Math 431 - Real Analysis
Solutions to Homework due September 5

Question 1. Let $a, b \in \mathbb{R}$.

(a) Show that if $a + b$ is rational, then a is rational or b is irrational.

(b) Use (a) to show that if $a + b$ is rational, then a and b are both rational or both irrational.

Solution 1.

(a) We will instead prove the contrapositive statement, which is "if a is irrational and b is rational, then $a + b$ is irrational. Assume, to the contrary, that $a + b$ is rational. Then, since b is rational, we have that $-b$ is also rational. Since the sum or rational numbers is rational, we get that

$$a = (a + b) - b \in \mathbb{Q}.$$

This, of course contradicts that a is irrational. Since we have arrived at a contradiction, then our claim that $a + b$ is rational is false. Thus, $a + b$ is irrational. Having proven the contrapositive, our original statement "if $a + b$ is rational, then a is rational or b is irrational" is true. □

(b) Assuming that $a + b$ is rational, (a) tells us that we have two cases: (1) a is rational or (2) b is irrational. For the first case, we assume that a is rational. Thus $-a \in \mathbb{Q}$ and therefore

$$b = (a + b) - a \in \mathbb{Q}.$$

Therefore, b is irrational and therefore a and b are both rational. IN the second case we have that b is irrational. We wish to show that a is also irrational. Assume, to the contrary, that a is rational. Then, $-a \in \mathbb{Q}$ as well. Thus,

$$b = (a + b) - a \in \mathbb{Q},$$

which, of course, contradicts that b is irrational. Thus, a must be irrational. So, a and b are irrational. □

In class on Monday, we learned of boundedness, the supremum/infimum, and the Completeness Axiom. Given a bounded set $S \subset \mathbb{R}$, a number b is called a *supremum* or *least upper bound* for S if the following hold:

(i) b is an upper bound for S, and

(ii) if c is an upper bound for S, then $b \leq c$.

Similarly, given a bounded set $S \subset \mathbb{R}$, a number b is called an *infimum* or *greatest lower bound* for S if the following hold:

(i) b is a lower bound for S, and

(ii) if c is a lower bound for S, then $c \leq b$.

If b is a supremum for S, we write that $b = \sup S$. If it is an infimum, we write that $b = \inf S$.

We were also introduced to our tenth and final axiom, the *Completeness Axiom*. This axiom states that any non-empty set $S \subset \mathbb{R}$ that is bounded above has a supremum; in other words, if S is a non-empty set of real numbers that is bounded above, there exists a $b \in \mathbb{R}$ such that $b = \sup S$.

Question 2. Show that if a set $S \subset \mathbb{R}$ has a supremum, then it is unique. Thus, we can talk about *the* supremum of a set, instead of the *a* supremum of a set.

Solution 2. Let S be a set and assume that b is a supremum for S To show equality, assume as well that c is also a supremum for S and show that $b = c$. Since c is a supremum, it is an upper bound for S. Since b is a supremum, then it is the least upper bound and thus $b \leq c$. Similarly, since b is a supremum, it is an upper bound for S; since c is a supremum, it is a least upper bound and therefore $c \leq b$. Thus, $c \leq b$ and $b \leq c$, giving us that $b = c$. Thus, a supremum for a set is unique if it exists.

Question 3. Let S be a non-empty subset of $\mathbb{R}$.

(a) Let $-S = \{-x \in \mathbb{R} \,|\, x \in S\}$. Show that S has a supremum b if and only if $-S$ has an infimum $-b$.

(b) Use (a) to show that if T is a non-empty set that is bounded below, then T has an infimum.

Solution 3.

(a) Assume that $b = \sup S$. Then, $x \leq b$ for all $x \in S$. Multiplying both sides by -1, we get that $-b \leq -x$ for all $x \in S$. Thus, $-b$ is a lower bound for the set S. Now, assume that c is another lower bound for $-S$; we will show that $c \leq -b$. If not, then $-b < c$. Multiplying by -1, this would give us that $-c < b$. Notice that since c is a lower bound for $-S$, then $c \leq y$ for all $y \in -S$. Since $y \in -S$, then $y = -x$ where $x \in S$. So, we have taht $c \leq -x$ for all $x \in S$ and therefore $x < -c$ for all $x \in S$. So, $-c$ is an upper bound for S. Thus, $-c$ is an upper bound for S and $-c < b$, contradicting that b is a supremum for S.

The converse direction is an almost identical argument.

(b) Since T is bounded below, say by a, then $a \leq x$ for all $x \in T$. Multiplying by -1, we get that $-x \leq -a$ for all $x \in T$. This is equivalence to $y \leq -a$ for all $y \in -T$. Thus, $-T$ is non-empty and bounded above. Thus, by the Completeness Axiom, $-T$ has a supremum b. By (a), we have that $-(-T) = T$ has an infimum $-b$, as desired.

Question 4. Prove the following *Comparison Theorem*: Let $S, T \subset \mathbb{R}$ be non-empty sets such that $s \leq t$ for every $s \in S$ and $t \in T$. If T has a supremum, then so does S and,

$$\sup S \leq \sup T.$$

Solution 4. Let $\tau = \sup T$. Since τ is a supremum for T, then $t \leq \tau$ for all $t \in T$. Let $s \in S$ and choose any $t \in T$. Then, since $s \leq t$ and $t \leq \tau$, then $s \leq t$. Thus, τ is an upper bound for S. By the Completeness Axiom, S has a supremum, say $\sigma = \sup S$. We will show that $\sigma \leq \tau$. Notice that, by the above, τ is an upper bound for S. Since σ is the least upper bound for S, then $\sigma \leq \tau$. Therefore,

$$\sup S \leq \sup T.$$

□

Question 5. Consider the set

$$S = \left\{ \frac{1}{n} \middle| n \in \mathbb{Z}_+ \right\}.$$

(a) Show that $\max S = 1$.

(b) Show that if d is a lower bound for S, then $d \leq 0$. [Hint: A proof by contradiction might be helpful, as well as the Archimedean Property.]

(c) Use (b) to show that $0 = \inf S$.

Solution 5.

(a) Let $x = \dfrac{1}{n} \in S$, where $n \geq 1$. Since $1 \leq n$, we have that $x = \dfrac{1}{n} \leq 1$. Thus, fore very $x \in S$, $x \leq 1$ and 1 is an upper bound. Notice as well that $1 = \dfrac{1}{1} \in S$. Thus, $1 = \max S$. ☐

(b) Let d be a lower bound for S. Thus, for every $s \in S$, $d \leq s$. Assume, to the contrary, that $d > 0$. Using the Archimedean property, we know that there exists an $n \in \mathbb{Z}_+$ such that $1 < dn$. Since $n > 0$, this gives us that $\dfrac{1}{n} < d$. But, $\dfrac{1}{n} \in S$, and this contradicts the fact that d is a lower bound for S. Thus, we must conclude that $d \leq 0$. ☐

(c) Clearly 0 is a lower bound for S since $0 \leq \dfrac{1}{n}$ for all $n \in \mathbb{Z}_+$. If d is any other lower bound, then by (b), $d \leq 0$. Thus, 0 is greatest lower bound and so $0 = \inf S$. ☐

Question 6. Consider the set

$$T = \left\{ (-1)^n \left(1 - \frac{1}{n}\right) \,\middle|\, n \in \mathbb{Z}_+ \right\}.$$

(a) Show that 1 is an upper bound for T.

(b) Similar to 5b, show that if d is an upper bound for T, then $d \geq 1$.

(c) Use (a) and (b) to show that $\sup T = 1$.

Solution 6.

(a) We will show that for any $x \in T$, $x \leq 1$. Since $x \in T$, then $x = (-1)^n(1 - 1/n)$ for some $n \in \mathbb{Z}_+$. Since $\frac{1}{n} > 0$, then $1 - \frac{1}{n} < 1$. We argue our desired inequality in two cases. If n is even, then $x = (-1)^n(1 - 1/n) = 1 - 1/n < 1$. If n is odd, then $x = (-1)^n(1 - 1/n) = 1 - 1/n < 0 < 1$. In either case, $x \leq 1$ (in fact, < 1) and 1 is an upper bound for T. ☐

(b) Let d be an upper bound for T. Thus, $(-1)^n \left(1 - \frac{1}{n}\right) \leq d$ for all $n \in \mathbb{Z}_+$. Assume, to the contrary that $d < 1$. Thus, $1 - d > 0$. By the Archimedean Property, there exists an $n \in \mathbb{Z}_+$ such that $1 < (1 - d)n$. Since $n > 0$, we can re-write this as $\frac{1}{n} < 1 - d$, which is equivalent to

$$d < 1 - \frac{1}{n}.$$

If n is even, then $(-1)^n = 1$ and we have that

$$d < (-1)^n \left(1 - \frac{1}{n}\right) \in T,$$

contradicting the fact that d is an upper bound. If n is odd, then consider instead $n + 1$, which is even. Then, $(-1)^{n+1} = 1$ and

$$d < 1 - \frac{1}{n} < (-1)^{n+1} \left(1 - \frac{1}{n+1}\right) \in T.$$

This again contradicts that d is an upper bound for T. Either way, we reach a contradiction and therefore conclude that $d \geq 1$. ☐

(c) By (a), 1 is an upper bound for T. By (b), if d is any other upper bound, then $1 \leq d$. Thus, $\sup T = 1$.