

$$|F(x) - F_N(x)| < c \left\| \sum_N^\infty \psi_k \right\|_p |x|^{1-\frac{1}{q}} \tag{5}$$

and $F_N(x) \longrightarrow F(x)$ uniformly on any finite segment. But then F(x) is a continuous function. It easily also follows from (4) and (5) that $F \in S'$ and $F_N \longrightarrow F(S')$.

In this case $$F' = \sum_1^\infty \lambda_k' = \sum_1^\infty \psi_k = f,$$

where all operations (differentiation, summation, and Fourier transformation) are understood in the S'-sense.

We can prove (see the book by Zigmund [2], part II, Chapter XV, at end of section 3.14 for proof), that the function F(x) (understood as an ordinary function) almost everywhere has no derivative. But then the generalized derivative F' on any interval (a, b) is not a summable function, in other words, whatever be the interval (a, b), there does not exist a function $\alpha(x)$ summable on (a, b) such that

$$(F', \varphi) = \int_a^b \alpha(x) \varphi(x)\, dx \tag{6}$$

for all functions $\varphi \in S$ that have a carrier on $(a, b)$. In fact, if the function $\alpha$ did exist, then, integrating the right side of (6) by parts, we will obtain the equality

$$\int_a^b F(x)\varphi'(x)\, dx = -\int_a^b \int_a^x \alpha(t)\, dt\, \varphi'(x)\, dx,$$

i.e.,

$$\int_a^b \psi(x)\varphi'(x)\, dx = 0, \quad \psi(x) = F(x) - \int_a^x \alpha(t)\, dt,$$

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whatever be the function $\varphi \in S$ with carrier on $(a, b)$. But then $\psi(x) \equiv C$ is a constant F would be differentiable almost everywhere on $(a, b)$. The fact that $\psi$ = constant can be proven thusly. If it would not be so, then we could select such a constant $c_1$ that the function $\lambda(x) = \psi(x) + c_1$ would take on values of different signs at some two points. Suppose for definiteness $a < x_1 < x_2 < b$ and $\lambda(x_1) < 0 < \lambda(x_2)$. It is obvious for the functions $\varphi$ considered,

$$\int_a^b \lambda(x)\varphi'(x)\, dx = 0,$$

because $\int_a^b \varphi'(x)\, dx = 0$. Let us choose $\delta > 0$ sufficiently small that $\lambda(x) < 0$ on $(x_1 - \delta, x_1 + \delta)$ and $\lambda(x) > 0$ on $(x_2 - \delta, x_2 + \delta)$, and suppose that $\omega(x)$ is a function continuous on $(a, b)$, equal to zero, for $x < x_1 - \delta/2$ and $x > x_2 + \delta/2$ and such that $\omega'(x) = -1$ on $(x_1 - \delta/2, x_1 + \delta/2)$, $\omega'(x) = 0$ on $(x_1 + \delta/2, x_2 - \delta/2)$ and $\omega'(x) = 1$ on $(x_2 - \delta/2, x_2 + \delta/2)$. Its $\varepsilon$-averaging $\omega_\varepsilon(x)$ obviously belong to the class of the function $\varphi$ considered here, also because

$$0 = \int_a^b \lambda(x)\, \omega_\varepsilon'(x)\, dx \to \int_a^b \lambda(x)\, \omega'(x)\, dx > 0 \qquad (\varepsilon \to 0),$$

and we have reached a contradiction.

Let us set $$g(x) = \widetilde{(1)_\Delta} = \frac{1}{\pi} \int D_1(x - t) f(t)\, dt, \quad \Delta = (|x| < 1).$$

Since $f \in L_p$, then the function $g \in L_p$ is integral and of the exponential type 1. Its Fourier transform

$$\tilde{g} = (1)_\Delta \tilde{f}$$

is a generalized function equal to $\tilde{f}$ on $\Delta$. This means that

$$(\tilde{g}, \varphi) = (\tilde{f}, \varphi)$$

for all $\varphi \in S$ with carrier around $\Delta$. But then $\tilde{f}$, consequently, $\tilde{g}$, can now be represented in any single interval $(a, b) \subset \Delta$ by a summable function.

# CHAPTER IX LIOUVILLE CLASSES L

## 9.1. Introduction

We denote the Liouville classes by $L_p^r(R_n)$ ($r \geq 0$, $L_p^0(R_n) = L_p(R_n)$) in the isotropic case and by $L_p^{\mathbf{r}}(R_n)$ in the anisotropic. For integral $r$, $\mathbf{r}$, they coincide with the Sobolev classes $W$:

$$W_p^r(R_n) = L_p^r(R_n) \qquad (r = 0,\ 1,\ \ldots),$$
$$W_p^{\mathbf{r}}(R_n) = L_p^{\mathbf{r}}(R_n) \qquad (r_j = 0,\ 1,\ 2,\ \ldots;\ j = 1,\ \ldots,\ n).$$

Generalizations are possible for the case when $p$ is vectorial.

The classes $L^r$, $L^{\mathbf{r}}$ for fractional $r$, $\mathbf{r}$ are the most natural extensions of the classes $W^r$, $W^{\mathbf{r}}$.

For orientation, even at this stage we will note their fundamental properties.

The functions $F \in L_p^r(R_n)$ are defined in the form of the integrals

$$F(x) = I_r f = \frac{1}{(2\pi)^{n/2}} \int G_r(|x - u|) f(u)\, du, \quad f \in L_p(R_n), \tag{1}$$

already familiar to the reader, where (cf 8.1) $G_r$ is a Bessel-Macdonald kernel. If $r$ is a natural number, then (cf 8.2) $F$ runs through the class $W_p^r(R_n)$, when $f$ runs through $L_p(R_n)$, where the isomorphism $I_r L_p(R_n) = W_p^r(R_n)$ obtains. For fractional $r$, equality (1) becomes the definition of class $L_p^r(R_n)$ (at least in this book, cf 9.2.3.), i.e., we assume that $F \in L_p^r(R_n)$, if and only if $F = I_r f$, where $f \in L_p(R_n)$ and we set

$$\|F\|_{L^r_p(R_n)} = \|f\|_{L_p(R_n)}.$$

For any $r > 0$ $L^r_p \subset H^r_p$, moreover "with an accuracy up to any small $\varepsilon$", the class $L^r_p$, just as $B^r_p$, coincide with $H^r_p$, namely

$$H^{r+\varepsilon}_p \to L^r_p \to H^r_p.$$

From these embeddings it follows that in any case, "with an accuracy up to $\varepsilon$" the same embedding theorems are valid for the class $L^r_p$ as for the class $H^r_p$, since, for example, $L^r_p(R_n) \to H^r_p(R_n) \to H_q^{r-\frac{n}{p}+\frac{m}{q}}(R_m) \to L_q^{r-\frac{n}{p}+\frac{m}{q}-\varepsilon}(R_m)$.

The embeddings (cf 9.3, $B^r_p = B^r_{pp}$)

$$B^r_p \to L^r_p \qquad (1 \leqslant p \leqslant 2), \quad L^r_p \to B^r_p \qquad (2 \leqslant p \leqslant \infty). \tag{2}$$

are valid. The coincidence of classes B in L obtains only when $p = 2$ ($B^r_2 = L^r_2$). When $p \neq 2$, they differ essentially from each other.

The classes $L^r_p$ are united by the same integral representation in terms of functions $f \in L_p$. The classes $B^r_p$ are united by the same representation, but now in terms of the function $f \in B^o_p$, where $B^o_p$ is a class essentially distinct from $L_p$; when $p > 2$ it includes generalized functions (cf 8.9.1).

The family of classes $L^r_p$ is also remarkable by being closed with respect to such embedding theorems where passage from one metric to another occurs. Thus, the embedding of different metrics

$$L^r_p(R_n) \to L^\rho_q(R_n) \tag{3}$$
$$\left(\rho = r - n\left(\frac{1}{p} - \frac{1}{q}\right) \geqslant 0,\ 1 < p < q < \infty\right).$$

obtains. Another, more general class is the embedding

$$L_p^r(R_n) \to L_q^\rho(R_m) \tag{4}$$
$$\left(\rho = r - \frac{n}{p} + \frac{m}{q} \geqslant 0,\ 1 < p < q < \infty\right).$$

where in addition to change of measure, there is the passage from one metric to another. Thus far as the embedding theorem is concerned where the number of measures changes without metric change, then the corresponding direct theorem reads thusly

$$\left(1 \leq m < n,\ 1 < p < \infty, \rho = r - \frac{n-m}{p} > 0\right):$$

$$L_p^r(R_n) \to B_p^\rho(R_m), \tag{5}$$

and the inverse thusly:

$$B_p^\rho(R_m) \to L_p^r(R_n). \tag{6}$$

Based on the foregoing, B can be replaced with L when p = 2 in (5) and (6); moreover, this can be done in (5), if $1 < p < 2$, and in (6), if $2 < p < \infty$. This substitution is invalid in the remaining cases. Thus, the embedding theorem of different measures is in general not closed with respect to the classes L.

The following situation obtains:

$$B_p^r(R_n) \to L_p^r(R_n) \to B_p^\rho(R_m) \to B_p^r(R_n) \qquad (1 < p \leqslant 2),$$
$$L_p^r(R_n) \to B_p^r(R_n) \to B_p^\rho(R_m) \to L_p^r(R_n) \qquad (2 \leqslant p < \infty),$$

showing that the two distinct classes $B_p^r(R_n)$ and $L_p^r(R_n)$ of functions defined in $R_n$ yields the same set of traces on $R_m$ (class $B_p$ $(R_m)$).

We must note that the embeddings

$$B_p^r(R_n) \rightleftarrows B_p^\rho(R_m)$$

for indicated m, n, p, r, and $\rho$, can be obtained as a consequence of embeddings (5) and (6). In fact,

$$B_p^r(R_n) \rightleftarrows L_p^{r+\frac{1}{p}}(R_{n+1}) \rightleftarrows B_p^r(R_m).$$

The facts that we present here are representative. In the anisotropic case similar facts obtain. They will also be proven in this chapter.

## 9.2. Definitions and Fundamental Properties of the Classes $L_p^r$ and $L_p^r$

Suppose $1 \leq m \leq n$, $x = (u, y)$, $u = (x_1, \ldots, x_m)$ $R_m$, and $y = (x_{m+1}, \ldots, x_n) \in R_{n-m}$. For the functions $\varphi(x) = \varphi(u, y)$ of the fundamental class S, we will denote their Fourier transform (direct and inverse) in the variable u with $\tilde{\varphi}^u$, $\hat{\varphi}^u$ ($\tilde{\varphi}^u = \tilde{\varphi}$, $\hat{\varphi}^u = \hat{\varphi}$ when $m = n$). For example,

$$\tilde{\varphi}^u(u, y) = \frac{1}{(2\pi)^{m/2}} \int \varphi(t, y) e^{-itu} dt. \tag{1}$$

The operations $\tilde{\varphi}^u$, $\hat{\varphi}^u$ map S into S and are weakly continuous (cf further 9.2.1); therefore for arbitrary generalized functions $f \in S'$ (defined on $R_n$), the corresponding Fourier transforms $\tilde{f}^u$ and $\hat{f}^u$ are correctly defined by the functionals

$$(\tilde{f}^u, \varphi) = (f, \tilde{\varphi}^u),$$
$$(\hat{f}^u, \varphi) = (f, \hat{\varphi}^u).$$

If $\lambda(u)$ is an infinitely differentiable function of polynomial group dependent only on u, then for $f \in S'$

$$\widehat{\lambda \tilde{f}^u}^u = \lambda \hat{f}, \quad \widetilde{\lambda \hat{f}^u}^u = \lambda \tilde{f},$$

which follows directly from the validity of these qualities for $\varphi \in S$.

Let us introduce the operation

$$F = I_{ur} f = \widehat{(1 + |u|^2)^{-r/2} \tilde{f}^u}^u = \widehat{(1 + |u|^2)^{-r/2} \tilde{f}} \tag{2}$$

($|u|^2 = \sum_1^m u_j^2$, $I_{ur} = I_r$, when $m = n$, $f \in S'$), corresponding to the real number r mapping S' onto S', mutually uniquely. When $m = 1$, when $R_m = R_{x_j}$

is the axis of coordinates $x_j$, we will denote it further with $I_{x_j r}$.

For the function $f \in L_p = L_p(R_n)(1 \leq p \leq \infty)$ this operation when $r > 0$ reduces to the convolution

$$f = I_{ur} f = \frac{1}{(2\pi)^{m/2}} \int_{R_m} G_r(|u - t|_m) f(t, y)\, dt \quad \left( |t|_m^2 = \sum_1^m t_i^2 \right), \tag{3}$$

where $G_r$ is a Bessel-Macdonald kernel, which is proven thusly.

For $f \in S'$ the equalities $\tilde{f}(x) = \hat{f}(-x) = \widehat{f(-x)}$,

obtain, which follow by means of ordinary "changeovers" [prebroski] from the fact that they obviously are valid for any $\varphi \in S$. Further, if $\hat{\Lambda} \in L$ and $f \in L_p$, then

$$\widetilde{\Lambda \tilde{f}} = \widehat{\Lambda \tilde{f}(-u)}(-x) = \frac{1}{(2\pi)^{n/2}} \int \hat{\Lambda}(-x-u) f(-u)\, du.$$

In particular, if $\hat{\Lambda}(u) = \hat{\Lambda}(-u)$ then $\widehat{\Lambda \tilde{f}} = \widehat{\Lambda \tilde{f}}$. Therefore, for $f \in L_p$, $\varphi \in S$, considering that $G_r(|u|_m) = G_r(|-u|_m) \in L_p(R_m)$, we get

$$\begin{aligned}
(I_{ur} f, \varphi) &= \left(f, \overline{(1 + |u|^2)^{-r/2} \tilde{\varphi}}\right) = \\
&= \left(f, \widehat{(1 + |u|^2)^{-r/2} \tilde{\varphi}}\right) = \left(f, \widehat{(1 + |u|^2)^{-r/2} \tilde{\varphi}^u}\right) = \\
&= \frac{1}{(2\pi)^{m/2}} \int f(u, y)\, du\, dy \int G_r(|u - t|_m) \varphi(t, y)\, dt = \\
&= \frac{1}{(2\pi)^{m/2}} \int \varphi(x)\, dx \int G_r(|t - u|_m) f(u, y)\, du \quad (dx = dt\, dy),
\end{aligned}$$

which in fact proves (3).

Let us introduce the functional classes $L_{up}^r = L_{up}^r(R_n)$,

$$L_{x_i p}^r = L_{x_i p}^r(R_n), \quad L_{up}^r = L_{up}^r(R_n), \quad r = (r_1, \ldots, r_n).$$

By definition, the function $F \in S'$ belongs to $L_{up}^r = L_{up}^r(R_n)$, $L_{x_i p}^r(R_n)$,

$1 \leq p \leq \infty$, $-\infty < r < \infty$, if it is representable, respectively, in the form

$$F = I_{ur} f, \quad F = I_{x_i r} f,$$

where $f \in L_p$. Here we introduce the norms $\|F\|_{L^r_{up}} = \|f\|_p$, in particular, $\|F\|_{L^r_{x_i p}} = \|f\|_p$, trivially indicating the isomorphisms

$$I_{ur}(L_p) = L^r_{up}, \quad I_{x_i r}(L_p) = L^r_{x_i p},$$

performing operations $I_{ur}$ and $I_{x_i r}$. The class $L^{\mathbf{r}}_{up} = L^{\mathbf{r}}_{up}(R_n)$ corresponding to an arbitrary real vector $\mathbf{r}$ is defined as the intersection

$$L^{\mathbf{r}}_{up} = \bigcap_{j=1}^{m} L^{r_j}_{x_j p}$$

with the norm

$$\|f\|_{L^{\mathbf{r}}_{up}} = \sum_{j=1}^{m} \|f\|_{L^{r_j}_{x_j p}}.$$

By virtue of the foregoing, the class $L^r_{up}$ can be defined further as a class of function representable (for almost all $\mathbf{y}$) as the integral (3), where $f(\mathbf{x}) = f(\mathbf{u}, \mathbf{y}) \in L_p(R_n)$.

Given the condition $1 < p < \infty$

(4)

$$L^r_{up} = L^{r, \ldots, r}_{up} \qquad (r \geq 0),$$

$$L^r_{up} = W^r_{up} = W^{r, \ldots, r}_{up} \qquad (r = 1, 2, \ldots), \qquad (5)$$

$$L^{\mathbf{r}}_{up} = W^{\mathbf{r}}_{up} \ (\mathbf{r} = (r_1, \ldots, r_m),$$

(6)

where $r_i$ are nonnegative integers).

Equalities (5) and (6) show that the classes $L^r_{up}$ and $L^{\mathbf{r}}_{up}$ can be considered as distributions on any real $r$, $\mathbf{r}$ of the Sobolev classes $W^r_{up}$ and $W^{\mathbf{r}}_{up}$. But we must bear in mind that the functions of classes $L^r_{up}$ and $L^{\mathbf{r}}_{up}$ were defined by us for the entire space $R_n$, while the functions of the Sobolev classes can be assigned on arbitrary open domain $g \subset R_n$.

The first equality (5), when $m = n$, is proven in 8.2. If however $m < n$, then suppose for the time being that $f \in S$ (class of fundamental functions). Then $f \in W_p^r(R_m)$ for any $y$, is also obvious $f \in L_p^r(R_m)$ for any $y$ if function $f(u, y)$ in $u$ belongs to $S = S(R_m)$ and if the operation $I_r$ (in u) maps it into the function of the class $S(R_m)$, and thus into $L_p(R_m)$. Therefore by the virtue of the relation already proven in 8.2

$$c_1 \| f \|_{W_p^r(R_m)} \leqslant \| I_{u(-r)} f \|_{L_p(R_m)} \leqslant c_2 \| f \|_{W_p^r(R_m)}, \tag{7}$$

where $c_1$ and $c_2$ do not depend on f and $y$.

Raising these equalities to the power p, applying the elementary inequality*), integration with respect to $y$, and application of another elementary inequality*) leads to the inequalities

$$c' \| f \|_{W_{up}^r} \leqslant \| I_{u(-r)} f \|_{L_p} \leqslant c'' \| f \|_{W_{up}^r} \tag{8}$$

even for just functions $f \in S$.

If now $f \in W_p^r(R_n)$ then we define the sequence of finite functions $f_l (l = 1, 2, \ldots)$ such that $\| f_l - f \|_{W_p^r(R_n)} \to 0 \ (l \to \infty)$.

From (8) it follows that

$$\begin{aligned} c' \| f_k - f_l \|_{W_{up}^r} &\leqslant \| \varphi_k - \varphi_l \|_{L_p} \leqslant \\ &\leqslant c'' \| f_k - f_l \|_{W_{up}^r} \to 0 \ (k, \ l \to \infty) \quad (\varphi_k = I_{u(-r)} f_k), \end{aligned}$$

*) We must bear in mind the inequalities

$$c \left| \sum a_k \right|^p \leqslant \sum a_k^p \leqslant c_1 \left| \sum a_k \right|^p,$$

where numbers $a_k > 0$ and $c$, $c_1$ depend only on p and on the number (finite) of terms under the sign $\sum$.

and by virtue of the completeness of $W^r_{up}$ and the fact that (cf (3))

$$f_l(u, y) = \frac{1}{(2\pi)^{m/2}} \int G_r(|u - t|_m) \varphi_l(t, y)\, dt,$$

where $G_r(|u|_m) \in L_p(R_m)$, the second inequality (8) obtains where

$$\| I_{u(-r)} f \|_{L_p} = \| f \|_{L^r_{up}}.$$

If however $f \in L^r_{up}$ and $I_{u(-r)} f = \varphi$, then we can select a sequence of finite (belonging to S') functions $\varphi_l$ such that $\|\varphi_k - \varphi\|_{L_p} \to 0$, therefore by virtue of the first inequality of (8) and the completeness of $W^r_{up}$, we arrive at the first inequality of (8). By this we have proven the first equality of (5).

From the first equality of (5) applied to each axis $R_{x_j}$ ($j = 1, \ldots, n$) obviously follows (6).

Let us proceed to the proof of (4). Suppose $F \in L^{r,\ldots,r}_{up}$ ($r \geqslant 0$), then

$$\psi = \overbrace{\sum_{j=1}^{m} (1 + u_j^2)^{r/2} \tilde{F}} \in L_p, \quad \|\psi\|_p \ll \|F\|_{L^{r,\ldots,r}_{up}}$$

and, since the function

$$(1 + |u|^2)^{r/2} \left( \sum_{j=1}^{m} (1 + u_j^2)^{r/2} \right)^{-1}$$

is a Marcinkiewicz multiplier (cf 1.5.5, example 9 provided $r = r_j$, $\sigma = 1$; here also consider and below the note 1.5.4.1), then

$$I_{u(-r)} F = \overbrace{(1 + |u|^2)^{r/2} \tilde{F}} \in L_p,$$

$$\|F\|_{L^r_{up}} = \|I_{u(-r)} F\|_p \ll \|\psi\|_p \ll \|F\|_{L^{r,\ldots,r}_{up}},$$

from whence it follows that $L^{r,\ldots,r}_{up} \to L^r_{up}$. Conversely, if $F \in L^r_{up}$, then

$$f=\widehat{(1+|u|^2)^{r/2}\tilde{F}}\in L_p,\quad \|f\|_p=\|F\|_{L^r_{up}}$$

and, since the function

$$(1+|u|^2)^{-r/2}(1+u_j^2)^{r/2}\qquad (r\geqslant 0)$$

is a Marcinkiewicz multiplier (cf 1.5.5, example 3), then

$$f_j=\widehat{(1+u_j^2)^{r/2}\tilde{F}}\in L_p,\quad \|f_j\|_p\ll\|f\|_p=\|F\|_{L^r_{up}},$$

therefore $L^r_{up}\to L^{r,\ldots,r}_{up}$ and (6) is proven.

From the foregoing it follows that

$$W^r_{up}\leftrightarrows L^r_{up}\leftrightarrows L^{r,\ldots,r}_{up}\leftrightarrows W^{r,\ldots,r}_{up}\qquad (r=0,\ 1,\ \ldots),$$

which entails the second equality of (5). Its nontrivial part is the embedding $W^{r,\ldots,r}_{up}\to W^r_{up}$ $(1<p<\infty)$, expressing that if the function $f\in L_p$ does have nonmixed derivatives of the order r with respect to the variables $x_1,\ \ldots,$ $x_m$ taken separately, belonging to $L_p$, then it also has any mixed derivatives of the order r with respect to these variables belonging also to $L_p$. Examples are existed showing that this embedding provided $p=1$ and $p=\infty$ does not obtain (B. S. Mityagin [1]).

9.2.1. The weak continuity of the operation $\tilde{\varphi}^u$ $(\varphi\in S)$ follows from the following considerations. We will write $\tilde{\varphi}$ instead of $\tilde{\varphi}^u$. Let us assign a natural number l and an integral nonnegative vector $k=s+\rho$, where $s=(k_1,\ldots,$ $k_m,\ 0,\ \ldots,\ 0)$, $\rho=(0,\ \ldots,0,\ k_{m+1},\ \ldots,\ k_n)$. Then, obviously, the derivative

$$D^k\tilde{\varphi}=D^s\widetilde{\varphi^{(\rho)}}.$$

We have (explanations below)

$$
\begin{aligned}
(1+|x|^2)^l |D^{(k)}\tilde{\varphi}| &\leqslant (1+|y|^2)^l (1+|u|^2)^l |D^s \widetilde{\varphi^{(\rho)}}| \leqslant \\
&\leqslant c(1+|y|^2)^l \sum_{(l', s') \in \mathfrak{F}_{l, s}} \max_u (1+|u|^2)^{l'} |D^{(s')}\varphi^{(\rho)}(u, y)| = \\
&= c(1+|y|^2)^l \sum (1+|u_0|^2)^{l'} |D^{(s'+\rho)}\varphi(u_0, y)| \leqslant \\
&\leqslant c \sum (1+|y|^2+|u_0|^2)^{2l} |D^{(s'+\rho)}\varphi(u_0, y)| \leqslant \\
&\leqslant c \sum \varkappa(2l, s'+\rho, \varphi).
\end{aligned} \tag{1}
$$

The second inequality follows from 1.5(4); in the third term the constant c depends on $l$ and $k$, but not on $\varphi$ and $y$; the sum of the third and successive terms is extended over some (dependent on $l$ and $\rho$) finite set $\mathfrak{F}_l$, of pairs $(l', s')$ and natural numbers $l'$ and nonnegative $s'$. In the fourth term, $u_0 \in R_m$ does in fact depend on the corresponding term and also on $y$; this $u_0$ is the maximum point (with respect to $u$) of the corresponding term for fixed $y$. The weak continuity of $\tilde{\varphi}^u$ follows from the derived inequalities (1).

9.2.2. Theorem on derivatives. Suppose $1 < p < \infty$, $F \in L_p^r = L_p^r(R)$, $R = R_n$, $r = (r_1, \ldots, r_n) > 0$ $(r_j > 0)$ and $l = (l_1, \ldots, l_n) \geqslant 0$ is an integral vector ($l_j \geqslant 0$ are integers) for which

$$\varkappa = 1 - \sum_{j=1}^{n} \frac{l_j}{r_j} \geqslant 0. \tag{1}$$

Further suppose

$$\rho = \varkappa r. \tag{2}$$

Then the derivative

$$F^{(l)} = (ix)^l \hat{F} \in L_p^\rho \tag{3}$$

and

$$\|F^{(l)}\|_{L_p^\rho} \leqslant c \|F\|_{L_p^r}. \tag{4}$$

Proof. By the condition $F \in L_p^r$, therefore

$$\psi = \widehat{\Lambda \widetilde{F}} \in L_p, \quad \Lambda = \sum_{j=1}^{n} (1 + x_j^2)^{r_j/2}, \quad \|\psi\|_p \leqslant c \|F\|_{L_p^r}. \tag{5}$$

To prove (3) and (4), we must establish that for any s = 1, ..., n, and

$$\widehat{(1 + x_s^2)^{\frac{\varkappa r_s}{2}} \widetilde{F^{(l)}}} = \widehat{(1 + x_s^2)^{\varkappa r_s} (ix)^l \widetilde{F}} \in L_p,$$

but this follows from (5) if we notice that the functions

$$_{1-}V_l(xl)\, \frac{z}{{}^s\varkappa}({}^s_z x + 1).$$

are Marcinkievicz multipliers (cf 1.5.5, example 6).

The proof in theorem 9.2.2 is in a certain sense analogous to the theorem 5.6.3 B-classes. However, theorem 9.2.2 is valid when $1 < p < \infty$ and $\varkappa \geqslant 0$, while theorem 5.6.3 is valid for $1 \leq p \leq \infty$ but when $\varkappa > 0$.

Example. Suppose f is a function defined on the circle $\sigma = \{\rho^2 = x^2 + y^2 \leq 1\}$ by the equalities

$$f = xy \ln \rho^2 \; (\rho > 0), \quad f = 0 \; (\rho = 0) \tag{1}$$

and is extended over the entire plane R such that it together with its partial derivatives up to the second order inclusively are bounded and continuous on the domain $\rho > \frac{1}{2}$ (cf theorem 3 in notes to 4.3.6).

It is easy to verify that f, $\partial f/\partial x$, and $\partial f/\partial y$ are continuous and bounded, while $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$ are bounded on R; at the same time $\partial^2 f/\partial x \partial y$ is continuous for $\rho > 0$, but is unbounded in the neighborhood of the zero point.

This example shows that when $p = \infty$, theorem 9.2.2 is in general invalid.

9.2.3. Note on derivatives of fractional order. In this book we will deal with expressions of the form

$$\widehat{(ix)^\alpha \widetilde{f}} = f^{(\alpha)} \qquad (f \in S') \tag{1}$$

only for the case of integral vectors $\alpha$. If $\alpha \geqslant 0$, then $f^{(\alpha)}$ is a derivative of $f \in S'$ of order $\alpha$. The function $(ix)^\alpha$ for integral $\alpha$ is infinitely differentiable and of polynomial growth, therefore expression (1) correctly defines $f^{(\alpha)} \in S'$.

If the real number $\alpha$ is nonintegral, then the function $(it)^{\alpha}$ $(-\infty < t < \infty)$ is multivalued, but we can agree to understand this expression to refer to the unique branch of this function

$$(it)^{\alpha} = |t|^{\alpha} \exp\left\{\frac{1}{2}\pi i\alpha \operatorname{sign} t\right\},$$

then when $\alpha$ is natural

$$(it)^{\alpha} = \underbrace{(it) \ldots (it)}_{\alpha \text{ раз}}.$$

If further, $\alpha$ and $\beta$ are real numbers, then

$$(it)^{\alpha+\beta} = (it)^{\alpha}(it)^{\beta}.$$

If now $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are real vectors and $\mathbf{x} = (x_1, \ldots, x_n)$ is a real vectorial variable, then we set

$$(ix)^{\alpha} = (ix_1)^{\alpha_1} \ldots (ix_n)^{\alpha_n},$$

then, obviously, the equality will be satisfied

$$(ix)^{\alpha}(ix)^{\beta} = (ix)^{\alpha+\beta}.$$

Now it is natural to define the derivative $f^{(\alpha)}$ of order $\alpha$ for any, not necessarily, integral vector $\alpha$ by means of expression (1). However, here we face the difficulty that for fractional $\alpha$ the function $(\mathbf{ix})^{\alpha}$ is not infinitely differentiable; it also is not a Marcinkievicz multiplier and, thus, is inapplicable in the sense of the definition considered in this book, even if $f \in L_p$. The way out of the dilemma is found in the fact that instead of S, we will consider another class $\Omega$ of fundamental functions consisting of functions that are orthogonal polynomials (P. I. Lizorkin [5]).

A class of functionals (generalized functions) $\Omega'$ is defined over $\Omega$. The concept $(\mathbf{ix})^{\alpha}\hat{f}$ for fractional vector $\alpha$ is meaningful in the terms of $\Omega'$. Here we have proven that the class $L^{r}_{x_j p}$, where $r > 0$ is generally fractional, can be defined as consisting of the functions f for which the norm

$$\|f\|_{L_p} + \|\widehat{(ix_j)^r \tilde{f}}\|_{L_p}.$$

is finite. This norm is equivalent to our introduced norm $\|f\|_{L^r_{x_j p}}$.

## 9.3. Interrelationships of Liouville and Other Classes

We will assume that R = $R_n$, L = L(R), H = H(R), ... and

$$B^r_{pp} = B^r_p, \quad B^{\mathbf{r}}_{pp} = B^{\mathbf{r}}_p.$$

The following embeddings ($r \geqslant 0$, $\mathbf{r} \geqslant 0$) are valid:

$$L^r_p \to H^r_p, \quad L^{\mathbf{r}}_p \to H^{\mathbf{r}}_p \qquad (1 \leqslant p \leqslant \infty), \tag{1}$$

$$B^r_p \to L^r_p, \quad B^{\mathbf{r}}_p \to L^{\mathbf{r}}_p \qquad (1 \leqslant p \leqslant 2)^*), \tag{2}$$

$$L^r_p \to B^r_p, \quad L^{\mathbf{r}}_p \to B^{\mathbf{r}}_p \qquad (2 \leqslant p \leqslant \infty,\ B^{\mathbf{r}}_\infty = B^{\mathbf{r}}_{\infty\infty} = H^{\mathbf{r}}_\infty)^*). \tag{3}$$

From (2) and (3) it follows that

$$B^r_2 = L^r_2, \quad B^{\mathbf{r}}_2 = L^{\mathbf{r}}_2 \tag{4}$$

and, in particular,

$$B^r_2 = W^r_2, \quad B^{\mathbf{r}}_2 = W^{\mathbf{r}}_2 \qquad (r,\ r_j = 0,\ 1,\ \ldots). \tag{5}$$

Thus, the value of parameter p = 2 is exclusive -- for it the corresponding classes B and L, and for natural r, $\mathbf{r}$, and W coincide.

Let us present the proof to (1) - (3) for n = 1, for the time being. In this case the embeddings appearing in each of the pairs (1), (2), and (3) correspondingly coincide.

Let function $f \in L_p = L_p(R_1)$ and $\sigma_N(f)$ be its de la Valleè-Poussin sum. Then

---

*) O. V. Besov [5] -- integral r, $\mathbf{r}$, $1 < p < \infty$; P. I. Lizorkin [6] -- general case, $1 < p < \infty$. Cf further note to 9.3.

$$\|\sigma_{2^0}(f)\|_p < M\|f\|_p,$$
$$\|\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f)\|_p < 2M\|f\|_p,$$

and this shows that (cf 8.9.1(1))

$$\|f\|_{H_p^0} = \sup_k \{\|\sigma_{2^0}(f)\|_p, \|\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f)\|_p\} < 2M\|f\|_p,$$

i.e., $L_p \to H_p^0$, but since the operation $I_r$ performs the isomorphisms:

$$I_r(L_p) = L_p^r, \quad I_r(H_p^0) = H_p^r,$$

then

$$L_p^r \to H_p^r.$$

Let us prove (3). The case $p = \infty$ was already considered. Suppose $2 \leq p < \infty$ and function $f \in L_p = L_p(-\infty, \infty)$ is, consequently, regular in the $L_p$-sense.

Then, considering that $\beta_k(f)$ has the same meaning as in 8.10.1, we get (explanations below)

$$\|f\|_p^p \gg \left\|\left\{\sum_0^\infty \beta_k(f)^2\right\}^{1/2}\right\|_p^p \geq \left\|\left\{\sum_0^\infty |\beta_k(f)|^p\right\}^{1/p}\right\|_p^p =$$
$$= \int_{-\infty}^{\infty} \sum_0^\infty |\beta_k(f)|^p\, dx = \sum_0^\infty \|\beta_k(f)\|_p^p = \|f\|_{B_p^0}^p.$$

The first inequality follows from 1.5.6.1, the second from 3.3.3, and finally the last from theorem 8.10.1. Consequently, $L_p \to B_p^0$.

Now suppose for the same notation $f \in B_p^0$, $1 < p \leq 2$. Then (explanations below)

$$\|f\|_{B_p^0}^p = \sum_0^\infty \|\beta_k(f)\|_p^p = \int_{-\infty}^{\infty} \sum |\beta_k(f)|^p\, dx \geq$$
$$\geq \int \left\{\sum \beta_k(f)^2\right\}^{p/2} dx \gg \|f\|_p^p.$$

The first equality follows from 8.10.1, the next to the last from 3.3.3, and the last from 1.5.6.1. Consequently, $B_p^0 \to L_p$.

When $p = 1$, let us reason in a different fashion. Suppose the function $f \in B_1^0$; then it is regular in the L-sense and is represented as the de la Valleé-Poussin series 8.9(3)

$$f = \sum_0^\infty q_s$$

weakly convergent in the S'-sense, where $\sum_0^\infty \| q_s \|_L < \infty$. But then, obviously, $f \in L$ and

$$\| f \|_L \leq \| f \|_{B_1^0}.$$

We have proven (1) - (3) for $n = 1$. But then the following embeddings are also valid

$$L^r_{x_1 p}(R_n) \to H^r_{x_1 p}(R_n), \tag{6}$$

$$B^r_{x_1 p}(R_n) \to L^r_{x_1 p}(R_n) \qquad (1 \leq p \leq 2), \tag{7}$$

$$L^r_{x_1 p}(R_n) \to B^r_{x_1 p}(R_n) \qquad (2 \leq p \leq \infty). \tag{8}$$

In fact, it is immediately clear from the definition of the H- and B-classes that if the function $F(\mathbf{x}) = F(x_1, \mathbf{y})$, $\mathbf{y} = (x_2, \ldots, x_n)$, belongs to $H^r_{x_1 p}(R_n)$, $B^r_{x_1 p}(R_n)$, then for almost all $\mathbf{y}$ it as a function of $x_1$ belongs, respectively, to $H^r_p(R_{x_1})$ and to $B^r_p(R_{x_1})$ where $R_{x_1}$ is the $x_1$ axis; analogously, if $F(\mathbf{x}) \in L_{x_1 p}(R_n)$, then for almost all $\mathbf{y}$ it belongs to $L^r_p(R_{x_1})$(this follows from the integral representation 9.2(3) of the functions of the class $L^r_{x_1 p}(R_{x_1})$).

The equalities

$$\| F \|_{\Lambda^r_{x_1 p}(R_n)} = \left( \int \| F(x_1, y) \|^p_{\Lambda^r_p(R_{x_1})} dy \right)^{1/p}, \tag{9}$$

$$\| F \|_{L^r_{x_1 p}(R_n)} = \left( \int \| f(x_1, y) \|^p_{L_p(R_{x_1})} dy \right)^{1/p}, \tag{10}$$

are valid here with an accuracy up to equivalency, where instead of $\Lambda$ we can substitute H or B, and in (10) F and f are related by equality 9.2(3). The inequalities defining embedding (6) - (8) then follow the corresponding inequalities between the norm under the integrals in (9) and (10) that were already proven for the one-dimensional case.

From (6) - (8), where $x_1$ can be further replaced by $x_i$ ($i = 1, \ldots, n$), embeddings (1) - (3) follow trivially, if we consider (in the proof of the first embeddings (1) - (3)), that

$$H_p^r = H_p^{r, \ldots, r} \quad (1 \leqslant p \leqslant \infty), \quad B_p^r = B_p^{r, \ldots, r} \quad (1 \leqslant p \leqslant \infty),$$
$$L_p^r = L_p^{r, \ldots, r} \quad (1 < p < \infty).$$

## 9.4. Integral Representation of Anisotropic Classes

In this section we will be concerned with studying the operations

$$F = \widehat{\Lambda_r \hat{f}} = I_r f, \tag{1}$$

$$\Lambda = \Lambda_r = \left\{ \sum_{j=1}^{n} (1 + x_j^2)^{r_j \sigma/2} \right\}^{-\frac{1}{\sigma}} \tag{2}$$
$$(\sigma > 0, \ r = (r_1, \ldots, r_n) > 0),$$

dependent on the positive vector $r$ and parameter $\sigma$. It is analogous to the already studied operation $I_r$($r$ is a number) and in the one-dimensional case, these operations, provided $r = r_1$ and $\sigma = 1$, coincide. For the case $n > 1$, $r_1 = \ldots = r_n = r$, the operations $I_r$ and $I_r$ even when $\sigma = 1$ do not coincide; however, they do have analogous properties, which, for example, is evident from the fact that the function

$$(1 + |x|^2)^{r/2} \Lambda_r(x) \tag{3}$$

and the quantity that is inverse to it for any $\sigma > 0$ is a Marcinkievicz multiplier (when $1 < p < \infty$, cf 1.5.5, examples 8 and 9).

We will further write

$$I_{-r} F = f. \tag{4}$$

Since the multiplier $\Lambda_r$ is an infinitely differentiable function of polynomial growth, just as the quantity that is its inverse, then $I_r$ transforms mutually uniquely S' onto S'.

The operation $I_r$ is remarkable in that it performs the isomorphism

$$L_p^r = I_r(L_p) \qquad (1 < p < \infty). \tag{5}$$

In fact, if $f \in L_p$, then

$$\| I_{-r_l} F \|_p \ll \| f \|_p,$$

which follows from the fact that the functions

$$(1 + x_l^2)^{r_l/2} \Lambda_r(x) \qquad (l = 1, \ldots, n) \tag{6}$$

are Marcinkievicz multipliers (cf 1.5.5, example 10). Therefore $F \in L_p^r$ and

$$\| F \|_{L_p^r} \ll \| f \|_p.$$

Conversely, if $F \in L_p^r$, then

$$\| f \|_p \ll \| F \|_{L_p^r}.$$

This follows from the fact that the function (cf 1.5.5, example 11)

$$\Lambda_r^{-1}(x) \left\{ \sum_{j=1}^{n} (1 + x_j^2)^{r_j/2} \right\}^{-1} \tag{7}$$

is a Marcinkievicz multiplier.

Suppose $r > 0$, $\lambda, \delta > 0$, then, as we have proven, the isomorphism

$$I_{(\lambda+\delta) r}(L_p) = L_p^{(\lambda+\delta) r} \qquad (1 < p < \infty). \tag{8}$$

obtains. Remarkably, even though the operation $I_{\lambda r} I_{\delta r}$

is generally distinct from the operation $I_{(\lambda+\delta)r}$, they are equivalent in the sense that in addition to (8) isomorphism

$$I_{\lambda r} I_{\delta r}(L_p) = L_p^{(\lambda+\delta)r} \qquad (1 < p < \infty).$$

obtains. This follows from the fact that the functions $\mu, \mu^{-1}$ considered in the example (1.5.5, example 12) are Marcinkievicz multipliers.

9.4.1. Estimates of anisotropic kernels. Let us assign $\mathbf{r} = (r_1, \ldots, r_n) > 0$ and $\mathbf{l} = (l_1, \ldots, l_n)$, and suppose $\sigma > 0$ is so large that the inequalities

$$\sum_1^n \frac{l_j}{r_j} < \sigma - \sum_1^n \frac{1}{r_j} + \frac{1}{r_i} \qquad (i = 1, \ldots, n).$$

are satisfied. Our most immediate aim will be shown that in this case the Fourier transform (cf 9.4(2))

$$\hat{\Lambda} = \hat{\Lambda}_r(x) = G_r(\boldsymbol{x}) \tag{1}$$

is an ordinary function exhibiting the "derivative"*)

$$I_{-l} G_r = \widehat{\prod_1^n (1 + x_j^2)^{l_j/2} \Lambda}$$

(an ordinary function), subject to the inequalities

$$|I_{-l} G_r(\boldsymbol{x})| \leqslant c \begin{cases} \left\{ \sum_1^n |x_j|^{r_j \varkappa} \right\}^{-1} & \left( \varkappa = \sum_1^n \frac{1 + l_j}{r_j} - 1 > 0, \right) \\ \ln\left(\frac{1}{|\boldsymbol{x}|} + 1\right) & (\varkappa = 0), \\ 1 & (\varkappa < 0), \end{cases} \tag{2}$$

$$|I_{-l} G_r(\boldsymbol{x})| \leqslant c e^{-c_1 |\boldsymbol{x}|} \qquad (|\boldsymbol{x}| > 1), \tag{3}$$

where $c_1 > 0$ is sufficiently small and $\sigma$ appears only in constant c, from whence it follows that $G_r \in L$. This, in particular, shows that

*) We can show that this assertion is preserved if in it the operation $I_{-l}$ for integral $\mathbf{l}$ is replaced by the operation of the derivative $D^{\mathbf{l}} G_r = \widehat{(ix)^{\mathbf{l}} \Lambda}$. (P. I. Lizorkin [10]).

$$I_r f = \int G_r(x-u) f(u)\,du \tag{4}$$

is an ordinary convolution for $f \in L_p (1 \leq p \leq \infty)$.

9.4.2. Let $\Omega$ stand for the complex plane with the excision $-\infty < x \leq 0$ and $\rho = \lambda + i\mu$ is a complex number. We will in the following assume without explanation that $z^\rho$ is a single-valued branch, defined on $\Omega$, of the multi-valued function $z^\rho$, equal to $x^\rho = x^\lambda e^{i\mu \ln x}$ on the ray $0 < x < \infty$. In other words, if $z = x + iy$, then it is always assumed that

$z^\rho = |z|^\rho e^{i\rho \arg z}$, where $|\arg z| < \pi$.

Lemma 1. Suppose $0 < \alpha \leq 1$. Then

$$|z^\alpha - A^\alpha| \leq M |z - A|^\alpha \quad (z \in \Omega,\ A \geq 0), \tag{1}$$

where M does not depend on z and A.

Proof. Let us first consider the single-valued analytic function

$$f(z) = \frac{z^\alpha - 1}{(z-1)^\alpha} \tag{2}$$

on the domain $\Omega^*$ of the complex plane with two excisions $-\infty < x \leq 0$, $1 \leq x < \infty$, equal to

$$f(x) = \frac{x^\alpha - 1}{(x-1)^\alpha}$$

on the upper edge of the excision $1 \leq x < \infty$.

In order to construct this function, we assume that the function z appearing in the numerator of (1) is defined by the formula $z^\alpha = \rho^\alpha e^{i\alpha\theta}$ $(z = \rho e^{i\theta}, \rho > 0, -\pi < \theta < \pi)$; i.e., that $z^\alpha$ (in the numerator) is understood as the single-valued branch of $z^\alpha$, defined on $\Omega$, equal to $x^\alpha$ for $0 < x < \infty$; as far as the function $(z - 1)$ in the denominator is concerned, then it is understood in the sense of $(z-1)^\alpha = r^\alpha e^{i\alpha\varphi}$ $(z - 1 = re^{i\varphi}, r > 0, 0 \leq \varphi \leq 2\pi)$.

The function f(z) defined this way has the limit $\lim_{z \to \infty} f(z) = 1$; moreover, it is bounded on all edges of both excisions. Thus, it is bounded over the entire boundary of $\Omega^*$ and, by the principle of the maximum, is bounded on $\Omega^*$:

$$M \geqslant \left|\frac{z^a-1}{(z-1)^a}\right| = \frac{|z^a-1|}{|(z-1)^a|} = \frac{|z^a-1|}{|z-1|^a},$$

and we have proven inequality (1) when A = 1 and for all $z \in \Omega^*$, but then also for $z \in \Omega$, because for real $z = x > 1$, the inequality

$$|x^a-1| \leqslant |x-1|^a \qquad (0<a\leqslant 1)$$

is well known.

If now A is an arbitrary positive number, then for $z \in \Omega$

$$|z^a-A^a| = A^a\left|\left(\frac{z}{A}\right)^a-1\right| \leqslant MA^a\left|\frac{z}{A}-1\right|^a = M|z-A|^a,$$

and we have proven (1).

Lemma 2. Suppose $\alpha \geqslant 1$, then for $z \in \Omega$ and any $A > 0$,

$$|z^a-A^a| \leqslant M|z-A|(A^{a-1}+|z|^{a-1}), \tag{3}$$

obtains, where M does not depend on z and A.

Proof. In fact, let us connect points A and z with the segment c:

$$\zeta = A+t(z-A) \qquad (0\leqslant t\leqslant 1),$$

obviously belonging to $\Omega$. Then

$$z^a - A^a = a\int_c z^{a-1}\,dz = a\int_0^1 [A+t(z-A)]^{a-1}(z-A)\,dt,$$

from whence also follows (3) ($(a+b)^\beta \leqslant c(a^\beta + b^\beta)$, $\beta > 0$, and c does not depend on a and b).

9.4.3. Let us introduce the notation

$$V = \sum_1^n (1+u_j^2)^{\frac{r_j a}{2}}, \quad U = \sum_1^n (1+u_j^2)^{\frac{r_j}{2r_n}} \qquad (r_j>0) \tag{1}$$

and take note of the inequality

$$U'^{r_n} \leqslant cV, \tag{2}$$

where c does not depend on V

$$|u_n| \leqslant (1+u_n^2)^{1/2} = (1+u_n^2)^{\frac{r_n}{2r_n}} \leqslant U. \tag{3}$$

Here (2) follows from the fact that for $\beta > 0$ and any $x_j > 0$

$$\left(\sum_1^n x_j^\beta\right)^{1/\beta} \leqslant c_\beta \sum_1^n x_j,$$

and $c_\beta$ does not depend on $x_j$.

Let us introduce the curve $L_{u'}$ in the plane of the complex variable $w_n = u_n + iv_n$:

$$u_n + ikU \ (0 < k < 1, \ -\infty < u_n < \infty), \tag{4}$$

dependent on the constant k and the vectorial parameter $u' = (u_1, \ldots, u_{n-1})$. Let us moreover introduce the curve $L^*_{u'}$. Suppose

$$B = k\left(\sum_1^{n-1} (1+u_j^2)^{\frac{r_j}{2r_n}} + 1\right), \tag{5}$$

where k is a constant. If $B \leq 1$, then we will assert that $L^*_{u'} = L_{u'}$, but if $B > 1$, then when $L_{u'}$ lies above point i, then suppose

$$L^*_{u'} = L_{u'} + l_{u'},$$

where $l_{u'}$ is the twice-transversed segment $[\bar{i}, i\bar{B}]$. More exactly, we assert that the oriented curve $L^*_{u'}$ is obtained by the following motion: first the point $L^*_{u'}$ transverses the left piece of $L_{u'}$ corresponding to an increment in $u_n$ over the interval $(-\infty, 0)$, then it descends along the segment $L_{u'}$ downward to i, envelopes i, ascends up to $L_{u'}$, and departs to $+\infty$ along the right piece of $L_{u'}$.

We let $E_{u'}$ stand for the set point $w_n$ filling part of the complex plane $w_n$ between the real axis $u_n$ and the curve $L^*_{u'}$.

Since $\Lambda = V^{-1/\sigma}$ is an infinitely differentiable function of polynomial growth, then $\hat{\Lambda} \in S'$ is meaningful. For small $r_j$ the integral

$$\hat{\Lambda}(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ixu} \Lambda(u)\, du$$

even in any case cannot converge absolutely, because study of the function $\hat{\Lambda}$ will proceed by the circuitous route of introducing the auxiliary function

$$V^{-\frac{\rho}{\sigma}} = \Lambda_{\rho, r, \sigma} \quad (\rho = \lambda + i\mu, \ \lambda > 0, \quad \Lambda_{1, r, \sigma} = \Lambda_r = \Lambda) \tag{6}$$

with complex parameter $\rho$. For sufficiently large $\lambda$, the direct notation in terms of the Lebesgue (absolutely convergent) integral*)

$$I_{-l}\hat{\Lambda}_{\rho, r, \sigma} = \frac{1}{(2\pi)^{n/2}} \int \frac{\prod_1^n (1 + u_j^2)^{l_j/2} e^{ixu}\, du}{V^{\rho/\sigma}} = \frac{1}{(2\pi)^{n/2}} \int e^{ix'u'}\, du' \int \frac{\prod_1^n (1 + u_j^2)^{l_j/2} e^{ix_n u_n}}{V^{\rho/\sigma}}\, du_n$$
$$(x' = (x_1, \ldots, x_{n-1}),\ u' = (u_1, \ldots, u_{n-1})). \tag{7}$$

is meaningful. Along with (7), we will further consider when $x_n > 0$ the function

$$\mu_\rho^{(l)}(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix'u'}\, du' \int_{L^*_{u'}} \frac{\prod_1^n (1 + u_j^2)^{l_j/2} e^{ix_n u_n}}{V^{\rho/\sigma}}\, du_n. \tag{8}$$

If $x_n < 0$, then $\mu_\rho^{(1)}(x)$ is determined analogously, but the curve in the complex plane symmetrical to it relative to the real axis would be taken as $L^*_{u'}$. The consideration of the second integral, which we will also denote with $\mu_\rho^{(1)}(x)$, is analogous and leads to analogous results.

*) on following page.

When $x_n > 0$ for real $\rho > \rho_0$, where $\rho_0$ is sufficiently large, the inner integrals (7) and (8) are equal to each other. In fact, in (8) the complex term

$$(1+z^2)^{\frac{r_n\sigma}{2}} = [1+(x+iy)^2]^{\frac{r_n\sigma}{2}} = (\xi+i\eta)^{\frac{r_n\sigma}{2}} \qquad (z \in E_{u'}),$$
$$\xi = 1+x^2-y^2, \quad \eta = 2xy.$$

appears in V.

The number $\xi + i\eta$ can belong to the excision $-\infty < \xi \leq 0$ if and only if $x = 0$, $y^2 \geq 1$, i.e., if

$$z = iy, \quad y^2 \geqslant 1. \tag{9}$$

But points of the formula (9) do not belong to $E_{u'}$, which shows that V is a single-valued analytic function of $w_n$ on $E_{u'}$. In the following (cf 9.4.6(7)) it will be shown that here, for sufficiently small k

$$-\pi < \arg V < \pi \tag{10}$$

(if we assume a priori that $|\arg V| \leq \pi$), which shows that when $w_n$ transverses $E_{u'}$, the point V belongs to $\Omega$ (a plane with the excision $-\infty < x \leq 0$), but then $V^{\rho/\sigma}$ is also (for real $\sigma$ and complex $\rho$) a single-valued analytic function.

Thus, a single-valued function, analytic in the domain $E_{u'}$, appears under the sign of the interrals of (7) and (8). Their equality follows from the fact that the integral over the segment $c_\xi$ of points $\xi + i\eta$ $(0 \leq \eta \leq kU)$ tends to zero as $|\xi| \to \infty$:

$$\left|\int_{c_\xi} \frac{(1+u_n^2)^{l_n/2} e^{ix_n u_n}}{V^{\rho/\sigma}} du_n\right| \leqslant \int_0^{kV} \frac{e^{-x_n\eta}\, d\eta}{|1+(\xi+i\eta)^2|^{\frac{r_n\rho-l_n}{2}}} \leqslant$$
$$\leqslant \frac{1}{(|\xi|-1)^{\frac{r_n\rho-l_n}{2}}} \int_0^{\infty} e^{-x_n\eta}\, d\eta \to 0.$$

*) In considering the operation $D^l\hat{\Lambda}_\rho$ in (7) and (8), the products $\prod_1^n (1 = x^2)^{l_j/2}$ is replaced with $(ix)^l$.

We can similarly prove the equality of the inner integrals in (7) and in an expression corresponding to (8), when $x_n < 0$. This shows that

$$I_l \hat{\Lambda}_{\rho, r, \sigma}(x) = \mu_\rho^{(l)}(x) \qquad (x_n \neq 0, \ \rho > \rho_0), \tag{11}$$

if $\rho_0 > 0$ is sufficiently large.

Estimates for the function $\mu_\rho^{(1)}(x)$ will be obtained in 9.4.6.

Based on these estimates and the analytic properties of $I_{-1}\hat{\Lambda}_{,r,}$ and $\mu_\rho^{(1)}$, we succeed in showing (cf 9.4.7) that equality (11) actually does obtain for all complex $\rho = \lambda + i\mu$, in particular for $\rho = 1$. Thus, the generalized function $\hat{\Lambda}$ is the ordinary function $\hat{\Lambda}(x) = \mu_1(x) = \mu_1^{(0)}$ summable on $R_n$. Estimates that will be obtained for $\mu_1^{(1)}(x)$ are directly transferred to $I_{-1}\hat{\Lambda}^{(1)}(x)$, which in fact leads to the inequality 9.4.1(1), (2).

9.4.4. Let us begin with estimation of the n-th integral ($r_j$, $s > 0$, $l = (l_1, \ldots, l_n) \geq 0$, explanations below)

$$\int \frac{\prod_1^n (1+u_j^2)^{l_j/2}}{V^{\frac{s}{\sigma}}} du = \int \frac{\prod_1^n (1+u_j^2)^{l_j/2} du}{\left\{\sum_1^n (1+u_j^2)^{\frac{r_j\sigma}{2}}\right\}^{\frac{s}{\sigma}}} =$$

$$= \int_{|u|<1} + \int_{|u|>1} \ll 1 + \int_{\substack{|u|>1 \\ u_j>0}} \frac{\prod_1^n (1+u_j^2)^{l_j/2}}{\left\{\sum_1^n u_j^{r_j\sigma}\right\}^{s/\sigma}} du \ll$$

$$\ll 1 + \int_{\substack{|\xi|>\beta>0 \\ \xi_j>0}} \frac{\prod_{j=1}^n (1+\xi_j^{2/r_j})^{l_j/2} \xi_j^{\frac{1}{r_j}-1}}{\left(\sum_1^n \xi_j\right)^s} d\xi \ll$$

$$\ll 1 + \int_\beta^\infty \frac{\rho^{\sum_1^n \frac{1+l_j}{r_j} - n} \rho^{n-1} d\rho}{\rho^s} = 1 + \int_\beta^\infty \frac{d\rho}{\rho^{1+\left(s - \sum_1^n \frac{1+l_j}{r_j}\right)}} < \infty, \tag{1}$$

if

$$s > \sum_1^n \frac{1+\gamma_j}{r_j}. \tag{2}$$

In the third relation the estimate of the integral on $\{|u|>1\}$ is reduced to the estimate on $\{|u| > 1,\ u_j > 0;\ j = 1, \ldots, n\}$ owing to the symmetrical properties of the function V. In the fourth, the change of variables $u_j^{r_j} = \xi_j$ with the Jacobian appearing in the numerator under the integral and the fifth term is introduced; here, we further make use of the inequality

$$\Big(\sum_1^n \xi_j^\sigma\Big)^{1/\sigma} \gg \sum_1^n \xi_j \quad (\sigma > 0);$$

here $\beta > 0$ is a sufficiently small number that the sphere $|u| < 1$ and closes the sphere $|\xi| < \beta$. In the fifth we introduce the conversion to polar coordinates. From (1) and (2) it follows that when $\rho = \lambda + i\mu$, $1 = 0$ and

$$\lambda > \sum_1^n \frac{1}{r_j} \tag{3}$$

integral 9.4.3(7) converges absolutely and can be written as

$$\int_R = \int_{R'} du' \int du_n, \quad u' = (u_1, \ldots, u_{n-1}),$$

where the inner integral (with respect to $u_n$) converges absolutely for any $u'$ ($|V^\rho| = |V|^\lambda > |u_n|^{\lambda r_n}$, $\lambda r_n > 1$, cf (3)).

9.4.5. Other show that whatever the $\varphi \in S$, the function

$$\Phi(\rho) = (I_{-l}\hat{\Lambda}_{\rho, r, \sigma}, \varphi) \qquad (\rho = \lambda + i\mu,\ \lambda > 0), \tag{1}$$

is analytic on $\{\lambda > 0\}$. We obviously have

$$\Phi(\rho) = (\Lambda_{\rho, r, \sigma}, \psi) = \int \frac{\psi(u)\,du}{V^{\rho/\sigma}} \quad \Big(\psi = \prod_1^n (1 + u_j^2)^{l_j/2}\hat{\varphi} \in S\Big). \tag{2}$$

The derivative of $\Phi$ is formally of the form

$$\Phi'(\rho) = -\frac{1}{\sigma}\int \frac{\psi \ln V}{V^{\rho/\sigma}}\,du. \tag{3}$$

Continuous functions ($V \geqslant 1$) are found under the integrals in (2) and (3) and, moreover,

$$\left|\frac{\psi(u)}{V^{\rho/\sigma}}\right| = \frac{|\psi(u)|}{V^{\lambda/\sigma}} \leqslant |\psi(u)| \in L,$$

$$\left|\frac{\ln V\psi(u)}{V^{\rho/\sigma}}\right| = \frac{|\ln V\psi(u)|}{V^{\lambda/\sigma}} \ll |\psi(u)| \in L,$$

where the right sides do not depend on $\rho$. This proves that differentiation (3) is legitimate and that $\Phi'(\rho)$ is continuous, and consequently, $\Phi$ is analytic when $\lambda > 0$.

9.4.6. Below it will be proven that if the parameter $\sigma$ is sufficiently large (more exactly, $r_n(\mathcal{H} - \sigma) < 1$, $\mathcal{H} = \sum_1^n \frac{1 + l_j}{r_j} - \lambda$*)), then the integral

(cf 9.4.3(8))

$$\mu_\rho^{(n)}(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix'u'}\,du' \int_{L_{u'}} \frac{\prod_1^n (1+u_j^2)^{l_j/2} e^{ix_n u_n}}{V^{\rho/\sigma}}\,du_n \tag{1}$$

($\rho = \lambda + i\mu$, $\lambda > 0$, кроме того**), $|\mu| < 1$)

($\rho = \lambda + i\mu$, $\lambda > 0$, moreover**), $|\mu| < 1$)

is a function continuous with respect to $(\rho, x)$ on the set $\{\lambda > 0, |\mu| < 1, x_n \neq 0\}$, analytic with respect to $\rho$ and the estimates

$$|\mu_\rho^{(n)}(x)| \ll \begin{cases} |x_n^{-r_n \varkappa}| & (\varkappa > 0), \\ |\ln|x_n|| + 1 & (\varkappa = 0), \ (|x_n| < 1), \\ 1 & (\varkappa < 0), \end{cases} \tag{2}$$

$$|\mu_\rho^{(n)}(x)| \ll e^{-c|x_n|} \quad (c > 0, |x_n| > 1). \tag{3}$$

are valid.

*) Here $\mathcal{H} = \mathcal{H}(\lambda)$, but $\mathcal{H}(1)$ approaches the value $\mathcal{H}$ considered in 9.4.1(2).
**)The restriction $|\mu| < 1$ is actually not essential.

Let us use $V_*$ to stand for the result of replacing $u_n$ in V with the complex variable $u_n + i\eta_n \in E_{u'}$, where $E_{u'}$ is the domain between $L_{u'}$ and the axis $u_n = 0$. Obviously

$$V_* = V + \omega,$$

where ($\eta_n = \eta$)

$$\omega = (1+(u_n+i\eta)^2)^{\frac{r_n\sigma}{2}} - (1+u_n^2)^{\frac{r_n\sigma}{2}} =$$
$$= (1+u_n^2+2u_n\eta i-\eta^2)^{\frac{r_n\sigma}{2}} - (1+u_n^2)^{\frac{r_n\sigma}{2}}.$$

Let us estimate $\omega$ from above. If $0 < r_n\sigma \leqslant 2$, then by 9.4.2(1) (explanations below)

$$|\omega| \leqslant M|2u_n\eta i-\eta^2|^{\frac{r_n\sigma}{2}} \ll ||u_n|\eta+\eta^2|^{\frac{r_n\sigma}{2}} \ll$$
$$\ll (|u_n|kU)^{\frac{r_n\sigma}{2}} + k^{r_n\sigma}U^{r_n\sigma} \ll k^{\frac{r_n\sigma}{2}}U^{r_n\sigma} \ll k^{\frac{r_n\sigma}{2}}V. \tag{4}$$

Use of inequality 9.4.2(1) is legitimate because as explained in 9.4.3, the complex point in the first brackets defining $\omega$ belongs to $\Omega$ (plane with the excision $-\infty < u_n \leqslant 0$).

We assume that the constants appearing in the inequality $\ll$ do not depend on k. The third inequality follows from $(x+y)^a \ll x^a + y^a$ $(x, y > 0)$; the next to last from the fact that $|u_n| \leqslant U$ (cf 9.4.3(3)), and the last from 9.4.3(2).

But if $r_n\sigma \geqslant 2$, then (cf 9.4.2(3))

$$|\omega| \ll |2u_n\eta i-\eta^2|\left[(1+u_n^2)^{\frac{r_n\sigma}{2}-1} + |2u_n\eta i-\eta^2|^{\frac{r_n\sigma}{2}-1}\right] \ll$$
$$\ll kV^{r_n\sigma} \ll kV, \tag{5}$$

because cf (9.4.3(2)) $\quad |2u_n\eta i-\eta^2| \ll |u_nkU| + k^2U^2 \leqslant kU^2$

and (cf 9.4.3(3) , $1 \leq U$)

$$(1+u_n^2)^{\frac{r_n\sigma}{2}-1} \leq U^{r_n\sigma-2}.$$

If follows from (4) and (5) that for sufficiently small k we can attain the result that for all $u_n + i\eta \in E_{u'}$

$$|\omega| < \gamma V \qquad \left(\gamma < \frac{1}{2}\right), \tag{6}$$

when $\gamma$ can be as small as we wish, hence

$$(1-\gamma)V \leq |V_*| \leq (1+\gamma)V. \tag{7}$$

Inequalities (6) and (7), in particular, are satisfied on the curve of $L^*_{u'}$ (upper bound of $E_{u'}$). The following estimate obtains for the differential of the length of the arc $L^*_{u'}$:

$$\text{на } L_{u'} : dL^*_{u'} = \sqrt{1+\left(k\frac{\partial\eta}{\partial u_n}\right)^2}\,du_n = \\ = \sqrt{1+(ku_n)^2(1+u_n^2)^{-1}}\,du_n \leq \sqrt{2}\,du_n, \tag{8}$$

on $l_{u'}$ : $dL^*_{u'} = |d\eta|$.

Argument $V_*$ (i.e., V on $E_{u'}$) for sufficiently small k designated thusly (assuming a priori, that $|\arg V_*| \leq \pi$):

$$|\arg V_*| < \left|\frac{\operatorname{Im} V_*}{\operatorname{Re} V_*}\right| < \frac{|\omega|}{V-|\omega|} < \frac{\gamma V}{(1-\gamma)V} < 1. \tag{9}$$

By this we have proven inequality 9.4.3(10), which we needed to in order to show that $V^{\rho/\sigma}$ is a single-valued analytic function of the complex variable $w_n \in E_{u'}$. From (9) and (7) it also follows that

$$|V_*^{\rho/\sigma}| = \left|(|V_*|e^{i\arg V_*})^{\frac{\lambda+i\mu}{\sigma}}\right| = |V_*|^{\frac{\lambda}{\sigma}} e^{-\frac{\mu}{\sigma}\arg V_*} \geqslant$$
$$\geqslant e^{-\frac{1}{\sigma}} |V_*|^{\frac{\lambda}{\sigma}} \geqslant cV^{\frac{\lambda}{\sigma}} \qquad (|\mu| < 1), \tag{10}$$

where, thus, c does not depend on $\lambda > 0$ and $|\mu| < 1$.

We have

$$\mu_\rho^{(l)}(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix'u'} du' \times$$
$$\times \left( \int_{-\infty}^{\infty} \frac{\prod_1^n (1+u_j^2)^{l_j/2} e^{ix_n(u_n+ikU)} \sqrt{1+ku_n(1+u_n^2)^{-1}}\, du_n}{V_*^{\rho/\sigma}} + \right.$$
$$\left. + \int_{l_{u'}} \frac{\prod_1^n (1+u_j^2)^{l_j/2} e^{ix_n w_n}}{V^{\rho/\sigma}} dw_n \right) = I_1^{(l)} + I_2^{(l)},$$

(11)

where $V_*$ is understood here as V on $L_{u'}$. The second integral develops if and only if

$$k\left(\sum_1^{n-1}(1+u_j^2)^{\frac{r_j}{2r_n}} + 1\right) > 1.$$

The modulo of the integrand in $I_1^{(l)}$ does not exceed

$$c\frac{\prod_1^n (1+u_j^2)^{l_j/2} e^{-kx_nU}}{V^{\lambda\sigma}} = \alpha(\lambda, x_n, u), \tag{12}$$

where c does not depend on $u$, $x_n$, $\rho = \lambda + i\mu(\lambda > 0, |\mu| < 1)$. The integral with respect to $u \in R_n$ of (12) we will then estimate. We would see that it is finite for any $l \geq 0$, $x_n > 0$, $\lambda > 0$. If we assume that $\alpha(\lambda, x_n, u)$ increases with decrease in $x_n$ and $\lambda$ $(V \geq 1)$, then we have

$$\alpha(\lambda, x_n, u) \leqslant \alpha(\lambda_0, x_n^0, u) \in L(R_n) - L$$
$$(\lambda \geqslant \lambda_0 > 0, \ x_n \geqslant x_n^0 > 0).$$

Moreover, the integrand $I_1^{(1)}$ is continuous with respect to $(\rho, x, u)$. In this case, based on the Weierstrass characteristic $I_1^{(1)} = I_1^{(1)}(\rho, x)$ is a continuous function of $(\rho, x)$. If the integrand in $I_1^{(1)}$ is differentiated with respect to (complex) $\rho$, then the module of the resulting derivative will be equal to, with an accuracy up to the constant coefficient,

$$\frac{|u^l| e^{-kx_n U} |\ln V_0|}{|V_0^{\rho/\sigma}|} \leqslant c \frac{|u^l| e^{-kx_n^0 U}}{V^{\frac{\lambda_0 - \varepsilon}{\sigma}}} = \alpha(\lambda_0 - \varepsilon, x_n^0, u) \tag{13}$$
$$\left(\lambda > \lambda_0 - \frac{\varepsilon}{2} > \lambda_0 - \varepsilon > 0; \ |\mu| < 1, \ x_n \geqslant x_0 > 0\right)$$

and, since the right side of (13) with respect to $u$ belongs to L, then owing to the Weierstrass characteristic of uniform convergence of the integral we can state that for specified $\rho$ and $x$ there exists the derivative $\partial\rho I_1^{(1)}$. continues with respect to $(\rho, x)$. This shows that the function $I_1^{(1)}(\rho, x)$ is analytic with respect to $\rho(l \geq 0, \lambda > 0, |\mu| < 1, x_n > 0)$.

Let us note that the constants in inequalities (12) and (13) (just as in the preceding equality in the estimate of $I_2^{(1)}$) depend continuously on $\rho$.

For small $x_n$ (explanations are the same of those in 9.4.4)

$$|I_1^{(n)}| \ll \int \frac{\prod_1^n (1+u_j^2)^{l_j/2} e^{-kx_n U}}{V^{\lambda/\sigma}} du \ll$$

$$\ll \int_{|u|<1} + \int_{|u|>1} \frac{\prod_1^n (1+u_j^2)^{l_j/2} e^{-kx_n \sum_1^n u_j^{r_j/r_n}}}{\left(\sum_1^n u_j^{r_j\sigma}\right)^{\lambda/\sigma}} du \ll$$

$$\ll 1 + \int_{\substack{|\xi|>\beta>0 \\ \xi_j>0}} \frac{\prod_1^n \left(1+\xi_j^{2/r_j}\right)^{l_j/2} \xi^{\frac{1}{r_j}-1} e^{-kx_n \sum_1^n \xi_j^{1/r_n}}}{|\sum \xi_j|^{\lambda}} d\xi \ll$$

$$\ll 1 + \int_\beta^\infty \rho^{\varkappa-1} e^{-cx_n \rho^{\frac{1}{r_n}}} d\rho \ll 1 + \int_{\gamma x_n}^\infty e^{-cz} z^{r_n\varkappa-1} dz \frac{1}{x_n^{r_n\varkappa}} \ll$$

$$\ll \begin{cases} x_n^{-r_n\varkappa} & (\varkappa > 0), \\ 1 & (\varkappa < 0), \\ |\ln x_n| + 1 & (\varkappa = 0) \end{cases} \quad (0 < x_n < 1). \tag{14}$$

We must bear in mind that the integral in the next to the last of the relations, taken over $(1,\infty)$ converges, but over $(\gamma x_n, 1)$ it does not exceed

$$\int_{\gamma x_n}^1 z^{r_n\varkappa-1} dz \ll \begin{cases} 1 & (\varkappa > 0), \\ x_n^{r_n\varkappa} & (\varkappa < 0), \\ |\ln x_n| + 1 & (\varkappa = 0) \end{cases} \left(0 < x_n < \frac{1}{\gamma}\right). \tag{15}$$

But for large $x_n > 1$

$$
\begin{aligned}
|I_1^{(l)}| &\ll \int\limits_{|u|<1} e^{-kx_n}\,du + \int\limits_{\substack{|u|\geq 1\\ u_j>0}} (1+u_j^2)^{\frac{l_j}{2}} e^{-kx_n\sum_1^n u_j^{r_j/r_n}}\,du \ll \\
&\ll e^{-kx_n} + \int\limits_{\substack{|\xi|\geq\beta\\ \xi_j>0}} \prod_{j=1}^{n}\left(1+\xi_j^{\frac{2r_n}{r_j}}\right)^{l_j/2}\cdot \xi_j^{\frac{r_n}{r_j}-1} e^{-kx_n\sum_1^n \xi_j}\,d\xi \ll \\
&\ll e^{-kx_n} + \int_\beta^\infty \rho^{r_n(\varkappa+\lambda)-1} e^{-cx_n\rho}\,d\rho \ll \\
&\ll e^{-kx_n} + \int_\beta^\infty e^{-\frac{c}{2}x_n\rho} \ll e^{-c_1x_n} \quad (c_1>0).
\end{aligned} \tag{16}
$$

Let us proceed to the estimate of $I_2^{(1)}$. Here the inner integral is taken along the section $(i,\ iB)$, where

$$
B = k\left(\sum_1^{n-1}(1+u_j^2)^{\frac{r_j}{2r_n}}+1\right).
$$

The number B depends on $u'$; when $u' = 0$ it is minimum and equal to $kn$. If $kn > 1$, then in computing $I_2^{(1)}$ the outer integration is performed with respect to all $u' \in R_{n-1}$, however if $kn<1$ then integration with respect to $u'$ proceeds along the external of some bounded neighborhood of the point $u' = 0$. We have $u_n = iy$ $(1 \leq y \leq B)$ on $(i,\ iB)$; here the term $(1 + u_n^2)^{r_n\sigma/2}$ along one margin of $l_{u'}$ appearing in V must be understood as $(y^2-1)^{r_n\sigma/2} e^{ir_n\sigma\pi}$, and on the other margin, as $(y^2-1)^{r_n\sigma/2} e^{-ir_n\sigma\pi/2}$. The corresponding V value on different margins $l_{u'}$ are complexly adjoint to each other; consequently, their product is equal to the square of their modules, and the inner integral in $I_2^{(1)}$ is equal to

$$
-\int_1^B \prod_1^{n-1} (1+u_j^2)^{l_j/2} y^{l_n} e^{-x_n y} \times
$$

$$
\times \left\{ \frac{1}{\left\{ A + \left[ (y^2-1) e^{-i\pi} \right]^{\frac{r_n \sigma}{2}} \right\}^{\rho/\sigma}} - \frac{1}{\left\{ A + \left[ (y^2-1) e^{i\pi} \right]^{\frac{r_n \sigma}{2}} \right\}^{\mu/\sigma}} \right\} dy,
$$

$$
A = \sum_1^{n-1} (1+u_j^2)^{\frac{r_j \sigma}{2}} \qquad (\rho = \lambda + i\mu,\ \lambda > 0). \tag{17}
$$

The module of the expression in the braces does not exceed (explanations below)

$$
\frac{\left| \left( A + \left[ (y^2-1) e^{i\pi} \right]^{\frac{r_n \sigma}{2}} \right)^{\rho/\sigma} - \left( A + \left[ (y^2-1) e^{-i\pi} \right]^{\frac{r_n \sigma}{2}} \right)^{\rho/\sigma} \right|}{\left| A - (y^2-1)^{\frac{r_n \sigma}{2}} \right|^{2\lambda/\sigma}} =
$$

$$
= A^{-\frac{\lambda}{\sigma}} \frac{\left| \left\{ 1 + \left( \tau e^{i\frac{\pi}{2}} \right)^{r_n \sigma} \right\}^{\mu/\sigma} - \left\{ 1 + \left( \tau e^{-i\frac{\pi}{2}} \right)^{r_n \sigma} \right\}^{\mu/\sigma} \right|}{\left| 1 - \tau^{r_n \sigma} \right|^{2\lambda/\sigma}} \ll
$$

$$
\ll a^{-r_n \lambda} \tau^{r_n \sigma} \left| \sin \frac{r_n \sigma \pi}{2} \right| \ll a^{-r_n \lambda} \tau^{r_n \sigma},
$$

where the constants in the inequality in any case can be assumed to be locally independent of $\rho = \lambda + i\mu$; here

$$
\tau = a^{-1} \sqrt{y^2 - 1}, \qquad a = A^{\frac{1}{r_n \sigma}}, \tag{18}
$$

and since $1 < y < B$, then

$$0 < \tau < a^{-1}\sqrt{B^2 - 1} \ll a^{-1}B = \frac{k\left\{\sum_{1}^{n-1}(1+u_j^2)^{\frac{r_j}{2r_n}} + 1\right\}}{\left\{\sum_{1}^{n-1}(1+u_j^2)^{\frac{r_j}{2}}\right\}^{\frac{1}{r_n^0}}} \leqslant ck = \omega < 1, \tag{19}$$

where $\omega$ can be assumed to be smaller than unity, given sufficiently small k. On this ground, we drop the denominator in the second term, restricting from below the positive constant. The function under the sign of the module in the numerator is analytic on the interval $|\tau| < 1$, equal to zero when $\tau = 0$. The theorem on the mean can be applied to it. Thus, the integrand in $I_2^{(1)}$ does not exceed, based on the module,

$$c\prod_{1}^{n-1}(1+u_j^2)^{r_j/2} y^{l_n} e^{-x_n y} a^{-r_n\lambda} \tau^{r_n^0}, \tag{20}$$

where c does not depend on $\mathbf{u}'$, y, $x_n > 0$ and in any case locally on $\lambda > 0$.

We will show that function (20) is summable over the domain $(\mathbf{u}'\ y)$ of definition of the integral $I_2^{(1)}$ for any specified $\mathbf{x}, \rho$; moreover, it is immediately clear that increases with decrease in $x_n$ and $\lambda$. This leads to the fact that $I_2^{(1)}(\rho, \mathbf{x})$ is continuous with respect to the specified $(\rho, \mathbf{x})$ and is a real derivative (with respect to $\mathbf{x}$) of order $\mathbf{l}$ of $I_2$. Finally, if we differentiate the integrand in (17) with respect to $\rho$, we get

$$\prod_{1}^{n-1} (1+u_j^2)^{l_j/2} y^{l_n} e^{-x_n y} \frac{1}{\sigma} \left\{ \frac{\ln\left(A + [(y^2-1)e^{-i\pi}]^{\frac{r_n\sigma}{2}}\right)}{\left(A - [(y^2-1)e^{-i\pi}]^{\frac{r_n\sigma}{2}}\right)^{\rho/\sigma}} - \right.$$

$$\left. - \frac{\ln\left(A + [(y^2-1)e^{i\pi}]^{\frac{r_n\sigma}{2}}\right)^{\rho/\sigma}}{\left(A + [(y^2-1)e^{i\pi}]^{\frac{r_n\sigma}{2}}\right)^{\rho/\sigma}} \right\}.$$

If the expression in the braces is reduced to a common denominator, its module taken, and A is everywhere removed from within the braces, then, as we know, in estimating (from above) the denominator can be dropped as a positive constant bounded from below; as for the numerator, it obviously can be estimated from above by the following means:

$$\ln A \left| \left\{1 + \left(\tau e^{+i\frac{\pi}{2}}\right)^{r_n\sigma}\right\}^{\rho/\sigma} - \left\{1 + \left(\tau e^{-\frac{i\pi}{2}}\right)^{r_n\sigma}\right\}^{\rho/\sigma} \right| +$$

$$+ \left| \left\{1 + \left(\tau e^{\frac{i\pi}{2}}\right)^{r_n\sigma}\right\}^{\rho/\sigma} \ln\left\{1 + \left(\tau e^{-\frac{i\pi}{2}}\right)^{r_n\sigma}\right\} - \right.$$

$$\left. - \left\{1 + \left(\tau e^{-\frac{i\pi}{2}}\right)^{r_n\sigma}\right\}^{\rho/\sigma} \ln\left\{1 + \left(\tau e^{\frac{i\pi}{2}}\right)^{r_n\sigma}\right\} \right| \ll (\ln A + 1)\tau^{r_n\sigma} =$$

$$= (\ln a^{r_n\sigma} + 1)\tau^{r_n\sigma} \ll a^{\varepsilon}\tau^{r_n\sigma} \quad (\varepsilon > 0,\ a \geqslant 1).$$

The constant in the right side depends on (arbitrary small) $\varepsilon$; but we can assume that it does not depend on $\rho$ from some small neighborhood of $\rho_0$. As a result,

we find that that integrand continues with respect to ($u'$, $y$, $\rho$, $x_n$), $x_n > 0$,

$\lambda > 0$), differentiated with respect to $\rho$, does not exceed as to module the function analogous to (20),

$$c\prod_{1}^{n-1} (1+u_j^2)^{l_j/2} y^{l_n} e^{-x_n y} a^{\varepsilon - r_n\lambda} \tau^{r_n\sigma} \in L. \tag{20'}$$

This shows that function $I_2^{(1)}(\rho, x)$ is analytic with respect to $\rho$.

And thus (explantions below)

$$\begin{aligned}
|I_2^{(1)}| &\ll \int \prod_1^{n-1} (1+u_j^2)^{l_j/2} a^{-r_n\lambda} \int_1^B y^{l_n} e^{-x_n y} \tau^{r_n\sigma} dy \ll \\
&\ll \int \prod_1^{n-1} (1+u_j^2)^{l_j/2} a^{1-r_n\lambda} du' \int_0^\omega (1+a^2\tau^2)^{l_n/2} \times \\
&\times e^{-x_n\sqrt{1+a^2\tau^2}} \tau^{r_n\sigma} d\tau \ll 1 + \int\limits_{\substack{|\xi|>\beta>0 \\ \xi_j>0}} \prod_1^{n-1} \left(1+\xi_j^{\frac{2r_n}{r_j}}\right)^{\frac{l_j}{2}} \times \\
&\times \xi_j^{\frac{r_n}{r_j}-1} a^{1-r_n\lambda} d\xi \int_0^\omega (1+a^2\tau^2)^{l_n/2} e^{-x_n\sqrt{1+a^2\tau^2}} \tau^{r_n\sigma} d\tau \ll \\
&\ll 1 + \int_0^\infty \tau^{r_n\sigma} d\tau \int_\beta^\infty \rho^{l_n+1-r_n\lambda+r_n\sum_1^{n-1}\frac{1+l_j}{r_j}-1} e^{-x_n\sqrt{1+c^2\rho^2\tau^2}} d\rho = \\
&= 1 + \int_0^\infty \tau^{r_n\sigma} d\tau \int_\beta^\infty \rho^{r_n\varkappa-1} e^{-x_n\sqrt{1+c^2\rho^2\tau^2}} d\rho \ll \\
&\ll 1 + \frac{1}{x_n^{r_n\varkappa}} \int_0^\infty \frac{d\tau}{\tau^{r_n(\varkappa-\sigma)}} \int_{\beta x_n\tau}^\infty \zeta^{r_n\varkappa-1} e^{-c\zeta} d\zeta \qquad (c>0).
\end{aligned}$$

(21)

In the first relation we employ the estimate of (20); in the second we replace $y$ with $\tau$ from formula (18) in the inner integral; we also took inequality (19) into account; in the third relation, the integral with respect to $u'$ was decomposed into two: with respect to $|u'| < 1$ and with respect to $|u'| > 1$; of which the first, obviously, is bounded; moreover, we consider the symmetric properties with respect to $u'$ of the integrand; the problem was reduced to integration with respect to $u_j > 0$. Here the substitution of variable

$$\xi_j = u_j^{r_j/r_n}, \quad du_j = \frac{r_n}{r_j} \xi_j^{\frac{r_n}{r_j}-1} d\xi_j \qquad (j = 1, \ldots, n-1)$$

was made; in the fourth relation, we change the order of integration; the polar coordinates ( $|\xi| = \rho$ ) was introduced into the space $\xi$ ; we use the fact that the variables a and $\rho$ have the same order:

$$\rho = \left(\sum_1^{n-1} u_j^{2r_j/r_n}\right)^{1/2} \ll \left\{\sum_1^{n-1}(1+u_j^2)^{\frac{r_j\rho}{2}}\right\}^{\frac{1}{r_n\sigma}} = a \ll \rho,$$

and we employ the inequality $(1+\rho^2\tau^2)^{l_n/2} \ll \rho^{l_n} (\rho > B,\ 0 < \tau < \omega)$.

Finally, in the last relation we use the ineuqality $\sqrt{1 + c^2\rho^2\tau^2} > c\rho\tau$ and introduce the change $x_n\rho\tau = \xi$, $x_n\tau\, d\rho = d\xi$.

Let $\varkappa > 0$ and let the parametor $\sigma$ be chosen so that $r_n(\varkappa - \sigma) < 1$; then the integral in $\zeta$ in the right side of (21) is finite with respect to the interval $(0, \infty)$, just as the integral with respect to $\tau$ is finite; therefore

$$|I_2^{(1)}| \ll x_n^{-r_n\varkappa}.$$

If $\varkappa = 0$, then integral in $\zeta$ is, for $x_n$, of the order $(x_n\tau)$; therefore when $\sigma > 0$, we will have

$$|I_2^{(1)}| \ll |\ln x_n| + 1.$$

Finally, when $\varkappa < 0$ the integral in $\zeta$ for small $x_n$ is of the order $x_n^{r_n\varkappa}$ ; so when $\sigma > 0$

$$|I_2^{(1)}| \ll 1.$$

We have proven that (for the appropriate $\sigma$)

$$|I_2^{(1)}| \ll \begin{cases} x_n^{-r_n\varkappa}, & \varkappa > 0, \\ 1, & \varkappa < 0, \\ \ln \frac{1}{x_n}, & \varkappa = 0 \end{cases} \qquad (0 < x_n < 1).$$

Finally, to get the estimate of $I_2^{(1)}$ for large $x_n$, let us decompose the integral $I_2^{(1)}$ into two integrals: with respect to $|u'| < 1$ and with respect to $|u'| > 1$. The first integral (cf third member in formula (21)) is of the order $e^{-x_n} (x_n > 1)$. Estimating the second, let us use the next to last integral (21). Then we get

$$|I_2^{(l)}| \ll e^{-x_n} + \int_0^{\omega} \tau^{r_n\sigma}\, d\tau \int_{\beta}^{\infty} \rho^{r_n\varkappa-1} e^{-x_n\sqrt{1+c^2\rho^2\tau^2}}\, d\rho =$$
$$= e^{-x_n} + \int_0^{\omega} \frac{d\tau}{\tau^{r_n(\varkappa-\sigma)}} \int_{\beta\tau}^{\infty} \xi^{r_n\varkappa-1} e^{-x_n\sqrt{1+c^2\xi^2}}\, d\xi \leqslant$$
$$\leqslant e^{-x_n} + \int_0^{\omega} \frac{d\tau}{\tau^{r_n(\varkappa-\sigma)}} \left( \int_{\beta\tau}^{\beta\omega} \xi^{r_n\varkappa-1} d\xi\, e^{-x_n} + \int_{\beta\omega}^{\infty} e^{-x_n c_1 \xi} d\xi \right) \ll e^{-c_2 x_n}.$$

$(r_n(\varkappa-\sigma) < 1;\ c_1, c_2, \sigma > 0)$.

For the case $x_n < 0$, the curve of $L^*_{u'}$ (cf (1)) if the complex plane $u_n$ is taken as symmetrical with respect to the axis $u_n = 0$; the proof in this case is analogous.

Thus, we have proven inequalities (2) and (3); here we already noted that the constant in these expressions depends continuously on $\rho$.

9.4.7. In defining the function $\mu_\rho(\mathbf{x}) = \mu_\rho^{(0)}$ by formula 9.4.3(8), the role of the variable $u_n$ was emphasized. Bearing this in mind, let us set $\mu_\rho(\mathbf{x}) = \mu_{\rho n}(\mathbf{x})$. We can with equal success introduce the function $\mu_{\rho j}(\mathbf{x})$ ($j = 1, \ldots, n$), where the role of $u_n$ is played by $u_j$. If $\mathbf{x} = (x_1, \ldots, x_n)$ is a point of which $x_i \neq 0$, $x_j \neq 0$, then

$$\mu_{\rho i}(\mathbf{x}) = \mu_{\rho j}(\mathbf{x}),$$

because this equality obtains in any case for large real $\rho$, then also for any complex $\rho = \lambda + i\mu$ ($\lambda > 0$), owing to the analyticity of both functions with respect to $\rho$ for fixed $\mathbf{x}$. For a given j, function $\mu_{\rho j}(\mathbf{x})$ is defined and the continuous (with respect to $\mathbf{x}$) at any point $\mathbf{x}$ that has the coordinate $x_j \neq 0$. From the foregoing it is clear that $\mu_{\rho j}(\mathbf{x})$ can be extended by continuity to any point $\mathbf{x} \neq 0$, and then

$$\mu_\rho(\mathbf{x}) = \mu_{\rho 1}(\mathbf{x}) = \ldots = \mu_{\rho n}(\mathbf{x}) \qquad (\mathbf{x} \neq 0). \tag{1}$$

Here, there exists for the vector $\mathbf{l} \geqslant 0$ such a $\sigma_0 > 0$ that for $\sigma > \sigma_0$ the function $\mu_\rho^{\mathbf{l}}(\mathbf{x})$ is continuous and

$$|\mu_\rho^{(l)}(x)| \ll \begin{cases} |x_j|^{-r_j\varkappa} & (\varkappa > 0), \\ |\ln|x_j|| + 1 & (\varkappa = 0), \\ 1 & (\varkappa < 0) \end{cases} \tag{2}$$

$$\left(\varkappa = \sum_1^n \frac{1+l_j}{r_j} - \lambda,\ |x_j| < 1,\ \rho = \lambda + i\mu,\ \lambda > 0\right),$$

$$|\mu_\rho^{(l)}(x)| \ll e^{-c|x_j|} \qquad (|x_j| > 1,\ c > 0).$$

Hence the estimates

$$|\mu_\rho^{(l)}(x)| \leqslant c \begin{cases} \left\{\sum_1^n |x_j|^{r_j\varkappa}\right\}^{-1} & (\varkappa > 0), \\ |\ln|x|| + 1 & (\varkappa = 0), \qquad (|x| < 1), \\ 1 & (\varkappa < 0), \end{cases} \tag{3}$$

$$|\mu_\rho^{(l)}(x)| \leqslant ce^{-c'|x|} \qquad (|x| > 1,\ c > 0), \tag{4}$$

follow at once, where the constants c and c' appearing in the inequalities depend continuously on $\rho$.

When $l = 0$, it follows from the estimates that

$$\mu_\rho(x) = \mu_\rho^{(0)}(x) \in L = L(R_n).$$

In fact, when $\varkappa \leq 0$, this is obvious; but if $\varkappa > 0$, then by the fact that $\lambda > 0$,

$$\frac{1}{\varkappa}\sum_1^n \frac{1}{r_j} - 1 = \frac{\lambda}{\varkappa} > 0$$

and consequently (explanations as in 9.4.4),

$$\int |\mu_\rho(x)|\,dx = \int_{|x|<1} \left\{ \sum_1^n |x_j|^{r_j n} \right\}^{-1} dx +$$

$$+ \int_{|x|>1} e^{-c|x|} dx < \int_{\substack{|\xi|<\rho \\ \xi_j>0}} \left( \sum_1^n \xi_j \right)^{-1} \prod_1^n \xi_j^{\frac{1}{r_j n}-1} d\xi + 1 <$$

$$< \int_0^\rho \rho^{\frac{1}{n}\sum_1^n \frac{1}{r_j}-2} d\rho + 1 < 1.$$

Now let $\varphi \in S$, then the expression ($\mu_\rho \in L$)

$$(\mu_\rho, \varphi) = \int \mu_\rho(x)\varphi(x)\,dx \quad (\lambda > 0). \tag{5}$$

is meaningful.

Let $\mu_\rho^{(1)}(x)$ stand for the right side of (3) without multiplier $c = c(\rho)$. It can easily be seen by virtue of the monotone properties of the function $\varkappa$ (with respect to $\lambda$) and the continuity of $c(\rho)$ (with respect to $\rho$), that for any $\rho_0 = \lambda_0 + i\mu_0$ ($\lambda_0 > 0$, $|\mu_0| < 1$) a $\delta > 0$ can be found such that if

$|\rho - \rho_0| < \delta$ ($\rho = \lambda + i\mu$, $\lambda > 0, |\mu| < 1$), then

$$\begin{aligned} |\mu_\rho^{(1)}(x)| &\leq c\mu^{(1)}_{\lambda_0-\delta,\,\cdot}(x) \in L\,(|x|<1), \\ |\mu_\rho^{(1)}(x)| &\leq ce^{-c_1|x|} \in L\,(|x|>1), \end{aligned} \tag{6}$$

where c and $c_1$ do not depend on the specified $\rho$. Therefrre the Weierstrass characteristic of uniform convergence (locally with respect to $\rho$) of integral (5) is satisfied and $(\mu_\rho, \varphi)$ depends continuously on the complex $\rho$ ($\lambda > 0$). The derivative with respect to $\varphi\mu_\rho$ over $\rho$ is also continuous over $(\rho, x)$, and the same estimates as in (6) obtain for it (cf 9.4.6 (13) and (20)). This shows that the function $(\mu_\rho, \varphi)$ is differentiable, and so analytic with respect to $\rho$.

9.4.8. For any complex $\rho = \lambda + i\mu$ ($\lambda > 0$) and consequently for $\rho = 1$, given sufficiently large $\sigma$ ($r_n$ $(\varkappa - \sigma) < 1$) the equality

$$\hat{\Lambda}_{\rho,r,\sigma} = \mu_\rho(x). \tag{1}$$

obtains. Actually, the functions

$$(\hat{\Lambda}_{\rho,r,\sigma}, \varphi) \text{ и } (\mu_\rho, \varphi) \quad (\varphi \in S)$$

of analytic with respect to $\rho(\lambda > 0)$ and coincide for real and sufficiently large $\rho$; therefore they coincide any $\rho$, but also for any $\varphi \in S$, which entails (1).

9.4.9. Other estimates of anisotropic kernels. In the lemma below the differences of the kernel $G_r$ in the metric $L_p(R_{n-1})$ are estimated. The estimate will be used in proving embedding theorem. Let us introduce the notation:

$$x = (\eta, \zeta), \quad \eta = (x_1, \ldots, x_{n-1}), \quad x_n = \zeta.$$

Lemma*) Let $r = (r_1, \ldots, r_n) > 0$, $1 < p < \infty$, and let a nonintegral positive number L be given such that for a certain j, j = 1, ..., n-1, the inequalities

$$1 - \frac{1}{r_n} < \frac{L}{r_j} < \min_l \left\{ \sigma - \sum_1^n \frac{1}{r_k} - \frac{1}{r_l} \right\}. \tag{1}$$

are fulfilled. Then

$$\int_{R_{n-1}} \left| \Delta_{x_j h}^{s+1} G_r(\eta, \zeta) \right| d\eta \leqslant c |h|^L |\zeta|^{-r_n\left(\frac{1}{r_n} + \frac{L}{r_j} - 1\right)}, \tag{2}$$

where s is the integral part of L, i.e., L = s + l, $0 \leqslant l < 1$, s is an integer.

Without violating generality, we will take h > 0, j = 1, $G_r = G$ and introduce the kernel

$$K_\nu(t) = (1 + t^2)^{-\nu/2} \qquad (-\infty < t < \infty).$$

We will write

$$G(t) = G(t, x_2, \ldots, x_n),$$
$$I_{-L} G(t) = \psi(t);$$

We will understand the s-th difference with pitch h of the function $\varphi(\zeta)$ in the sense

$$\Delta_h \varphi = \varphi(\zeta + h) - \varphi(\zeta), \quad \Delta_h^{s+1} \varphi = \Delta_h \Delta_h^s \varphi.$$

From the following estimates it will be clear that $\psi$ is a summable function of t in any case for almost all $x_2, \ldots, x_{n+1}$.

We have

*) P. I. Lizorkin [10].

$$G(t)=\int K_{s+l}(t-\xi)\psi(\xi)\,d\xi,$$

$$\Delta_h^{s+1}G(t)=\left\{\int_{-\infty}^{t}+\int_{t}^{t+(s+1)h}+\int_{t+(s+1)h}^{\infty}\right\}\Delta_h^{s+1}K_{s+l}(t-\xi)\psi(\xi)\,d\xi=I_1+I_2+I_3,$$

$$\int|I_1|\,dt\leqslant\int_{-\infty}^{\infty}|\psi(\xi)|\,d\xi\int_{\xi}^{\infty}|\Delta_h^{s+1}K_{s+l}(t-\xi)|\,dt\leqslant \tag{3}$$

$$\leqslant\|\psi\|_L\int_0^{\infty}|\Delta_h^{s+1}K_{s+l}(t)|\,dt,\qquad L=L(-\infty,\infty).$$

But (explanations below)

$$\int_0^{\infty}|\Delta_h^{s+1}K_{s+l}(t)|\,dt=$$

$$=\int_0^{\infty}dt\int_0^h\ldots\int_0^h\left|K_{s+l}^{(s+1)}\left(t+\sum_1^{s+1}t_k\right)\right|dt_1\ldots dt_{s+1}\ll$$

$$\ll\int_0^{\infty}dt\int_0^h\ldots\int_0^h\frac{dt_1\ldots dt_{s+1}}{\left(t+\sum_1^{s+1}t_k\right)^{1-(s+l)+s+1}}\ll\int_0^{\infty}dt\int_0^{ch}\frac{\rho^s\,d\rho}{(t+\rho)^{2-l}}\ll$$

$$\ll\int_0^{ch}\rho^{s+l-1}\,d\rho\leqslant h^{s+l}.$$

In the second relation (inequality), we used the (third) estimate 8.1(7); in the third relation, polar coordinates were introduced into the space $t_1, \ldots, t_{s+1}$ and we consider that $\sum_1^{s+1} t_k \geqslant \left(\sum_1^{s+1} t_k^2\right)^{\frac{1}{2}}$; in the fourth, the order of integration was changed.

The integral $I_3$ is estimated analogously:

$$\int|I_j|\,dt\leqslant ch^{s+l}\|\psi\|_L\qquad(j=1,\,3). \tag{4}$$

Further

$$\int_{-\infty}^{\infty} |I_2|\,dt = \int_{-\infty}^{\infty} dt \left| \int_{t}^{t+(s+1)h} \Delta_h^{s+1} K_{s+l}(t-\xi)\,\psi(\xi)\,d\xi \right| \leqslant$$

$$\leqslant \int_{-\infty}^{\infty} |\psi(\xi)|\,d\xi \int_{\xi-(s+1)h}^{\xi} |\Delta_h^{s+1} K_{s+l}(t-\xi)|\,dt =$$

$$= \|\psi\|_L \int_{-(s+1)h}^{0} |\Delta_h^{s+1} K_{s+l}(t)|\,dt =$$

$$= \|\psi\|_L \int_{-(s+1)h}^{0} dt \left| \int_0^h \dots \int_0^h \left\{ K_{s+l}^{(s)}\left(t + \sum_1^s t_k + h\right) - \right.\right.$$

$$\left.\left. - K_{s+l}^{(s)}\left(t + \sum_1^s t_k\right) \right\} dt_1 \dots dt_s \right| \leqslant$$

$$\leqslant 2\|\psi\|_L \int_{-(s+1)h}^{(s+1)h} dt \int_0^h \dots \int_0^h \left| K_{s+l}^{(s)}\left(t + \sum_1^s t_k\right) \right| dt_1 \dots dt_s \ll$$

$$\ll \|\psi\|_L \int_{-(s+1)h}^{(s+1)h} dt \int_0^{ch} \frac{\rho^{s-1}\,d\rho}{|t+\rho|^{1-l}} =$$

$$= \|\psi\|_L \int_0^{ch} \rho^{s+l-1}\,d\rho \int_{-(s+1)\frac{h}{\rho}}^{\frac{s+1}{\rho}h} \frac{du}{|1+u|^{1-l}} \leqslant$$

$$\leqslant 2\|\psi\|_L \int_0^{ch} \rho^{s+l-1}\,d\rho \int_0^{\frac{s+1}{\rho}h+1} \frac{dv}{v^{1-l}} \ll$$

$$\ll \|\psi\|_L \int_0^{ch} \rho^{s+l-1} \left[\left(\frac{h}{\rho}\right)^l + 1\right] d\rho \ll \|\psi\|_L\, h^{l+s}.$$

(5)

From (3) - (5) it follows after the additional integration of the inequality with respect to $(x_1, \dots, x_{n-1})$ that

$$\int_{R_{n-1}} |\Delta_{x_j h}^s G_r(\eta, \xi)|\,d\eta \ll h^{s+l_j} \int_{R_{n-1}} |I_{x_j, -(s+l_j)} G_r|\,d\eta$$

$$(s = 0, 1, \dots;\ j = 1, \dots, n-1).$$

Using estimates 9.4.1(2) ( considering that $\mathcal{H} = \dfrac{1+s+l_1}{r_1} + \sum_2^n \dfrac{1}{r_j} - l > 0$), we get

$$\int_{R_{n-1}} |I_{x_1} -(s+l_1) G_r| d\eta \ll \int_{\xi_j>0} \frac{d\eta}{\xi^{r_n\varkappa} + \sum_1^{n-1} x_j^{r_j\varkappa}} \ll$$

$$\ll \int \frac{\prod_1^{n-1} \lambda_j^{\frac{1}{r_j\varkappa}-1} d\lambda}{\xi^{r_n\varkappa} + \sum_1^{n-1} \lambda_j} \ll \int \frac{\rho^{\sum_1^{n-1} \frac{1}{r_j\varkappa}-1}}{\xi^{r_n\varkappa} + \rho} d\rho =$$

$$= \frac{1}{\xi^{r_n\varkappa}} \int_{\rho<\xi^{r_n\varkappa}} \rho^{\sum_1^{n-1} \frac{1}{r_j\varkappa}-1} d\rho + \int_{\xi^{r_n\varkappa}<\rho} \rho^{\sum_1^{n-1} \frac{1}{r_j\varkappa}-2} d\rho \ll$$

$$\ll \frac{1}{\xi^{r_n\varkappa}} \xi^{r_n \sum_1^{n-1} \frac{1}{r_j}} + \xi^{r_n\left(\sum_1^{n-1} \frac{1}{r_j} - \varkappa\right)} = \xi^{-r_n\left(\frac{s+l_1}{r_1} + \frac{1}{r_n} - 1\right)};$$

The first inequality follows from the first inequality 9.4.1(2) ($\varkappa > 0$) and from the symmetry of its right part; the substitution $x_j^{r_j\varkappa} = \lambda_j$ was made in the second inequality; polar coordinates are introduced in the third; and in the last must be considered that

$$\varkappa - \sum_1^{n-1} \frac{1}{r_j} = \frac{s+l_1}{r_1} + \frac{1}{r_n} - 1 > 0.$$

## 9.5. Embedding Theorem

9.5.1. Theorem of different measures. The embedding

$$L_p^r(R_n) \to B_p^\rho(R_m), \tag{1}$$

$$\rho = (\rho_1, \ldots, \rho_m), \quad \rho_i = r_i\varkappa \ (i = 1, \ldots, m), \tag{2}$$

$$1 < p \leq \infty, \quad B_\infty^\rho = H_\infty^\rho, \quad 1 \leq m < n,$$

$$\varkappa = 1 - \frac{1}{p} \sum_{m+1}^n \frac{1}{r_j} > 0.$$

is valid.

Given the condition $2 \leq p \leq \infty$, $L_p^r \to B_p^r$ obtains (cf 9.3(3)) and the theorem follows from the corresponding theorem for the B--classes (cf 6.5).

Therefore, it is essential to prove it for $1 \leqslant p < 2$. However, the proof presented below is suitable for any finite p.

Proof. It is sufficient to conduct the proof for the case $m = n - 1$, because if $m < n - 1$, then we can proceed from n to n-1 by using embedding (1), and the transition from n-1 to m can be made by using the corresponding theorem for the B-classes (cf 6.5). This is possible owing to the transitivity of relations (2)(cf 7.1).

And thus, we need only prove the embedding

$$L_p^r(R_n) \to B_p^\rho(R_{n-1}), \tag{3}$$

$$\rho = (\rho_1, \ldots, \rho_{n-1}), \quad \rho_i = r_i \varkappa, \tag{4}$$

$$\varkappa = 1 - \frac{1}{pr_n} > 0. \tag{5}$$

Three relations (cf 9.3(1))

$$L_p^r(R_n) \to H_p^r(R_n) \to H_p^\rho(R_{n-1}) \to L_p(R_{n-1}).$$

obtain. This shows that the arbitrary function $f \in L_p^r(R_n)$ has the trace $g(x) = f|_{R_{n-1}}$ on $R_{n-1}$, belonging to $L_p(R_{n-1})$, and that the inequality

$$\|g\|_{L_p(R_{n-1})} \leqslant c\|f\|_{L_p^r(R_n)}. \tag{6}$$

is satisfied. We will assume that $y = (x_1, \ldots, x_{n-1}) \in R_{n-1}$, $z = x_n$, and let (as always) $\bar{\rho}_1$ be the largest integer less than $\rho_1$, and let $f \in L_p^r(R_n)$ by theorem 9.2.2

$$\frac{\partial^{\bar{\rho}_1} f}{\partial x_1^{\bar{\rho}_1}} \in L_p^{r'}(R_n),$$

where

$$r' = \varkappa' r, \qquad \varkappa' = 1 - \frac{\bar{\rho}_1}{r_1} > 0$$

(in fact, $\bar{\rho}_1 < \rho_1 < r_1$) and

$$\left\| \frac{\partial^{\bar{\rho}_1} f}{\partial x_1^{\bar{\rho}_1}} \right\|_{L_p^{r'}(R_n)} \leqslant c\|f\|_{L_p^r(R_n)}.$$

Therefore the representation

$$\frac{\partial^{\bar{\rho}_1} f}{\partial x_1^{\bar{\rho}_1}} = \int G_{r'}(y-\eta,\ z-\zeta)\, v(\eta,\ \zeta)\, d\eta\, d\zeta\ (v \in L_p(R_n)) \tag{7}$$

and

$$\| v \|_{L_p(R_n)} = \left\| \frac{\partial^{\bar{\rho}_1} f}{\partial x_1^{\bar{\rho}_1}} \right\|_{L_p^{r'}(R_n)} \leqslant c \| f \|_{L_p^{r}(R_n)}. \tag{8}$$

obtains. Let us suppose

$$w(y) = \int G_{r'}(y-\eta,\ \zeta)\, v(\eta,\ \zeta)\, d\eta\, d\zeta = \left. \frac{\partial^{\bar{\rho}_1} f}{\partial x_1^{\bar{\rho}_1}} \right|_{R_{n-1}} \tag{9}$$

(considering the evenness of $G_{r'}$). The explanation of the fact that the formal change $z = 0$ and (7) leads to the trace $\frac{\partial^{\bar{\rho}_1} f}{\partial x_1^{\bar{\rho}_1}}$ on $R_m$ will be made at the end of the proof.

Let

$$\Lambda(y,\ z) = \Delta^2_{x_1 h} G_{r'}(y,\ z)$$

be the second difference $G_{r'}$ with pitch h in the direction of the $x_1$ axis. Then

$$\Delta^2_{x_1 h} w = \int_{-\infty}^{\infty} \left( \int_{R_{n-1}} \Lambda(y-\eta,\ \zeta)\, v(\eta,\ \zeta)\, d\eta \right) d\zeta =$$

$$= \int_{-\infty}^{\infty} \left( \int_{R_{n-1}} \Lambda(\eta,\ \zeta)\, v(y-\eta,\ \zeta)\, d\eta \right) d\zeta,$$

from whence, by using the Minkowski inequality twice

$$\| \Delta^2_{x_1 h} w \|_{L_p(R_{n-1})} \leqslant$$

$$\leqslant \int_{-\infty}^{\infty} \left\{ \int_{R_{n-1}} dy \left| \int_{R_{n-1}} \Lambda(\eta,\ \zeta)\, v(y-\eta,\ \zeta)\, d\eta \right|^p \right\}^{1/p} d\zeta \leqslant$$

$$\leqslant \int_{-\infty}^{\infty} d\zeta \int_{R_{n-1}} |\Lambda(\eta,\ \zeta)|\, d\eta \left( \int | v(y-\eta,\ \zeta)|^p\, dy \right)^{1/p} =$$

$$= \int_{-\infty}^{\infty} I(h,\ \zeta) \| v(\eta,\ \zeta) \|_{L_p(R_{n-1})}\, d\zeta, \tag{10}$$

$$I(h, \zeta) = \int_{R_{n-1}} |\Delta^2_{x_1 h} G_{r'}(\eta, \zeta)|\, d\eta. \tag{11}$$

Let us set

$$\alpha_1 = \rho_1 - \bar{\rho}_1$$

and let us note that

$$\frac{1}{r_{n'}} + \frac{\alpha_1}{r_1'} = \frac{1}{1 - \frac{\bar{\rho}_1}{r_1}} \left( \frac{1}{r_n} + \frac{\alpha_1}{r_1} \right) = \frac{\frac{1}{r_n} + \frac{\alpha_1}{r_1}}{\frac{1}{pr_n} + \frac{\alpha_1}{r_1}} > 1 \ (p > 1),$$

therefore we can define $l_1$ satisfying the inequality $0 < l_1 < \alpha_1$ and such that

$$\frac{1}{r_n'} + \frac{l_1}{r_1'} > 1.$$

In this case, by virtue of estimates 9.4.9(2), for the kernel $G_{r'}$

$$I(h, \zeta) \ll |h|^{1+l_1} |\zeta|^{\beta'}, \tag{12}$$

$$I(h, \eta) \ll |h|^{l_1} |\zeta|^{\beta} \tag{13}$$

(the absolute magnitude of the second difference was replaced by the sum of the absolute magnitudes of the first differences exceeding it), where

$$\beta = -r_n' \left( \frac{1}{r_n'} + \frac{l_1}{r_1'} - 1 \right),$$

$$\beta' = -r_n' \left( \frac{1}{r_n'} + \frac{l_1 + 1}{r_1'} - 1 \right) = \beta - \frac{r_n}{r_1}.$$

Let us further introduce the numbers

$$\alpha = \frac{r_n'}{r_1'} p (l_1 - \alpha_1) - 1 < -1,$$

$$\alpha' = \frac{r_n'}{r_1'} p (l_1 - \alpha_1 + 1) - 1 > -1, \quad \alpha' = \alpha + \frac{r_n}{r_1} p.$$

The numbers $\alpha, \alpha'$, $\beta$, and $\beta'$ are associated by the following relations:

Below we use Hardy's inequality:

$$\alpha + p + p\beta = -1 - \frac{r_n' \alpha_1 p}{r_1'} + r_n' p = -1 - \frac{r_n \alpha_1 p}{r_1} + r_n p\left(1 - \frac{\beta_1}{r_1}\right) =$$
$$= -1 - \frac{r_n \alpha_1 p}{r_1} + r_n p\left(1 - \frac{\rho_1 - \alpha_1}{r_1}\right) = -1 + r_n p\left(1 - \frac{\rho_1}{r_1}\right) =$$
$$= -1 + r_n p\left(1 - 1 + \frac{1}{r_n p}\right) = 0,$$
$$\alpha' + p + p\beta' = 0.$$

Its proof, thus, reduces to the change of variable $\zeta$ = tu and then to the use of Minkowski's inequality*). The inequality

$$\int_0^\infty t^\alpha \left( \int_{|\zeta| < t} \varphi(\zeta)\, d\zeta \right)^p dt = \left\{ \int_0^\infty dt \left( \int_{|u|<1} \varphi(tu)\, t^{\frac{\alpha}{p}+1}\, du \right)^p \right\}^{1/p} \leqslant$$
$$\leqslant \int_{|u|<1} \left( \int_0^\infty \varphi(tu)^p\, t^{\alpha+p}\, dt \right)^{1/p} du = c \left( \int_{-\infty}^\infty \varphi(\zeta)^p\, |\zeta|^{\alpha+p}\, d\zeta \right)^{1/p},$$
$$c = \int_0^1 \frac{du}{u^{1+\frac{\alpha+1}{p}}} < \infty, \quad \text{если} \quad \alpha < -1, \quad 1 \leqslant p < \infty.$$

is similarly proven.

Now we have (setting t = $h^{r_1/r_n}$ in the third relation)

$$\int_0^\infty t^\alpha \left( \int_{|\zeta| > t} \varphi(\zeta)\, d\zeta \right)^p dt \leqslant c_1 \int_{-\infty}^\infty \varphi(\zeta)^p\, |\zeta|^{\alpha+p}\, d\zeta,$$
$$c_1 = \int_1^\infty \frac{du}{u^{1+\frac{\alpha+1}{p}}} \qquad (\alpha > -1, \ 1 \leqslant p < \infty).$$

*) Cf book by Hardy, Littlewood, and Polya [1].

$t = h^{r_1/r_n})$

$$\|f\|^p_{b^{\rho_1}_{x_1 p}(R_{n-1})} = \int_0^\infty h^{-1-p\alpha_1}\|\Delta^2_{x_1 h} w\|^p_{L_p(R_{n-1})}\,dh =$$

$$= \int_0^\infty h^{-1-p\alpha_1}\,dh\left\{\left[\int_{|\zeta|<h^{r_1/r_n}} + \int_{|\zeta|>h^{r_1/r_n}}\right]\times\right.$$

$$\left.\times\, l(\eta,\zeta)\|v(\eta,\zeta)\|_{L_p(R_{n-1})}\,d\zeta\right\}^p \ll$$

$$\ll \int_0^\infty h^{-1-p\alpha_1}\,dh\left(\int_{|\zeta|<h^{r_1/r_n}} h^{l_1}\zeta^{\beta}\|v(\eta,\zeta)\|_{L_p(R_{n-1})}\,d\zeta +\right.$$

$$\left.+ \int_{|\zeta|>h^{r_1/r_n}} h^{l_1+1}\zeta^{\beta'}\|v(\eta,\zeta)\|_{L_p(R_{n-1})}\,d\zeta\right)^p =$$

$$= \int_0^\infty t^{\alpha}\,dt\left(\int_{|\zeta|<t}\zeta^{\beta}\|v(\eta,\zeta)\|_{L_p(R_{n-1})}\,d\zeta\right)^p +$$

$$+ \int_0^\infty t^{\alpha'}\,dt\left(\int_{|\zeta|>t}\zeta^{\beta'}\|v(\eta,\zeta)\|_{L_p(R_{n-1})}\,d\zeta\right)^p \ll$$

$$\ll \int_{-\infty}^{\infty}\|v(\eta,\zeta)\|^p_{L_p(R_{n-1})}\,d\zeta = \|v\|^p_{L_p(R_n)}.$$

Since in the inequality obtained $x_1$ can be replaced with $x_j$, then we have proven (cf also (8)) that

$$\|f\|_{b^{\rho_j}_{x_j p}(R_n)} \ll \|v\|_{L_p(R_n)} \qquad (j = 1, \ldots, n-1),$$

from whence (cf further (6)) for $z = 0$, we have

$$\|f(y,z)\|_{\ldots_p(R_{n-1})} \ll \|f\|_{L^r_p(R_n)}. \tag{14}$$

It obviously is valid for any z not necessarily equal to zero, which is similarly proven. By this we have proved (3).

The function $f \in L^r_p(R_n)$ can be written as

$$f(y,z) = \int G_r(y-\eta, z-\zeta)\lambda(\eta,\zeta)\,d\eta\,d\zeta,$$
$$\|f\|_{L^r_p(R_n)} = \|\lambda\|_{L_p(R_n)}.$$

Therefore

$$f(y, z+h)-f(y, z)=$$
$$=\int G_r(y-\eta, z-\zeta)[\lambda(\eta, \zeta+h)-\lambda(\eta, \zeta)]\,d\eta\, d\zeta$$

and by (14)

$$\left\|\frac{\partial^{\bar{\rho}_1}}{\partial x_1^{\bar{\rho}_1}}[f(y, z+h)-f(y, z)]\right\|_{L_p(R_{n-1})} \ll$$
$$\ll \|f(y, z+h)-f(y, z)\|_{B_p^{\rho}(R_{n-1})} \ll$$
$$\ll \|f(y, z+h)-f(y, z)\|_{L_p^r(R_n)} =$$
$$=\|\lambda(\eta, \zeta+h)-\lambda(\eta, \zeta)\|_{L_p(R_n)} \to 0 \quad (h \to 0) \quad (1 \leqslant p < \infty).$$

This shows that if we specify z in $\frac{\partial^{\bar{\rho}_i}}{\partial x_i^{\bar{\rho}_i}} f(y, z)$, then we get a function of $y \in R_{n-1}$, which is the trace of $\frac{\partial^{\bar{\rho}_1} f}{\partial x_1^{\bar{\rho}_1}}$ on the subspace $x_n=z$.

9.5.2. Inverse theorem of different measures. Suppose $1 \leqslant p < \infty$ and suppose that the positive numbers $r_i$ $(i = 1, \ldots, n)$ and possible vector with nonnegative integral coordinates

$$\lambda=(\lambda_{m+1}, \ldots, \lambda_n),$$

for which

$$\rho_i^{(\lambda)}=r_i\left(1-\sum_{j=m+1}^{n} \frac{\lambda_j}{r_j}-\frac{1}{p} \sum_{j=m+1}^{n} \frac{1}{r_j}\right)=r_i \varkappa>0 \tag{1}$$
$$(i=1, \ldots, m).$$

be given.

Further let the function

$$\varphi_{(\lambda)}(u)=\varphi_{(\lambda)}(x_1, \ldots, x_m) \in B_p^{\rho^{(\lambda)}}(R_m).$$

be brought into correspondence with each vector .

Then we can construct the function $f(x) = f(x_1, \ldots, x_n)$ of n variables exhibiting the following properties:

$$(2)$$

$$f \in L_p^r(R_n),$$

$$\|f\|_{L_p^r(R_n)} \leqslant c \sum_\lambda \|\varphi_{(\lambda)}\|_{B_p^{\rho(\lambda)}(R_m)}, \tag{3}$$

$$f^{(\lambda)}|_{R_m} = \varphi_{(\lambda)}(u). \tag{4}$$

Proof. Let us show that we can take the function

$$f = \sum_\lambda f_{(\lambda)}, \tag{5}$$

already defined in 6.8, where the sum is extended over all possible admissible vectors $\lambda$ and

$$f_{(\lambda)} = \sum_{s=0}^{\infty} \varphi_{(\lambda)}^s \prod_{j=m+1}^{n} b^{-\frac{s\lambda_j}{r_j\varkappa}} \Phi_{\lambda_j}\left(b^{\frac{s}{r_j\varkappa}} x_j\right). \tag{6}$$

as f.

If it is essential to note that the function

$$q_s = \varphi_{(\lambda)}^s = g_{b^{\frac{s}{r_1\varkappa}}, \ldots, b^{\frac{s}{r_m\varkappa}}} \qquad (s = 0, 1, \ldots)$$

are integral and of exponential type $b^{s/r_j\varkappa}$ with respect to $x_j (j = 1, \ldots, m)$

$$\varphi_{(\lambda)} = \sum_{s=0}^{\infty} \varphi_{(\lambda)}^s$$

and that

$$\|\varphi_{(\lambda)}\|_{B_p^{\rho(\lambda)}(R_m)} = \left(\sum_{s=0}^{\infty} b^{sp} \|q_s\|_{L_p(R_m)}^p\right)^{1/p} < \infty.$$

As for the functions $\Phi_{\lambda_j}(t)$, they can be assumed to be equal to

$$\Phi_{\lambda_j}(t) = \frac{T_{\lambda_j}(A_j t)}{t^s}, \tag{7}$$

where $T_{\lambda_j}$ are suitably chosen trigonometric polynomials and $A_j$ are numbers. Here $\Phi_{\lambda_j}$ are integral functions of the exponential type 1. In contrast to 6.8, $t^3$ (instead of $t^2$) is inserted in the denominator of (7), which is not essential; on the other hand, here the functions $\Phi_{\lambda_j}$ together with their first-order derivatives belong to $L = L(-\infty,\infty)$.

The fact that $f \in B_p^r(R_n)$ and that the boundary conditions (4) are satisfied is proven in theorem 6.8. It remains to prove the properties (2) and (3).

Let $R_{n-1}$ stand for the subspace of points $(x_1, \ldots, x_{n-1})$. The inequality (explanations below)

$$\begin{aligned} \| I_{x_i(-r_i)} f_{(\lambda)}(x) \|_{L_p(R_{n-1})} &< \\ &< \sum_s \| q_s \|_{L_p(R_m)} b^{\frac{s}{\varkappa}\left(1 - \sum_{m+1}^{n} \frac{\lambda_j}{r_j} - \frac{1}{p} \sum_{m+1}^{n-1} \frac{1}{r_j}\right)} |\psi_s(x_n)| = \\ &= \sum_s \lambda_s a^{s/p} |\psi_s(x_n)|, \end{aligned} \tag{8}$$

where

$$\lambda_s = \| q_s \|_{L_p(R_m)} b^s, \quad a = b^{\frac{1}{r_n \varkappa}} \qquad (a > 1), \tag{9}$$

$$\psi_s(t) = \Phi(a^s t) \quad \text{when } i = 1, \ldots, n-1 \tag{10}$$

$$\psi_s(t) = a^{-r_n s} I_{-r_n} \Phi(a^s t) \quad \text{when } i = n, \ \Phi = \Phi_{\lambda_n}. \tag{11}$$

is valid.

The norm in the metric $L_p(R_{n-1})$ of each member of series (6) is equal to the product of $L_p$ norms of the cofactors of which it is constituted (in the corresponding subspaces of the variable on which these cofactors depend). Here we must consider that

$$\|l_{x_i}(-r_i) q_s\|_{L_p(R_m)} \ll b^{s\varkappa}\|q_s\|_{L_p(R_m)}$$
$$(i = 1, \ldots, m;\ s = 0, 1, \ldots)$$

(cf 8.7)

$$\left| l_{x_i}(-r_i) \Phi_{\lambda_i}\left(b^{\frac{s}{r_i\varkappa}} x_i\right)\right|_{L_p(R_{x_i})} \ll b^{\frac{s}{\varkappa}} \left|\Phi_{\lambda_i}\left(b^{\frac{s}{r_i\varkappa}} x_i\right)\right|_{L_p(R_{x_i})};$$
$$\left|\Phi_{\lambda_i}\left(b^{\frac{s}{r_i\varkappa}} x_i\right)\right|_{L_p(R_{x_i})} = c_i b^{-\frac{s}{pr_i\varkappa}} \qquad (i = m+1, \ldots, n-1),$$

where $R_{x_i}$ is the $x_i$ axis and $c_i$ does not depend on $s = 0, 1, 2, \ldots$

From (8) it follows (explanations below) that

$$\| l_{x_i}(-r_i) f_{(\lambda)}(x)\|_{L_p(R_{n-1})} \ll \left(\int_{-\infty}^{\infty}\left|\sum_s \lambda_s a^{s/p}\psi_s(y)\right|^p dy\right)^{1/p} \ll$$
$$\ll \left(\sum_s \lambda_s^p\right)^{1/p} = \left(\sum_s b^{s\rho}\|q_s\|^p_{L_p(R_m)}\right)^{1/p} = |\varphi_{(\lambda)}|_{B_p^{\rho(\lambda)}(R_m)},$$

which proves (2) and (3). But in these relations we must validate the second inequality.

Let us note the inequalities

$$|\psi_s(t)| < A, \tag{12}$$
$$|\psi_s(t)| < \frac{A}{a^s|t|} \qquad (a^s|t| > 1), \tag{13}$$
$$a^s\int_{-\infty}^{\infty}|\psi_s(t)|\,dt < A, \tag{14}$$

where the constant A does not depend on the series of the standing multipliers. In the case (10) these inequalities follow at once from the fact that $\phi(t)$ is an integral function representable in the form (7). But in the case (11), this requires an explanation. The function $\phi(t)$ is integral and of the exponential type 1 and belongs to L together with its derivatives; therefore its Fourier transform $\sqrt{2\pi}\mu(x)$ has a continuous derivative and a compact carrier on (-1, +1).

Thus, $\mu(1) = \mu(-1) = 0$ and, consequently ($r = r_n$),

$$\Phi(t) = \int_{-1}^{+1} \mu(\lambda) e^{i\lambda t} \, d\lambda, \tag{15}$$

$$\Phi(a^s t) = \int_{-1}^{+1} \mu(\lambda) e^{i\lambda a^s t} \, d\lambda = a^{-s} \int_{-a^s}^{a^s} \mu(a^{-s}\xi) e^{i\xi t} \, d\xi,$$

$$|I_{t(-r)} \Phi(a^s t)| = a^{-s} \left| \int_{-a^s}^{a^s} (1+\xi^2)^{r/2} \mu(a^{-s}\xi) e^{i\xi t} \, d\xi \right| =$$

$$= \left| \frac{a^{-s}}{t} \int_{-a^s}^{a^s} [\mu'(a^{-s}\xi) a^{-s} (1+\xi^2)^{r/2} +\right.$$

$$\left. + r(1+\xi^2)^{r/2-1} \xi \mu(a^{-s}\xi)] e^{i\xi t} \, d\xi \right| \ll$$

$$\ll \frac{a^{-s}}{|t|} a^s (a^{-s} a^{rs} + a^{(r-2)s} a^s) = \frac{a^{(r-1)s}}{|t|}.$$

We have proven (13) (cf (11)). Further, if it is considered that $\Phi(a^s t)$ is integral and of the exponential type $a^s$, we get

$$|\psi_s(t)| \leqslant a^{-r_n s} a^{r_n s} \max |\Phi(a^s t)| < A,$$

$$\int |\psi_s(t)| \, dt \leqslant a^{-r_n s} a^{r_n s} \int |\Phi(a^s t)| \, dt < A a^{-s},$$

i.e., (12) and (14).

Now we have

$$\int_0^\infty \left| \sum_s \lambda_s a^{s/p} \psi_s(y) \right|^p dy \ll \Lambda_1 + \Lambda_2 + \Lambda_3, \tag{16}$$

where

$$\Lambda_1 = \sum_{m=0}^{\infty} \int_{a^{-m-1}}^{a^{-m}} \left| \sum_{s=0}^{m} \lambda_s a^{s/p} \psi_s(y) \right|^p dy,$$

$$\Lambda_2 = \sum_{m=0}^{\infty} \int_{a^{-m-1}}^{a^{-m}} \left| \sum_{s=m+1}^{\infty} \lambda_s a^{s/p} \psi_s(y) \right|^p dy,$$

$$\Lambda_3 = \int_1^{\infty} \left| \sum_{s=0}^{\infty} \lambda_s a^{s/p} \psi_s(y) \right|^p dy.$$

But (cf (12), 1/p + 1/q = 1)

$$\Lambda_1 \ll \sum_{m=0}^{\infty} a^{-m} \left( \sum_{s=0}^{m} \lambda_s a^{\frac{s(1-\varepsilon)}{p}} a^{\frac{s\varepsilon}{p}} \right)^p \leqslant$$

$$\leqslant \sum_{m=0}^{\infty} a^{-m} \sum_{s=0}^{m} \lambda_s^p a^{s(1-\varepsilon)} \left( \sum_{s=0}^{m} a^{\frac{s\varepsilon q}{p}} \right)^{p/q} \leqslant \sum_{m=0}^{\infty} a^{-m} \sum_{s=0}^{m} \lambda_s^p a^{s(1-\varepsilon)} a^{m\varepsilon} =$$

$$= \sum_{s=0}^{\infty} \lambda_s^p a^{s(1-\varepsilon)} \sum_{m=s}^{\infty} a^{-m(1-\varepsilon)} \ll \sum_{s=0}^{\infty} \lambda_s^p a^{s(1-\varepsilon)} a^{s(\varepsilon-1)} = \sum_0^{\infty} \lambda_s^p$$

(when p = 1, the third member in this chain can be omitted).

Further (cf (14) , explanations below)

$$\Lambda_2 \leqslant \sum_{m=0}^{\infty} \int_{a^{-m-1}}^{a^{-m}} \sum_{s=m+1}^{\infty} \lambda_s^p a^s |\psi_s(y)| \sum_{s=m+1}^{\infty} |\psi_s(y)|^{p-1} dy \ll$$

$$\ll \sum_{m=0}^{\infty} \sum_{s=m+1}^{\infty} \lambda_s^p a^s \int_{a^{-m-1}}^{a^{-m}} |\psi_s(y)|\, dy = \sum_{s=0}^{\infty} \lambda_s^p a^s \sum_{m=0}^{s} \int_{a^{-m-1}}^{a^{-m}} |\psi_s|\, dy =$$

$$= \sum_{s=0}^{\infty} \lambda_s^p a^s \int_0^1 |\psi_s|\, dy \ll \sum_{s=0}^{\infty} \lambda_s^p,$$

because (cf (13))

$$\sum_{s=m+1}^{\infty} |\psi_s(y)| \ll \frac{1}{|y|} \sum_{s=m+1}^{\infty} a^{-s} \ll a^m a^{-m-1} \ll 1 \,(a^{-m-1} < y). \tag{17}$$

Finally (cf (14))

$$\Lambda_3 \leqslant \int_1^{\infty} \sum_{s=0}^{\infty} \lambda_s^p a^s |\psi_s(y)| \left( \sum_{s=0}^{\infty} |\psi_s(y)| \right)^{p-1} dy \ll$$

$$\ll \sum_{s=0}^{\infty} \lambda_s^p a^s \int_1^{\infty} |\psi_s| \, dy \ll \sum_s \lambda_s^p.$$

because

$$\sum_{s=0}^{\infty} |\psi_s(y)| = |\psi_0(y)| + \sum_1^{\infty} |\psi_s(y)| \ll 1 \quad (1 \leqslant y < \infty)$$

(cf (12) and (17) when $m = 0$).

We have proven that integral (16) does not exceed $\sum \lambda_s^p$. This fact is analogously proven also for the integral extended over $\{-\infty < y < 0\}$.

9.5.3. If it is considered, as we have stipulated, that $B_{\infty}^{\rho} = H_{\infty}^{\rho}$, then theorem 9.5.2 when $p = \infty$ ceases to be valid. In fact, the arbitrary function

$$f(x, y) \in W_{\infty}^{1,1}(R_2) = L_{\infty}^{1,1}(R_2) \subset H_{\infty}^{1,1}(R_2) \subset H_{\infty}^{\alpha,\alpha}(R_2),$$
$$0 < \alpha < 1,$$

is uniformly continuous (after suitable modification by the multiplier of planar measures zero). It satisfies on $R_2$, therefore also on the $R_1$ axis, the Lipshits condition of degree 1. However, the function $\varphi(x_1) \in H_{\infty}^1(R_1)$ not satisfying the Lipshits condition (it is even nowhere-differentiable, cf note to 5.6.2-5.6.3) can be defined on the $R_1$ axis. So there does not exist the function $f(x_1, x_2) \in W_{\infty}^{1,1}(R_2)$ which would extend $\varphi$ from $R_1$ onto $R_2$.

9.5.4*) From theorem 9.5.1 and 9.5.2, as a consequence, we can get the embedding theorems:

$$B_p^r(R_n) \rightleftarrows B_p^\rho(R_m), \tag{1}$$

where $\rho = \varkappa r$, $\varkappa = 1 - 1/p \sum\limits_{m+1}^{n} \dfrac{1}{r_j} > 0$, $1 < p < \infty$. In fact (explanations below),

$$B_p^r(R_n) \to L_p^{\frac{r_1}{\varkappa_1}, \ldots, \frac{r_n}{\varkappa_1}, r_{n+1}}(R_{n+1}) \to B_p^{\frac{\varkappa_2}{\varkappa_1} r_1, \ldots, \frac{\varkappa_2}{\varkappa_1} r_m}(R_m) = \tag{2}$$

$$= B_p^\rho(R_m),$$

where

$$\varkappa_1 = 1 - \frac{1}{p r_{n+1}} > 0, \quad \varkappa_2 = 1 - \frac{1}{p} \sum_{m+1}^{m} \frac{\varkappa_1}{r_j} - \frac{1}{p r_{n+1}} = \varkappa_1 \varkappa;$$

$$B_p^\rho(R_m) \to L_p^{\frac{\rho_1}{\varkappa_2}, \ldots, \frac{\rho_m}{\varkappa_2}, \frac{r_{m+1}}{\varkappa_1}, \ldots, \frac{r_n}{\varkappa_1}, r_{n+1}}(R_{n+1}) \to B_p^r(R_n). \tag{3}$$

The first embedding in (2), just as in (3), follows from theorem 9.5.2, and the second in (2) and (3) -- from theorem 9.5.1.

## 9.6. Embedding Theorem With Limiting Exponent

9.6.1. Lemma. Suppose $g \in L_{q'}(R_m)$, $f \in L_p(R_n)$,

$$1 \leqslant m \leqslant n, \quad 1 < p < q < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

$$x = (x_1, \ldots, x_n) \in R_n, \quad r(\xi) = \left\{ \sum_1^n |\xi_j|^{2/\varkappa_j} \right\}^{1/2}, \quad \varkappa_j > 0,$$

$$\xi = (\xi_1, \ldots, \xi_n), \quad \lambda = \frac{1}{p'} \sum_1^n \varkappa_j + \frac{1}{q} \sum_1^m \varkappa_j. \tag{1}$$

*) This remark is owed to V. I. Burenkov.

Then the inequality*)

$$\left| \int_{R_m} dx \int_{R_n} \frac{g(x) f(y)\, dy}{r^{\lambda}(x-y)} \right| \leqslant c \| f \|_{L_p(R_n)} \| g \|_{L_{q'}(R_m)}, \tag{2}$$

where c does not depend on f, g, and $(x_{m+1}, \ldots, x_n) \in R_{n-m}$ is valid; from which it follows that

$$\left\| \int_{R_n} \frac{f(y)\, dy}{r^{\lambda}(x-y)} \right\|_{L_q(R_m)} \leqslant c \| f \|_{L_p(R_n)}. \tag{3}$$

Proof. In the one-dimensional case ($n = m = 1$, $\varkappa_1 = 1$) (2) is the Hardy-Littlewood inequality

$$\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(\xi) f(\eta)\, d\xi\, d\eta}{|\xi - \eta|^{\frac{1}{p'} + \frac{1}{q}}} \right| \leqslant c \| f \|_{L_p(R_1)} \| g \|_{L_{q'}(R_1)}. \tag{4}$$

We would not prove it here**). The fact that (3) follows from (2) is the F. Riecz theorem (cf Banach [1]), stating that if a function F measurable on $R_m$ is such that the Lebesgue integral

$$\int_{R_m} Fg\, dx$$

exists for any $g \in L_{q'}(R_m)$, then

$$\| F \|_{L_q(R_m)} = \sup_{\| g \|_{L_{q'}(R_m)} < 1} \int_{R_m} Fg\, dx.$$

and

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*) Hardy and Littlewood [1], case n = m = 1; V. I. Il'yin [8], general case.

**) The proof is found in the book by Hardy, Littlewood, and Polya [1], page 346.

Let us write the integral estimated as

$$I = \int_{R_m} g(x)\,dx \int_{R_m} dy' \left[ \int_{R_{n-m}} \frac{f(y)\,dy''}{r^{\lambda}} \right], \tag{5}$$

where $\mathbf{y}' = (y_1, \ldots, y_m)$, $\mathbf{y}'' = (y_{m+1}, \ldots, y_n)$. Hölder's inequality

$$|[\cdot]| \leqslant \left( \int_{R_{n-m}} |f(y)|^p \, dy'' \right)^{1/p} \left( \int_{R_{n-m}} \frac{dy''}{r^{\lambda p'}} \right)^{1/p'} = P(y')\,Q(y').$$

can be applied to the integral appearing in the brackets. But (explanations below)

$$Q^{p'} = \int_{R_{n-m}} \frac{dy_{m+1} \ldots dy_n}{\left\{ H^2 + \sum\limits_{m+1}^{n} |y_j|^{2/\varkappa_j} \right\}^{1/2\left(\sum\limits_{m+1}^{n} \varkappa_j + \varepsilon\right)}} =$$

$$= \frac{1}{H^{\varepsilon}} \int_{R_{n-m}} \frac{du_{m+1} \ldots du_n}{\left\{ 1 + \sum\limits_{m+1}^{n} |u_j|^{2/\varkappa_j} \right\}^{1/2\left(\sum\limits_{m+1}^{n} \varkappa_j + \varepsilon\right)}} = \frac{c}{H^{\varepsilon}} =$$

$$= \frac{c}{\left\{ \sum\limits_{1}^{m} |x_j - y_j|^{2/\varkappa_j} \right\}^{\frac{p'}{2} \sum\limits_{1}^{m} \varkappa_j \left( \frac{1}{p'} + \frac{1}{q} \right)}} \leqslant \frac{c}{\left\{ \prod\limits_{j=1}^{m} |x_j - y_j| \right\}^{p' \left( \frac{1}{p'} + \frac{1}{q} \right)}}.$$

Above we used the following notation:

$$H^2=\left\{\sum_1^m |x_j-y_j|^{2/\varkappa_j}\right\},$$

$$\varepsilon=\lambda p'-\sum_{m+1}^n \varkappa_j-\sum_1^m \varkappa_j+\frac{p'}{q}\sum_1^m \varkappa_j>0.$$

The substitution

$$u_j=\frac{y_j}{H^{\varkappa_j}} \qquad (j=1,\ \ldots,\ m-1);$$

was introduced into the second equality; the integral in the third term was denoted with c; its finiteness on the unit sphere $R_{n-m}$ is obvious; but outside it, if we set $u_j^{2/\varkappa_j}=\xi_j$, confining ourselves to positive $u_j$ and introducing polar coordinates for $\xi=(\xi_{m+1},\ \ldots,\ \xi_n)$, then the corresponding integral is estimated thusly:

$$\int\limits_{|\xi|>\beta>0}\frac{\prod\limits_{m+1}^n \xi_j^{\frac{\varkappa_j}{2}-1}\,d\xi}{\left(\sum\limits_{m+1}^n \xi_j\right)^{\frac{1}{2}\left(\sum\limits_{m+1}^n \varkappa_j+\varepsilon\right)}}\ll\int\limits_\beta^\infty \rho^{-1-\frac{\varepsilon}{2}}\,d\rho<\infty$$

$$(\varepsilon>0).$$

The last inequality is obtained from the obvious inequalities ($\xi_j=|x_j-y_j|$)

$$\xi_j=\xi_j^{\frac{2}{\varkappa_j}\cdot\frac{\varkappa_j}{2}}\leqslant\left(\sum_{s=1}^n \xi_s^{2/\varkappa_s}\right)^{\varkappa_j/2} \qquad (j=1,\ \ldots,\ m),$$

which remain to be multiplied and raised to the power $p'(1/p'+1/q)$.

Consequently,

$$|I|\ll\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\frac{dx_1\,dy_1}{|x_1-y_1'|^{\frac{1}{p'}+\frac{1}{q}}}\cdots\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\frac{|g(x)|\,|P(y)|}{|x_m-y_m|^{\frac{1}{p'}+\frac{1}{q}}}\,dx_m\,dy_m,$$

from whence follows (2) by successive m-fold application of the one-dimensional inequality (4).

9.6.2. Generalization of Sobolev embedding theorem*).

Theorem. Given the condition $1 < p < q < \infty$, $1 \leqslant m \leqslant n$,

$$r=(r_1, \ldots, r_n) \geqslant 0, \quad \varkappa = 1-\left(\frac{1}{p}-\frac{1}{q}\right)\sum_1^m \frac{1}{r_j} - \frac{1}{p}\sum_{m+1}^n \frac{1}{r_j} \geqslant 0 \tag{1}$$

the embedding

$$L_p^r(R_n) \to L_q^\rho(R_m), \tag{2}$$
$$\rho=(\rho_1, \ldots, \rho_m), \quad \rho_j = \varkappa r_j \quad (j=1, \ldots, m).$$

obtains.

Proof. We will let $I_{-r}$ stand for the operation that is the inverse of $I_r$ ($r \geqslant 0$, $I_0$ is the unit operator) and we will consider that operations $I_r$, $I_{r'}$, $I_{-r}$, and $I_{-r'}$ are commutative. Let $f \in L_p^r(R_n)$ ($r > 0$), then

$$f = I_r g \quad (g \in L_p(R_n))$$

and consequently,

$$f = I_\rho I_{r(1-\varkappa)} h,$$

where

$$h = I_{-\rho} I_{-r(1-\varkappa)} I_r g, \quad \|h\|_{L_p(R_n)} \leqslant c \|g\|_{L_p(R_n)},$$

because the function

$$\left\{\sum_1^m (1+u_j^2)^{\frac{r_j\varkappa}{2\sigma}}\right\}^\sigma \left\{\sum_1^n (1+u_j^2)^{\frac{r_j(1-\varkappa)}{2\sigma}}\right\}^\sigma \left\{\sum_1^n (1+u_j^2)^{r_j/2\sigma}\right\}^{-\sigma}$$

is a Marcinkievicz multiplier (cf 1.5.5, example 12 and note at end of 1.5.5).
And so

$$f = I_\rho u, \tag{3}$$
$$u = \int G_{(1-\varkappa)r}(x-y)\, h(y)\, dy \tag{4}$$

*) Cf note to 6.1 and 9.6.2.

and for sufficiently large parameter $\sigma$, the inequalities

$$|G_{(1-\varkappa)r}(x)| \ll \left\{\sum_1^n |x_j|^{r_j(1-\varkappa)\left(\sum_{s=1}^n \frac{1}{r_s(1-\varkappa)}-1\right)}\right\}^{-1} =$$
$$= \left\{\sum_1^n |x_j|^{r_j\left(\sum_{s=1}^n \frac{1}{r_s}-1+\varkappa\right)}\right\}^{-1} <$$
$$< \left\{\sum_1^n |x_j|^{2r_j}\right\}^{-1/2\left(\sum_{s=1}^n \frac{1}{r_s}-1+\varkappa\right)} = r(x)^{-\lambda}. \tag{5}$$

are satisfied. Here we must bear in mind that $\varkappa < 1$ and

$$\lambda = \sum_{s=1}^n \frac{1}{r_s} - 1 + \varkappa = \frac{1}{p'}\sum_1^n \frac{1}{r_j} + \frac{1}{q}\sum_1^m \frac{1}{r_j}, \tag{6}$$

and we can use the estimate (first) 9.4.1(2). In the last inequality we employed the ordinary estimate
$\left(\sum_1^n \xi_j^{\beta}\right)^{1/\beta} \leqslant c \sum_1^n \xi_j$, where $c = c_n$ is a constant).

From (4), (5), and (6) by virtue of lemma 9.6.1 (cf formula 9.6.1(3), where we must assume $\varkappa_j = 1/r_j$), we get

$$\|u\|_{L_q(R_m)} \leqslant c\|h\|_{L_p(R_n)}.$$

But from (3) it follows that $f \in L_q(R_m)$ for any fixed $x_{m+1}, \ldots, x_n$ and

$$\|f\|_{L_q^{\varrho}(R_m)} = \|u\|_{L_q(R_m)} \leqslant c_2\|h\|_{L_p(R_n)} \leqslant$$
$$\leqslant c_3\|g\|_{L_p(R_n)} = c_3\|f\|_{L_p^r(R_n)},$$

and the theorem is proven.

9.6.3*) From theorem 9.6.2, if we take note of the theorem 9.5.1 and 9.5.2 we can as a consequence obtain an analogous theorem with the spaces B:

$$B_p^r(R_n) \to B_q^\rho(R_m) \tag{1}$$

given the conditions $1 < p < q < \infty$, $1 \leq m \leq n$, $r > 0$, $\rho = \varkappa r$, $\varkappa > 0$ ($\varkappa$ cf 9.6.2(1)). In fact (explanations below),

$$\begin{aligned} B_p^r(R_n) &\to L_p^{\frac{r_1}{\varkappa_1},\dots,\frac{r_n}{\varkappa_1}, r_{n+1}, r_{n+2}}(R_{n+2}) \to \\ &\to L_q^{\frac{\varkappa_2}{\varkappa_1}r_1,\dots,\frac{\varkappa_2}{\varkappa_1}r_n, \varkappa_2 r_{n+1}}(R_{n+1}) \to \\ &\to B_q^{\frac{\varkappa_3\varkappa_2}{\varkappa_1}r_1,\dots,\frac{\varkappa_3\varkappa_2}{\varkappa_1}r_m}(R_m) = B_q^\rho(R_m), \end{aligned} \tag{2}$$

where $r_{n+1}$, $r_{n+2} > 0$ are selected sufficiently large that

$$\varkappa_1 = 1 - \frac{1}{p}\left(\frac{1}{r_{n+1}} - \frac{1}{r_{n+2}}\right) > 0,$$

$$\varkappa_2 = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\left(\sum_1^n \frac{\varkappa_1}{r_j} + \frac{1}{r_{n+1}}\right) - \frac{1}{pr_{n+2}} > 0.$$

Here

$$\varkappa_3 = 1 - \frac{1}{q}\left(\sum_{m+1}^n \frac{\varkappa_1}{\varkappa_2 r_j} + \frac{1}{\varkappa_2 r_{n+1}}\right) = \frac{\varkappa\varkappa_1}{\varkappa_2}.$$

Embeddings (2) follow successively from 9.5.2, 9.6.2, and 9.5.1.

## 9.7. Nonequivalence of the Classes $B_p^r$ and $L_p^r$

*) This note belongs to V. I. Burenkov.

In conclusion let us show that the classes $B_p^r$ and $L_p^r$ for $1 \leqslant p < \infty$, $p \neq 2$, are not equivalent (are essentially distinct). Let us confine ourselves to considering the one-dimensional case.

First let $1 < p < \infty$. Let us look at the sequence of functions

$$\Phi_N(t) = \sum_0^N \phi_k(t) = \varkappa \sum_0^N \cos[(2^k+1)t] \frac{\sin t}{t} \quad (N = 1, 2, \ldots),$$

$$\varphi_k(t) = \begin{cases} 1 & (2^k < |t| < 2^k + 2), \\ 0 \text{ for remaining } t \end{cases}$$

(cf 1.5.7 (10).

Let us note that provided $1 < p < \infty$

$$\int_{-\infty}^{\infty} |\phi_k(t)|^p \, dt \ll \int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right|^p dt = A < \infty, \tag{1}$$

$$\int_{-\infty}^{\infty} |\phi_k(t)|^p \, dt \gg \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} |\cos(2^k+1)t|^p \, dt > B > 0 \tag{2}$$

(on $(\pi/3, \pi/2)$, the function $|t^{-1} \sin t|$ is restricted from below by the positive constant), where B does not depend on k.

Obviously (cf 1.5.6.1),

$$\beta_k(\Phi_N) = \widehat{(1)_{\Lambda_k} \check{\Phi}_N} = \widehat{\varphi_k(t)}, \quad \Lambda_k = \{2^k \leqslant |t| \leqslant 2^{k+1}\}$$

and

$$\|\Phi_N\|_p \ll \left\| \left( \sum_0^N \phi_k^2 \right)^{1/2} \right\|_p \ll \|\Phi_N\|_p, \tag{3}$$

obtains, where the constants appearing in the inequalities here and below do not depend on N.

From the first inequality in (3) it follows that

$$\|\Phi_N\|_p \ll \left(\int_{-\infty}^{\infty} N^{p/2}\left|\frac{\sin t}{t}\right|^p dt\right)^{1/p} \ll N^{1/2}. \tag{4}$$

Further, from the second inequality (3), when $2 \leqslant p < \infty$ (cf 3.3.3), it follows that

$$\|\Phi_N\|_p \gg \left\|\left(\sum_0^N |\hat{\varphi}_k|^p\right)^{1/p}\right\|_p = \left(\int_{-\infty}^{\infty} \sum_0^N |\hat{\varphi}_k|^p dt\right)^{1/p} > $$
$$> (NB)^{1/p} \gg N^{1/p} \tag{5}$$

and provided $1 < p \leq 2$, by means of the generalized Minkowski inequality 1.3.2 (1), with the exponent $\alpha = 2/p \geq 1$

$$\|\Phi_N\|_p^p \gg \left\|\left(\sum_0^N \hat{\varphi}_k^2\right)^{1/2}\right\|_p^p = \int \left(\sum_0^N \hat{\varphi}_k^2\right)^{p/2} dt \geqslant$$
$$\geqslant \left\{\sum_0^N \left(\int |\hat{\varphi}_k|^p dt\right)^{2/p}\right\}^{p/2} \gg \left(\sum_0^N B^{2/p}\right)^{p/2} = BN^{p/2}. \tag{6}$$

From (4), (5), and (6) it follows that

$$N^{1/2} \ll \|\Phi_N\|_p \ll N^{1/2} \quad (1 < p < \infty). \tag{7}$$

On the other hand (cf 8.9(5)), by virtue of (1) and (2) the quantity

$$\|\Phi_N\|_{B_p^0} = \left(\sum_{k=0}^N \|\varphi_k\|_p^p\right)^{1/p} \approx N^{1/p}, \tag{8}$$

i.e., has the rigorous order $N^{1/p}$.

We have seen that the orders of the quantities (7) and (8) provided $p \neq 2$ are different. This shows that the zero classes $L^0_p = L_p$ and $B^0_p$ and, consequently, the classes $L^r_p$ and $B^r_p$ for any r are not equivalent.

Using functions $\varphi_N$, we similarly prove that even for any $\theta \neq 2$, the class $B^0_{p\theta}$ is not equivalent to $L_p$ (cf O. V. Besov [5], to whom belongs the above argumentation). When $\theta = 2$, $p \neq 2$, nonequivalency also obtains, however it is proven in a different fashion (cf K. K. Golovkin [1]).

Let us proceed to the case $p = 1$. The one-dimensional de la Valleė-Poussin kernel (cf 8.6(5), (10), and (11))

$$V_N(t) = \frac{1}{N} \cdot \frac{\cos Nt - \cos 2Nt}{t^2}$$

has the Fourier transform $\tilde{V}_N = \sqrt{\pi/2} \cdot \mu^*_N(t)$, where

$$\mu^*_N(t) = \sqrt{\frac{\pi}{2}} \begin{cases} 1 & (|x| < N), \\ \frac{1}{N}(2N - x) & (N < |x| < 2N), \\ 0 & (2N < |x|). \end{cases}$$

If k and N are natural numbers and $k \leqslant N$, then

$$\mu^*_{2^k}(t)\, \mu^*_{2^N}(t) = \mu^*_{2^k}(t).$$

Therefore the k-th de la Valleė-Poussin sum of the function $V_{2^N}$ is equal to

$$\sigma_{2^k}\left(V_{2^N}, x\right) = \frac{2}{\pi}\left(V_{2^k} * V_{2^N}\right) = \frac{2}{\pi}\, \widehat{\tilde{V}_{2^k} \tilde{V}_{2^N}} = V_{2^k}(x)$$

and, consequently, the expansion of $V_{2^N}$ in a series in de la Valleė-Poussin sums is of the form

$$V_{2^N} = V_{2^0} + \sum_1^N \left(V_{2^k} - V_{2^{k-1}}\right).$$

From whence

$$\|V_{2^N}\|_{B^0_1} = \|V_{2^0}\|_L + \sum_{k=1}^N \|V_{2^k} - V_{2^{k-1}}\|_L \to \infty, \quad (N \to \infty)$$

because (after change of variable $u = 2^{k-1}t$)

$$\|V_{2^k} - V_{2^{k-1}}\|_L = \int \left| \frac{\cos 2u - \cos 4u}{2u^2} - \frac{\cos u - \cos 2u}{u^2} \right| du = c > 0,$$

where c does not depend on k = 1, 2, ... On the other hand, the norm of $V_2N$ in the metric L

$$\|V_{2^N}\|_L = \int \frac{|\cos u - \cos 2u|}{u^2} du = c_1 < \infty$$

is bounded. This shows that the embedding $B_1^0 \longrightarrow L$ is irreversible.

## NOTES

### To Chapter I

1.1-1.4. Familiar facts are presented, often without proof, from the theory of functions of a real variable and the theory of Banach spaces in order that reference can later be made then and that the reader familiarize himself with the notation adopted. These facts can be found in the books: P. S. Aleksandrov and A. N. Kolmogorov [1], A. N. Kolmogorov and S. V. Fomin [1], Banach [1], L. A. Lyusternik and V. I. Sobolev [1], I. P. Natanson [1], V. I. Smirnov [1], and S. L. Sobolev [3].

1.5. We presented with proof elementary background information (only that which is essential for this book) from the theory of generalized functions for the class S, as this class is defined by L. Schwartz [1]. Let us note the articles by S. L. Sobolev [1 & 2], where the concept of the generalized function is introduced, and the Russian-language books on the theory of generalized functions of Halperin [1], V. S. Vladimirov [1], and by I. M. Gel'fand and G. E. Shilov [1].

Let us also mention the book by Hormander [1], where far-ranging results on multiplicators are derived. The multiplier $\mu \in S'$ could not early be defined as a bounded measurable function, but it was assumed that $f \in S'$ and displays the property that

$$\|\tilde{f}\mu\|_p \leq c_p \|f\|_p \text{ for all } f \in S.$$ Hormander showed

that such a generalized $\mu$ function must be a bounded measurable function $\mu(x)$.

1.5.2. Inequalities (6) are proven in the works by Littlewood and Paley [1]. The theorems presented for the periodic one-dimensional case are found in Chapter XV of the book by Zigmund [2], and cf Marcinkievicz for the two-dimensional case [1].

1.5.3. The Marcinkievicz theorem in the periodic two-dimensional case was proven in his article cited [1]. The transition to the periodic case was made by S. G. Mikhlin [1]. Further development is to be found in P. I. Lizorkin [5]. The condition introduced in 1.5.4 that $D^k\lambda$ in each coordinate closed juncture be continuous at any point x with $x_i \neq 0$, $i \in e_k$, is used, for example, in example 1.5.5(5), which is employed in subsequent theory.

1.5.9. Operation $I_r$ was studied in a number of works, which include those by L. Schwartz [1], Calderon [1], Aronszajn and Smith [2], P. I. Lizorkin [1 & 8], Nikol'skiy, Lions, and Lizorkin [1], and Taibleson [1].

1.5.10. The concept of a generalized function that is regular in the $L_p$-sense is to be found in S. M. Nikol'skiy [17 & 18].

To Chapter II

The information presented in 2.1-2.5 is familiar and is auxiliary in purpose. In particular, cf, for example, the books by V. L. Goncharov [1] and A. F. Timan [1] about interpolation. We also note the books by N. I. Akhiezer [1], A. Zigmund [2], and I. P. Natanson[2], where, just as in the above-cited books, detailed background of trigonometric polynomials of one variable is given.

To Chapter III

3.1. Cf the book by N. I. Akhiezer [1] on integral functions of a single variable of the exponential type, bounded on a real axis. In particular, this book derives the criteria 3.1 (5), (6) for integral functions of the exponential type and a complete proof of the facts pertaining to the theory of Borel integrals, which we omitted in our exposition.

3.2. In deriving interpolation formula (4) for functions of the exponential type, we followed the approach presented in the article by Civin [1]. But the line of reasoning (cf 3.2.1) proceeds with the involvement of generalized functions as was done by P. I. Lizorkin [8]. We have somewhat imrpoved them.

3.2.2. Interpolation formula (2) for an arbitrary function f(z) of the exponential type, bounded on a real axis, is to be found in the problem book of Polya and Sege [1], on the assumption that it is already known that $|f(z)| \leq Ae^{\sigma|y|}$ $(z = x + iy)$. A complete derivation is to be found in the book by N. I. Akhiezer[1], section 84.

3.2.3. The approach used to obtain inequalities of the Bernshteyn inequality type for the case of general norms is indicated in the book by N. I. Akhiezer [1](section 81, theorem 3). We add to the conditions 1) and 2) listed there condition 3).

3.3-3.5. S. M. Nikol'skiy [3] obtained the inequalities 3.4.3(2), (3), and (4) for trigonometric polynomials, along with the analogous inequalities for integral functions of the exponential type (3.3.2(2), 3.3.5(1), and 3.4.2 (1)). The case 3.4.3(3) when n = 1, $p' = \infty$ for trigonometric polynomials was known even to Jackson [2]. Inequalities 3.4.3(2) for trigonometric polynomials in the case $n = p = 1$ derive from the results of S. M. Lozinskiy [1].

The constants of the inequalities presented are exact in the sense of order, but they are not exact absolutely. In some cases more exact or absolutely exact values of these constants are known. Accordingly, we point to the book by I. I. Ibragimov [1]; see further N. K. Bari [1].

Inequality 3.4.1(1) on the assumption that $g_\nu(x) = g_\nu(u, y)$ is an integral function of the exponential type $\nu$ with respect to all $x_1, \ldots, x_n$ is to be found in the work by S. M. Nikol'skiy [3]. Here the more general case when $g$ is an integral function only with respect to $u$ is considered.

3.3.7. V. I. Burenkov directed my attention to the inequality $\|f\|_{L_{(p_1, \ldots, p_n)}} \leq \|f\|_{L_{(q_1, \ldots, q_n)}}$, where $q_1, \ldots, q_n$ is the permutation of the numbers $p_1, \ldots, p_n$ placed in nondescending order.

To Chapter IV

4.1. In addition to the work by Beppo Levi [1] and S. L. Sobolev [1-5], generalized derivatives have been studied by Tonelli [1], Evans [1, 2], Nikodym [1], Calkin [1], Morrey [1], S. M. Nikol'skiy [3, 5], and Deny and Lions [1, 2], where a further bibliography on this topic are to be found.

4.2. Cf S. B. Stechkin [1] for formula (2) and inequality (6) for periodic functions of a single variable.

4.3. Fractional classes $W_p^r(\Omega) = B_p^r(\Omega)$ ($r$ is a fraction) emerged in a natural fashion as classes of traces of functions of integral classes $W_p^l(g)$ on the manifold $\Omega \subset g$ or the boundary of $g$ of measure $m$ less than the measure of domain $g$. First this problem on traces was solved for $p = 2$ in the works by Aronszajn [1], V. M. Babich and L. N. Slobodetskiy [1], and then for $m = n - 1$ by Gagliardo [1], L. N. Slobodetskiy [1], and for arbitrary $m$ and $l$ by O. V. Besov [2]. In the latter case not only are fractional B classes, but also integral B classes required.

Zigmund [1] directed attention to the fact that from several standpoints, for example, from the viewpoint of the problem of the order of the best approximations of functions using trigonometric polynomials, the class of periodic, with period $2\pi$, measurable functions of one variable satisfying the condition

$$\left(\int_0^{2\pi} |\Delta_h^2 f(x)|^p \, dx\right)^{1/p} \leq M|h|, \tag{1}$$

where $k > 1$ more naturally supplements classes of functions of x with period $2\pi$ satisfying the condition

$$\left(\int_0^{2\pi} |f(x+h)-f(x)|^p dx\right)^{1/p} \leqslant M|h|^\alpha \qquad (0<\alpha<1), \tag{2}$$

then the class of functions for which (2) is satisfied when $\alpha = 1$.

The theory of embeddings of H- and B-classes yields a number of new examples confirming this finding.

4.3.4. At present a good many equivalent methods of defining spaces $B^r_{p\theta}$ are known. Many are collected in section 2 of the review by V. I. Burenkov [3].

4.3.6. Let us introduce the concept of the open set $g \subset R_n$ with Lipshits boundary. If the set g is bounded, then its boundary $\Gamma$ is called a Lipshits boundary if, whatever be the point $x^0 \in \Gamma$, an orthogonal system of coordinates $\xi = (\xi_1, \ldots, \xi_n)$ can be found with origin at $x^0$ and a cubo

$$\Delta = \{|\xi_j| < \eta;\ j = 1, \ldots, n\}, \tag{1}$$

excising from $\Gamma$ the portion $\gamma = \Gamma\Delta$ described by the equation

$$\begin{gathered}\xi_n = \psi(\lambda);\ \lambda = (\xi_1, \ldots, \xi_{n-1}),\\ \lambda \in \Delta' = \{|\xi_j| \leqslant \eta;\ j = 1, \ldots, n-1\},\end{gathered} \tag{2}$$

can be found, where $\psi(\lambda)$ satisfies on $\Delta'$ the Lipshits condition, i.e., that there exists such a constant M that

$$\begin{gathered}|\psi(\lambda') - \psi(\lambda)| \leqslant M|\lambda' - \lambda|,\\ \lambda,\ \lambda' \in \Delta'.\end{gathered} \tag{3}$$

If the set g is not bounded, then its boundary $\Gamma$ is called a Lipshits boundary if there exist positive numbers $\eta$ and M, not dependent on $x^0 \in \Gamma$, and a finite set e of orthogonal coordinate system obtained by rotation of the given orthogonal system $(x_1, \ldots, x_n)$, such that whatever be the point $x^0 \in \Gamma$, an orthogonal system of coordinates $\xi = (\xi_1, \ldots, \xi_n)$ with origin at $x^0$ can be found, parallel to one of the systems in the set e, and cube (1) can be found, excising from $\Gamma$ the portion $\gamma = \Gamma\Delta$ described by equation (2), where $\psi(\lambda)$ satisfies the Lipshits conditim (3) on $\Delta'$.

Theorem 1. Suppose the open set $\Omega \subset R_n$ has a Lipshits boundary. Any of the classes $W^l_p(\Omega)$ (l is an integer, $1 < p < \infty$), $H^r_p(\Omega)$($r > 0$, $1 \leq p \leq \infty$), $B^r_{p\theta}(\Omega)$ ($r > 0$, $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$) can be extended in linear fashion beyond the limits of $\Omega$ on $R_n$ with norm preserved.

Domains $\Omega$ beyond whose limits the extension of functions of anisotropic classes is possible depends essentially on the defining class of vectors /illegible/, more exactly, on the proportion in which its components are found.

Let us assign the positive vector $r > 0$ ($r_j > 0$; $j = 1, \ldots, n$), $\delta > 0$, and further the positive vector $a > 0$. Let $P(r) = P(r, \rho, a, \delta)$ stand for the set (of horns with apex at the zero point) of points $x = (x_1, \ldots, x_n) \in R_n$ subject to the conditions

$$a_j h < x_j^{r_j} < (a_j + \delta) h \qquad (j = 1, \ldots, n), \quad 0 < h < \rho \tag{4}$$

or any set obtained from (4) by mirror mappings (possibly, several times) with respect to (n - 1)-dimensional coordinate planes. Thus, an arbitrary horn P(r) can be further described by the inequalities

$$a_j h < |x_j|^{r_j} < (a_j + \delta) h, \quad \operatorname{sign} x_j = \text{const}, \quad 0 < h < \rho. \tag{5}$$

Let the symbol $g_1 + g_2$

stand for the vector sum of the set $g_1$ and $g_2 \subset R_n$, i.e., the set of all possible sum $x + y$, where $x \in g_1$, $y \in g_2$.

We will state that the open set $\Omega \in A$ (r) ($\varepsilon > 0$), if : 1) it can be represented in the form of the two sums

$$\Omega = \bigcup_1^N U^k = \bigcup_1^N U^k_\varepsilon, \tag{6}$$

where $U^k_\varepsilon$ is the set of points $x \in U^k$ located at a distance from the boundary $\Omega - U^k$ greater than $\varepsilon$, and 2) there exist $\rho$, a, and $\delta$ such that the horn $P^k = P^k(r, \rho, a, \delta)$ can be brought into correspondence with each k, such that

$$U^k + P^k \subset \Omega \qquad (k = 1, \ldots, N). \tag{7}$$

relation (7) expresses the situation that whatever the point $x \in U^k$, if the horn $P^k$ is translated parallel to itself in order that its apex coincide with $x$, then the horn thus shifted belongs to $\Omega$.

Let us note that for the case when $r_1 = \ldots = r_n = r$, the horn P is a cone resting on some polyhedron with its apex at the zero point. It can be proved that in this case concepts of a domain with a Lipshits boundary and a domain of the class $A_\varepsilon(r, \ldots, r)$ coincide.

Theorem 2. Suppose that the domain $\Omega \in A_\varepsilon(r)$ and the classes with norms

$$\|f\|_{W_p^l(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{j=1}^{n} \left\| \frac{\partial^{l_j} f}{\partial x_j^{l_j}} \right\|_{L_p(\Omega)} \tag{8}$$

($l_j$ are integers, $1 < p < \infty$),

$$\|f\|_{B_{p\theta}^r(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{j=1}^{n} \left( \int_0^H \left\| \Delta_{x_j h}^{k_j} \frac{\partial^{\rho_j} f}{\partial x_j^{\rho_j}} \right\|_{L_p(\Omega_{k_j h})}^{\theta} \frac{dh}{h^{1+\theta(r_j-\rho_j)}} \right)^{1/\theta} \tag{9}$$

$$(k_j > r_j - \rho_j > 0, \ 1 \leqslant \theta < \infty, \ 1 \leqslant p \leqslant \infty),$$

$$\|f\|_{H_p^r(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{j=1}^{n} \sup_h \frac{\left\| \Delta_{x_j h}^{k_j} \frac{\partial^{\rho_j} f}{\partial x_j^{\rho_j}} \right\|_{L_p(\Omega_{k_j h})}}{h^{r_j-\rho_j}} \tag{10}$$

$$(1 \leqslant p \leqslant \infty).$$

are given. Any of the classes can be extended beyond $\Omega$ on $R_n$ linearly with norm preserved.

This theorem was proven for $W_p^l(\Omega)$ when $l_1 = \ldots = l_n = l$ by Smith [1], who amplified the result of Calderon [2], who had proven it for the stronger norm

$$\|f\|_{L_p(\Omega)} + \sum_{|\alpha| \leqslant l} \|f^{(\alpha)}\|_{L_p(\Omega)} = \|f\|_{*W_p^l(\Omega)}.$$

Here the embeddings

$$W_p^l(\Omega) \to W_p^{l,\ldots,l}(\Omega) \to W_p^{l,\ldots,l}(R_n) \to W_p^l(R_n) \to$$
$$\to {}^*W_p^l(R_n) \to {}^*W_p^l(\Omega) \to W_p^l(\Omega), \tag{11}$$

obtain, where the first and the two last are trivial, the second has already been the results of Smith noted above, and the third is proven in 9.2. Theorem 1 follows from (11) for the space $W_p^l(\Omega)$.

Given dissimilar natural $l_j$, this theorem was proven simultaneously and independently for $W_p^l(\Omega)$ by O. V. Besov [9, 10] and by V. P. Il'yin [6].

This theorem, and thus, also theorem 1) was proven for the norm $B_{p\theta}^r$ $(1 \leq \theta \leq \infty)$ by O. V. Besov [9, 10], cf also O. V. Besov and V. P. Il'yin [1].

V. I. Burenkov [4, 5] showed that a domain not belonging to $A_\varepsilon(r)$ for which theorem 2 on extension no longer satisfied can be specified for each $r > 0$. For example, any horn $P(k)$, where $k \neq cr$, is such a domain.

In this work cited it is also proven that if theorem 2 does obtain for the spaces $W_p^r(\Omega)$ and $B_p^r(\Omega)$ for classes of domains of the form $A_\varepsilon(r)$, then this is true if and only if instead of the horn $P(r)$ the horn $P(r, p) = P(r, p, \rho, \alpha, \delta)$ defined as the set of points $x \in R_n$ subject to the inequalities

$$a_i h < x_i^{\rho_i} < (a_i + \delta) h \qquad (i = 1, \ldots, n), \quad 0 < h < \rho,$$

where

$$\rho_i = \frac{r_i}{\varkappa_i}, \quad \varkappa_i = 1 - \sum_{j=1}^{n} \frac{1}{r_j}\left(\frac{1}{p_j} - \frac{1}{p_i}\right).$$

is under consideration.

Let us note yet another theorem stemming from theorems 1 and 2.

Theorem 3. If $g \subset R_n$ is an arbitrary open set and $g_1 \subset \bar{g}_2 \subset g$ is another bounded open set, then functions of any class cited in theorems 1 and 2, which we represent by $\Lambda(g)$, can be extended from $g_1$ onto $R_n$ linearly with norm (with respect to $g$) preserved. This must be understood in the sense that to each function $f \in \Lambda(g)$ its extension $f \in \Lambda(R_n)$ with $g_1$ (not with $g$) can be brought into correspondence, such that

$$\|f\|_{\Lambda(R_n)} \leq c\,\|f\|_{\Lambda(g)}.$$

where c does not depend on f and the dependence of $\bar{f}$ on f is linear.

In fact, let us assign the orthogonal grid dividing R into cubelets, and let $\Omega$ be a set consisting of the cubelets of the grid containing the points $\bar{g}_1$. The boundary of $\Omega$, obviously, satisfies the condition $A_\varepsilon(r)$ for any r and if the grid sufficiently dense, then $g_1 \subset \Omega \subset g$. Functions $f \in \Lambda(g) \subset \Lambda(\Omega)$ can be extended by employing the theorem given above, with preservation of the norm from $\Omega$ onto R: for the corresponding extensions of $\bar{f}$ the following relation obtains:

$$\|\bar{f}\|_{\Lambda(R_n)} \ll \|f\|_{\Lambda(\Omega)} \ll \|f\|_{\Lambda(g)}.$$

This theorem 3 can be proven by a simpler approach, setting $\bar{f} = f\varphi$, where $\varphi$ is the "cap", i.e., a function infinitely differentiable on $R_n$, equal to unity on $g_1$ and to zero outside of g (for the class $H_p^r(g)$, cf S. M. Nikol'skiy [5]).

Particular cases of theorems 1 and 2 pertaining to the extnesion beyond the domain with sufficiently smooth boundary and beyond the limits of a segment have been considered in earlier work by S. M. Nikol'skiy [4, 7], V. K. Dzyadyk [1], and O. V. Besov [4]; cf further V. M. Babich [1].

4.4.1-4.4.3. Cf S. M. Nikol'skiy [11] for investigations of this kind. Inequality 4.4.3(4) is discussed in the book by S. L. Sobolev [4].

4.4.5. If the derivative $\partial f/\partial x_1$ is understood in the Sobolev sense, then this lemma is at once proven. In fact, let there be assigned on g two sequences of functions $f_k$ and $\lambda_k \in L_p(g)$ such that

$$\int f_k \frac{\partial \varphi}{\partial x_1}\,dx = -\int \lambda_k \varphi\,dx \qquad (k = 1, 2, \ldots) \tag{1}$$

for all continuously differentiable functions $\varphi$ that are finite on g. If here $f_k \to f$, $\lambda_k \to \lambda$ in the $L_p(g)$-sense, then it obviously follows from (1) that

$$\int f \frac{\partial \varphi}{\partial x_1}\,dx = -\int \lambda \varphi\,dx, \quad f, \quad \lambda \in L_p(g)$$

for all specified $\varphi$, i.e., $\lambda$ is a derivative of f with respect to $x_1$ on g in the Sobolev sense. S. L. Sobolev [4] made extensive use of this lemma. Here it is proven, starting from the fundamental definition of the generalized derivative adopted in this book (cf beginning of section 4.1).

4.4.9. Cf S. M. Nikol'skiy [5] on this theorem.

4.8. This theorem was proven in the periodic case by Hardy and Littlewood [1]; it was formulated without proof by A. A. Dezin [1]; the proof for $p = 2$ is presented in the dissertation of A. S. Fokht [1].

## To Chapter V

5.2. S. N. Bernshteyn [2], pages 421-432 studied the method of approximation 5.2.1(4); he proved theorem 5.2.1(7) for $p = \infty$, $m = 1$. Cf S. M. Nikol'skiy [3] for the case $m = 1$, $1 \leq p \leq \infty$. Here we consider the more general case $m \leq n$ of approximation by integral functions of the spherical exponential type.

Inequality 5.2.1(7) in itself when $m = n = 1$, $1 \leq p < \infty$ was obtained by another method by N. I. Akhiyezer [1].

The periodic inequalities 5.3.2(2) were first derived ($n = k = 1$, $p = \infty$) by Jackson [1]. Cf investigations by Quade [1] and N. I. Akhiyezer [1] for the case $n = 1$, $1 \leq p \leq \infty$. The representations 5.3.1(11) (analogous of 5.2.1(4)) are to be found in S. B. Stechkin [1]. Inequality 5.3.2(5) in the case of functions satisfying Lipshits condition for Fejer sums ($p = \infty$) is to be found in A. Zigmund [3], section 4.7.9, and in the general case ($p = \infty$) in S. B. Stechkin [1].

The theorem on approximation 5.2.4 and its periodic analog presenting in 5.3.3 for the case $p = p_1 = \ldots = p_n = \infty$ is to be found in S. N. Bernshteyn [2], pages 421-432, and if the numbers $p_1, \ldots, p_n$ are generally different -- in S. M. Nikol'skiy [10]. Cf Nikol'skiy [6] for inequality 5.3.2(6) for exponential continuity modules.

5.4. The inverse theorem of S. N. Bernshteyn on approximation with algebraic and trigonometric polynomials ($p = \infty$, $n = 1$) was proven in his work [1], pages 11-104. It is refined in the periodic case (for nonintegral $r$) by dela Valleé-Poussin [1] and Zigmund [1] (for integral $r$).

5.4.4. Ya. S. Bugrov [3, 4] also obtained similar inequalities for polyharmonic functions in a circle and semiplane and applied them in studying differontial properties of these functions all the way to the boundary.

5.5.3. The equivalence of the norms $\|\cdot\|_H$ for different admissible pairs $(k, \rho)$ can be proven directly, without resorting to approximation methods (cf Marchoud [1]. Cf K. K. Golovkin [1, 2] for more general investigations in this area. Equivalence was proven in the periodic one-dimensional case by the approximation method by Zigmund [1]. He emphasized equivalence for integral $r$ of norms of the form ${}^1\|\cdot\|_{H^*}$ with the norm ${}^2\|\cdot\|_{H^*}$, expressed in terms of the best approximations. Cf S. N. Bernshteyn [2], pages 421-432 in the aperiodic case. Here this problem is explored for approximations with integral functions of the exponential spherical type.

5.5.4. In the periodic one-dimensional case when $p = \infty$, this is the classical theorem proven in the works of S. N. Bernshteyn [1], pages 11-104, Jackson [1], dela Valleé-Poussin [1], and Zigmund [1]. When $1 \leq p < \infty$, cf Quade [1] and Zigmund [1]. Cf N. I. Akhiyezer [1] for the aperiodic one-dimensional case when $1 \leq p \leq \infty$. Here generalization to the case of approximation with integral functions of the exponential spherical type is given.

5.5.8. Many results pertaining to this problem are available in the periodic case.

5.6-5.6.1. In presenting the sections, we made heavy use of the work of O. V. Besov [5], and in the case of 5.6.1, even the work of T. I. Amanov [3]. O. V. Besov made available to me a new (presented in the text) variant of the proof of the embedding ${}^4B' \to {}^5B$. This procedure has its advantage that it is freely transferable to more general cases of theorems of this kind (cf K. K. Golovkin [2]).

Among different equivalent norms $\|\cdot\|_B$ (in particular, $\|\cdot\|_H$), we introduced the norms ${}^2\|\cdot\|_B$ and ${}^4\|\cdot\|_B$ expressed in terms of directionwise derivatives. In the isotropic case there have an advantage, and in any case, a technical advantage — instead of the sum of integrals corresponding to all possible particular derivatives of orders s with $|s| = \rho$, a single integral is taken. In the case $\rho = 0$, these norms are used infrequently in the literature.

The equivalence of the classes ${}^1B^r_{p\theta}(R_n)$ and ${}^5B^r_{p\theta}(R_n)(1 \leq \theta < \infty)$ was proven by O. V. Besov [3, 5]; and by A. A. Konyoshkov [1] and P. L. Ul'yanov [1] in the periodic one-dimensional case. The equivalence of ${}^1B^r_{p\theta}(R_n)$ and ${}^3B^r_{p\theta}(R_n)$ was proven in the same works of O. V. Besov, and when $\theta = p$, $r_i$ are integers, in the work by S. V. Uspenskiy [3].

In particular case, $\theta = p = 2$ (for the admissible pairs $(\bar{r}, 1)$, $(\bar{r}, 2)$), the norm ${}^3\|\cdot\|$ were introduced and was studied in the earlier works of Aronshain [1], V. M. Babich and L. I. Slobodetskiy [1], and for $p = \theta \neq 2$ for nonintegral r by Gagliardo [1] and L. I. Slobodetskiy [1].

Expansions of functions f in the form of series 5.6(7) with norm ${}^6\|\cdot\|_B$ are to be found in O. V. Besov [3]; the norms ${}^6\|\cdot\|_B$ were used explicitly in the work by T. I. Amanov [3].

5.6.2–5.6.3. Suppose function $f(x)$ has the derivatives $\partial^{\bar{r}_j} f/\partial x_j^{\bar{r}_j}$ $(j = 1, \ldots, n)$, satisfying with respect to $x_j$ on the bounded domain $\Omega$ the Lipshits condition of degree $\alpha_j (0 < \alpha_j \leq 1, r_j = \bar{r}_j + \alpha_j)$ uniformly with respect to the remaining variables, and let $\rho = (\rho_1, \ldots, \rho_n)$ be a vector (integral) for which $\sum_1^n \frac{\rho_j}{r_j} < 1$. S. N. Bernshteyn (1911 г., cf [1], pages 96-104) when $r = r_1 = \ldots = r_n$ and Montel (1918 г., cf [1]) for any $r_j > 0$ shows that in this case a mixed continuous derivative $f^{(\rho)}$ bound on any domain $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega$ exists on $\Omega$. S. N. Bernshteyn showed further that the derivatives $f^{(\rho)} (|\rho| = \bar{r} = r - \alpha, r = r_j)$ satisfy the Lipshits condition of degree $\alpha' < \alpha$ on $\Omega_1$.

Theorem 5.6.3 amplifies these results, specifying exactly the class to which $f^{(\rho)}$ belongs and extending (when $\Omega = R_n$) the result to the case of the metric $L_p (1 \leq p \leq \infty)$ of the B-classes as well. This theorem was proven by S. N. Bernshteyn for the H-classes when $p = \infty$ and $r = r_1 = \ldots = r_n$ ([2], pages 426-432) and simultaneously independently for arbitrary $r_j (j = 1, \ldots, n)$ by S. M. Nikol'skiy [2], who even showed its unimprovability in terms of H-classes, and by this same author [5] in the metric $L_p$. It was proven for the B-classes by O. V. Besov [5].

The question of the extension of theorem 5.6.3 to the case of the domains $\Omega \subset R_n$ is solved by employing theorems on extension (cf 4.3.6). In the Montel works cited above, a square with sides parallel to the coordinate axes was actually considered as the $\Omega$: $\rho$ is not necessarily an integral vector, and then $f^{(\rho)}$ is a mixed particular derivative in the Liouville sense.

Theorem 5.6.2 on equivalence

$$B_{p\theta}^{r,\ldots,r}(R_n) = B_{p\theta}^{r}(R_n),$$

for nonintegral r follows from theorem 5.6.3. In the general case its proof is given (by other methods) by V. A. Solonnikov [1]; cf further S. M. Nikol'skiy, G. Lions and P. I. Lizorkin [1] for the H-class. When $R_n$ in (1) is replaced with $\Omega$, property 5.6.2(1) is finitely preserved for the domain $\Omega \subset R_n$ for which the theorem on the extension of functions of the classes

$B^{r,\dots,r}_{p\theta}(\Omega)$ is valid (cf 4.3.6 and the note to 4.3.6). V. I. Burenkov [2] investigated domains for which the equivalence 5.6.2(1) does not hold.

Suppose $B^r_p = B^r_{pp}(R_n)$, $H^r_p = B^r_{p\infty}(R_n)$, $L_p = L_p(R_n)$. The condition for theorem 5.6.3 when $\mathcal{H} = 0$ and $1 \leq p \leq 2$ $(\theta = p)$ entails $f^{(1)} \in L_p$. This follows from 9.2.2 and 9.3(2). When $2 \leq p < \infty$, this no longer is the case: if, for example, $f \in B^1_p(R_1)$, then hence it in general does not follow that $f^{(1)}$ exists and belongs to $L_p$ (cf 9.7).

Theorem 5.6.3 for the class $H^r_p = B^r_{p\infty}$ in the case $\mathcal{H}= 0$ is also invalid. In fact, the function

$$f(\theta) = \sum_{s=1}^{\infty} b^{-s} \cos b^s\theta \qquad (b > 1)$$

nowhere has a derivative (Weierstrass, Hardy, and cf Zigmund [2], Chapter II, sections 4.8-4.11), while at the same time it belongs to the periodic class $H^1_{p*}$. The last assertion is proven thusly. It is easy to verify that

$$\|\cos b^s x\|_{L^*_p} = \|\cos b^s x\|_{L_p(0,2\pi)} < K,$$

where K does not depend on $s = 1, 2, \dots$ Let us assign $h > 0$ $(0 < h < 1)$ and select a natural N such that

$$b^{-(N+1)} < h \leq b^{-N},$$

then

$$\|\Delta_h^2 f\|_{L^*_p} \ll \sum_{s=1}^{N} b^{-s} \|\Delta_h^2 \cos b^s\theta\|_{L^*_p} + \sum_{N}^{\infty} b^{-s} \ll$$
$$\ll h^2 \sum_{s=1}^{N-1} b^{-s} b^{2s} + b^{-N} \ll h^2 b^N + b^{-N} \ll h,$$

where inequality 4.4.4(3) is used for trigonometric polynomials (cf reference on text page 202 [latter half of section 4.4].

5.6.4. The example is given by Yu. S. Nikol'skiy [1].

5.6.5. Properties 5.6.5(1), (2) expressed normwise continuity in the corresponding spaces $W^r_p$ and $B^r_{p\theta}$. P. I. Lizorkin directed my attention to this property in the case of B-classes.

To Chapter VI

6.1. The addition of V. P. Il'yin [2] to the embedding theorem (1) applies to the case of the limiting exponent (when $\rho = 0, 1, 2, \ldots$) for $m < n$. This assertion in individual cases of a limiting exponent was known to V. I. Kondrashov [1]; he also investigated several cases when $m < n$ and when $\rho$ is nonintegral.

S. L. Sobolev also proved that in theorem (1) when $m = n$, we can assert that $p - 1$.

In the one-dimensional case, the problem of traces does not arise and we can speak only about the "pure" theorem of different measures. It was proven by Hardy and Littlewood [1]; see further the book by Hardy, Littlewood, and Polya [1].

6.4. Suppose $\Gamma \subset R_n$ is a sufficiently smooth surface of measure $m < n$. The trace $f|_\Gamma$ of the function $f \in H_p^r(R_n)$ is correctly defined for it in any case given the condition $r - \frac{n-m}{p} > 0$. Provided this condition holds, the direct embedding and the inverse embedding to it $H_p^r(R_n) \rightleftarrows H_p^{r-\frac{n-m}{p}}(\Gamma)$ obtain (cf S. M. Nikol'skiy [5]). Cf O. V. Besov [11] for the corresponding generalization to the B-classes.

6.7. Ya. S. Bugrov [4] showed that the embedding

$$H_p^{r'}(R_m) \to H_p^r(R_n), \qquad r' = r - \frac{n-m}{p},$$

is valid not only when $r' > 0$, but also when $r' = 0$, if $H_p^0(R_m)$ in its left member is replaced with $L_p(R_m)$.

Different amplifications of the theorem on extension can be obtained if we require that the extending theorem satisfies additional properties.

L. D. Kudryavtsev [2] showed that in theorem 6.6 the function $f \in H_p^r(R_n)$ extending on $R_n$ the function

$$\varphi \in H^{r-\frac{n-m}{p}}(R_m) \qquad \left(1 \leqslant m < n,\ r - \frac{n-m}{p} > 0\right),$$

can be constructed so that it is infinitely differentiable on $R_n - R_m$ and that the properties

$$\int_{R_n} \rho^{p(s-\alpha)+\varepsilon} \left| \frac{\partial^{\bar{r}+s} f}{\partial x_1^{s_1} \ldots \partial x_n^{s_n}} \right|^p dR_n < \infty,$$

$$\varepsilon > 0, \quad \sum_1^n s_k = \bar{r} + s \quad (1 \leqslant p < \infty), \quad \rho^2 = \sum_2^n x_j^2. \tag{1}$$

are satisfied. These inequalities ceased to be valid when $\varepsilon = 0$. A similar result was obtained by him for $p = \infty$. These facts point to a certain relationship of the classes considered here with the so-called weighting classes of function whose derivatives (or their differences) are integrable in the p-th degree with weight. If we proceed not from H-, but the B-classes ($\theta < \infty$), then similar findings obtain even when $\varepsilon = 0$ (S. V. Uspenskiy [3]).

Systematic study of weighting classes was begun in the work by L. D. Kudrashov [2] referred to above; cf further A. A. Vasharin [1], S. V. Uspenskiy [3], I. A. Kipriyanov [1], and A. Kufner [1]. Cf V. I. Burenkov [3], L. D. Kudryavtsev [3], I. Necas [1], and S. M. Nikol'skiy [12] for Bibliographies on this topic.

Ya. S. Bugrov [1] proved for the unit circle $\sigma$ on the plane in terms of the classes H the limiting exact theorem.

Suppose functions with period $2\pi$

$$\varphi_k(\theta) \in H_{p^*}^{r+l-k-1} \qquad (k = 0, 1, \ldots, l-1, \ 1 \leqslant p \leqslant \infty, \ r > 0).$$

Then the polyharmonic function u ($\rho$, $\theta$) of polar coordinates ($0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$) solving in unit circle $\sigma$ the boundary-value problem

$$\Delta^l u(\rho, \theta) = 0, \quad \left. \frac{\partial^k u}{\partial \rho^k} \right|_{\rho=1} = \varphi_k(\theta) \qquad (k = 0, 1, \ldots, l-1)$$

($\Delta$ is the Laplace operator) belong to the class $H_p^{r+l+\frac{1}{p}-1}(\sigma)$.

A similar result was obtained for the semiplane (Ya. S. Bugrov [3]). The exact results of N. M. Gyunter [1] and Kellog [1] for the three-dimensional domain with smooth boundary when $p = \infty$ and when r is nonintegral precede these theorems; O. V. Besov [1] for the semispace when $1 \leq p \leq \infty$ and for nonintegral r; N. I. Mozzherova [1] for the three-dimensional domain with smooth boundary when $1 \leq p < \infty$ and nonintegral r; S. M. Nikol'skiy [4, 9] for a circle with $p = 2$ and any r. Cf, further, T. I. Amanov [2]. At the present time there are many results of this kind with estimates of solutions of different boundary-value problems in terms of the classes examined here (cf, for example, V. A. Solonnikov [1], Nechas [1, 2], and I. N. Vekua [1]).

6.9. Suppose $\mathbf{k} = (k_1, \ldots, k_n) \geqslant 0$ (i.e., $k_j \geqslant 0$ for all j) is an integral vector and $\mathbf{h} = (h_1, \ldots, h_n)$ is an arbitrary vector ($h_j \neq 0$, $j = 1, \ldots, n$). By definition

$$\Delta_h^k f = \Delta_{h_1}^{k_1} \ldots \Delta_{h_n}^{k_n} f,$$

where $\Delta_{x_j h_j}^{k_j} f$ is the difference of f of order $k_j$ with pitch $h_j$ in the direction $x_j$ ($\Delta_{x_j h_j}^{0} f = f$). Let us assign the vector $\mathbf{r} = (r_1, \ldots, r_n) \geqslant 0$ and assume that e is any sub-set of the set of natural numbers $e_n = \{1, \ldots, n\}$, and $\mathbf{r}^e = (r_1^e, \ldots, r_n^e)$ is a vector whose complnents are governed by the condition

$$r_j^e = \begin{cases} r_j, & j \in e, \\ 0, & j \notin e. \end{cases}$$

Let us set

$$\overline{r^e} = (\overline{r_1^e}, \ldots, \overline{r_n^e}),$$

$$\alpha^e = r^e - \overline{r^e} = (\alpha_1^e, \ldots, \alpha_n^e),$$

where, if $r_j^e > 0$, then $\overline{r_j^e}$ is the largest integer less than $r_j^e$, and if $r_j^e = 0$, then $\overline{r_j^e} = 0$.

Let us further introduce the vector $\omega = (1, \ldots, 1)$. By definition the function $f(x) = f(x_1, \ldots, x_n)$ belongs to the class $S_p^r H = S_p^r$ if the norm

$$\|f\|_{S_p^r H} = \sum_e \sup_h \left\| \frac{\Delta_h^{2\omega} f^{(\overline{r^e})}(x)}{h^{\alpha^e}} \right\|_{L_p(R_n)}$$

is finite, where the sum is extended over all sub-sets e, $h^{\alpha^e} = h_1^{\alpha_1^e} \ldots h_n^{\alpha_n^e}$ and $f^{(\overline{r^e})}$ is a partial derivative of order $\overline{\mathbf{r}^e}$. This sum contains the term

$\|f\|_{L_p(R_n)}$ corresponding to the empty set e.

Theorem on representation. Suppose $r > 0$. For $f(x) \in S_p^r(H)$, it is necessary and sufficient that the representation

$$f(x) = \sum_{k \geqslant 0} Q_k(x), \tag{1}$$

obtain, where $Q_k(x) = Q_{2^{k_1 r_1}, \ldots, 2^{k_n r_n}}(x)$ are integral functions of the exponential type $2^{k_j r_j}$, respectively, with respect to $x_j$ for which

$$\|Q_k\|_{L_p(R_n)} \leqslant c2^{-kr} \left(kr = \sum_{j=1}^{n} k_j r_j\right) \tag{2}$$

and c does not depend on k.

The embeddings

$$S_p^r(R_n) \to S_{p'}^{\rho}(R_n) \left(1 \leqslant p < p' \leqslant \infty,\ \rho = r - \left(\frac{1}{p} - \frac{1}{p'}\right)\varkappa > 0\right) \tag{3}$$

$$S_p^r(R_n) \to S_p^{\overline{r^{e_m}}}(R_m)\ (e_m = (1, \ldots, m);\ 1 \leqslant m < n;$$
$$r_j - \frac{1}{p} > 0,\quad j = m+1, \ldots, n). \tag{4}$$

obtain. In fact, if $f \in S_p^r(R_n)$, then (1) and (2) are valid.

$$\|Q_k\|_{L_{p'}(R_n)} \leqslant c_1 2^{-k\rho} \qquad (\rho > 0)$$

and $f \in S_{p'}^{\rho}(R_n)$. Further

$$\|Q_k\|_{L_p(R_m)} \leqslant c_1 2^{-kr + \frac{1}{p}\sum_{m+1}^{n} k_j r_j}.$$

and if we set in (1) $x_{m+1} = \ldots = x_n = 0$, then we obtain for the trace of f on $R_m$ the representation

$$\varphi(x_1, \ldots, x_m) = f(x_1, \ldots, x_m, 0, \ldots, 0) = \sum_{k^{e_m} \geq 0} q_{k^{e_m}},$$

where the sum is extended to m-dimensional vectors $k^{(m)} \geqslant 0$ and

$$q_{k^{e_m}} = \sum_{\substack{k_j \geq 0 \\ m+1 \leq j \leq n}} Q_k(x_1, \ldots, x_m, 0, \ldots, 0),$$

where $(r_j - 1/p > 0)$

$$|q_{k^{e_m}}| \leq 2^{-\sum_1^m k_j r_j} \sum_{\substack{k_j \geq 0 \\ m+1 \leq j \leq n}} 2^{-\sum_{m+1}^n k_j\left(r_j - \frac{1}{p}\right)} \leq 2^{-\sum_1^m k_j r_j'},$$

which entails (4) by virtue of the inverse theorem on representation.

Let us introduce the space

$$S^{r^1, \ldots, r^N} = S_p^{r^1, \ldots, r^N} = \bigcap_1^N S_p^{r^j}$$

with the norm

$$\|f\|_{S_p^{r^1, \ldots, r^N}} = \sum_{j=1}^N \|f\|_{S_p^{r^j}}.$$

for vectors $r^1, \ldots, r^N$. the theorem on interpolation is valid:

$$S^{r^1, \ldots, r^N} \to S^{\sum_1^N \lambda_k r^k} \quad \left(\lambda_k \geq 0, \ \sum_1^N \lambda_k \leq 1\right). \tag{5}$$

If N n and $r^i = (0, \ldots, 0, r_i, 0, \ldots, 0)$, then

$$S^{r^1, \ldots, r^N} = S^{r^1, \ldots, r^N} H = H_p^{r_1, \ldots, r_n}.$$

These results were proven by S. M. Nikol'skiy [15, 16].

Let us note the work of N. S. Bakhvalov [1], where he independently proved one aspect (the necessity) of the theorem of representation of functions of the periodic class $S_p^r H$: if $f \in S_p^r H$, then (1) and (2) obtain, where $Q_s$ are trigonometric polynomials. Extensions of these theorems from H- to B-classes belong to T. I. Amanov [3] and (by other methods) to A. D. Dzhabrailov [1].

6.10.2. This note on mean functions was communicated to me by O. V. Besov.

To Chapter VII

7.2. Inequalities between the norms of partial derivatives with parameter $\varepsilon$ and multiplicative inequalities are found in the work of V. P. Il'yin [7] and in his later works, V. A. Solonnikov [1], K. K. Golovkin [1], V. P. Il'yin and V. A. Solonnikov [1], and others.

Inequalities containing $\varepsilon$ are employed in the theory of differential equations when it is desired that one of the terms of the form

$$\varepsilon^{a}\|f\|+\frac{1}{\varepsilon^{b}}\|f^{(k)}\|$$

be smaller than a pre-specified number.

It follows from the results of S. M. Nikol'skiy [11] pertaining to more general embedding theorems that the inequalities between the semi-norms

$$\|f\|_{h_{p'}^{r'}(g)} \leq c\|f\|_{w_p^r(g)}, \qquad (1)$$

$$\|f\|_{w_{p'}^{l}(g)} \leq c\|f\|_{w_p^r(g)} \qquad (2)$$

$$\left(1 \leq p < p' \leq \infty,\ r' = r-\left(\frac{1}{p}-\frac{1}{p'}\right)n > 0,\ l < r'\right)$$

obtain for the arbitrary domain $g \subset R_n$, without the pre-assumption that $\|f\|_{L_p(g)}$ is finite, if and only if $\bar{r} \leq r' < r$ (in the case of (1)) and $\bar{r} < l < r' < r$ (in the case of (2)).

The inequality

$$\|f\|_{w_p^{l-\frac{1}{p}}(R_{n-1})} \leq c\|f\|_{w_p^{l}(R_n)} \qquad (1 < p < \infty)$$

follows from the work of L. D. Kudryavtsev [4] and Yu. S. Nikol'skiy [1] on weighting spaces, along with confirmation of the possibility of extending functions $\varphi \in W_p^{1-\frac{1}{p}}(R_{n-1})$ onto $R_n$, so that for their extending functions, the following relation

$$\|f\|_{w_p^l(R_n)} \leqslant c \|\varphi\|_{w_p^{l-\frac{1}{p}}(R_{n-1})} \qquad (1 < p < \infty)$$

obtains without the presumption that the norms $\|f\|_{L_p(R_n)}$ and $\|\varphi\|_{L_p(R_{n-1})}$ are finite. In these two theorems, just as above, $0 < 1 - 1/p < 1$.

7.3. Extrenum functions are introduced and studied in the works of S. M. Nikol'skiy [2,3], T. I. Amanov [1], and P. Pilika [1]. The accuracy (unimprovability) of the inequalities presented here between H-norms was established by means of these works.

7.7. Many investigations, beginning with the works of Ascoli [1] and Arzelà [1], deal with problem of the compactness of classes of functions. The fundamental Arzelà theorem on the compactness pertains to the class of continuous functions. In the $L_p$ metric the Kolmogorov theorem [1] (for $p > 1$) corresponds to it, and the Tulaykov theorem [1] (for $p = 1$). Investigations on the problem of the compactness of classes of differentiable functions include the works of Rellich [1], I. G. Petrovskiy and K. N. Smirnov [1], V. I. Kondrashov [1], Picone [1], Pucci [1], L. D. Kudryavtsev [1], O. V. Besov [12], V. P. Il'yin [9], and others.

The theorem presented here for the classes $H_p^r$ and $W_p^r$ can be found in detail in S. M. Nikol'skiy [8]. In essence, here we are concerned with weak compactness: from a sequence bounded in the metric $H_p^r$ or $W_p^r$, we can separate a subsequence convergent in the $H_p^{r-}$ sense ($\varepsilon > 0$) to some function $f \in H_p^r$, $W_p^r$.

The theorems 7.7.1-7.7.5 are already concerned with the compactness of a set in the metric of the space to which it belongs. In particular, it encompasses the theorem on compactness in $L_p$ (cf S. L. Sobolev [4], Chapter I, section 4.3).

Theorems 7.7.2-7.7.5 were proven by P. I. Lizorkin and S. M. Nikol'skiy.

O. V. Besov [12] studied problems of the compactness of sets of functions f in the H-classes by imposing additional conditions on f. For example, in the case $H_p^r$ ($r < 1$) it is assumed that

$$\|f(x+h) - f(x)\| \leqslant \alpha(h)\,|h|^r$$
$$(\alpha(h) \to 0,\ |h| \to 0).$$

To Chapter VIII

8.1. Operation $I_l$ corresponds to some extent to the Weyl operation (cf Zigmund [2], Chapter XII, page 8)

$$f(x) = I_l^{*}\varphi = \frac{1}{\pi}\int_{-\pi}^{\pi} K_l(x-t)\,\varphi(t)\,dt, \tag{1}$$

$$K_l(t) = \sum_{\nu=1}^{\infty} \frac{\cos\left(\nu t + \frac{l\pi}{2}\right)}{\nu^l} \qquad (l > 0),$$

$$\int_{-\pi}^{\pi} \varphi(t)\,dt = 0, \qquad \varphi \in L.$$

It is intimately involved with the (aperiodic) operation

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x} (x-t)^{\alpha-1}\varphi(t)\,dt$$

of Liouville. Kernels $I_l$, $I_l^*$ have for t the same singularity $|t|^{\alpha-1}$ (we have in mind the one-dimensional case, compare 8.1 (6), (13) and Zigmund [1], Chapter V, section 2.1). The Liouville kernel $(x - t)^{\alpha-1}$ when $t = x$ has the same singularity.

Estimates of the form 8.1(7) for partial derivatives of $G_r(|x|)$ are found in Aronszajn and Smith [1].

8.2. The theorem on the isomorphism of the classes $W_p^l(R_n)$ is proven in the works of Calderon [3] and Lions and Magenes [1].

8.3. Estimates of differences of derivatives of $G_r(|x|)$ are to be found in Aronszajn and Smith [1], Nikol'skiy, Lions, and Lizorkin [1], Chapter I, and S. M. Nikol'skiy (with an addition by Ye. Nosilovskiy [18], lemma 6.

8.4. Inequality (2), cf. Nikol'skiy, Lions, and Lizorkin [1], Chapter I, and S. M. Nikol'skiy [18], lemma 6.

In the periodic case when $n = 1$, $p = \infty$, it was known to I. P. Natanson [2], pp 119 - 120, if it is assumed that $E^*_\nu(f)_\infty$ is the best approximation of the function f using trigonometric polynomials with a period-based mean equal to zero, and it was known to S. B. Stechkin [2] in the ordinary meaning of $E^*_\nu(f)_\infty$ ; see also Sung Yung-sheng [1].

Inequality (4) is an analog of the corresponding one-dimensional Favard inequality [1] in the periodic case. It is used in deriving inequality 8.6(16) $(r > 0)$ and here the counsel of my colleague S. A. Telyakovskiy proved useful to me.

8.6. N. I. Akhiyezer and B. M. Levitan studied kernels more general than $V_N(t)$ for other purposes; these kernels corresponded to the more general trigonometric dela Vallee-Poussin sums $\frac{1}{p+1}(D^*_{N-p} + \ldots + D^*_N)$, where $D^*_k$ are Dirichlet kernels.

Cf S. M. Nikol'skiy [17] on expansion of functions regular in the $L_p$-sense in dela Valleé-Poussin sums.

8.8-8.92. The findings presented here, based on the understanding of a generalized function regular in the $L_p$ $(1 \leq p \leq \infty)$-sense, and its expansion in weakly convergent dela Valleé-Poussin series can be found in S. M. Nikol'skiy [17, 18]. In themselves, the concepts of the zero classes $B^0_{p\theta}$, the isomorphism $B^r_{p\theta}$ for different r, and the integral representations $B^r_{p\theta}$ in terms of zero classes and negative classes $B^r_{p\theta}$ where established from different considerations in the works of Calderon [3], Aronszajn, Mulla and Szeptycki [1], Taibleson [1, 2], S. M. Nikol'skiy, Lions, and P. I. Lizorkin [1].

8.9. The collection $S'_p = S'_p(R_n)$ of all generalized functions regular in the sense of $L_p (1 \leq p \leq \infty)$ (cf 1.5.10) can be further defined as the sum

$$S'_p = \bigcup_k H^{r_k}_p \tag{1}$$

$(H^r_p = H^r_p(R_n))$, where $\{r_k\}$ is an arbitrary sequence of real numbers tending to $-\infty$. In fact, if $f \in S'_p$, then for some $\rho \geq 0$, $I_\rho f \in L_p$ obtains (cf 1.5.10), threefore (cf 8.2) $I_{\rho+1} f \in W^1_p \subset H^1_p$, but then $f \in H^{-\rho}_p \rightarrow H^{r_k}_p$, if k is such that $r_k < \cdots$. Conversely, if $f \in H^{r_k}_p$ for some k, then $I_{-r_k+1} f \in H^r_p \subset L_p$.

Clearly, in (1) H can be replaced by B or L (cf 6.1).

Let us agree to state that the function $f \in S'_p$ has a spectrum in the domain $G \subset R_n$ if its Fourier transform $\tilde{f}$ as a carrier on G, i.e., $\tilde{f} = 0$ outside G.

From the foregoing it follows that if the function $f \in S'_p$, then f also belongs to $H^r_p$ for certain r and can be expanded in the series

$$f = \sum_0^\infty q_s \tag{2}$$

with the following properties: 1) $q_s \in L_p$ and has a spectrum in $\Delta_{s+1} - \Delta_{s-1}$ $(s = 1, 2, \ldots)$, $\Delta_0$ (when $s = 0$), where $\Delta_s = \{|x_j| < a^s$, $a > 1$; b) the inequalities

$$\|q_s\|_{L_p} < Ma^{-sr} \qquad (s = 0, 1, \ldots), \tag{3}$$

are satisfied where M does not depend on s.

In fact, we can take the corresponding dela Valleé-Poussin sums of the function f as the $q_s$ (cf 8.9). In the case $1 < p < \infty$ property a) can be replaced by the following: a) $q_s \in L_p$ and it has a spectrum in $\Delta_s - \Delta_{s-1}$ $(s = 1, 2, \ldots)$, $\Delta_0$ (when $s = 0$) (cf 8.10.1).

If the function f is represented in the form of the series (2) weakly convergent to it with indicated properties a) and b) for some real r, then we can state that the series is the regular expansion of f.

Lemma. An arbitrary formally constructed series

$$\sum_0^\infty u_s, \tag{4}$$

whose members satisfy the properties: a*) $u_s \in L_p$ and which has a spectrum outside $\Delta_{n_s}$ ($n_s = \varkappa s$, $\varkappa > 0$, is a constant not dependent on s) and b)

$$\|u_s\|_{L_p} < a^{-sr} \qquad (s = 0, 1, \ldots) \tag{5}$$

weakly converges to some function $f \in S'_p$. The functions $u_s$ per se, thus, form a set convergent weakly to zero.

In particular, the series in the right side of (2) with properties a) and b) converges weakly to some function $f \in S'_p$, more exactly, $f \in H^r_p$.

Proof. Let $\mu\rho > -r$, then (cf 8.4(4)) $\|I_\rho u_s\|_{L_p} \ll a^{-s(\mu\rho+r)}$, and because the series

$$\sum_0^\infty I_\rho u_s = F$$

converges in the $L_p$-sense, and consequently, weakly to some $F \in L_p$, but then series (1) converges weakly to $f \in I_{-\rho} F \in S'_p$.

Let us note that when $r > 0$ series (1) converges to $L_p$ on the assumption that condition b) is satisfied (without a*)).

The embeddings

$$L^r_p(R_n) \to L^\rho_{p'}(R_n) \qquad (1 < p < p' < \infty), \tag{6}$$

$$B^r_{p\theta}(R_n) \to B^\rho_{p'\theta}(R_n) \quad (1 \leqslant p \leqslant p' \leqslant \infty;\ 1 \leqslant \theta \leqslant \infty,\ B^r_{p\infty} = H^r_p),$$
$$\rho = r - n\left(\frac{1}{p} - \frac{1}{p'}\right), \tag{7}$$

where r is an arbitrary real number are valid.

In fact, both $\Lambda^r_p$ denotes one of the classes appearing in the left members of (1) and (2), and let k be such number that $k + \rho > 0$, then (cf 8.2, 8.7, 9.62, and 6.2)

$$I_k(\Lambda^r_p) = \Lambda^{r+k}_p \to \Lambda^{\rho+k}_{p'},$$

from whence

$$\Lambda^r_p \to I_{-k}(\Lambda^{\rho+k}_{p'}) = \Lambda^\rho_{p'},$$

and we have proven (6) and (7).

The situation is more involved with theorems of embeddings of different measures, as will be clear below.

Let us set $x = (u, v)$, $u = (x_1, \ldots, x_m)$, $v = (x_{m+1}, \ldots, x_n)$ $(1 \leq m < n)$ and let $R_m(v^0) = R_m$ denote the linear sub-set $R_n$ of points $(u, v^0)$, where $v^0$ is fixed and $u$ is arbitrary.

Definition. Suppose the function $f \in S'_p = S'_p(R_n)$, $1 \leq p \leq \infty$ and

$$f(u, v) = \sum_{s=0}^{\infty} q_s(u, v) \tag{8}$$

is its regular expansion, displaying the property that for any $s$, spectrum $q_s$ belongs to the spectrum of $f$. (We note that the terms in the dela Valleé-Poussin series are governed by this property).

If, whatever be the regular expansion of $f$ defined above, the series

$$f(u, v^0) - \sum_{s=0}^{\infty} q_s(u, v^0) \tag{9}$$

converges weakly with respect to $u$ (in the sense of $S(R_m)$) to some function $f(u, v^0)$, not dependent on the expansion of $f$, then this function (of $u$) is called the trace of $f$ on $R_m$.

Let us note that if the function $f(u, v)$ is integral and the exponential type, then any regular expansion of it is a finite sum (8) and, obviously, its trace on $R_m$ is $f(u, v^0)$.

Below we present several confirmations without proof.

Theorem. Traces of the function $f(u, v)$ on $R_m$ in the sense of the definition given above and in the sense of the definition in 6.3 coincide.

Let $\mathfrak{M}_\lambda$ stand for any set of points $x = (u, v)$ of the form

$$\mathfrak{M}_\lambda = A + \{|v| < |u|^\lambda\} \qquad (\lambda > 0),$$

where $A$ is a bounded set in $R_n$ belonging to the cube $\Delta_M = \{|x_j| < a^M,\ a > 1\}$.

Theorem. If the function $f(u, v) \in S'_p(R_n)(1 \leq p \leq \infty)$ has a spectrum belonging to the set $\mathfrak{M}_\lambda$, then it has the trace $f(u, v^0) \in S'_p(R_m)$.

More exactly, embeddings with constants dependent on $M$ and $\lambda$ obtain for classes of the functions $H^r_p(R_n)$ that have a spectrum in $\mathfrak{M}_\lambda$:

$$H_p^r(R_n) \to \begin{cases} H_p^{\left(r-\frac{n-m}{p}\right)\lambda}(R_n) & \left(r-\frac{n-m}{p}<0,\ \lambda>1\right), \\ H_p^{-\varepsilon}(R_m) & \left(r-\frac{n-m}{p}=0,\ \lambda\geqslant 1\right), \\ H_p^{\left(r-\frac{\lambda(n-m)}{p}\right)}(R_m) & (0<\lambda\leqslant 1 \text{ except for the case} \\ & \left. r-\frac{n-m}{p}=0,\ \lambda=1\right). \end{cases} \tag{10}$$

Inverse theorem. The function

$$\varphi(u) \in H_p^{\left(r-\frac{\lambda(n-m)}{p}\right)}(R_m) \qquad (0<\lambda\leqslant 1)$$

or

$$\varphi(u) \in H_p^{\left(r-\frac{n-m}{p}\right)\lambda}(R_m) \qquad (\lambda>1)$$

can be extended on $R_n$ such that the extended function $f(u, v) \in H_p^r(R_n)$ has a spectrum belonging to the set of the form $\mathfrak{M}_\lambda$, and its trace is $f(u, 0) = \varphi(u)$. Here the embeddings (with the corresponding inequalities, cf 6.0(13))

$$H_p^{\left(r-\frac{\lambda(n-m)}{p}\right)}(R_m) \to H_p^r(R_n) \qquad (0<\lambda\leqslant 1). \tag{11}$$

$$H_p^{\left(r-\frac{n-m}{p}\right)\lambda}(R_m) \to H_p^r(R_n) \qquad (\lambda>1). \tag{12}$$

obtain. Embedding (11), when $\lambda = 1$ and $r - \frac{n-m}{p} > 0$, is already familiar to us (cf 6.5), but here it is given a stronger formulation, including the assertion on the nature of these spectra of the standing function. When $\lambda > 1$ and $r - \frac{n-m}{p}$ 0, the (inverse) embedding (12) and the corresponding (direct) embedding (10) are no longer mutually inverse.

Let us emphasize that in relations (11) and (12), no restrictions were imposed whatever on the spectrum of functions of the original (embedded) classes.

For the cases $0 < \lambda \leq 1$, an ascending function $f(u, v)$ satisfying the condition of the theorem can be defined in the form of the weakly convergent series

$$f(u, v) = \sum_{s=0}^{\infty} Q_s(u, v),$$

$$Q_s(u, v) = \varphi_s(u) \prod_{j=m+1}^{M} F(2^{(s-k)\lambda} v_j), \quad F(t) = 4\left(\frac{\sin\frac{t}{2}}{t}\right)^2,$$

where

$$\varphi(u) = \sum_{s=0}^{\infty} \varphi_s(u),$$

$$\varphi_0(u) = \sigma_{2^0} f, \quad \varphi_s(u) = (\sigma_{2^s} - \sigma_{2^{s-1}})\varphi$$

(cf 8.9).

But in the case $\lambda > 1$, the extending function $f(u, v)$ is defined by the weakly convergent series

$$f(u, v) = \sum_{s=0}^{\infty} q_{n_s}(u, v),$$

$$q_{n_s}(u, v) = \varphi_s(u) \prod_{j=m+1}^{M} \alpha_s(v_j), \quad \alpha_s(\xi) = \cos 3 \cdot 2^{n_s - 1} \xi F(2^{n_s - 1} \xi),$$

where $n_s$ ($s = 0, 1, \ldots$) is an ascending sequence of natural numbers such that $\frac{n_s}{s} \to 1$ ($s \to \infty$), and functions $\varphi_s$ are meaningful in their former sense.

Function $\psi(x, y)$ of two variables, with the Fourier transform

$$\hat{\psi} = \begin{cases} u^{-1} v^{-1} & (u, v > 2), \\ 0 & \text{for the remaining } (u, v) \end{cases}$$

belong to $H_2^{\frac{1}{2}}(R_2)$ and at the same time do not have a trace on the axis $v = 0$.

Proof. Let us adopt the series

where
$$\varphi = \sum_{s=1}^{\infty} \varphi_s,$$
$$\varphi_s(x, y) = \frac{1}{2\pi} \int\int_{\Delta_s - \Delta_{s-1}} u^{-1} v^{-1} e^{i(xu+yv)}\, du\, dv,$$

as its regular expansion and
$$\|\varphi_s\|^2_{L_2(R_2)} < 2 \int_{2^{s-1}}^{2^s} u^{-2}\, du \int_2^{2^s} v^{-2}\, dv < 2^{-s},$$

therefore $\psi \in H_2^{\frac{1}{2}}(R_2)$. Further

$$S_N(x) = \sum_1^N \varphi_s(x, 0) = \frac{1}{2\pi} \int\int_{\Delta_N} u^{-1} v^{-1} e^{ixu}\, du\, dv =$$
$$= \frac{1}{2\pi} \int_2^{2^N} v^{-1}\, dv \int_2^{2^N} u^{-1} e^{ixu}\, du = cN \int_2^{2^N} u^{-1} e^{ixu}\, du.$$

The function $S_N(x)$ does not converge weakly, because for example, the function $\varphi$ is such that $\tilde{\varphi} = e^{-x^2} \in S'$ obtains:

$$(S_N, \varphi) = (\tilde{S}_N, \tilde{\varphi}) = cN \int_2^{2^N} u^{-1} e^{-u^2}\, du \to \infty \qquad (N \to \infty).$$

It is possible to construct an example showing that in (10) $\varepsilon > 0$ cannot be replaced by $\varepsilon = 0$.

The assertions stated above can be extended from the classes $H_p^r$ to $B_{p\theta}^r$.

8.10 - 8.10.1. The fact presented here, pertaining to the expansion of functions of the classes $B_{p\theta}^r$ in series in Dirichlet sums for the case $1 < p < \infty$ are close to the results of P. I. Lizorkin [7], and also to those of M. D. Ramazanov [1], who investigated classes of functions somewhat distinct from $B_{p\theta}^r$ from this point of view.

## To Chapter IX

9.1. Suppose

$$1 < p < q < \infty, \tag{1}$$

$$1 \leqslant m \leqslant n, \ \rho = r - \frac{n}{p} + \frac{m}{q} > 0$$

(2)

and r is an integer. Then from 9.1(4) follows the embedding

$$W_p^r(R_n) = L_p^r(R_n) \to L_q^\rho(R_m) \to W_q^{[\rho]}(R_m)$$

(S. L. Sobolev with addition by V. I. Kondrashov and V. I. Il'yin, cf 6.1). When $\rho$ is a noninteger and p = q, the following embedding is valid (cf 9.3 (1), 6.2(4), and 6.5(1')):

$$L_p^r(R_n) \to H_p^r(R_n) \to H_p^\rho(R_m) \to W_p^{[\rho]}(R_m).$$

The case when $\rho = 0$, $p = 1 < q < \infty$ is interesting; here it was proven for a natural r($L_p^r$ -- $W_p^r$) space (by S. L. Sobolev [4] when m = n and by E. Gagliardo [2] when m < n) that embedding (3) also remains valid.

9.2.2. The theorem on derivatives was proven for the case p = 2 by S. N. Bernshteyn [1], page 98, for $l_1 = l_2 = 2$ and by L. N. Slobodetskiy [3] in the general case; when $1 < p < \infty$ and for integral $l = l_1 = \dots = l_n$, by A. I. Koshelev [1], and for any $l > 0$ -- by P. I. Lizorkin [10]; and for arbitrary l in the periodic case, by Yu. L. Bessonov [1, 2].

9.4 - 9.6. The results set for here, pertaining to anisotropic classes $L_p^r$, belongs generally to P. I. Lizorkin, who published them without proof in note [10]. He made available to me some manuscript materials that were used as the basis of my exposition. Everywhere I reduced the issue to the $I_r$ operation, while P. I. Lizorkin applied the "pure" Liouville derivatives in the corresponding cases (cf 9.2.3). The main goal of these investigations was to obtain integral representations for (functions) of anisotropic classes $L_p^r$ for any $r \geqslant 0$, and on the basis to construct a complete system of embedding theorems for these classes. Integral representationsof this kind were obtained for isotropic classes in the preceding chapter. The necessary estimates were reached in this preceding chapter from the facts applying to the theory of Bessel-Macdonald kernels. In the anisotropic case, the corresponding kernels become more involved. Cf course, embedding theorems in the isotropic case can be obtained from the corresponding anisotropic theory if we set $r_1 = \dots = r_n = r$. For integral r, r, we obtained corresponding results for the W-classes in particular, the embedding theorems of S. L. Sobolev with which this multi-dimensional theory historically began.

9.4.1. Estimates (2) and (3) for $I_{-1}G_r(x)$ are equivalent in the case $r_1 = \dots = r_n = r$ to the isotropic estimates 8.1(7).

9.5.1 - 9.5.2. Theorems 9.5.1 and 9.5.2 ignore their completeness where obtained by P. I. Lizorkin [10]. They include a number of antecedent results pertaining to the case of integral $r(W_p^r = L_p^r)$ and arbitrary $r$ when p = 2 of Aronszajn [1], L. N. Slobodetskiy [1] (cf further, V. M. Babich and L. N. Slobodetskiy [1]), Gagliardo [1], O. V. Besov [2], P. I. Lizorkin [9], and S. V. Uspenskiy [1] (cf review by S. M. Nikol'skiy [12])for more details.

Here we also include the corresponding results for the isotropic classes $L_p^r = L_p^{r,\dots,r}$ belonging to Stein [1], Aronszajn, Mulle, and Szeptycki [1], and to P. I. Lizorkin [3] (cf V. I. Burenkov [3] for a more detailed treatment).

These results were obtained by different methods.

In this book, when a function was extended from $R_m$ to $R_n$, the method of expanding it in a series in integral functions of the exponential type and the successive increment of its term with special functions (S. M. Nikol'skiy [5]) was employed. Other authors also used another technique for these purposes, based on Steklov averaging of the function (cf A. A. Dezin [1] and Galeyado [1].

Let us note if a very simple direct proof of the theorem for the embedding of different measures in the anisotropic case for integral classes $L_p^r = W_p^r$, belonging to V. A. Solonnikov [1].

9.6.2. The S. L. Sobolev embedding theorem (with additions by V. I. Kondrashov and V. P. Il'yin) (cf 6.1 and denote 6.1) are part of theorem 9.6.2 as a particular case and are maximally accurate in terms of (integral) classes $W_p^l$.

In the isotropic case of fractional l, theorem 9.6.2 was proven by Stein [1] and P. I. Lizorkin [5], and by P. I. Lizorkin [10] in the anisotropic case (presented in the text).

In proofreading, we became acquaintod with an article by Sadosky and Cotlar [1], which define for rational vectors $r \geq 0$ classes equivalent to the classes $L_p^r$, and for which several embedding theorems are proven.

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