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Combinatoire des tresses
Combinatorics of braids

Thèse
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devant le jury composé de

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Résumé

Les groupes et les monoïdes de tresses forment une extension naturelle des groupes symétriques. Ils apparaissent dans des contextes variés, qui vont de la la topologie à la théorie des structures automatiques. Dans cette thèse, nous nous intéressons aux propriétés combinatoires des tresses, ainsi que de certaines structures algébriques qui leur sont apparentées, et nous nous penchons plus spécifiquement sur la notion de complexité d'une tresse et sur l'étude des tresses de complexité élevée.

Nous commençons par un rappel de nombreux résultats concernant aussi bien les points de vue algébrique que géométrique que l'on peut avoir sur les tresses, et faisant partie de l'état de l'art. Cela nous laisse ensuite le loisir de nous intéresser de plus près à des problèmes spécifiques concernant les tresses.

Dans la première moitié de cette thèse, nous optons pour un point de vue géométrique sur les tresses, et définissons des notions de complexité géométrique adaptées à ce point de vue. La complexité géométrique des tresses est à l'origine de la construction d'une forme normale de relaxation pour les groupes de tresses. Nous étudions les propriétés algébriques de cette forme normale, et montrons que, malgré son origine purement géométrique, la forme normale de relaxation est rationnelle.

Nous étudions ensuite la croissance des groupes de tresses vis-à-vis de leur complexité géométrique, et concevons des algorithmes permettant de compter les tresses ayant une complexité donnée. Nous calculons également de manière exacte la fonction génératrice associée à la complexité géométrique, dans le cas des groupes de tresses à deux et à trois brins. Enfin, de manière plus générale, nous procédons à une étude asymptotique du nombre de tresses ayant une complexité géométrique donnée.

Dans la seconde moitié de cette thèse, nous changeons d'approche et étudions les tresses du point de vue algébrique, c'est-à-dire en tant que sous-groupe quotient d'un groupe libre. Nous étudions la croissance des marches aléatoires dans les groupes et les monoïdes de tresses, et montrons que les formes normales de Garside à gauche et à droite sont stables. Ce faisant, nous démontrons l'existence d'une limite non triviale pour la forme normale de Garside à gauche lors des marches aléatoires.

Enfin, nous étudions le problème de la génération uniforme dans les monoïdes d'Artin–Tits de type FC, une classe qui généralise à la fois les monoïdes de tresses et de traces. Nous définissons et caractérisons la notion de mesure de Bernoulli uniforme sur des limites projectives de monoïdes d'Artin–Tits de type FC, et démontrons des résultats de convergence des mesures uniformes usuelles sur les tresses et les traces d'une taille donnée.

Abstract

Braid groups and monoids form a natural extension of symmetric groups. They arise in various contexts, that range from the theory of automatic structures to topology. In this thesis we focus on studying combinatorial properties of braids, as well as of some related algebraic structures, with an emphasis of the notion of complexity of a braid and on the study of braids of large complexity.

After recalling state-of-the-art results about braids in both algebraic and geometric contexts, we focus on specific problems about braids.

In the first half of this thesis, we consider braids from a geometric point of view, and define adequate notions of *geometric complexity* of braids. The geometric complexity of braids leads to the construction of a *relaxation normal form* on braid groups. We study the algebraic properties of this normal form and show that, despite its purely geometric nature, the relaxation normal form is regular.

Then, we study the growth of braid groups with respect to their geometric complexity, and provide algorithms for counting braids of a given complexity. In addition, we provide an exact formula for the generating function associated with the geometric complexity, in the case of the groups of braids with two or three strands, and more generally we focus on asymptotic studies of the number of braids with a given geometric complexity.

In the second half of this thesis, we change our approach and we adopt the algebraic point of view on braids, which we view as quotient subgroups of free groups. We study the growth of random walks in braid groups and monoids, and prove that the left and right Garside normal forms are *stable*, thereby constructing a non-trivial limit for the left Garside normal form of a random walk.

Finally, we study the problem of *uniform generation* in *Artin–Tits monoids of FC type*, a class that encompasses braid and heap monoids. We define and characterise the notion of *uniform Bernoulli measure* on a projective limit of Artin–Tits monoids of FC type, and we prove convergence results for standard uniform measures on braids and heaps of a given size.

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Chapitre 1

Introduction (Français)

Cette thèse a pour sujet l'étude des propriétés combinatoires des tresses. Bien que les tresses aient été des objets bien connus depuis des milliers d'années, comme en témoigne la Figure 1.1, leur étude en tant qu'objets mathématiques ne date que du début du vingtième siècle, avec les travaux d'Artin [8, 9]. Une définition intuitive des tresses est la suivante.

FIGURE 1.1 – Les tresses dans l'antiquité gauloise

Soit un ensemble de n brins, où n est un entier naturel non nul, tel que chaque brin ait une extrémité supérieure et une extrémité inférieure. Supposons que les extrémités supérieures de nos n brins soient collées à un axe horizontal, donc contraintes à rester immobiles, et que les extrémités inférieures puissent être accrochées à un autre axe horizontal : on peut les décrocher de manière temporaire, de manière à les mouvoir, avant de les raccrocher à l'axe horizontal.

Tresser ces n brins revient à répéter un nombre fini de fois le mouvement de tressage élémentaire décrit comme suit, et qu'illustre la Figure 1.2. Choisissons deux extrémités inférieures situées l'une à côté de l'autre, décrochons-les, échangeons leurs positions, et raccrochons-les à l'axe horizontal. Ce faisant, nous devrons faire passer une des extrémités au premier plan, tandis que l'autre passera à l'arrière-plan. Par la suite, les brins sont libres de bouger individuellement, si tant est que leurs extrémités supérieures et inférieures restent immobiles, et que les brins ne se touchent pas mutuellement : nous considérons donc les brins à isotopie près.

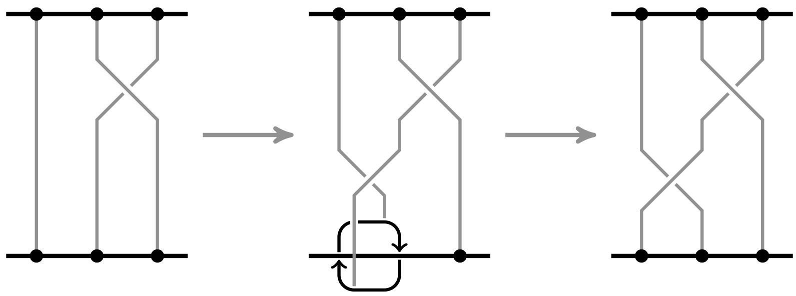


FIGURE 1.2 – Mouvement de tressage élémentaire

Pour tirer de cette description une définition précise, Artin introduit la classe des diagrammes de tresses (que nous reprendrons en Définition 2.4), puis identifie chaque tresse à une classe d'isotopie de diagrammes de tresses. La multiplication des tresses se définit alors de manière naturelle à partir de la concaténation des diagrammes de tresses. Cette description formelle fut à l'origine de deux points de vue sur les tresses.

Le premier point de vue, de nature algébrique, consiste à voir les tresses comme un groupe finiment présenté. S'il est aisé de montrer que les tresses forment un groupe finiment engendré, dont les générateurs sont les mouvements de tressage élémentaires, montrer que les tresses admettent une présentation finie n'est pas évident. Selon le second point de vue, plus géométrique, les tresses forment le mapping class group d'un disque épointé, c'est-à-dire le quotient d'un certain groupe d'homéomorphismes du disque épointé par la relation d'isotopie.

Dès lors que l'on considère au moins deux brins, le groupe de tresses associé est infini, ce qui est à l'origine de plusieurs questions importantes d'ordre algorithmique. Est-il algorithmiquement faisable, et à quel coût, de tester si elles sont égales ? De définir des représentants canoniques des tresses et de les manipuler ? Ces questions sont au cœur de cette thèse.

Y répondre requiert de procéder en plusieurs étapes. Une première étape est de formuler ces questions de manière rigoureuse, et de définir les outils adéquats pour trouver une telle formulation. De tels outils incluent diverses notions de complexité, qui peuvent être liées à différents aspects des tresses et donc être adéquates dans des contextes variés. En effet, et bien que la notion de complexité d'une tresse puisse être intuitivement liée à l'idée qu'une tresse est complexe si ses descriptions sont complexes, les points de vue algébrique et géométrique sur les tresses nous conduisent à des types de descriptions divers, donc à des notions de complexité diverses également.

Une seconde étape consiste à concevoir des formes normales, c'est-à-dire des représentations canoniques des tresses. Selon le point de vue algébrique, les tresses sont des classes d'équivalence de mots, et choisir une forme normale consiste donc à choisir, pour chaque tresse, un mot qui la représente. Idéalement, ces formes normales devraient elles devraient être facile à calculer, c'est-à-dire que, étant données les formes normales respectives $\mathbf{NF}(\alpha)$ et $\mathbf{NF}(\beta)$ de deux tresses α et β, le calcul de la forme normale du produit $\alpha\beta$ devrait être algorithmiquement peu coûteux. Il est donc raisonnable de rechercher des formes normales ayant des propriétés variées telles que la géodicité (les mots choisis devraient être aussi courts que possible), la rationalité (l'ensemble des mots de la forme $\mathbf{NF}(\alpha)$ devrait être rationnel, c'est-à-dire accepté par un automate fini) ou l'automaticité (pour chaque générateur σ du groupe de tresses, l'ensemble des couples de la forme $(\mathbf{NF}(\alpha), \mathbf{NF}(\sigma\alpha))$ devrait être rationnel, de même que l'ensemble des couples de la forme $(\mathbf{NF}(\alpha), \mathbf{NF}(\sigma\alpha)))$. De même, on peut souhaiter que la longueur du mot $\mathbf{NF}(\alpha)$ et le coût du calcul de $\mathbf{NF}(\alpha\beta)$ soient corrélés avec les complexités des tresses α et β.

Une troisième étape consiste en l'étude des tresses de grande complexité. Cette analyse inclut des sujets tels que le dénombrement des tresses d'une complexité donnée et la génération uniforme des tresses d'une complexité donnée. Un outil clé dans l'étude de ces deux sujets est la notion de fonction génératrice. Les fonctions génératrices permettent l'utilisation de l'analyse réelle ou complexe pour en déduire des estimations, plus ou moins précises, exactes ou bien asymptotiques, du nombre de tresses d'une complexité donnée.

Parallèlement, une question naturelle est la suivante : dans quelle mesure peut-on réduire le problème de la génération aléatoire uniforme de tresses de complexité k au problème de la génération aléatoire uniforme de tresses de complexité $k+1$ (et vice-versa) ? Cela nous mène à prendre en compte d'autres manières de tirer des tresses au hasard, comme les marches aléatoires, et dans lesquelles une tresse de complexité $k+1$ est simplement vue comme le produit d'un générateur élémentaire par une tresse de complexité k.

Contributions principales de cette thèse

Nous présentons ci-dessous une description détaillée de chaque chapitre ainsi que des contributions principales qu'il contient.

Le chapitre 2 est pour une large part bibliographique, et consiste en la revue de plusieurs notions faisant partie de l'état de l'art et qui seront utilisées par la suite. Ce chapitre se compose de deux parties. La première partie est centrée sur le point de vue algébrique sur les tresses, ainsi que sur d'autres structures algébriques, dont les relations sont esquissées par la Figure 1.3.

Afin d'identifier des cadres dans lesquels nous pourrons étudier les propriétés des monoïdes de tresses (donc des groupes de tresses), nous étudions d'abord deux généralisations des monoïdes de tresses que sont les monoïdes d'Artin–Tits et les monoïdes de Garside. Ces classes ne se généralisent pas l'une l'autre, ce qui nous pousse à introduire la classe des monoïdes d'Artin–Tits de type sphérique, qui sont à l'intersection des monoïdes d'Artin–Tits et des monoïdes de Garside. Diverses propriétés algébriques, centrées notam-ment sur les formes normales de Garside, sont prouvées dans le cadre des monoïdes de Garside. Puis, dans le contexte plus restreint des monoïdes d'Artin–Tits de type sphérique, certaines de ces propriétés sont précisées, ce qui nous mène à la notion de forme normale de Garside symétrique. Nous étudions ensuite les monoïdes de traces, qui forment une structure algébrique analogue aux monoïdes de tresses. De surcroît, outre ces structures algébriques déjà largement étudiées [9, 20, 37, 42, 43, 44, 53], nous étudions les monoïdes d'Artin–Tits de type FC, qui regroupent à la fois les monoïdes de traces et les monoïdes d'Artin–Tits de type sphérique.

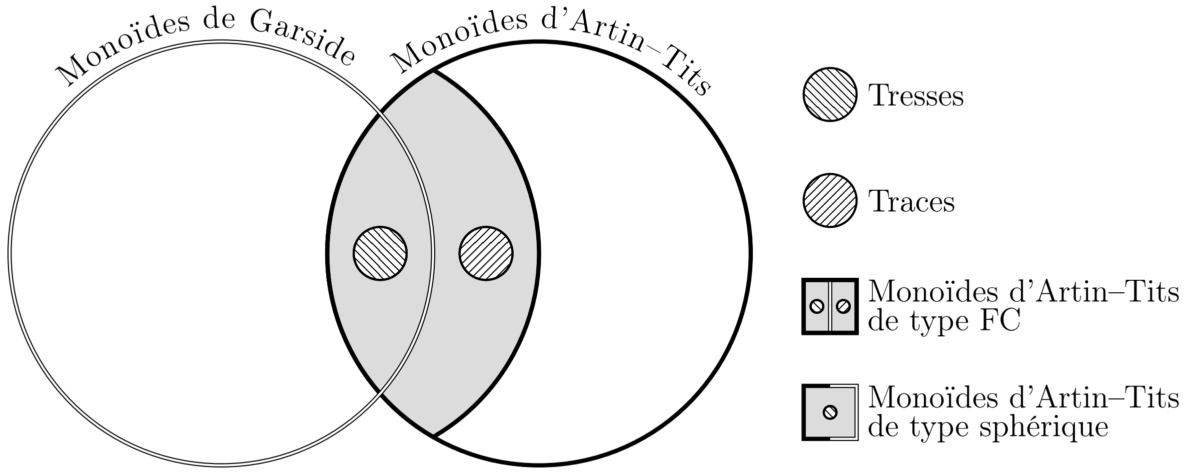


FIGURE 1.3 – Structures algébriques étudiées dans le chapitre 2

La seconde partie du chapitre 2 est centrée sur l'étude des aspects géométriques des groupes de tresses, notamment la représentation des tresses par des laminations du disque épointé et des diagrammes de courbes, qui sont des représentations géométriques classiques des tresses [13, 39, 48]. Nous déduisons de ces représentations deux notions de complexité géométrique, dont nous montrons qu'elles sont intimement liées l'une à l'autre, et qui seront au centre de l'attention des chapitres 3 et 4.

Le chapitre 3 est dédié à l'étude de la forme normale de relaxation, qui provient du point de vue géométrique sur les tresses. Cette forme normale a été introduite et étudiée dans [25], et est un cas particulier d'une famille de formes normales introduites dans [46].

Nous définissons d'abord de nouvelles structures discrètes, de nature combinatoire, que nous appelons arbres de laminations, arbres d'arcs et cartes cellulaires, liées aux représentations géométriques des tresses. Nous prouvons plusieurs propriétés topologiques et combinatoires simples de ces objets, et montrons en particulier que les arbres d'arcs associés à une tresse sont des arbres enracinés unaires-binaires. Cette étude est à l'origine de la contribution principale du chapitre 3, qui consiste à montrer que, malgré sa nature géométrique,

La forme normale de relaxation est rationnelle.

Nous procédons en calculant explicitement un automate déterministe qui reconnaît la forme normale de relaxation. Il s'agit donc d'une nouvelle forme normale rationnelle, qui vient s'ajouter aux formes normales de Garside [37, 53], de Birman-Ko-Lee [15] et de Bressaud [19]. Notons en revanche que les autres formes normales étudiées dans [46] ne sont pas rationnelles a priori.

Nous prouvons ensuite d'autres résultats sur la forme normale de relaxation et sur l'automate susmentionné. Nous démontrons d'abord que notre automate et l'automate déterministe minimal de la forme normale de relaxation ont des tailles comparables. Plus précisément, si on note C et $C_{\min}$ leurs nombres d'états respectifs, alors $C_{\min} \leqslant C \leqslant 2^{60} C_{\min}^{40}$, les constantes 2^{60} et 40 n'étant pas optimales. Nous mettons ensuite en évidence une connexion entre la forme normale de relaxation et l'ordre de Dehornoy sur les tresses, et prouvons que la forme normale de relaxation associe l'ensemble des tresses positives pour l'ordre de Dehornoy à un ensemble de mots rationnel. Nous concluons enfin ce chapitre en conjecturant l'automaticité de la forme normale de relaxation, conjecture que nous démontrons dans le cas des groupes de tresses à trois brins.

Le contenu de ce chapitre provient en grande partie de [63], et est soumis pour publication.

Le chapitre 4 est consacré au problème du dénombrement des tresses ayant une complexité géométrique donnée. Si la question du dénombrement a été abondamment étudiéé dans le cadre de la complexité algébrique, issue de la représentation des tresses sous la forme de factorisations en un nombre minimal de générateurs d'Artin, elle ne semble pas avoir été considérée précédemment pour la complexité géométrique.

Nous commençons par généraliser la notion de diagrammes de courbes, puis nous introduisons une bijection entre les diagrammes de courbes généralisés et un système de coordonnées à valeurs entières. Nous mettons alors en évidence des critères simples caractérisant les collections d'entiers qui sont les coordonnées d'un diagramme de courbes (au sens usuel). Nous définissons ensuite la fonction génératrices associée à la complexité géométrique du groupe à n brins, c'est-à-dire la fonction $\mathcal{B}_n(z) := \sum_{k \geqslant 0} b_{n,k} z^k$ telle que $b_{n,k}$ est le nombre de tresses à n brins et de complexité géométrique k. Les critères susmentionnés nous permettent alors de calculer les fonctions $\mathcal{B}_2(z)$ et $\mathcal{B}_3(z)$ et, plus précisément, de montrer que

La fonction $\mathcal{B}_3(z)$, qui n'est ni rationnelle ni même holonome, et les entiers $b_{3,k}$ sont donnés par :

$$\begin{aligned}
\mathcal{B}_3(z) &= 2\frac{1+z^2-z^4}{z^2(1-z^4)}\left(\sum_{n \geqslant 3} \varphi(n) z^n\right) + \frac{z^2(1-3z^4)}{1-z^4}; \\
b_{3,k} &= \mathbf{1}_{k=0} + 2\left(\varphi(k+2) - \mathbf{1}_{k \in 2\mathbb{Z}} + 2\sum_{i=1}^{\lceil k/2 \rceil} \varphi(k+3-2i)\right)\mathbf{1}_{k \geqslant 1},
\end{aligned}$$

où φ est l'indicatrice d'Euler.

Il est intéressant de constater que, alors que la complexité algébrique est a priori beaucoup plus difficile à calculer que la complexité géométrique, la série associée à la complexité algébrique pour le groupe de tresses à trois brins est rationnelle, c'est-à-dire beaucoup plus simple que la série associée à la complexité géométrique pour le groupe de tresses à trois brins.

Nous calculons également des équivalents simples des entiers $b_{3,k}$ quand $k \to +\infty$, et identifions des bornes supérieures et inférieures non triviales pour les entiers $b_{n,k}$ quand $n \geqslant 4$.

Ce chapitre est majoritairement issu de [64] et a été publié dans *Journal of Knot Theory and Its Ramifications.*

Le chapitre 5 est centré sur l'étude et la convergence des marches aléatoires dans les monoïdes et groupes d'Artin–Tits de type sphérique irréductibles. Nous commençons par étudier le cadre plus simple des marches aléatoires dans les monoïdes et les groupes de traces irréductibles. Les résultats mentionnés existent déjà sous diverses formes [54, 77], et nous en donnons ici une démonstration unifiée. En nous appuyant sur la notion de trace bloquante, nous montrons que les préfixes des formes normales de Garside des éléments obtenus lors d'une marche aléatoire à droite sur un monoïde ou un groupe de traces irréductible convergent presque sûrement, c'est-à-dire que les formes normales de Garside de deux traces successives obtenues lors de la marche aléatoire ont des suffixes communs arbitrairement longs.

Nous quittons ce cadre déjà bien connu et étudions les marches aléatoires dans des monoïdes et groupes d'Artin–Tits de type sphérique irréductibles. La situation générale est analogue mais les monoïdes et les groupes de tresses ont un centre non trivial, ce qui complique considérablement les choses : si la question de la stabilisation de la forme normale de Garside a été posée par Vershik dès le début des années 2000 [89, 90], elle n'avait pas de réponse jusqu'à aujourd'hui. Nous proposons une telle réponse dans le contexte des groupes d'Artin–Tits de type sphérique irréductibles.

Nous introduisons tout d'abord la notion de graphe de Garside bilatère, graphe qui accepte l'ensemble des mots qui sont à la fois des formes normales de Garside à gauche et à droite. Nous prouvons que le graphe de Garside bilatère est fortement connexe et nous en déduisons une notion de mot de tresses bloquant analogue aux traces bloquantes, emprunté à [26]. Puis, en usant d'arguments de sous-additivité tels que le lemme ergodique de Kingman, nous démontrons que

Les préfixes des formes normales de Garside à droite des éléments obtenus lors d'une marche aléatoire à droite sur un monoïde d'Artin–Tits de type sphérique irréductible convergent presque sûrement.

Nous montrons aussi dans quelles conditions ce résultat peut être appliqué au cadre des groupes d'Artin–Tits de type sphérique.

Nous étudions ensuite la limite des formes normales de Garside à gauche associée à la notion de convergence présentée ci-dessus et démontrons que, sous des hypothèses assez larges, cette limite est ergodique. Nous en déduisons d'autres résultats concernant la densité de sous-mots dans la limite et sur la distance de pénétration dans la limite et dans des mots obtenus après avoir effectué un grand nombre de pas, ainsi que sur la vitesse de convergence des formes normales vers cette limite.

Le contenu de ce chapitre est majoritairement issu d'un article en cours de rédaction, écrit en collaboration avec Jean Mairesse.

Le chapitre 6 est centré sur l'étude et le calcul systématique du diamètre du graphe de Garside bilatère des groupes d'Artin–Tits de type sphérique irréductibles. Sous des hypothèses plus relâchées que celles considérées à la fin du chapitre 5, la vitesse de convergence des formes normales de Garside vers leur limite est minorée en fonction, entre autres, du diamètre du graphe de Garside bilatère du groupe considéré. Le calcul du diamètre de ce graphe est donc une question naturelle qui permet d'obtenir de meilleures garanties sur la vitesse de convergence des formes normales de Garside.

En considérant séparément les différents types de groupes de Coxeter associés aux groupes d'Artin–Tits de type sphérique irréductibles, nous démontrons que

Le diamètre du graphe de Garside bilatère d'un groupe d'Artin–Tits de type sphérique irréductible de groupe de Coxeter $\mathbf{W}$ vaut

- 1 si $\mathbf{W} = I_2(a)$;
- 2 si $\mathbf{W} = F_4$, H_3 ou H_4 ;
- 3 si $\mathbf{W} = A_3$, B_3, B_4, D_4 ou E_n (avec $6 \leqslant n \leqslant 8$) ;
- 4 si $\mathbf{W} = A_n$ (avec $n \geqslant 4$), B_n (avec $n \geqslant 5$) ou D_n (avec $n \geqslant 6$).

Cette étude est effectuée directement dans le cas des familles infinies de groupes de Coxeter de type A_n, B_n, D_n, et à l'aide de l'ordinateur dans le cas des familles finies de groupes de Coxeter exceptionnels.

Le chapitre 7 est dédié à la construction de mesures uniformes sur les monoïdes d'Artin–Tits de type FC. De telles mesures uniformes ont déjà été étudiées dans le cadre des monoïdes de traces [2], et nous suivons ici un programme parallèle dans le contexte des monoïdes d'Artin–Tits de type FC, et en particulier des tresses.

Ces mesures uniformes sont apparentées à la mesure de Parry [72, 79] et à la mesure de Patterson-Sullivan [65, 81, 87]. La mesure de Parry est la mesure d'entropie maximale d'un système dynamique symbolique sofique, c'est-à-dire, de manière informelle, la mesure uniforme sur les chemins infinis dans un automate fini. Les éléments d'un monoïde d'Artin–Tits de type FC sont représentés par leur forme normale de Garside à gauche, qui est donc reconnue par un automate fini associé à une mesure de Parry, et nous mettons ici en lumière la structure combinatoire de cette mesure de Parry. La mesure de Patterson-Sullivan est également une mesure uniforme sur le bord à l'infini de certains groupes géométriques. Alors que la preuve de l'existence de cette mesure n'est pas constructive en général, nous proposons ici une construction explicite de la mesure de Patterson-Sullivan.

Nous commençons par définir une notion de monoïdes d'Artin–Tits de type FC étendus, qui sont des limites projectives de monoïdes d'Artin–Tits de type FC contenant à la fois des éléments finis et infinis. Nous étudions alors la notion de mesure uniforme paramétrée par une valuation et sur le cas particulier des mesures de Bernoulli, et nous identifions plusieurs caractérisations et paramétrages équivalents des mesures uniformes ainsi que des mesures de Bernoulli. Nous prouvons entre autres que l'ensemble des mesures de Bernoulli d'un monoïde d'Artin–Tits de type FC étendu est homéomorphe à un simplexe ouvert. Nous relions ensuite les mesures de Bernoulli aux mesures uniformes sur les sphères (la sphère de rayon k étant définie comme l'ensemble des éléments de longueur k) et démontrons que, pour tout monoïde d'Artin–Tits de type FC,

> La famille des mesures uniformes sur les sphères converge au sens faible vers une mesure de Bernoulli.

Ce résultat nous permet de démontrer la convergence en loi de nombreuses variables aléatoires, par exemple les préfixes de la forme normale de Garside à gauche d'une tresse choisie uniformément au hasard parmi les tresses positives de longueur k. Il nous permet en particulier de réfuter une conjecture de Gebhardt [55] et de démontrer une variante de cette conjecture.

À l'aide de la notion de graphe pondéré conditionné, nous établissons ensuite d'autres résultats de convergence plus fins, qui concernent les fonctions Garside-additives et les fonctions additives, dans le cadre des monoïdes d'Artin–Tits de type FC. Ces résultats incluent des variantes du théorème central limite, qui nous donnent des informations précises sur le comportement asymptotique de quantités telles que l'accélération moyenne due à la parallélisation d'une suite de calcul partiellement parallélisable. Il s'agit là d'une extension des résultats généraux de Hennion et Hervé [61] au cadre des monoïdes de tresses. Enfin, nous détaillons explicitement certains des calculs susmentionnés dans les cas spécifiques des monoïdes de tresses usuels et de modèles de dimères.

Chapter 1

Introduction (English)

This thesis is devoted to the study of combinatorial properties of braids. Although braids have been well-known objects for thousands of years, as illustrated by Figure 1.1, their study as mathematical objects dates only from the early twentieth century, with the seminal work of Artin [8, 9]. An intuitive definition of braids is as follows.

Figure 1.1 – Braids in the ancient Gaulish society

Consider a set of n strands, where n is a positive integer, such that each strand has one upper and one lower end. We assume that the upper ends of our n strands are glued to some horizontal axis, hence cannot move, while the lower ends are clipped to some horizontal axis: we may temporarily release some of the clips, in order to move the lower ends, before reattaching the clips to the horizontal axis.

Braiding these n strands consists in repeating a finite number of times the following *elementary braiding move*, illustrated in Figure 1.2. Choose two consecutive lower ends, unclip them, then exchange their positions, and reclip them to the lower horizontal axis. While doing so, we will have to make one of the lower ends go to the foreground, while the other one must go to the background. Afterwards, the strands may still move individually, provided that their upper and lower ends remain motionless, and that the strands do not touch each other: the strands are considered up to isotopy.

Making the above description precise, Artin introduces the class of *braid diagrams* (see Definition 2.4), then identifies braids with an *isotopy set* of braid diagrams. The concatenation of braid diagrams leads then to a notion of multiplication of braids. This formal mathematical description of braids subsequently led to two points of view on braids.

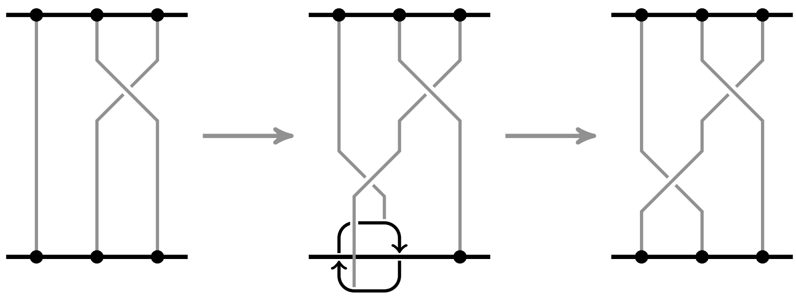


Figure 1.2 – Elementary braiding move

In the first one, which is of algebraic nature, braids form a finitely presented group; showing that braids form a finitely generated group, whose generators are the elementary braiding moves, is straightforward, but showing that the group of braids may admits a *finite presentation* is not so obvious. In the second point of view, which has a more geometric flavour, braids form a *mapping class group* of a punctured open disk, i.e. are the quotient of some group of homeomorphisms of the punctured open disk by the isotopy relation.

As soon as the number of strands considered is greater than one, braid groups are infinite, and as such raise important questions of algorithmic nature. Is it computationally possible, and at which cost, to test braid equality? To define canonical representatives of braids and to handle them? These questions lie at the core of this thesis.

Answering these questions requires several steps. A first step consists in making the question statement precise, and defining adequate tools for tackling it. Such tools include defining several notions of complexity, which may enlighten different aspects of braids and be meaningful in various contexts. Indeed, and although the notion of *complexity* of a braid may be intuitively bound to the idea that “a braid is complex if it has only complex descriptions”, the algebraic and geometric points of view on braids lead to descriptions of different kinds, and thus to different notions of complexities.

A second step resides in designing *normal forms*, i.e. *canonical* representations of braids. In the algebraic viewpoint, braids are equivalence classes of words, and therefore choosing a normal form consists in choosing, for each braid, a word that represents this braid. A crucial point of normal forms is that they should be easy to compute, i.e. that, given the respective normal forms $\mathbf{NF}(\alpha)$ and $\mathbf{NF}(\beta)$ of two braids α and β, computing the normal form of the braid $\alpha\beta$ should be algorithmically inexpensive. Consequently, it makes sense to look for various properties of normal forms, such as geodicity (the chosen words should be as short as possible), regularity (the set of words of the form $\mathbf{NF}(\alpha)$ should be regular, i.e. recognised by a finite-state automaton) or automaticity (for each generator σ of the braid group, the sets of pairs of the form $(\mathbf{NF}(\alpha), \mathbf{NF}(\sigma\alpha))$, respectively of the form $(\mathbf{NF}(\alpha), \mathbf{NF}(\sigma\alpha))$, should be regular sets). Alternatively, we might wish that the length of the word $\mathbf{NF}(\alpha)$ and the cost of computing $\mathbf{NF}(\alpha\beta)$ be correlated with the complexities of the braids α and β.

A third step consists in a study of braids of large complexity. This analysis includes topics such as counting braids of a given complexity or generating *uniformly* braids of a given complexity. A key ingredient in the study of both these topics is the use of generating functions. Indeed, generating functions allow using real and complex analysis for deriving more or less precise, exact or asymptotic, estimations of the number of braids of a given complexity.

In parallel, we might wish to design an algorithm in which, provided that we be able to draw braids of complexity k uniformly at random, would allow us to draw braids of complexity $k + 1$ uniformly at random. This leads us to consider alternative manners of drawing braids such as random walks, in which a braid of complexity $k + 1$ is just seen as the product of an *elementary generator* by a braid of complexity k.

Main Contributions of this Thesis

We present below a detailed description of each chapter and of the main contributions contained in each of them.

Chapter 2 is mostly bibliographical, and consists in an overview of several state-of-the-art notions that will be used subsequently. This chapter is divided in two parts. The first part is focused on the algebraic point of view on braids, as well as on related algebraic structures, whose relations are depicted in Figure 1.3.

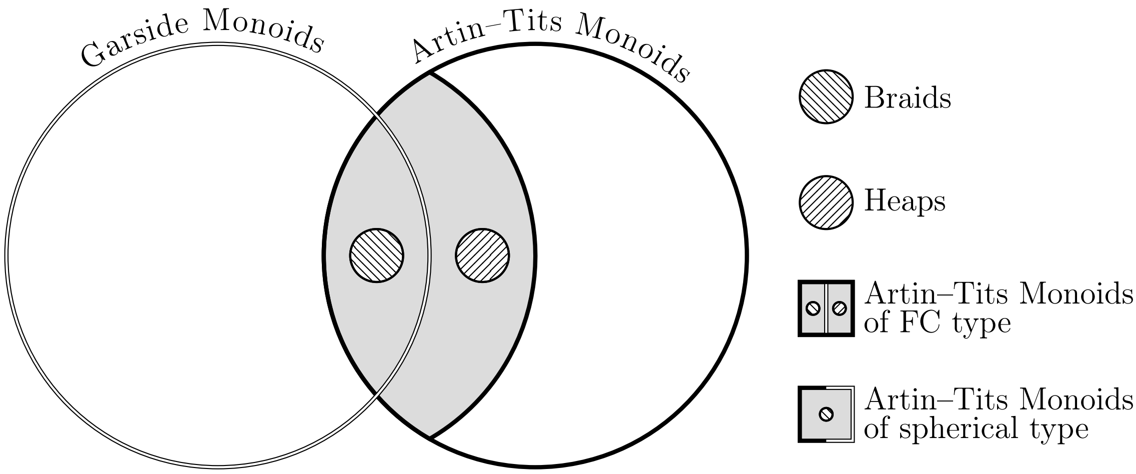


Figure 1.3 – Algebraic structures studied in Chapter 2

Aiming first to identify general frameworks in which we will be able to study the properties of braid monoids, and therefore of braid groups, we first focus on two generalisations of braids monoids, which are Artin–Tits monoids and Garside monoids. Neither class generalises the other class, and therefore we also identify the class of *Artin–Tits monoids of spherical type*, which lies at the intersection between Artin–Tits monoids and Garside monoids. Various algebraic properties, with a remarkable emphasis on *Garside normal forms*, hold in the context of Garside monoids. Later, when considering the restricted framework of Artin–Tits monoids of spherical type, some of these properties are refined, which leads to the notion of *symmetric Garside normal form*. We focus then on *heap monoids*, which are an algebraic structure analogous to braid monoids. Furthermore, in addition to these well-known structures [9, 20, 37, 42, 43, 44, 53], we also focus on the notion of *Artin–Tits monoid of FC type*, which encompasses both heap monoids and Artin–Tits monoids of spherical type.

The second part of Chapter 2 is focused on geometric aspects of braid groups, and in particular on the representation of braids in terms of *laminations of the punctured disk* and of *curve diagrams*, which are standard geometric representations of braids [13, 39, 48]. We derive from these representations two notions of geometric complexities, which we show to be deeply connected to each other, and which will be the focal points of Chapters 3 and 4.

Chapter 3 is focused on the study of the *relaxation normal form*, -which is a normal form stemming from the geometric point of view of braids. This normal form was introduced and studied in [25] and belongs to a larger class of normal forms introduced in [46].

We begin by defining new discrete, combinatorial structures, which we call *lamination trees*, *arc trees* and *cell maps*, in connection with geometric representations of braids. We show that these structures enjoy simple combinatorial and topological properties, and prove in particular that the *arc trees* associated with a braid are unary-binary rooted trees. This study leads to the main contribution of Chapter 3, which consists in proving that, in spite of its geometric nature,

> The relaxation normal form is regular.

We do so by computing explicitly a deterministic automaton that recognises the relaxation normal form. Hence, we extend the class of known regular normal forms, which already contains the normal forms of Garside [37, 53], of Birman-Ko-Lee [15] and of Bressaud [19]. However, the other normal forms studied in [46] do not seem to be regular.

Later on, we prove additional results on the relaxation normal form and on the automaton computed above. We first prove that our automaton and the minimal deterministic automaton of the relaxation normal form have state spaces of comparable sizes. More precisely, if their state spaces are of respective cardinalities C and $C_{\min}$, we prove that $C_{\min} \leqslant C \leqslant 2^{60} C_{\min}^{40}$, where the constants 2^{60} and 40 are not meant to be optimal. Second, we draw strong connections between the relaxation normal form and the order of Dehornoy on braids, and prove that the relaxation normal form maps the set of Dehornoy-positive braids to a regular set of words. Finally, we conjecture that the relaxation normal form is synchronously automatic, and prove that our conjecture holds in the case of the three-strand braid group.

Most of the content of this chapter appeared in [63], and was submitted for publication.

Chapter 4 is devoted to the problem of counting braids with a given geometric complexity. Whereas the problem of counting braids with a given algebraic complexity (i.e. braids that are a product of a given number of Artin generators) was abundantly studied, the problem of counting braids with a given geometric complexity does not seem to have been considered yet. We begin with defining a generalisation of curve diagrams, then design a bijection between generalised curve diagrams and a system of integer *coordinates*, and we identify simple criteria that characterise which collections of integers are the coordinates of a (standard) curve diagram. Then, we introduce the generating functions associated with the geometric complexity in the n-strand braid group, i.e. the functions $\mathcal{B}_n(z) := \sum_{k \geqslant 0} b_{n,k} z^k$ such that $b_{n,k}$ is the number of n-strand braids and geometric complexity k. The above-mentioned criteria allow us to compute closed expressions of the functions $\mathcal{B}_2(z)$ and $\mathcal{B}_3(z)$ and, more precisely, to show that

The function $\mathcal{B}_3(z)$, which is not rational, nor even holonomic, and the integers $b_{3,k}$ are given by:

$$\begin{aligned}
\mathcal{B}_3(z) &= 2\frac{1+z^2-z^4}{z^2(1-z^4)}\left(\sum_{n\geqslant 3}\varphi(n)z^n\right) + \frac{z^2(1-3z^4)}{1-z^4}; \\
b_{3,k} &= \mathbf{1}_{k=0} + 2\left(\varphi(k+2) - \mathbf{1}_{k\in 2\mathbb{Z}} + 2\sum_{i=1}^{\lceil k/2 \rceil}\varphi(k+3-2i)\right)\mathbf{1}_{k\geqslant 1},
\end{aligned}$$

where φ denotes the Euler totient.

Whereas the algebraic complexity of braids is much harder to compute than the geometric complexity, the generating function associated with the algebraic complexity for the group of braids with three strands is rational, i.e. much more *simple* than the generating function associated with the algebraic complexity for the group of braids with three strands.

In addition, we compute simple equivalents of the integers $b_{3,k}$ when $k \to +\infty$, and find non-trivial lower and upper bounds on the integers $b_{n,k}$ when $n \geqslant 4$.

Most of the content of this chapter appeared in [64] and was published in the *Journal of Knot Theory and Its Ramifications.*

Chapter 5 is focused on the study and the convergence of random walks in irreducible braid groups and monoids. We first recall some results for random walks in irreducible heap monoids and irreducible heap groups, which are scattered in several papers [54, 77], and for which we provide here unified proofs. Using the notion of *blocking heap*, we show that the Garside normal forms of the elements obtained during random walks on irreducible heap monoids and on irreducible heap groups are almost surely prefix-convergent, i.e. that the Garside normal forms of two successive heaps obtained during the random walk have common prefixes of arbitrarily large sizes.

Then, we change our framework and study random walks in irreducible Artin–Tits monoids of spherical type and groups. The overall situation is analogous but, due to the fact that braid monoids and groups have a non-trivial centre, the problem is much harder: although Vershik had asked in the 2000s the question of the stabilisation of Garside normal forms in braid groups [89, 90], no answer had yet been provided. We provide such an answer in the context of irreducible Artin–Tits groups of spherical type.

We first introduce a notion of *bilateral Garside automaton*, which accepts the words that are both left and right Garside normal words. We prove that the bilateral Garside automaton is strongly connected and we derive a notion of *blocking braid word* analogous to blocking heaps, which we borrow from [26]. Then, using sub-additivity arguments such as Kingman's ergodic lemma, we prove that

> The right Garside normal forms of the elements obtained during right random walks on irreducible Artin–Tits monoids of spherical type are almost surely prefix-convergent.

We also show in which conditions this result can be lifted to the framework of Artin–Tits groups of spherical type.

Subsequently, we study the limit of the left Garside normal forms associated with the above notion of suffix-convergence and prove that, under mild assumptions on the random walk, this limit is *ergodic*. From this result, we derive additional results concerning the *density* of subwords of the limit and on the *penetration distance* into the limit and into words attained after having performed a large number of steps, as well as the speed of convergence of normal forms towards their limit.

The content of this chapter is aimed to be published in an article written in collaboration with Jean Mairesse.

Chapter 6 consists in the study and the systematic computation of the diameter of the bilateral Garside automaton of irreducible Artin–Tits groups of spherical type. Under more relaxed hypotheses than those considered at the end of Chapter 5, the speed of convergence of Garside normal forms towards their limit is bounded below by parameters that depend, among others, of the diameter of the bilateral Garside automaton. Therefore, computing this diameter is a natural question in order to derive better guarantees on the speed of convergence of the Garside normal forms.

By treating separately the different types of Coxeter groups associated with irreducible Artin–Tits groups of spherical type, we show that

> The diameter of the bilateral Garside automaton of an irreducible Artin–Tits group of spherical type with Coxeter group $\mathbf{W}$ is
>
> - 1 if $\mathbf{W} = I_2(a)$;
> - 2 if $\mathbf{W} = F_4$, H_3 or H_4;
> - 3 if $\mathbf{W} = A_3$, B_3, B_4, D_4 or E_n (with $6 \leqslant n \leqslant 8$);
> - 4 if $\mathbf{W} = A_n$ (with $n \geqslant 4$), B_n (with $n \geqslant 5$) or D_n (with $n \geqslant 6$).

This study was performed directly for infinite families of Coxeter groups of type A_n, B_n, D_n, and with the help of computers for finite families of exceptional Coxeter groups.

Chapter 7 is devoted to the construction of *uniform measures* on Artin–Tits monoids of FC type. Such uniform measures have already been studied for heap monoids [2], and we develop here analogous arguments in the context of Artin–Tits monoids of FC type, and in particular in the context of braid monoids.

These uniform measures are similar to the Parry measure [72, 79] and to the Patterson-Sullivan measure [65, 81, 87]. The Parry measure is the measure of maximal entropy of a sofic subshift, i.e., informally, the "uniform" measure on infinite paths in a finite-state automaton. The elements of an Artin–Tits monoid of FC type are represented by their Garside normal form, which is recognised by a finite-state automaton associated with a Parry measure. The Patterson-Sullivan measure is also a uniform measure on the border at infinity of some geometric groups. Although the proof of the existence of this measure is not constructive in general, we propose here an explicit construction of the Patterson-Sullivan measure.

We first define the class of *extended Artin–Tits monoids of FC type*, which are projective limits of Artin–Tits monoids of FC type containing both finite elements and infinite elements. We first explore the notion of *uniform measure parametrised by a valuation*, with a special emphasis on the concept of *Bernoulli measure*, and find equivalent characterisations and parametrisations of uniform measures and of Bernoulli measures. In particular, we prove that the set of Bernoulli measures on an extended Artin–Tits monoid of FC type is homeomorphic to an open simplex. Then, we relate Bernoulli measures with *uniform measures on spheres* (the "sphere of radius k" is defined as the set of elements of length k) and prove that, for all Artin–Tits monoids of FC type, The family of uniform measures on spheres converges weakly towards a Bernoulli measure.

This statement allows us to prove a wide range of convergence results, for instance on the leftmost letter of left Garside normal words of braids chosen uniformly at random among the set of braids of length k. In particular, we disprove a conjecture of Gebhardt [55] and prove a variant of this conjecture.

Later, using the notion of conditioned weighted graphs, we derive finer convergence results about *Garside-additive functions* and additive functions in Artin–Tits monoids of FC type. Such results include central limit theorems, which provide us with detailed informations about the asymptotic behaviour of quantities such as the *mean speed-up* of the parallelisation of partially commutative computations. This is an extension of general result of Hennion and Hervé [61] to the framework of braid monoids. Finally, we detail explicitly some of the above-mentioned computations in the specific cases of the standard braid monoids and of the dimer models.

Chapter 2

Preliminaries

Résumé

Nous énonçons ici des définitions et des résultats préliminaires concernant les groupes de tresses et faisant partie de l'état de l'art. Ces définitions et résultats suivent deux approches. La première approche est algébrique. Elle concerne aussi bien les groupes de tresses que les groupes d'Artin–Tits de type sphérique. Elle consiste à voir les groupes de tresses comme des groupes finiment présentés, donc à identifier chaque tresse à une classe d'équivalence de mots pour une certaine relation de congruence. Nous étudions égalemnt la notion de monoïde d'Artin–Tits de type FC, qui inclut à la fois les tresses et les traces et permet d'unifier les formes normales de Garside et de Cartier-Foata.

La deuxième approche est géométrique, et ne concerne que les groupes de tresses, excluant donc les groupes d'Artin–Tits de type sphérique. Elle consiste à voir les groupes de tresses comme des groupes d'isotopie sur diverses structures topologiques, notamment le disque épointé avec bord immobile, donc à identifier chaque tresse à une classe d'équivalence d'homéomorphismes du disque unité complexe.

Abstract

We state here the preliminary definitions and state-of-the-art results related to braid groups. These definitions and results follow two approaches on braids. The first approach is algebraic. It concerns braid groups, but also Artin–Tits groups of spherical type. It is based on viewing braid groups as finitely presented groups, hence identifying each braid with an equivalence class of words under some congruence relation. We also study the notion of *Artin–Tits monoid of FC type*, which includes both braids and traces and allows unifying the Garside normal form and the Cartier-Foata normal form.

The second approach is geometric, and concerns only braid groups, not all Artin–Tits groups of spherical type. It is based on viewing braid groups as isotopy groups of some topological structures, namely the *punctured disks with fixed border*, hence identifying each braid with an equivalence class of homeomorphisms of the unit complex disk.

2.1 Some Notations About Words

Throughout the entire document, we will frequently deal with words, whose letters will be chosen from various (finite or infinite) alphabets. Consequently, we begin with introducing some of the notations that we will frequently use in the following pages.

Let $\mathcal{A}$ be such an alphabet. We denote by $\mathcal{A}^*$ the set of finite words with letters in $\mathcal{A}$, whose elements will be denoted by finite sequences of the form $w_1 \cdot w_2 \cdot \ldots \cdot w_k$ or $w_{-k} \cdot \ldots \cdot w_{-2} \cdot w_{-1}$. In particular, we have $w_i = w_{i-1-k}$ for all $i \in \{1, \ldots, k\}$.

We will have to manipulate several kinds of products. Hence, from this point on, we reserve the use of the $\cdot$ product for concatenating letters into a word, or for concatenating words. In particular, in case $\mathcal{A}$ is a monoid or a group, then the notation $w_1 w_2$ will denote the product of the elements w_1 and w_2 in $\mathcal{A}$, whereas the notation $w_1 \cdot w_2$ will denote the 2-letter-word whose letters are w_1 and w_2.

In order to emphasise the difference between elements of $\mathcal{A}^*$ and elements of $\mathcal{A}$, we also choose to denote each (finite or infinite) word with an underlined letter, whereas underlined letters will never be used for denoting objects that are not words. For example, if $\underline{\mathbf{a}} = a_1 \cdot a_2 \cdot \ldots \cdot a_k$ is an element of $\mathcal{A}^*$, we will denote the product element $a_1 a_2 \ldots a_k$ by $\langle \underline{\mathbf{a}} \rangle$, or simply $\mathbf{a}$ if the context is clear.

Let $\underline{\mathbf{w}} := w_1 \cdot w_2 \cdot \ldots \cdot w_k$ and $\underline{\mathbf{x}} := x_1 \cdot x_2 \cdot \ldots \cdot x_\ell$ be two elements of $\mathcal{A}^*$. We denote by $|\underline{\mathbf{w}}|$ the length of the word $\underline{\mathbf{w}}$, i.e. the integer k. We also denote by $\underline{\mathbf{w}} \cdot \underline{\mathbf{x}}$ the concatenation of the word $\underline{\mathbf{w}}$ with the word $\underline{\mathbf{x}}$, i.e. the word $w_1 \cdot w_2 \cdot \ldots \cdot w_k \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_\ell$.

If $\underline{\mathbf{w}}$ is a finite sub-word of $\underline{\mathbf{x}}$, i.e. there exists integers i and j, with the same sign, such that $i \leqslant j + 1$ and that $\underline{\mathbf{w}} = x_i \cdot x_{i+1} \cdot \ldots \cdot x_j$, then we denote the word $\underline{\mathbf{w}}$ by $\underline{\mathbf{x}}_{i\ldots j}$. If $i > j + 1$, we also denote by $\underline{\mathbf{x}}_{i\ldots j}$ the empty word. In addition, if $\underline{\mathbf{w}}$ is a prefix of $\underline{\mathbf{x}}$, i.e. if $i = 1$ and $0 \leqslant j$, we also denote $\underline{\mathbf{w}}$ by $\mathbf{pre}_j(\underline{\mathbf{x}})$. Likewise, if $\underline{\mathbf{w}}$ is a suffix of $\underline{\mathbf{x}}$, i.e. if $i \leqslant 0$ and $j = -1$, we also denote $\underline{\mathbf{w}}$ by $\mathbf{suf}_{|i|}(\underline{\mathbf{x}})$.

Moreover, we will write $\underline{\mathbf{w}} \lhd \underline{\mathbf{x}}$ when $\underline{\mathbf{w}}$ is a prefix of $\underline{\mathbf{x}}$, and $\underline{\mathbf{w}} \rhd \underline{\mathbf{x}}$ when $\underline{\mathbf{x}}$ is a suffix of $\underline{\mathbf{w}}$. If $\underline{\mathbf{w}} \lhd \underline{\mathbf{x}}$, we also denote by $\underline{\mathbf{w}}^{-1} \cdot \underline{\mathbf{x}}$ the suffix of $\underline{\mathbf{x}}$ such that $\underline{\mathbf{w}} \cdot (\underline{\mathbf{w}}^{-1} \cdot \underline{\mathbf{x}}) = \underline{\mathbf{x}}$. If $\underline{\mathbf{x}} \rhd \underline{\mathbf{w}}$, we denote by $\underline{\mathbf{x}} \cdot \underline{\mathbf{w}}^{-1}$ the prefix of $\underline{\mathbf{x}}$ such that $(\underline{\mathbf{x}} \cdot \underline{\mathbf{w}}^{-1}) \cdot \underline{\mathbf{w}} = \underline{\mathbf{x}}$.

Finally, if $\varphi : \mathcal{A} \mapsto \mathcal{A}$ is an endomorphism of monoids, we denote by $\varphi(\underline{\mathbf{w}})$ the word $\varphi(w_1) \cdot \varphi(w_2) \cdot \ldots \cdot \varphi(w_k)$; hence, $\langle \varphi(\underline{\mathbf{w}}) \rangle = \varphi(\langle \underline{\mathbf{w}} \rangle)$. If $\psi : \mathcal{A} \mapsto \mathcal{A}$ is an anti-endomorphism of monoids, i.e if $\psi(\mathbf{bc}) = \psi(\mathbf{c})\psi(\mathbf{b})$ for all $\mathbf{b}, \mathbf{c} \in \mathcal{A}$, we denote by $\psi(\underline{\mathbf{w}})$ the word $\psi(w_k) \cdot \psi(w_{k-1}) \cdot \ldots \cdot \psi(w_1)$; hence, $\langle \psi(\underline{\mathbf{w}}) \rangle = \psi(\langle \underline{\mathbf{w}} \rangle)$.

2.2 Braids, Configuration Spaces and Braid Diagrams

The following seminal definition of braid groups and monoids is due to Artin [9].

Definition 2.1 (Braid monoid and braid group).
Let n be some positive integer. The braid monoid on n strands *is the monoid* $\mathbf{B}_n^+$ *presented as follows:*

$$\mathbf{B}_n^+ := \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geqslant 2\rangle^+ .$$

The braid group on n strands *is the group* $\mathbf{B}_n$ *presented as follows:*

$$\mathbf{B}_n := \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geqslant 2\rangle .$$

Such groups and monoids are the central object on which we will focus. In his seminal work [9], Artin showed that braid groups are isomorphic to the fundamental groups of configuration spaces, as follows.

Definition 2.2 (Configuration space).
Let n be some positive integer. Consider the set $\mathcal{F}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \forall i \neq j, z_i \neq z_j\}$ of n-tuples of pairwise distinct complex numbers.

Observe that the symmetric group $\mathfrak{S}_n$ acts freely on $\mathcal{F}_n(\mathbb{C})$. We call configuration space with n complex points*, and denote by $\mathcal{C}_n(\mathbb{C})$, the orbit space of the action of $\mathfrak{S}_n$ over $\mathcal{F}_n(\mathbb{C})$.*

Alternatively, one may define $\mathcal{C}_n(\mathbb{C})$ as the set of subsets of $\mathbb{C}$ with cardinality n, i.e. $\mathcal{C}_n(\mathbb{C}) := \{S \subseteq \mathbb{C} : |S| = n\}$.

Theorem 2.3.
Let n be some positive integer. The braid group $\mathbf{B}_n$ is isomorphic to the fundamental group of the set $\mathcal{C}_n(\mathbb{C})$, i.e. $\mathbf{B}_n \simeq \pi_1(\mathcal{C}_n(\mathbb{C}))$.

Each path in the configuration space $\mathcal{C}_n(\mathbb{C})$ is commonly represented graphically by using *braid diagrams*, as illustrated in Fig. 2.5.

Definition 2.4 (Semi-group of braid diagrams).
A braid diagram on n strands *consists in n strands with fixed upper endpoints and mobile lower endpoints. The upper endpoints are aligned from left to right on some (upper) horizontal line, and the lower endpoints are aligned from left to right on some (lower) horizontal line. Then, the strands may be intertwined. Each intertwining consists in exchanging two successive lower endpoints, by applying a half-twist that may be either clockwise or anti-clockwise when seen from above.*

Like paths in $\mathcal{C}_n(\mathbb{C})$, braid diagrams can be concatenated as follows: the concatenation of two braid diagrams $\mathbf{D}_1$ and $\mathbf{D}_2$ is the braid diagram $\mathbf{D}_1 \cdot \mathbf{D}_2$ obtained by merging the k-th lower endpoint of $\mathbf{D}_1$ with the k-th upper endpoint of $\mathbf{D}_2$, for each $k \in \{1, \ldots, n\}$. Similarly, the "reversal" of a braid diagram is its vertical symmetric diagram. Hence, the semi-group of braid diagrams is isomorphic to a semi-group of paths in $\mathcal{C}_n(\mathbb{C})$, and the isotopy group of braid diagrams is isomorphic to $\pi_1(\mathcal{C}_n(\mathbb{C}))$.

In particular, Theorem 2.3 is then equivalent to the following result.

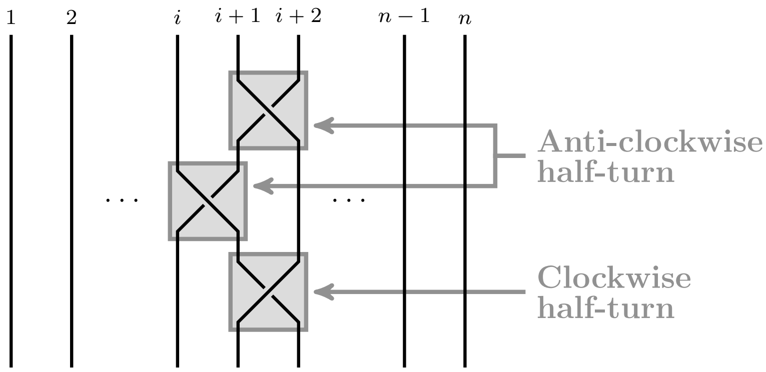


Figure 2.5 – Braid diagram with three half-turns

Theorem 2.6.
Let n be some positive integer. The braid group $\mathbf{B}_n$ is isomorphic to the isotopy group of the set $\mathcal{D}_n$ of n-strand braid diagrams.

The isomorphism $\mathbf{I}$ between $\mathbf{B}_n$ and isotopy classes of $\mathcal{D}_n$ is illustrated in Fig. 2.7. The generator σ_i is mapped to the anti-clockwise half-turn that exchanges the i-th and $(i+1)$-st lower endpoints of the diagram, and σ_i^{-1} is mapped to the clockwise half-turn that exchanges the i-th and the $(i+1)$-st lower endpoints. In particular, when a braid $\sigma_i^{\pm 1}$ acts onto a braid diagram, what was the i-th lower endpoint moved in position $i+1$, thereby becoming the $(i+1)$-st lower endpoint, and vice-versa.

This induces a mapping ι from braid words to braid diagrams. Then, since each braid $\mathbf{b}$ is a set of braid words, the isomorphism $\mathbf{I}$ maps $\mathbf{b}$ to a set $\{\iota(\underline{\mathbf{w}}) : \underline{\mathbf{w}} \in \mathbf{b}\}$ of braid diagrams. According to Theorem 2.6, this set is in fact an isotopy class of braid diagrams, i.e. an element of the isotopy group of $\mathcal{D}_n$.

The connections between braid groups, braid diagrams and configuration spaces will lead to the geometric approach we are to explore later, but also have deep implications in the algebraic approach we are to follow now.

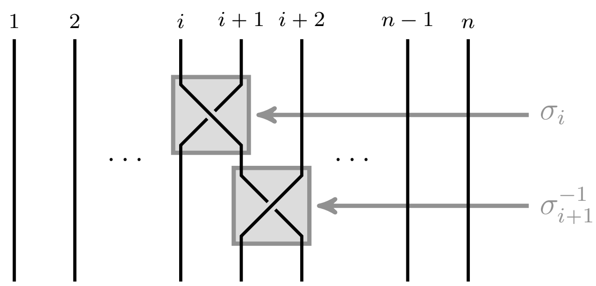


Figure 2.7 – Braid diagram associated with the braid $\sigma_i \sigma_{i+1}^{-1}$

2.3 An Algebraic Approach to Braids

Let us focus now on the definition of braid groups through finite presentations. We present below standard results about braids and about some algebraic structures that either generalise braids or are related to braids. These structures are Artin–Tits monoids and groups, Garside monoids and groups, Artin–Tits monoids of spherical type and trace monoids. They have been studied for a long time, hence they are the subject of an abundant literature [14, 20, 24, 27, 36, 37, 43, 44, 47, 70, 78, 91].

In that context, we focus on several results that will be useful in the subsequent chapters of this thesis, which mainly concern the properties of the above-mentioned structures as quotients of free monoids or groups. Doing so, we rewrite some of the proofs that can be found in the literature, and that will give us a foretaste of the original proofs that we will write later on. In addition, in Section 2.3.4, we also propose an original notion of *Artin–Tits monoid of FC type*. This class is a subclass of Artin–Tits monoids, and encompasses both Artin–Tits monoid of spherical type and trace monoids, thereby unifying the associated notions of Garside normal form and Cartier-Foata normal form.

2.3.1 From Braid Monoids to Garside Monoids

Within this approach, there are various fruitful ways to generalise braid groups. One such generalisation is due to Brieskorn and Saito [20].

Let u and v be two letters, chosen from some alphabet, and let ℓ be some non-negative integer. Hereafter, we denote by $[uv]^\ell$ the word $u \cdot v \cdot u \cdot v \cdot \ldots$ with ℓ letters, i.e.

$$[uv]^0 = \underline{\varepsilon},\ [uv]^1 = u \text{ and } [uv]^{\ell+2} = u \cdot v \cdot [uv]^\ell .$$

We also denote by $[uv]^{-\ell}$ the mirror of the word $[vu]^\ell$, i.e.

$$[uv]^0 = \underline{\varepsilon},\ [uv]^{-1} = u \text{ and } [uv]^{-\ell-2} = \cdot [uv]^{-\ell} \cdot v \cdot u.$$

For instance, we have

$$[uv]^4 = u \cdot v \cdot u \cdot v,\ [uv]^5 = u \cdot v \cdot u \cdot v \cdot u,\ [uv]^{-4} = v \cdot u \cdot v \cdot u \text{ and } [uv]^{-5} = u \cdot v \cdot u \cdot v \cdot u.$$

Definition 2.8 (Artin–Tits monoid and Artin–Tits group).
An Artin–Tits monoid *is a monoid* $\mathbf{A}^+$ *with a presentation of the form*

$$\mathbf{A}^+ := \langle \sigma_1, \ldots, \sigma_n \mid [\sigma_i\sigma_j]^{m_{i,j}} = [\sigma_j\sigma_i]^{m_{i,j}} \ \textit{if } i \neq j \rangle^+ ,$$

where the $m_{i,j}$ *are elements of the set* $\{2, 3, 4, \ldots, \infty\}$ *such that* $m_{i,j} = m_{j,i}$ *for all* $i \neq j$, *and where the equality* $[\sigma_i\sigma_j]^\infty = [\sigma_j\sigma_i]^\infty$ *denotes the absence of relation.*

The associated Artin–Tits group *is the group* $\mathbf{A}$ *presented as follows:*

$$\mathbf{A} := \langle \sigma_1, \ldots, \sigma_n \mid [\sigma_i\sigma_j]^{m_{i,j}} = [\sigma_j\sigma_i]^{m_{i,j}} \ \textit{if } i \neq j \rangle .$$

Note that Artin–Tits groups and monoids indeed generalise braid groups and monoids, as it suffices to choose $m_{i,j} = 3$ if $|i - j| = 1$ and $m_{i,j} = 2$ if $|i - j| \geqslant 2$.

The relations used to define Artin–Tits groups and monoids have two important features:

1. they are invariant under word reversal, i.e. under the mapping that sends the word $\mathbf{b} := \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ to the *reversed word* $\mathbf{b}^* := \sigma_{i_k}\ldots\sigma_{i_1}$;
2. they are length-preserving (we also say that they are *homogeneous*).

Invariance under word reversal proves that the left and right-divisibility orderings, i.e. the orderings $\leqslant_\ell$ and $\geqslant_r$ such that $\mathbf{a} \leqslant_\ell \mathbf{ab}$ and $\mathbf{ab} \geqslant_r \mathbf{b}$ for all elements $\mathbf{a}$ and $\mathbf{b}$, play dual roles in the monoid $\mathbf{A}^+$, and will enjoy similar properties.[1] Homogeneity is at the origin of the notion of Artin length of a braid.

Definition 2.9 (Artin length)**.**
Let $\mathbf{A}$ be an Artin–Tits group, with generators $\sigma_1, \ldots, \sigma_n$. The mapping $\lambda : \mathbf{A} \mapsto \mathbb{Z}$ such that $\lambda(\sigma_i) = 1$ for all $i \in \{1, \ldots, n\}$ extends to a group homomorphism, which we call Artin length*.*

Additional properties of Artin–Tits groups and monoids come from studying their associated Coxeter groups, which we define now.

Definition 2.10 (Coxeter group)**.**
Let $\mathbf{A}$ and $\mathbf{A}^+$ be an Artin–Tits group and the associated Artin–Tits monoid, such as introduced in Definition 2.8. The Coxeter group *associated with $\mathbf{A}$ and $\mathbf{A}^+$ is defined as the group $\mathbf{W}$ presented as follows:*

$$\mathbf{W} := \left\langle \sigma_1, \ldots, \sigma_n \mid \sigma_i^2 = \mathbf{1}, [\sigma_i\sigma_j]^{m_{i,j}} = [\sigma_j\sigma_i]^{m_{i,j}} \text{ if } i \neq j \right\rangle,$$

where $\mathbf{1}$ denotes the neutral element of the group $\mathbf{W}$.

In particular, we denote hereafter the Coxeter group associated with the braid group $\mathbf{B}_n$ by $\mathbf{W}_n$. The group $\mathbf{W}_n$ is obtained from $\mathbf{B}_n$ by identifying the generators σ_i and σ_i^{-1} for all $i \in \{1, \ldots, n-1\}$. Hence, $\mathbf{W}_n$ is isomorphic to the isotopy set of braid diagrams in which clockwise half-turns have been identified with anti-clockwise half-turns: let us call such diagrams *orientation-free diagrams*, and let $\tilde{\mathcal{D}}_n$ denote the set of these diagrams.

In an orientation-free diagram, the only important feature of a half-turn is which endpoints i and $i+1$ it exchanges: the orientation of the half-turn has become irrelevant. Hence, each isotopy class of $\tilde{\mathcal{D}}_n$ is characterised by which pairs of upper and lower endpoints are linked to each other by some strand, i.e. can be identified with a permutation of the set $\{1, \ldots, n\}$. In addition, the isomorphism from $\mathbf{B}_n$ to the isotopy group of $\mathcal{D}_n$ induces an isomorphism from $\mathbf{W}_n$ to the isotopy group of $\tilde{\mathcal{D}}_n$.

Theorem 2.11.
Let n be a positive integer. The Coxeter group $\mathbf{W}_n$ of the braid group $\mathbf{B}_n$ is isomorphic to the symmetric group $\mathfrak{S}_n$ on n elements.

[1] For the sake of readability, we avoid using symbols such as $\geqslant_\ell$ or $\leqslant_r$.

Several results follow from the study of Coxeter groups associated with Artin–Tits groups and monoids [42, 78].

Definition 2.12 (Simple elements)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid, let $\mathbf{W}$ be the associated Coxeter group, and let $\pi : \mathbf{A}^+ \mapsto \mathbf{W}$ be the canonical projection.

We denote by $\mathcal{S}$ the set $\{\mathbf{b} \in \mathbf{A}^+ : \forall \mathbf{a} \in \mathbf{A}^+, \pi(\mathbf{a}) = \pi(\mathbf{b}) \Rightarrow \lambda(\mathbf{b}) \leqslant \lambda(\mathbf{a})\}$, and call its elements simple elements *of $\mathbf{A}^+$.*

Definition 2.13 (Lower semilattice and conditional upper semilattice)**.**
Let $(S, \leqslant)$ be an ordered set. We say that S is a lower semilattice *if every pair of elements of S has a greatest lower bound. We also say that S is a* conditional upper semilattice *if every pair of elements of S with an upper bound has a least upper bound.*

Theorem 2.14.
Each Artin–Tits monoid $\mathbf{A}^+$ is left and right-cancellative. Moreover, the canonical projection $\pi : \mathbf{A}^+ \mapsto \mathbf{W}$ induces a bijection from the set $\mathcal{S}$ of simple elements to $\mathbf{W}$, and the orderings $\leqslant_\ell$ and $\geqslant_r$ induce lower semilattice and conditional upper lattice structures on $\mathbf{A}^+$ and on $\mathcal{S}$.

In addition, the following statements are equivalent:

1. *the ordering $\leqslant_\ell$ induces a lattice structure on $\mathbf{A}^+$;*
2. *the ordering $\geqslant_r$ induces a lattice structure on $\mathbf{A}^+$;*
3. *the set $\{\sigma_1, \ldots, \sigma_n\}$ admits some common $\leqslant_\ell$-multiple;*
4. *the set $\{\sigma_1, \ldots, \sigma_n\}$ admits some right $\leqslant_\ell$-multiple;*
5. *the Coxeter group $\mathbf{W}$ is finite.*

If they are satisfied, let Δ be the $\leqslant_\ell$-LCM of the set $\{\sigma_1, \ldots, \sigma_n\}$. We have $\{\mathbf{b} \in \mathbf{A}^+ : \mathbf{b} \leqslant_\ell \Delta\} = \mathcal{S}$.

Some lower and upper bounds in $\mathbf{A}^+$ are easy to characterise. Hence, we denote by $\mathbf{GCD}_{\leqslant_\ell}$ the greatest lower bound for $\leqslant_\ell$ in $\mathbf{A}^+$, and by $\mathbf{LCM}_{\leqslant_\ell}$ the lowest upper bound for $\leqslant_\ell$ in $\mathbf{A}^+$. Similarly, we use the notations $\mathbf{GCD}_{\geqslant_r}$ and $\mathbf{LCM}_{\geqslant_r}$.

Lemma 2.15.
Let $\mathbf{A}^+$ be an Artin–Tits monoid, and let σ_i and σ_j be two Artin generators of $\mathbf{A}^+$. If $m_{i,j} < +\infty$, then $[\sigma_i \sigma_j]^{m_{i,j}} = \mathbf{LCM}_{\leqslant_\ell}(\sigma_i, \sigma_j) = \mathbf{LCM}_{\geqslant_r}(\sigma_i, \sigma_j)$.

For all elements $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of $\mathbf{A}^+$, we have $\mathbf{GCD}_{\leqslant_\ell}(\mathbf{xy}, \mathbf{xz}) = \mathbf{x}\mathbf{GCD}_\ell(\mathbf{y}, \mathbf{z})$ and $\mathbf{GCD}_{\geqslant_r}(\mathbf{yx}, \mathbf{zx}) = \mathbf{GCD}_{\geqslant_r}(\mathbf{y}, \mathbf{z})\mathbf{x}$. Furthermore, $\mathbf{y}$ and $\mathbf{z}$ have a common $\leqslant_\ell$-multiple in $\mathbf{A}^+$ if and only if $\mathbf{xy} = \mathbf{xz}$ have a common $\leqslant_\ell$-multiple in $\mathbf{A}^+$, and in this case we have $\mathbf{LCM}_{\leqslant_\ell}(\mathbf{xy}, \mathbf{xz}) = \mathbf{x}\mathbf{LCM}_{\leqslant_\ell}(\mathbf{y}, \mathbf{z})$. Similarly, $\mathbf{y}$ and $\mathbf{z}$ have a common $\geqslant_r$-multiple in $\mathbf{A}^+$ if and only if $\mathbf{yx} = \mathbf{zx}$ have a common $\geqslant_r$-multiple in $\mathbf{A}^+$, and in this case we have $\mathbf{LCM}_{\geqslant_r}(\mathbf{yx}, \mathbf{zx}) = \mathbf{LCM}_{\geqslant_r}(\mathbf{y}, \mathbf{z})\mathbf{x}$.

Proof. First, if $m_{i,j} < +\infty$, then the only factors (both for $\leqslant_\ell$ and for $\geqslant_r$) of $[\sigma_i\sigma_j]^{m_{i,j}}$ are the elements $[\sigma_i\sigma_j]^k$ and $[\sigma_j\sigma_i]^k$ with $0 \leqslant k \leqslant m_{i,j}$. Among those, only $[\sigma_i\sigma_j]^{m_{i,j}}$ is a multiple of both σ_i and σ_j, and therefore it is their lower common multiple.

The second part of Lemma 2.15 follows immediately from the fact that $\mathbf{A}^+$ is both left-cancellative and right-cancellative. □

From Theorem 2.14 follow many results concerning the set $\mathcal{S}$, which we identify with the Coxeter group $\mathbf{W}$.

Lemma 2.16.
Let $\mathbf{A}^+$ be an Artin–Tits monoid, let $\mathbf{a}$ be a simple element of $\mathbf{A}^+$ and let σ be a generator of $\mathbf{A}^+$. Then, $\mathbf{a}\sigma \notin \mathcal{S}$ if and only if $\mathbf{a} \geqslant_r \sigma$.

Proof. First, if $\mathbf{a} \geqslant_r \sigma$, let us write $\mathbf{a} = \mathbf{b}\sigma$. Since $\mathbf{a}\sigma = \mathbf{b}\sigma^2$, it follows that $\pi(\mathbf{b}) = \pi(\mathbf{a}\sigma)$ and that $\lambda(\mathbf{a}\sigma) = \lambda(\mathbf{b}) + 2$, hence, by definition, we have $\mathbf{a}\sigma \notin \mathcal{S}$.

Conversely, if $\mathbf{a}\sigma \notin \mathcal{S}$, consider the element $\mathbf{c}$ of $\mathcal{S}$ such that $\pi(\mathbf{c}) = \pi(\mathbf{a}\sigma)$. Observe that $\lambda(\mathbf{a}\sigma) \equiv \lambda(\mathbf{c}) \pmod 2$, whence $\lambda(\mathbf{c}) \leqslant \lambda(\mathbf{a}) - 1$. Since $\lambda(\mathbf{c}\sigma) \leqslant \lambda(\mathbf{a})$, $\pi(\mathbf{c}\sigma) = \pi(\mathbf{a})$ and $\mathbf{a} \in \mathcal{S}$, it follows that $\mathbf{c}\sigma \in \mathcal{S}$. Theorem 2.14 states that $\pi : \mathcal{S} \mapsto \mathbf{W}$ is bijective, hence $\mathbf{a} = \mathbf{c}\sigma$ and therefore $\mathbf{a} \geqslant_r \sigma$. □

Lemma 2.16 suggests that the membership to the set of simple braids is easy to characterise using the following notions of *left* and *right* sets.

Definition 2.17 (Left and right sets)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid, with generators $\sigma_1, \ldots, \sigma_n$, and let $\mathbf{a}$ be an element of $\mathbf{A}^+$. We call left set *of $\mathbf{a}$, and denote by $\mathbf{left}(\mathbf{a})$, the set $\{\sigma_i : \sigma_i \leqslant_\ell \mathbf{a}\}$. We also call* right set *of $\mathbf{a}$, and denote by $\mathbf{right}(\mathbf{a})$, the set $\{\sigma_i : \mathbf{a} \geqslant_r \sigma_i\}$.*

Indeed, Lemma 2.16 can then be reformulated as follows: Let $\mathbf{a}$ be a simple element of $\mathbf{A}^+$ and let σ be a generator of $\mathbf{A}^+$. Then, $\mathbf{a}\sigma \in \mathcal{S}$ if and only if $\sigma \notin \mathbf{right}(\mathbf{a})$ and $\sigma\mathbf{a} \in \mathcal{S}$ if and only if $\sigma \notin \mathbf{left}(\mathbf{a})$.

Corollary 2.18.
*Let $\mathbf{A}^+$ be an Artin–Tits monoid, with generators $\sigma_1, \ldots, \sigma_n$. The simple elements of $\mathbf{A}^+$ are those without factor of the form σ^2 (we say that they are σ^2-*free*), i.e.*

$$\mathcal{S} = \{\mathbf{a} \in \mathbf{A}^+ : \forall \mathbf{b}, \mathbf{c} \in \mathbf{A}^+, \forall i \in \{1, \ldots, n\}, \mathbf{a} \neq \mathbf{b}\sigma_i^2\mathbf{c}\}.$$

Proof. First, if $\mathbf{a} = \mathbf{b}\sigma_i^2\mathbf{c}$, then $\pi(\mathbf{a}) = \pi(\mathbf{bc})$ and $\lambda(\mathbf{b}) > \lambda(\mathbf{ac})$, hence Theorem 2.14 proves that $\mathbf{a} \notin \mathcal{S}$. Conversely, if $\mathbf{a} := \sigma_{i_1} \ldots \sigma_{i_k} \notin \mathcal{S}$, let m be the largest integer such that $\sigma_{i_1} \ldots \sigma_{i_m} \in \mathcal{S}$. Note that $1 \leqslant m < k$. Lemma 2.16 proves that $\sigma_{i_1} \ldots \sigma_{i_m} \geqslant_r \sigma_{i_{m+1}}$. Therefore, the braids $\mathbf{b} := (\sigma_{i_1} \ldots \sigma_{i_m})\,\sigma_{i_{m+1}}^{-1}$ and $\mathbf{c} = \sigma_{i_{m+2}} \ldots \sigma_{i_k}$ both belong to the monoid $\mathbf{A}^+$, and they obviously satisfy the relation $\mathbf{a} = \mathbf{b}\sigma_{m+1}^2\mathbf{c}$. □

In particular, it follows from the above characterisation of the set $\mathcal{S}$ that the GCD and (conditional) LCM operations in $\mathbf{A}^+$ and in $\mathcal{S}$ coincide.

Lemma 2.19.
Let $\mathbf{A}^+$ be an Artin–Tits monoid. The set $\mathcal{S}$ of simple elements of $\mathbf{A}^+$ is closed under $\leqslant_\ell$-division. Moreover, for all pairs $(\mathbf{a}, \mathbf{b})$ of elements of $\mathcal{S}$, their greatest $\leqslant_\ell$-divisors in $\mathbf{A}^+$ and in $\mathcal{S}$ are equal, and if they have a common $\leqslant_\ell$-multiple in $\mathbf{A}^+$, then their least $\leqslant_\ell$-multiples in $\mathbf{A}^+$ and in $\mathcal{S}$ exist and are equal.

Similar statements hold for the order $\geqslant_r$.

Proof. Consider two elements $\mathbf{x}$ and $\mathbf{y}$ of the monoid $\mathbf{A}^+$, and let us assume that $\mathbf{x} \notin \mathcal{S}$. There exists some $\mathbf{z} \in \mathbf{A}^+$ such that $\pi(\mathbf{x}) = \pi(\mathbf{z})$ and $\lambda(\mathbf{x}) > \lambda(\mathbf{z})$. It follows that $\pi(\mathbf{xy}) = \pi(\mathbf{zy})$ and that $\lambda(\mathbf{xy}) > \lambda(\mathbf{zy})$, hence that $\mathbf{xy} \notin \mathcal{S}$. This proves that $\mathcal{S}$ is closed under $\leqslant_\ell$-division.

Now, let $\mathbf{a}$ and $\mathbf{b}$ be elements of $\mathcal{S}$. Their greatest $\leqslant_\ell$-divisor in $\mathbf{A}^+$ must belong to $\mathcal{S}$, hence is also their greatest $\leqslant_\ell$-divisor in $\mathcal{S}$. Likewise, if they have a common $\leqslant_\ell$-multiple in $\mathcal{S}$, then they have a least $\leqslant_\ell$-multiple in both $\mathcal{S}$ and $\mathbf{A}^+$, and their least $\leqslant_\ell$-multiple in $\mathbf{A}^+$ must belong to $\mathcal{S}$, so that these least $\leqslant_\ell$-multiples coincide. It remains to prove that, if $\mathbf{a}$ and $\mathbf{b}$ have a common $\leqslant_\ell$-multiple in $\mathbf{A}^+$, they also have a common $\leqslant_\ell$-multiple in $\mathcal{S}$.

Let $\mathbf{z}$ be a common $\leqslant_\ell$-multiple of $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{A}^+$, and let $\mathbf{c} := \mathbf{GCD}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b})$.. We prove by induction on $\lambda(\mathbf{z}) - \lambda(\mathbf{c})$ that $\mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b}) \in \mathbf{S}$. If $\lambda(\mathbf{c}) = \lambda(\mathbf{z})$, then $\lambda(\mathbf{c}) \geqslant \max\{\lambda(\mathbf{a}), \lambda(\mathbf{b})\}$, hence $\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b})$. Hence, let us assume that $\lambda(\mathbf{c}) < \lambda(\mathbf{z})$. If $\mathbf{a} = \mathbf{c}$, then $\mathbf{b} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b})$. Similarly, if $\mathbf{b} = \mathbf{c}$, then $\mathbf{a} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b})$. Hence, we focus on the case where $\mathbf{c} <_\ell \mathbf{a}$ and $\mathbf{c} <_\ell \mathbf{b}$.

Consider two generators σ_i and σ_j of $\mathbf{A}^+$ such that $\mathbf{c}\sigma_i \leqslant_\ell \mathbf{a}$ and $\mathbf{c}\sigma_j \leqslant_\ell \mathbf{b}$. Since $\mathbf{c}^{-1}\mathbf{z}$ is a common $\leqslant_\ell$-multiple of σ_i and σ_j, it follows that $m_{i,j} < +\infty$ and that $[\sigma_i\sigma_j]^{m_{i,j}}$ is the least common $\geqslant_r$-multiple (and $\leqslant_\ell$-multiple) of σ_i and σ_j in $\mathbf{A}^+$. Hence, following Lemma 2.16, an immediate induction on k proves that $[\sigma_i\sigma_j]^k \in \mathcal{S}$ whenever $0 \leqslant k \leqslant m_{i,j}$. Indeed, if $k \in \{1, \ldots, m_{i,j}\}$ is a minimal integer such that $[\sigma_i\sigma_j]^k \notin \mathcal{S}$, then it follows that $\{\sigma_i, \sigma_j\} \subseteq \mathbf{right}([\sigma_i\sigma_j]^{k-1})$, which is impossible.

Likewise, if $k \in \{1, \ldots, m_{i,j}\}$ is a minimal integer such that $\mathbf{c}[\sigma_i\sigma_j]^k \notin \mathcal{S}$, it follows that $\{\sigma_i, \sigma_j\} \subseteq \mathbf{right}(\mathbf{c}[\sigma_i\sigma_j]^{k-1})$. This means that $\mathbf{c}[\sigma_i\sigma_j]^{k-1} \geqslant_r [\sigma_i\sigma_j]^{m_{i,j}}$, and therefore that $\mathbf{c} \geqslant_r \sigma_j$, which contradicts the fact that $\mathbf{c}\sigma_j$ belongs to $\mathcal{S}$. Hence, let $\mathbf{d} := \mathbf{c}[\sigma_i\sigma_j]^{m_{i,j}}$ and $\mathbf{e} := \mathbf{GCD}_{\leqslant_\ell}(\mathbf{a}, \mathbf{d})$. Both $\mathbf{c}\sigma_i$ and $\mathbf{c}\sigma_j$ belong to $\mathbf{S}$ and $\leqslant_\ell$-divide $\mathbf{z}$, hence so does $\mathbf{d} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{c}\sigma_i, \mathbf{c}\sigma_j)$. Since $\mathbf{c}\sigma_i \leqslant_\ell \mathbf{e}$, it follows that $\lambda(\mathbf{c}) < \lambda(\mathbf{c}\sigma_i) \leqslant \lambda(\mathbf{e})$, and the induction hypothesis states that $\mathbf{z_a} := \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{d})$ belongs to $\mathcal{S}$.

We show similarly that the element $\mathbf{z_b} := \mathbf{LCM}_{\leqslant_\ell}(\mathbf{b}, \mathbf{d})$ belongs to $\mathcal{S}$. Finally observe that $\mathbf{d} \leqslant_\ell \mathbf{GCD}_{\leqslant_\ell}(\mathbf{z_a}, \mathbf{z_b})$, and therefore the induction hypothesis states that the element $\mathbf{y} := \mathbf{LCM}_{\leqslant_\ell}(\mathbf{z_a}, \mathbf{z_b})$ belongs to $\mathcal{S}$. This shows that

$$\mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b}) = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b}, \mathbf{c}\sigma_i, \mathbf{c}\sigma_j) = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b}, \mathbf{d}) = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{z_a}, \mathbf{z_b}) = \mathbf{y}$$

belongs to $\mathcal{S}$, which completes the induction and the proof of Lemma 2.19. $\square$

The proof of Lemma 2.19 is illustrated in Fig. 2.20.

$$\begin{tikzcd}
 & & \mathbf{c}\sigma_i \arrow[r] & \mathbf{e} \arrow[r] \arrow[dr] & \mathbf{a} \arrow[r] & \mathbf{z_a} \arrow[dr] & & \\
\mathbf{1} \arrow[r] & \mathbf{c} \arrow[ur] \arrow[dr] & & & \mathbf{d} \arrow[ur] \arrow[dr] & & \mathbf{y} \arrow[r] & \mathbf{z} \\
 & & \mathbf{c}\sigma_j \arrow[urr] \arrow[rr] & & \mathbf{b} \arrow[r] & \mathbf{z_b} \arrow[ur] & &
\end{tikzcd}$$

Figure 2.20 – Having a common multiple in $\mathbf{A}^+ \Leftrightarrow$ having a common multiple in $\mathcal{S}$

Since the Coxeter groups $\mathbf{W}_n$ of braid groups $\mathbf{B}_n$ are finite, the second part of Theorem 2.14 applies to braid groups. This leads to two generalisations of braid monoids, that are *Artin–Tits monoids of spherical type* and *Garside monoids* [14, 27, 36, 37, 70, 78].

Definition 2.21 (Garside element and Garside monoid)**.**
Let $\mathbf{G}^+$ be a finitely generated monoid, equipped with the two divisibility orderings $\leqslant_\ell$ and $\geqslant_r$.

A Garside element *of the monoid $\mathbf{G}^+$ is an element Δ such that the equality $\{\mathbf{a} \in \mathbf{G}^+ : \mathbf{a} \leqslant_\ell \Delta\} = \{\mathbf{a} \in \mathbf{G}^+ : \Delta \geqslant_r \mathbf{a}\}$ holds, and such that the set $\{\mathbf{a} \in \mathbf{G}^+ : \mathbf{a} \leqslant_\ell \Delta\}$ generates the monoid $\mathbf{A}^+$. The divisors of Δ are called* simple elements *of the monoid $\mathbf{G}^+$ relatively to Δ, or just* simple elements *of $\mathbf{G}^+$ is the Garside element Δ is clear from the context.*

If $\mathbf{G}^+$ admits a Garside element and, in addition,

- *the monoid $\mathbf{G}^+$ is left and right-cancellative;*
- *there exists a super-additive* length *function, i.e. a function $\lambda : \mathbf{G}^+ \mapsto \mathbb{Z}$ such that $\lambda(\mathbf{1}) = 0$, $\lambda(m) > 0$ for all $m \neq \mathbf{1}$, and $\lambda(a) + \lambda(b) \leqslant \lambda(ab)$ for all $a, b \in \mathbf{G}^+$;*
- *the divisibility orderings $\leqslant_\ell$ and $\geqslant_r$ are lattices;*

then we say that $\mathbf{G}^+$ is a Garside monoid*.*

Artin–Tits monoids and Garside monoids generalise braid monoids in two different directions. Indeed, not all Artin–Tits monoids are Garside monoids, and not all Garside monoids are Artin–Tits monoids, as illustrated below.

Example 2.22.
The monoid $\mathbf{M}^+_{2,3} := \langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^3 \rangle^+$ is a Garside monoid, with Garside element $\Delta := \sigma_1^2 = \sigma_2^3$. However, this monoid is not homogeneous, hence is not an Artin–Tits monoid.

Conversely, the free monoid $\mathbb{N} * \mathbb{N} := \langle \sigma_1, \sigma_2 \rangle^+$ is an Artin–Tits monoid, but the elements σ_1 and σ_2 have no common multiple, hence $\mathbb{N} * \mathbb{N}$ is not a Garside monoid.

Definition 2.23 (Artin–Tits group or monoid of spherical type).
Let $\mathbf{A}^+$ be an Artin–Tits monoid, and let $\mathbf{W}$ be the Coxeter group associated with $\mathbf{A}^+$. We say that the monoid $\mathbf{A}^+$ is a Artin–Tits monoid of spherical type *(or, equivalently, has* finite Coxeter type*), and that the associated Artin–Tits group $\mathbf{A}$ is a* Artin–Tits group of spherical type*, if the group $\mathbf{W}$ is finite. Due to the similarity between such groups and monoids and their braid equivalents, we still call* braids *the elements of the monoid $\mathbf{A}^+$ and of the group $\mathbf{A}$.*

It follows immediately from Theorem 2.11 that braid monoids are indeed Artin–Tits monoids of spherical type. Moreover, Artin–Tits monoids of spherical type are themselves Garside monoids.

Proposition 2.24.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type, with generators $\sigma_1, \ldots, \sigma_n$. The element $\Delta := \mathbf{LCM}_{\leqslant_\ell}(\sigma_1, \ldots, \sigma_n)$ is a Garside element of the monoid $\mathbf{A}^+$, which is a Garside monoid. In addition, the element Δ is also equal to $\mathbf{LCM}_{\geqslant_r}(\sigma_1, \ldots, \sigma_n)$.

Proof. Recall the process of word reversal. It shows that $\Delta^* = \mathbf{LCM}_{\geqslant_r}(\sigma_1, \ldots, \sigma_n)$. Hence, Corollary 2.18 shows that

$$\{\mathbf{a} \in \mathbf{A}^+ : \mathbf{a} \leqslant_\ell \Delta\} = \{\mathbf{a} \in \mathbf{A}^+ : \mathbf{a} \text{ is } \sigma^2\text{-free}\} = \{\mathbf{a} \in \mathbf{A}^+ : \Delta^* \geqslant_r \mathbf{a}\}.$$

It follows that $\Delta^* \leqslant_\ell \Delta$ and that $\Delta^* \geqslant_r \Delta$, i.e. that $\Delta^* = \Delta$.

This proves that $\Delta = \mathbf{LCM}_{\geqslant_r}(\sigma_1, \ldots, \sigma_n)$ is a Garside element of the monoid $\mathbf{A}^+$. Furthermore, Theorem 2.14 already states that $\mathbf{A}^+$ is left and right-cancellative, and that the divisibility orderings $\leqslant_\ell$ and $\geqslant_r$ are lattices. Finally, the Artin length λ is an additive length function, which proves that $\mathbf{A}^+$ is a Garside monoid. □

In addition, not all Artin–Tits monoids of spherical type are braid monoids, and there even exists finite Coxeter groups that are not permutation sets. In particular, the class of finite Coxeter groups has been completely classified, as follows.

Definition 2.25 (Finite irreducible Coxeter system).
Let $\langle \sigma_1, \ldots, \sigma_n \mid \sigma_i^2 = \mathbf{1}, [\sigma_i \sigma_j]^{m_{i,j}} = [\sigma_j \sigma_i]^{m_{i,j}} \text{ if } i \neq j \rangle$ be a presentation of a Coxeter group $\mathbf{W}$. We say this presentation is a finite irreducible Coxeter system *if one of the following requirements is satisfied:*

- *$m_{i,i+1} = 3$, $m_{i,j} = 2$ otherwise, and $n \geqslant 0$ (we say that $\mathbf{W}$ is of type A_n);*
- *$m_{1,2} = 4$, $m_{i,i+1} = 3$ if $i \neq 1$, $m_{i,j} = 2$ otherwise, and $n \geqslant 3$ ($\mathbf{W}$ is of type B_n);*
- *$m_{1,3} = 3$, $m_{i,i+1} = 3$ if $i \neq 1$, $m_{i,j} = 2$ otherwise, and $n \geqslant 4$ ($\mathbf{W}$ is of type D_n);*
- *$m_{1,4} = 3$, $m_{i,i+1} = 3$ if $i \neq 1$, $m_{i,j} = 2$ otherwise, and $6 \leqslant n \leqslant 8$ ($\mathbf{W}$ is of type E_n);*
- *$m_{2,3} = 4$, $m_{i,i+1} = 3$ if $i \neq 2$, $m_{i,j} = 2$ otherwise, and $n = 4$ ($\mathbf{W}$ is of type F_4);*
- *$m_{1,2} = 5$, $m_{i,i+1} = 3$ if $i \neq 2$, $m_{i,j} = 2$ otherwise, and $3 \leqslant n \leqslant 4$ ($\mathbf{W}$ is of type H_n);*
- *$m_{1,2} = a$ for some integer $a \geqslant 3$, and $n = 2$ ($\mathbf{W}$ is of type $I_2(a)$).*

The above typology and type names follow the Cartan-Killing classification of simple Lie algebras. Hence, the Coxeter group associated to the braid group $\mathbf{B}_n$ is of type A_{n-1}, *not* of type B_n, which might be misleading. However, we will mainly focus on braid groups $\mathbf{B}_n$ in Chapters 3 and 4, while we will focus on arbitrary Artin–Tits groups of spherical type $\mathbf{A}_n$, using the typology of finite irreducible Coxeter systems, in Chapters 5, 6 and 7. This separation of focal points should avoid all possible confusions.

Note that these Coxeter systems are pairwise distinct, with the exception $A_2 = I_2(3)$. Moreover, the Coxeter systems A_0 and A_1 are not very interesting in themselves, since they are respectively associated with the trivial Artin–Tits monoid $\{0\}$ and the monoid $\mathbb{Z}_{\geqslant 0}$ of non-negative integers. Furthermore, finite irreducible Coxeter system are prototypical Coxeter groups [31, 32].

Theorem 2.26.
Let $\mathbf{W} = \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i^2 = \mathbf{1}, [\sigma_i\sigma_j]^{m_{i,j}} = [\sigma_j\sigma_i]^{m_{i,j}} \text{ if } i \neq j \rangle$ *be a Coxeter group. The group* $\mathbf{W}$ *is finite if and only if there a partition of* $\{1, \ldots, n\}$ *into sets* $\mathbf{I}_1, \ldots, \mathbf{I}_k$ *such that:*

- *for all* $i, j \in \{1, \ldots, n\}$*, if* i *and* j *belong to two distinct parts of the partition, then* $m_{i,j} = 2$*;*
- *for all* $\ell \in \{1, \ldots, k\}$*, and up to renumbering the generators* σ_i*, each presentation*

$$\left\langle (\sigma_i)_{i \in \mathbf{I}_\ell} \mid \sigma_i^2 = \mathbf{1}, [\sigma_i\sigma_j]^{m_{i,j}} = [\sigma_j\sigma_i]^{m_{i,j}} \text{ if } i \neq j \text{ and } i, j \in \mathbf{I}_\ell \right\rangle$$

  is a finite (non-trivial) irreducible Coxeter system.

In addition, the group $\mathbf{W}$ *is finite and irreducible (i.e. is not a direct product of non-trivial Coxeter groups) if and only* $k = 1$*, i.e. if the presentation of* $\mathbf{W}$ *is itself a finite irreducible Coxeter system.*

Figure 2.27 presents the complete classification of finite irreducible Coxeter system in terms of their Coxeter diagrams. Each generator σ_i is represented by the vertex with label i. Vertices i and j are linked by an unlabelled edge if $m_{i,j} = 3$, or by an edge with label k if $m_{i,j} = k$ for some integer $k \geqslant 4$.

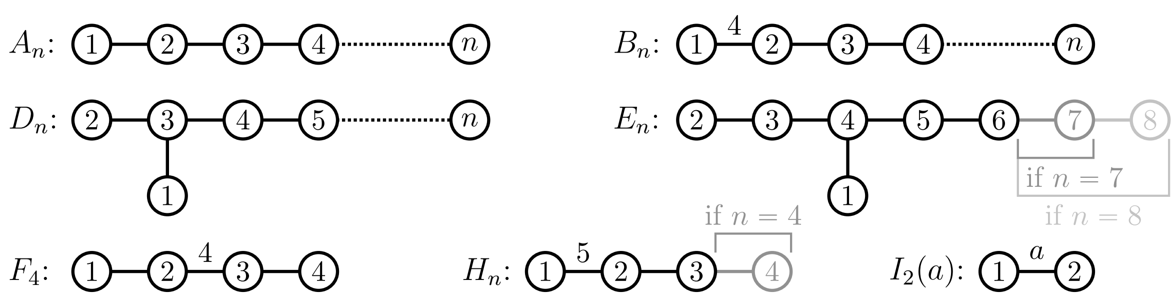


Figure 2.27 – Finite irreducible Coxeter systems

The classification of finite irreducible Coxeter systems leads to natural characterisations of the three infinite families A_n, B_n and D_n (see [16]). The first result is just a variant of Theorems 2.11 and 2.14.

Definition 2.28 (Symmetric group and positive descents).
Let n be a positive integer, and let $\mathfrak{S}_n$ be the symmetric group of the set $\{1,\ldots,n\}$. In addition, let β be an element of $\mathfrak{S}_n$. We call positive descents *of β the elements of the set $\mathbf{d}_{>0}(\beta) := \{k \in \{1,\ldots,n-1\} : \beta(k) > \beta(k+1)\}$.*

Proposition 2.29.
Let $\mathbf{A}^+$ be the Artin–Tits monoid of spherical type whose associated Coxeter group is A_n (i.e. $\mathbf{A}^+$ is the braid monoid $\mathbf{B}^+_{n+1}$). The group morphism $\iota : A_n \mapsto \mathfrak{S}_{n+1}$ such that $\iota : \sigma_k \mapsto (k \leftrightarrow k+1)$ for all $k \in \{1,\ldots,n\}$ is a group isomorphism.

In addition, let $\mathbf{b}$ be a simple braid of $\mathbf{A}^+$, and let $\beta = \iota \circ \pi(\mathbf{b})$ be the associated permutation, where $\pi : \mathbf{A}^+ \mapsto A_n$ is the canonical projection of $\mathbf{A}^+$ on its Coxeter group. We have $\mathbf{left}(\mathbf{b}) = \{\sigma_k : k \in \mathbf{d}_{>0}(\beta)\}$ and $\mathbf{right}(\mathbf{b}) = \{\sigma_k : k \in \mathbf{d}_{>0}(\beta^{-1})\}$.

Definition 2.30 (Signed symmetric group and non-negative descents).
Let n be a positive integer, and let $\mathfrak{S}^\pm_n$ be the signed symmetric group of order n, i.e. the set $\{\varphi : \varphi$ is a permutation of $\{-n,\ldots,n\}$ and $\varphi(-k) = -\varphi(k)$ for all $k \in \{-n,\ldots,n\}\}$. In addition, let β be an element of $\mathfrak{S}^\pm_n$. We call non-negative descents *of β the elements of the set $\mathbf{d}_{\geqslant 0}(\beta) := \{k \in \{0,\ldots,n-1\} : \beta(k) > \beta(k+1)\}$.*

Proposition 2.31.
Let $\mathbf{A}^+$ be the Artin–Tits monoid of spherical type whose associated Coxeter group is B_n. The group morphism $\iota : B_n \mapsto \mathfrak{S}^\pm_n$ such that $\iota : \sigma_1 \mapsto (1 \leftrightarrow -1)$ and $\iota : \sigma_k \mapsto (k-1 \leftrightarrow k)(1-k \leftrightarrow -k)$ for all $k \in \{2,\ldots,n\}$ is a group isomorphism.

In addition, let $\mathbf{b}$ be a simple braid of $\mathbf{A}^+$, and let $\beta = \iota \circ \pi(\mathbf{b})$ be the associated signed permutation, where $\pi : \mathbf{A}^+ \mapsto B_n$ is the canonical projection of $\mathbf{A}^+$ on its Coxeter group. We have $\mathbf{left}(\mathbf{b}) = \{\sigma_k : k-1 \in \mathbf{d}_{\geqslant 0}(\beta)\}$ and $\mathbf{right}(\mathbf{b}) = \{\sigma_k : k-1 \in \mathbf{d}_{\geqslant 0}(\beta^{-1})\}$.

Definition 2.32 (Positive signed symmetric group and twisted descents).
Let n be a positive integer, and let $\mathfrak{S}^{++}_n$ be the positive signed symmetric group of order n, i.e. the set $\{\varphi \in \mathfrak{S}^\pm_n : \prod_{k=1}^n \varphi(k) > 0\}$. In addition, let β be an element of $\mathfrak{S}^{++}_n$. We call twisted descents *of β the elements of the set $\mathbf{d}_{\mathrm{tw}}(\beta) := \{k \in \{1,\ldots,n-1\} : \beta(k) > \beta(k+1)\}$ if $\beta(1)+\beta(2) > 0$, and $\mathbf{d}_{\mathrm{tw}}(\beta) := \{0\} \cup \{k \in \{1,\ldots,n-1\} : \beta(k) > \beta(k+1)\}$ if $\beta(1)+\beta(2) < 0$.*

Proposition 2.33.
Let $\mathbf{A}^+$ be the Artin–Tits monoid of spherical type whose associated Coxeter group is D_n. The group morphism $\iota : D_n \mapsto \mathfrak{S}^\pm_n$ such that $\iota : \sigma_1 \mapsto (1 \leftrightarrow -2)(2 \leftrightarrow -1)$ and $\iota : \sigma_k \mapsto (k-1 \leftrightarrow k)(1-k \leftrightarrow -k)$ for all $k \in \{2,\ldots,n\}$ is a group isomorphism.

In addition, let $\mathbf{b}$ be a simple braid of $\mathbf{A}^+$, and let $\beta = \iota \circ \pi(\mathbf{b})$ be the associated (positive) signed permutation, where $\pi : \mathbf{A}^+ \mapsto D_n$ is the canonical projection of $\mathbf{A}^+$ on its Coxeter group. We have $\mathbf{left}(\mathbf{b}) = \{\sigma_k : k-1 \in \mathbf{d}_{\mathrm{tw}}(\beta)\}$ and $\mathbf{right}(\mathbf{b}) = \{\sigma_k : k-1 \in \mathbf{d}_{\mathrm{tw}}(\beta^{-1})\}$.

2.3.2 Garside Monoids, Groups and Normal Forms

Having proved that braid monoids, and even Artin–Tits monoids of spherical type, are in fact special instances of a larger class of Garside monoids, we now focus on the latter monoids.

Now, let us mention and prove several standard results about Garside monoids, which we will use in subsequent parts of this document. These results can be found from numerous sources, such as [14, 27, 36, 37, 47, 70, 78].

In this section, we will always consider a Garside monoid $\mathbf{G}^+$ generated by a finite set $\{\sigma_1, \ldots, \sigma_n\}$. We will also fix some Garside element Δ of $\mathbf{G}^+$, and will denote by $\mathcal{S}$ the set $\{\mathbf{a} \in \mathbf{G}^+ : \mathbf{a} \leqslant_\ell \Delta\} = \{\mathbf{a} \in \mathbf{G}^+ : \Delta \geqslant_r \mathbf{a}\}$.

In addition, we will consider the functions $\alpha_\ell : \mathbf{G}^+ \mapsto \mathbf{G}^+$ and $\alpha_r : \mathbf{G}^+ \mapsto \mathbf{G}^+$ such that $\alpha_\ell : \mathbf{a} \mapsto \mathbf{GCD}_{\leqslant_\ell}(\mathbf{a}, \Delta)$ and $\alpha_r : \mathbf{a} \mapsto \mathbf{GCD}_{\geqslant_r}(\mathbf{a}, \Delta)$, where $\mathbf{GCD}_{\leqslant_\ell}$ and $\mathbf{GCD}_{\geqslant_r}$ respectively denote the greatest lower bounds for the orderings $\leqslant_\ell$ and $\geqslant_r$.[2] These functions are crucial for defining the notions of Garside normal forms of $\mathbf{G}^+$, as follows, and which were originally introduced by Adian in the context of braids [3].

Definition 2.34 (Garside normal forms in the monoid $\mathbf{G}^+$)**.**
Let $\mathbf{G}^+$ be a Garside monoid. The left Garside normal form *of an element $\mathbf{a}$ of $\mathbf{G}^+$ is defined as the word $\mathbf{NF}_\ell(\mathbf{a}) := \mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k$ such that:*

- $\mathbf{a} = \mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_k$,
- $\mathbf{a}_1, \ldots, \mathbf{a}_k$ *are simple elements of* $\mathbf{G}^+$,
- *either* $k = 0$ *or* $\mathbf{a}_k \neq \mathbf{1}$, *and*
- $\mathbf{a}_i = \alpha_\ell(\mathbf{a}_i\mathbf{a}_{i+1} \ldots \mathbf{a}_k)$ *for all* $i \in \{1, \ldots, k\}$.

The right Garside normal form *of $\mathbf{a}$ is defined as the word $\mathbf{NF}_r(\mathbf{a}) := \mathbf{a}'_1 \cdot \mathbf{a}'_2 \cdot \ldots \cdot \mathbf{a}'_\ell$ such that:*

- $\mathbf{a} = \mathbf{a}'_1\mathbf{a}'_2 \ldots \mathbf{a}'_\ell$,
- $\mathbf{a}'_1, \ldots, \mathbf{a}'_\ell$ *are simple elements of* $\mathbf{G}^+$,
- *either* $\ell = 0$ *or* $\mathbf{a}'_1 \neq \mathbf{1}$, *and*
- $\mathbf{a}'_i = \alpha_r(\mathbf{a}'_1\mathbf{a}'_2 \ldots \mathbf{a}'_i)$ *for all* $i \in \{1, \ldots, \ell\}$.

An immediate induction on the length $\lambda(\mathbf{a})$ proves that the words $\mathbf{NF}_\ell(\mathbf{a})$ and $\mathbf{NF}_r(\mathbf{a})$ are well-defined. Indeed, they are well-defined (and identical) if $\mathbf{a} \in \mathcal{S}$, and therefore if $\lambda(\mathbf{a}) \leqslant 1$, and if $\lambda(\mathbf{a}) \geqslant 2$, we have $\mathbf{NF}_\ell(\mathbf{a}) := \alpha_\ell(\mathbf{a}) \cdot \mathbf{NF}_\ell(\alpha_\ell(\mathbf{a})^{-1}\mathbf{a})$ and $\mathbf{NF}_r(\mathbf{a}) := \mathbf{NF}_\ell(\mathbf{a}\alpha_r(\mathbf{a})^{-1}) \cdot \alpha_r(\mathbf{a})$.

[2] We stick to standard notations for handling braids and elements of Garside monoids, which leads us to identify the orders $\leqslant_\ell$ and $\geqslant_r$ with divisibility relations and to denote greatest lower bounds by **GCD**. Such standard notations include the notation $\inf(\mathbf{x})$, called *infimum* of $\mathbf{x}$, which is the largest integer u such that $\Delta^u \leqslant_\ell \mathbf{x}$, for all elements $\mathbf{x}$ of the monoid $\mathbf{G}^+$.

In addition, when defining the left Garside normal form, requiring that $\mathbf{a}_k \neq \mathbf{1}$ is equivalent to requiring that $\mathbf{a}_i \neq \mathbf{1}$ for all $i \in \{1, \ldots, k\}$. Indeed, if $\mathbf{a}_i = \mathbf{1}$ for some $i \leqslant k$, then the relation $\mathbf{a}_i = \alpha_\ell(\mathbf{a}_i \mathbf{a}_{i+1} \ldots \mathbf{a}_k)$ proves that $\mathbf{a}_i \mathbf{a}_{i+1} \ldots \mathbf{a}_k = \mathbf{1}$, and therefore that $\mathbf{a}_k = \mathbf{1}$. Similarly, when defining the right Garside normal form, requiring that $\mathbf{a}'_1 \neq \mathbf{1}$ is equivalent to requiring that $\mathbf{a}'_i \neq \mathbf{1}$ for all $i \in \{1, \ldots, \ell\}$.

Furthermore, in the sequel of the document, and due to the invariance of $\mathbf{G}^+$ under word reversal, we will often refer to the *left* variant of the Garside normal form. Therefore, we may also omit the word *left*, and just refer to the *Garside normal form.* In addition, we will say that a word is a Garside normal *word* if it is the Garside normal form of some element of $\mathbf{G}^+$.

There is a canonical way of choosing a length function for $\mathbf{G}^+$ and generators of $\mathbf{G}^+$.

Definition & Proposition 2.35 (Product length)**.**
Let $\mathbf{M}^+$ be a finitely generated monoid equipped with a super-additive length function.

For each element $\mathbf{a} \in \mathbf{M}^+$, we define the product length *of $\mathbf{a}$ as the largest integer $\chi(\mathbf{a}) \geqslant 0$ such that $\mathbf{a}$ can be written as a product of $\chi(\mathbf{a})$ non-trivial elements of $\mathbf{M}^+$.*

The product length is a super-additive length function on $\mathbf{M}^+$. Moreover, the set $\{\mathbf{a} \in \mathbf{M}^+ : \chi(\mathbf{a}) = 1\}$ is the smallest *generating set of $\mathbf{M}^+$, and it is a finite set.*

Proof. Let λ be a super-additive length function on $\mathbf{M}^+$. First, whenever $\mathbf{a}$ is written as a product $\mathbf{a}_1 \ldots \mathbf{a}_u$ of non-trivial elements of $\mathbf{G}^+$, we know that $\lambda(\mathbf{a}) \geqslant \lambda(\mathbf{a}_1) + \ldots + \lambda(\mathbf{a}_u) \geqslant u$. Hence, the product length $\chi(\mathbf{a})$ is well-defined, and such that $\chi(\mathbf{a}) \leqslant \lambda(\mathbf{a})$.

Second, if two elements $\mathbf{a}, \mathbf{b} \in \mathbf{G}^+$ have respective factorisations $\mathbf{a}_1 \ldots \mathbf{a}_u$ and $\mathbf{b}_1 \ldots \mathbf{b}_v$ into non-trivial elements of $\mathbf{G}^+$, then $\mathbf{a}_1 \ldots \mathbf{a}_u \mathbf{b}_1 \ldots \mathbf{b}_v$ is a factorisation of $\mathbf{ab}$. This shows that $\chi(\mathbf{a}) + \chi(\mathbf{b}) \leqslant \chi(\mathbf{ab})$, i.e. that χ is super-additive.

In particular, it follows immediately that the set $\{\mathbf{a} \in \mathbf{M}^+ : \chi(\mathbf{a}) = 1\}$ is contained in all the generating sets of $\mathbf{M}^+$, and is itself a generating set of $\mathbf{M}^+$. Hence, since $\mathbf{M}^+$ admits some finite generating set, the set $\{\mathbf{a} \in \mathbf{M}^+ : \chi(\mathbf{a}) = 1\}$ must be finite too. □

Henceforth, we implicitly assume that $\lambda = \chi$, i.e. that λ is the product length, and that the generators $\sigma_1, \ldots, \sigma_n$ of $\mathbf{G}^+$ that we consider are the elements of $\{\mathbf{a} \in \mathbf{G}^+ : \chi(\mathbf{a}) = 1\}$. Nevertheless, occasionally, we may still consider different lengths than the product length, and will make it explicit when such situations occur. This may happen, for instance, when looking for *additive* length functions, as in the following example.

Example 2.36.
The monoid $\mathbf{M}^+_{2,3} := \langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^3 \rangle^+$ is a Garside monoid, with Garside element $\Delta := \sigma_1^2 = \sigma_2^3$. The product length χ is *not* additive, since $\chi(\sigma_1) = 1$ and $\chi(\sigma_1^2) = 3$. However, there exists an *additive* length function $\lambda : \mathbf{M}^+_{2,3} \mapsto \mathbb{Z}$ defined by $\lambda(\sigma_1) = 3$ and $\lambda(\sigma_2) = 2$.

Lemma 2.37.
Let $\mathbf{G}^+$ *be a Garside monoid. For all elements* $\mathbf{a}$ *and* $\mathbf{b}$ *of the monoid* $\mathbf{G}^+$*, we have* $\alpha_\ell(\mathbf{ab}) = \alpha_\ell(\mathbf{a}\alpha_\ell(\mathbf{b}))$ *and* $\alpha_r(\mathbf{ab}) = \alpha_r(\alpha_r(\mathbf{a})\mathbf{b})$*.*

Proof. First, due to word reversal, we just need to prove that $\alpha_\ell(\mathbf{ab}) = \alpha_\ell(\mathbf{a}\alpha_\ell(\mathbf{b}))$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{G}^+$. Then, observe that α_ℓ is non-decreasing for the ordering $\leqslant_\ell$. Since $\mathbf{a}\alpha_\ell(\mathbf{b}) \leqslant_\ell \mathbf{ab}$, it follows that $\alpha_\ell(\mathbf{a}\alpha_\ell(\mathbf{b})) \leqslant_\ell \alpha_\ell(\mathbf{ab})$. Conversely, let us prove, by induction over the length $\lambda(\mathbf{a})$, that $\alpha_\ell(\mathbf{ab}) \leqslant_\ell \mathbf{a}\alpha_\ell(\mathbf{b})$.

If $\lambda(\mathbf{a}) = 1$, then $\mathbf{a}$ is simple. Hence, $\mathbf{a} \leqslant_\ell \alpha_\ell(\mathbf{ab})$, and the braid $\mathbf{c} := \mathbf{a}^{-1}\alpha_\ell(\mathbf{ab})$ belongs to $\mathbf{G}^+$. Since $\mathcal{S}$ is closed under divisibility, we know that $\mathbf{c} \in \mathcal{S}$. Moreover, note that $\mathbf{ac} = \alpha_\ell(\mathbf{ab}) \leqslant_\ell \mathbf{ab}$, whence $\mathbf{c} \leqslant_\ell \mathbf{b}$. It follows that $\mathbf{c} \leqslant_\ell \alpha_\ell(\mathbf{b})$, whence $\alpha_\ell(\mathbf{ab}) \leqslant_\ell \mathbf{a}\alpha_\ell(\mathbf{b})$. Now, if $\lambda(\mathbf{a}) \geqslant 2$, let us factor $\mathbf{a}$ as a product $\mathbf{a} = \sigma_i\mathbf{d}$, where σ_i is a generator of the monoid $\mathbf{G}^+$ (and therefore $\lambda(\sigma_i) = 1$. The induction hypothesis shows successively that $\alpha_\ell(\sigma_i\mathbf{db}) \leqslant_\ell \sigma_i\alpha_\ell(\mathbf{db})$, because $\lambda(\sigma_i) = 1$, and that $\alpha_\ell(\mathbf{db}) \leqslant_\ell \mathbf{d}\alpha_\ell(\mathbf{b})$, because $\lambda(\mathbf{d}) < \lambda(\mathbf{a})$. It follows that

$$\alpha_\ell(\mathbf{ab}) = \alpha_\ell(\sigma_i\mathbf{db}) \leqslant_\ell \sigma_i\alpha_\ell(\mathbf{db}) \leqslant_\ell \sigma_i\mathbf{d}\alpha_\ell(\mathbf{b}) = \mathbf{a}\alpha_\ell(\mathbf{b}),$$

which concludes the induction and completes the proof. □

Corollary 2.38.
Let $\mathbf{G}^+$ *be a Garside monoid, let* $\mathcal{S}$ *be the set of simple elements of* $\mathbf{G}^+$*, and* $\underline{\mathbf{b}} := b_1 \cdot b_2 \cdot \ldots \cdot b_k$ *be a word in* $\mathcal{S}\backslash\{\mathbf{1}\}$*, i.e. whose letters are non-trivial simple elements of* $\mathbf{G}^+$*. The word* $\underline{\mathbf{b}}$ *is a left Garside word if and only if the words* $b_i \cdot b_{i+1}$ *are left Garside words for all* $i \in \{1, \ldots, k-1\}$*. Analogously, the word* $\underline{\mathbf{b}}$ *is a right Garside word if and only if the words* $b_i \cdot b_{i+1}$ *are right Garside words for all* $i \in \{1, \ldots, k-1\}$*.*

Proof. Once again, we focus on proving the fact that $\underline{\mathbf{b}}$ is a left Garside word if and only if the words $b_i \cdot b_{i+1}$ are left Garside words for all $i \in \{1, \ldots, k-1\}$. First, if either the word $\underline{\mathbf{b}}$ or all of the words $b_i \cdot b_{i+1}$ are left Garside, then none of the letters b_i is equal to $\mathbf{1}$, for $i \in \{1, \ldots, k\}$. Second, if $\underline{\mathbf{b}}$ is left Garside, then we have

$$\alpha_\ell(b_i b_{i+1}) = \alpha_\ell(b_i\alpha_\ell(b_{i+1} \ldots b_k)) = \alpha_\ell(b_i \ldots b_k) = b_i$$

for all $i \in \{1, \ldots, k-1\}$, which proves that the words $b_i \cdot b_{i+1}$ are all left Garside. Conversely, if each word $b_i \cdot b_{i+1}$ is left Garside, then an immediate downward induction on $i \in \{1, \ldots, k-1\}$ proves that

$$\alpha_\ell(b_i \ldots b_k) = \alpha_\ell(b_i\alpha_\ell(b_{i+1} \ldots b_k)) = \alpha_\ell(b_i b_{i+1}) = b_i,$$

which completes the proof. □

Hence, when $\mathbf{a}$ and $\mathbf{b}$ are simple elements, we denote by $\mathbf{a} \longrightarrow \mathbf{b}$ the fact that $\mathbf{a} \cdot \mathbf{b}$ is a left Garside word, and by $\mathbf{a} \longleftarrow \mathbf{b}$ the fact that $\mathbf{a} \cdot \mathbf{b}$ is a right Garside word. Corollary 2.38 amounts to saying that the sets of left Garside words and of right Garside words are regular languages. They consist respectively in the words $b_1 \cdot \ldots \cdot b_k$ such that $b_1 \longrightarrow \ldots \longrightarrow b_k$ and such that $b_1 \longleftarrow \ldots \longleftarrow b_k$.

In addition, following Lemma 2.37, there exists simple characterisations of the relations $\mathbf{a} \longrightarrow \mathbf{b}$ and $\mathbf{a} \longleftarrow \mathbf{b}$.

Definition 2.39 (Left and right (outgoing) sets)**.**
Let $\mathbf{G}^+$ be a Garside monoid with generators $\sigma_1, \ldots, \sigma_n$, and let $\mathbf{a}$ be an element of $\mathbf{G}^+$. We call left set *of $\mathbf{a}$, and denote by $\mathbf{left}(\mathbf{a})$, the set $\{\sigma_i : \sigma_i \leqslant_\ell \mathbf{a}\}$. We also call* right set *of $\mathbf{a}$, and denote by $\mathbf{right}(\mathbf{a})$, the set $\{\sigma_i : \mathbf{a} \geqslant_r \sigma_i\}$.*

In addition, if $\mathbf{a}$ is a simple element, we call left outgoing set *of $\mathbf{a}$, and denote by $\overline{\mathbf{left}}(\mathbf{a})$, the set $\{\sigma_i : \sigma_i\mathbf{a} \notin \mathcal{S}\}$. We also call* right outgoing set *of $\mathbf{a}$, and denote by $\overline{\mathbf{right}}(\mathbf{a})$, the set $\{\sigma_i : \mathbf{a}\sigma_i \notin \mathcal{S}\}$.*

Lemma 2.40.
Let $\mathbf{G}^+$ be a Garside monoid, and let $\mathbf{a}$ and $\mathbf{b}$ be two non-trivial, simple elements of $\mathbf{G}^+$. We have $\mathbf{a} \longrightarrow \mathbf{b}$ if and only if $\mathbf{left}(\mathbf{b}) \subseteq \overline{\mathbf{right}}(\mathbf{a})$, and $\mathbf{a} \longleftarrow \mathbf{b}$ if and only if $\mathbf{right}(\mathbf{a}) \subseteq \overline{\mathbf{left}}(\mathbf{b})$.

Proof. Since both statements of Lemma 2.40 are dual to each other, we focus on proving that $\mathbf{a} \longrightarrow \mathbf{b}$ if and only if $\mathbf{left}(\mathbf{b}) \subseteq \overline{\mathbf{right}}(\mathbf{a})$. First, if $\mathbf{left}(\mathbf{b}) \nsubseteq \overline{\mathbf{right}}(\mathbf{a})$, consider some generator $\sigma_i \in \mathbf{left}(\mathbf{b}) \backslash \overline{\mathbf{right}}(\mathbf{a})$. Since $\mathbf{a}\sigma_i$ left-divides both Δ and $\mathbf{ab}$, it follows that $\mathbf{a}\sigma_i \leqslant \alpha_\ell(\mathbf{ab})$, whence $\mathbf{a} \neq \alpha_\ell(\mathbf{ab})$ and $\mathbf{a} \not\longrightarrow \mathbf{b}$.

Conversely, if $\mathbf{a} \not\longrightarrow \mathbf{b}$, i.e. $\mathbf{a} \neq \alpha_\ell(\mathbf{ab})$, recall that $\mathbf{a} \leqslant_\ell \alpha_\ell(\mathbf{ab})$. Therefore, there exists a generator σ_i such that $\mathbf{a}\sigma_i \leqslant_\ell \alpha_\ell(\mathbf{ab})$. This means both that $\mathbf{a}\sigma_i \leqslant_\ell \Delta$ and that $\mathbf{a}\sigma_i \leqslant_\ell \mathbf{ab}$, i.e. that $\sigma_i \in \mathbf{left}(\mathbf{b}) \backslash \overline{\mathbf{right}}(\mathbf{a})$. This completes the proof. □

A first consequence of Lemma 2.40 is a condition for concatenating Garside words.

Corollary 2.41.
Let $\mathbf{G}^+$ be a Garside monoid, and let $\mathbf{a}$ and $\mathbf{b}$ be two non-trivial elements of $\mathbf{G}^+$, with respective left Garside normal forms $\underline{\mathbf{a}}^\ell := a_1^\ell \cdot a_2^\ell \cdot \ldots \cdot a_u^\ell$ and $\underline{\mathbf{b}}^\ell := b_1^\ell \cdot b_2^\ell \cdot \ldots \cdot b_v^\ell$. The equality $\mathbf{NF}_\ell(\mathbf{ab}) = \underline{\mathbf{a}}^\ell \cdot \underline{\mathbf{b}}^\ell$ holds if and only if $a_u^\ell \longrightarrow b_1^\ell$ or, equivalently, if $\mathbf{left}(\mathbf{b}) \subseteq \overline{\mathbf{right}}(a_u^\ell)$.

Similarly, let $\underline{\mathbf{a}}^r := a_1^r \cdot a_2^r \cdot \ldots \cdot a_u^r$ and $\underline{\mathbf{b}}^r := b_1^r \cdot b_2^r \cdot \ldots \cdot b_v^r$ be the right Garside normal forms of $\mathbf{a}$ and $\mathbf{b}$. The equality $\mathbf{NF}_r(\mathbf{ab}) = \underline{\mathbf{a}}^r \cdot \underline{\mathbf{b}}^r$ holds if and only if $a_u^r \longrightarrow b_1^r$ or, equivalently, if $\overline{\mathbf{left}}(b_1^r) \supseteq \mathbf{right}(\mathbf{a})$.

Proof. The only point that does not follow directly from Corollary 2.38 is the fact that $\mathbf{left}(\mathbf{b}) = \mathbf{left}(b_1^\ell)$ and that $\mathbf{right}(\mathbf{a}) = \mathbf{right}(a_u^r)$. These equalities are direct consequences from the definitions of $b_1^\ell = \alpha_\ell(\mathbf{b})$ and $a_u^r = \alpha_r(\mathbf{a})$, and from the fact that each generator of the monoid $\mathbf{A}^+$ is both a left-divisor and a right-divisor of the Garside element Δ. □

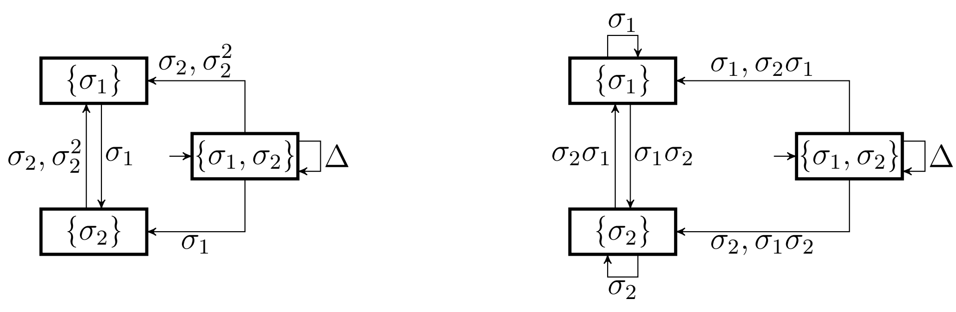
In addition, Lemma 2.40 provides us with an automaton that accepts the set of left Garside words.

Definition & Proposition 2.42 (Left Garside acceptor automaton)**.**
Let $\mathbf{G}^+$ *be a Garside monoid, with generators* $\sigma_1, \ldots, \sigma_n$*. We call* left Garside acceptor automaton *the finite-state automaton* $\mathcal{A}^{\mathbf{left}} := (A, V, \delta, i_s, V)$*, with*

- *alphabet* $A = \mathcal{S}\backslash\{\mathbf{1}\}$*;*
- *set of states* $V = \{\overline{\mathbf{right}}(\mathbf{a}) : \mathbf{a} \in A\}$*;*
- *transition function* δ *with domain* $\{(P, \mathbf{a}) : \mathbf{left}(\mathbf{a}) \subseteq P\}$ *and such that* $\delta : (P, \mathbf{a}) \mapsto \overline{\mathbf{right}}(\mathbf{a})$ *if* $\mathbf{left}(\mathbf{a}) \subseteq P$*;*
- *initial state* $i_s = \overline{\mathbf{right}}(\Delta)$*;*
- *set of accepting states* V*.*

The automaton $\mathcal{A}^{\mathbf{left}}$ *is the minimal deterministic automaton that accepts the set of left Garside words. In particular, the left Garside normal form is a* regular *normal form, i.e. the set of left Garside words is regular.*

Proof. First, since $\mathbf{left}(\mathbf{a}) \subseteq \{\sigma_1, \ldots, \sigma_n\} = i_s$ for all $\mathbf{a} \in \mathcal{S}\backslash\{\mathbf{1}\}$, it is clear that $\mathcal{A}^{\mathbf{left}}$ accepts the set of words $b_1 \cdot \ldots \cdot b_k$ such that $b_1 \longrightarrow \ldots \longrightarrow b_k$, i.e. the set of left Garside words. Second, each state of $\mathcal{A}^{\mathbf{left}}$ is accessible (in one step) from the state i_s, and is itself an accepting state. Hence, it is enough to prove that the sets $\{\mathbf{a} \in A : \delta(P, \mathbf{a}) \text{ is defined}\}$ are pairwise distinct, for all states P. In particular, if P and Q are distinct states of $\mathcal{A}^{\mathbf{left}}$, let us assume, without loss of generality, that some generator σ_i of the monoid $\mathbf{G}^+$ belongs to $P\backslash Q$. Since $\mathbf{left}(\sigma_i) = \{\sigma_i\}$, the pair (P, σ_i) belongs to the domain of δ, but the pair (Q, σ_i) does not belong to the domain of δ, which proves that $\{\mathbf{a} \in A : \delta(P, \mathbf{a}) \text{ is defined}\} \neq \{\mathbf{a} \in A : \delta(Q, \mathbf{a}) \text{ is defined}\}$. This completes the proof. □



Automaton $\mathcal{A}^{\mathbf{left}}$ of $\mathbf{M}^+_{2,3}$ Automaton $\mathcal{A}^{\mathbf{left}}$ of $\mathbf{B}^+_3$

Figure 2.43 – Left Garside acceptor automata of the monoids $\mathbf{M}^+_{2,3}$ and $\mathbf{B}^+_3$

Figure 2.43 presents the left Garside acceptor automata associated with the Garside monoid $\mathbf{M}^+_{2,3} = \langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^3 \rangle^+$ (introduced in Example 2.36) and the braid monoid $\mathbf{B}^+_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle^+$.

We might also have defined similarly a *right Garside acceptor automaton* that accepts the set of left Garside words.

Definition & Proposition 2.44 (Right Garside acceptor automaton).
Let $\mathbf{G}^+$ be a Garside monoid, with generators $\sigma_1, \ldots, \sigma_n$. We call right Garside acceptor automaton *the finite-state automaton $\mathcal{A}^{\mathbf{right}} := (A, V, \delta, i_s, V)$, with*

- *alphabet $A = \mathcal{S}\backslash\{\mathbf{1}\}$;*
- *set of states $V = \{\varnothing\} \cup \{\overline{\mathbf{right}}(\mathbf{a}) : \mathbf{a} \in A\}$;*
- *transition function δ with domain $\{(P, \mathbf{a}) : P \subseteq \overline{\mathbf{left}}(\mathbf{a})\}$ and such that $\delta : (P, \mathbf{a}) \mapsto \overline{\mathbf{right}}(\mathbf{a})$ if $P \subseteq \overline{\mathbf{left}}(\mathbf{a})$;*
- *initial state $i_s = \varnothing$;*
- *set of accepting states V.*

The automaton $\mathcal{A}^{\mathbf{right}}$ is the minimal deterministic automaton that accepts the set of right Garside words. In particular, the right Garside normal form is a regular *normal form, i.e. the set of right Garside words is regular.*

Proof. Since $i_s = \varnothing \subseteq \overline{\mathbf{left}}(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{S}\backslash\{\mathbf{1}\}$, it is clear that $\mathcal{A}^{\mathbf{right}}$ accepts the set of words $b_1 \cdot \ldots \cdot b_k$ such that $b_1 \longleftarrow \ldots \longleftarrow b_k$, i.e. the set of right Garside words.

Moreover, each state of $\mathcal{A}^{\mathbf{right}}$ is accessible (in at most one step) from the state i_s, and is itself an accepting state. Hence, it is enough to prove that the sets $\{\mathbf{a} \in A : \delta(P, \mathbf{a}) \text{ is defined}\}$ are pairwise distinct, for all states P. If P and Q are distinct states of $\mathcal{A}^{\mathbf{left}}$, let us assume, without loss of generality, that some generator σ_i of the monoid $\mathbf{G}^+$ belongs to $P\backslash Q$. The only elements of the set $\{\mathbf{x} : \Delta \geqslant_r \mathbf{x} \geqslant_r \sigma_i^{-1}\Delta\}$ are $\sigma_i^{-1}\Delta$ and Δ, which proves that $\overline{\mathbf{left}}(\sigma_i^{-1}\Delta) = \{\sigma_1, \ldots, \sigma_n\}\backslash\{\sigma_i\}$. It follows that $(Q, \sigma_i^{-1}\Delta)$ belongs to the domain of δ, but that the pair $(P, \sigma_i^{-1}\Delta)$ does not belong to the domain of δ, which proves that $\{\mathbf{a} \in A : \delta(P, \mathbf{a}) \text{ is defined}\} \neq \{\mathbf{a} \in A : \delta(Q, \mathbf{a}) \text{ is defined}\}$. This completes the proof. □

An alternative way of constructing the right Garside acceptor automaton would have been to use the Brzozowski minisation algorithm [22], i.e. to reverse the left Garside acceptor automaton, then to determinise it (retaining only the accessible and coaccessible part).

Moreover, the left and right Garside normal forms in Garside groups were seminal examples of the notion of *automatic monoid* [23, 45, 47]

Definition 2.45 (Synchronously automatic normal form).
Let $(G, \cdot)$ be a finitely generated monoid, with generating set $\Gamma := \{g_1, \ldots, g_n\}$, and let $\mathbf{NF} : G \mapsto \Gamma^*$ be a regular normal form, which maps each element of G to a finite word with letters in Γ. In addition, let ε be a letter that does not belong to the set Γ, and let us consider the extended alphabet $\Gamma_\varepsilon := \Gamma \cup \{\varepsilon\}$. We say that a word $\underline{\mathbf{w}} \in \Gamma_\varepsilon^*$ reduces *to a word $\underline{\mathbf{x}} \in \Gamma^*$, which we denote by $\underline{\mathbf{w}} \twoheadrightarrow \underline{\mathbf{x}}$, if $\underline{\mathbf{w}} \in \underline{\mathbf{x}} \cdot \{\varepsilon\}^*$.*

We say that the normal form $\mathbf{NF}$ is synchronously left-automatic *if, for all generators g_i of G, the set*

$$\mathbf{NF}_i^{\mathbf{left}} := \{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_\varepsilon^* \times \Gamma_\varepsilon^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists \gamma \in G \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}(g_i\gamma)\}$$

is regular. Similarly, we say that **NF** *is* synchronously right-automatic *if, for all generators g_i, the set*

$$\mathbf{NF}_i^{\mathbf{right}} := \{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_\varepsilon^* \times \Gamma_\varepsilon^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists \gamma \in G \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}(\gamma g_i)\}$$

is regular. Finally, we say that **NF** *is* synchronously automatic *if it is both synchronously left- and right-automatic.*

Synchronous automatic normal forms are crucial for algorithmic purposes. Indeed, if **NF** is a synchronously automatic normal form, computing the normal form of a product $g_i\gamma$ or γg_i, where γ is an element of the group G, can be performed in space and time linear in the length of the word $\mathbf{NF}(\gamma)$.

Note that being an automatic monoid does not depend on the generating set that we consider [45].

Proposition 2.46.
Let $\mathbf{G}^+$ be a Garside monoid. The left and right Garside normal forms of $\mathbf{G}^+$ are synchronously automatic.

Proof. Consider the generating set $\Gamma := \mathcal{S}\backslash\{\mathbf{1}\}$ of $\mathbf{G}^+$, to which we may add a supplementary letter $\mathbf{1}$, thereby obtaining the set $\Gamma_{\mathbf{1}} := \mathcal{S}$. We first prove that the left Garside normal form $\mathbf{NF}_\ell : \mathbf{G}^+ \mapsto \Gamma$ is synchronously left and right automatic.

For all simple elements $\beta \in \mathcal{S}\backslash\{\mathbf{1}\}$, the finite-state automaton $\mathcal{A}_{\mathbf{s}}^{\mathbf{left}} := (A, V, \delta, i_s, F)$, with

- alphabet $A = \mathcal{S} \times \mathcal{S}$;
- set of states $V = \mathcal{S}$;
- transition function δ with domain $\{(\gamma, (\mathbf{a}, \mathbf{b})) \in V \times A : \mathbf{b} = \alpha_\ell(\gamma\mathbf{a})\}$ and such that $\delta : (\gamma, (\mathbf{a}, \mathbf{b})) \mapsto \alpha_\ell(\gamma\mathbf{a})^{-1}\gamma\mathbf{a}$ if $\mathbf{b} = \alpha_\ell(\gamma\mathbf{a})$;
- initial state $i_s = \beta$;
- set of accepting states $F = \{\mathbf{1}\}$

recognises a set $\mathbf{S}$ such that $\mathbf{S} \cap \mathbf{NF}_\ell(\mathbf{G}^+)^2$ is equal to the set

$$\{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_{\mathbf{1}}^* \times \Gamma_{\mathbf{1}}^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists \gamma \in \mathbf{G}^+ \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}_\ell(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}_\ell(\beta\gamma)\},$$

and the finite-state automaton $\mathcal{A}_{\mathbf{s}}^{\mathbf{right}} := (A, V, \delta', i'_s, F')$, with

- alphabet $A = \mathcal{S} \times \mathcal{S}$;
- set of states $V = \mathcal{S}$;
- transition function δ' with domain $\{(\gamma, (\mathbf{a}, \mathbf{b})) \in V \times A : \mathbf{a} \leqslant_\ell \gamma\mathbf{b} \text{ and } \mathbf{a}^{-1}\gamma\mathbf{b} \in \mathcal{S}\}$ and such that $\delta' : (\gamma, (\mathbf{a}, \mathbf{b})) \mapsto \mathbf{a}^{-1}\gamma\mathbf{b}$ if $\mathbf{a} \leqslant_\ell \gamma\mathbf{b}$ and $\mathbf{a}^{-1}\gamma\mathbf{b} \in \mathcal{S}$;
- initial state $i'_s = \mathbf{1}$;
- set of accepting states $F' = \{\mathbf{s}\}$

recognises a set $\mathbf{S}'$ such that $\mathbf{S}' \cap \mathbf{NF}_\ell(\mathbf{G}^+)^2$ is equal to the set

$$\{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_{\mathbf{1}}^* \times \Gamma_{\mathbf{1}}^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists \gamma \in \mathbf{G}^+ \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}_\ell(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}_\ell(\gamma\beta)\}.$$

This proves that the left Garside normal form is synchronously automatic.

Mirrorring all the relations in the monoid $\mathbf{G}^+$ (and therefore replacing the Garside monoid $\mathbf{G}^+$ by its mirror monoid, which is also a Garside monoid) provides the same results for the right Garside normal form. □

Garside words have additional connections with the Garside element Δ, as we show below.

Definition & Proposition 2.47 (Conjugation by Δ)**.**
Let $\mathbf{G}^+$ be a Garside monoid, with Garside element Δ. There exists a conjugation *by Δ in the monoid $\mathbf{G}^+$, i.e. a function $\phi_\Delta : \mathbf{G}^+ \mapsto \mathbf{G}^+$ such that $\Delta\phi_\Delta(\mathbf{a}) = \mathbf{a}\Delta$ for all elements $\mathbf{a} \in \mathbf{G}^+$.*

The function ϕ_Δ is a morphism of monoids, is λ-invariant, and induces permutations of the sets $\{\sigma_1, \ldots, \sigma_n\}$, $\mathcal{S}$ and $\mathbf{G}^+$.

Proof. First, let $\mathbf{a}$ be a simple element of $\mathbf{G}^+$. Since $\mathbf{a} \leqslant_\ell \Delta$, there exists an element $\mathbf{b} \in \mathbf{G}^+$ such that $\mathbf{ab} = \Delta$. Then, $\Delta \geqslant_r \mathbf{b}$, and therefore $\mathbf{b} \in \mathcal{S}$. This proves that there exists a function $\partial_\Delta : \mathcal{S} \mapsto \mathcal{S}$ such that $\mathbf{a}\partial_\Delta(\mathbf{a}) = \Delta$. It follows immediately that $\Delta\partial_\Delta^2(\mathbf{a}) = \mathbf{a}\partial_\Delta(\mathbf{a})\partial_\Delta^2(\mathbf{a}) = \mathbf{a}\Delta$, and therefore that the function $\phi_\Delta = \partial_\Delta^2$ is well-defined on $\mathcal{S}$.

Second, let $\mathbf{b}$ be some element of $\mathbf{G}^+$, and let $\mathbf{b}_1 \ldots \mathbf{b}_k$ be a factorisation of $\mathbf{b}$ into simple elements of $\mathbf{G}^+$ (e.g., in generators σ_i of $\mathbf{G}^+$). It follows immediately that $\Delta\phi_\Delta(\mathbf{b}_1)\phi_\Delta(\mathbf{b}_2)\ldots\phi_\Delta(\mathbf{b}_k) = \mathbf{b}_1\Delta\phi_\Delta(\mathbf{b}_2)\ldots\phi_\Delta(\mathbf{b}_k) = \ldots = \Delta\mathbf{b}$. Hence, the above construction provides us with an element $\phi_\Delta(\mathbf{b}) := \phi_\Delta(\mathbf{b}_1)\phi_\Delta(\mathbf{b}_2)\ldots\phi_\Delta(\mathbf{b}_k)$ such that $\Delta\phi_\Delta(\mathbf{b}) = \mathbf{b}\Delta$. Since $\mathbf{G}^+$ is cancellative, the element $\phi_\Delta(\mathbf{b})$ is unique, and therefore does not depend on which factorisation of $\mathbf{b}$ we considered. In particular, the function $\phi_\Delta : \mathbf{b} \mapsto \phi_\Delta(\mathbf{b}_1)\phi_\Delta(\mathbf{b}_2)\ldots\phi_\Delta(\mathbf{b}_k)$ is well-defined on $\mathbf{G}^+$.

In addition, if $\mathbf{b}_1 \ldots \mathbf{b}_k$ and $\mathbf{c}_1 \ldots \mathbf{c}_\ell$ are two factorisations of two elements $\mathbf{b}, \mathbf{c} \in \mathbf{G}^+$ into simple elements, then $\phi_\Delta(\mathbf{bc}) = \phi_\Delta(\mathbf{b}_1)\ldots\phi_\Delta(\mathbf{b}_k)\phi_\Delta(\mathbf{c}_1)\ldots\phi_\Delta(\mathbf{c}_\ell) = \phi_\Delta(\mathbf{b})\phi_\Delta(\mathbf{c})$, which shows that ϕ_Δ is a morphism of monoids. As such, ϕ_Δ preserves the divisibility relations, and therefore is λ-invariant. Hence, ϕ_Δ maps the sets $\{\sigma_1, \ldots, \sigma_n\}$, $\mathcal{S}$ and $\mathbf{G}^+$ to themselves.

In addition, using word reversal, there must exist a function ϕ_Δ^{-1} such that $\Delta\mathbf{a} = \phi_\Delta^{-1}(\mathbf{a})\Delta$ for all $\mathbf{a} \in \mathbf{G}^+$, and ϕ_Δ^{-1} maps the sets $\{\sigma_1, \ldots, \sigma_n\}$, $\mathcal{S}$ and $\mathbf{G}^+$ to themselves. Since

$$\phi_\Delta^{-1}(\phi_\Delta(\mathbf{a}))\Delta = \Delta\phi_\Delta(\mathbf{a}) = \mathbf{a}\Delta \text{ and } \Delta\phi_\Delta(\phi_\Delta^{-1}(\mathbf{a})) = \phi_\Delta^{-1}(\mathbf{a})\Delta = \Delta\mathbf{a}$$

for all $\mathbf{a} \in \mathbf{G}^+$, it follows that ϕ_Δ^{-1} and ϕ_Δ are inverse bijections, and Proposition 2.47 follows. □

Lemma 2.48.
Let $\mathbf{G}^+$ be a Garside monoid, and let $\mathbf{a}$ be an element of the monoid $\mathbf{G}^+$. We have $\mathbf{NF}_\ell(\phi_\Delta(\mathbf{a})) = \phi_\Delta(\mathbf{NF}_\ell(\mathbf{a}))$ and $\mathbf{NF}_r(\phi_\Delta(\mathbf{a})) = \phi_\Delta(\mathbf{NF}_r(\mathbf{a}))$.

Proof. We focus on proving the first equality. The second one is similar. Observe that $\Delta\phi_\Delta(\Delta) = \Delta^2$, i.e. that $\phi_\Delta(\Delta) = \Delta$, and that ϕ_Δ is an automorphism of the ordered set $(\mathbf{G}^+, \leqslant_\ell)$. It follows that $\alpha_\ell(\phi_\Delta(\mathbf{a})) = \mathbf{GCD}_{\leqslant_\ell}(\phi_\Delta(\mathbf{a}), \phi_\Delta(\Delta)) = \phi_\Delta(\mathbf{GCD}_{\leqslant_\ell}(\mathbf{a}, \Delta)) = \phi_\Delta(\alpha_\ell(\mathbf{a}))$. An immediate induction on the word length $|\mathbf{NF}_\ell(\mathbf{a})|$ completes the proof of the equality $\mathbf{NF}_\ell(\phi_\Delta(\mathbf{a})) = \phi_\Delta(\mathbf{NF}_\ell(\mathbf{a}))$. □

In addition, divisibility relations involving powers of the Garside element Δ are tightly connected with the length of Garside words and occurrences of the letter Δ within these words.

Proposition 2.49.
Let $\mathbf{G}^+$ be a Garside monoid, let Δ be a Garside element of $\mathbf{G}^+$, and let $\mathbf{NF}_\ell$ and $\mathbf{NF}_r$ be the associated Garside normal forms. In addition, let u be a non-negative integer and let $\mathbf{b}$ be an element of the monoid $\mathbf{G}^+$. The following statements are equivalent:

1. *$\mathbf{b}$ is a product of u simple elements;*
2. *$\mathbf{b} \leqslant_\ell \Delta^u$;*
3. *$\Delta^u \geqslant_r \mathbf{b}$;*
4. *$|\mathbf{NF}_\ell(\mathbf{b})| \leqslant u$;*
5. *$|\mathbf{NF}_r(\mathbf{b})| \leqslant u$.*

In particular, Δ^u is a Garside element of $\mathbf{G}^+$. The following statements are also equivalent:

6. *$\Delta^u \leqslant_\ell \mathbf{b}$;*
7. *$\mathbf{b} \geqslant_r \Delta^u$;*
8. *$(\Delta)^u \lhd \mathbf{NF}_\ell(\mathbf{b})$;*
9. *$\mathbf{NF}_r(\mathbf{b}) \rhd (\Delta)^u$,*

where $(\Delta)^u$ denotes the word $\Delta \cdot \ldots \cdot \Delta$ with u letters.

Proof. First, the implication $4 \Rightarrow 1$ is immediate (recall that, since $\mathbf{1}$ is a simple elem. In addition, recall the function $\partial_\Delta : \mathcal{S} \mapsto \mathcal{S}$ such that $\mathbf{a}\partial_\Delta(\mathbf{a}) = \Delta$. If $\mathbf{b}$ is a product $\mathbf{b} := \mathbf{s}_1\mathbf{s}_2 \ldots \mathbf{s}_u$ with $\mathbf{s}_i \in \mathcal{S}$ for all $i \in \{1, \ldots, u\}$, let us define the elements $\mathbf{s}_i^* := \phi_\Delta^{u-i}\partial_\Delta(\mathbf{s}_i)$. It immediately comes that $\mathbf{b}\mathbf{s}_u^*\mathbf{s}_{u-1}^* \cdot \ldots \cdot \mathbf{s}_1^* = \Delta^u$, which proves that $1 \Rightarrow 2$.

Second, we show by induction on $|\mathbf{NF}_\ell(\mathbf{b})|$ that $|\mathbf{NF}_\ell(\mathbf{b})| \leqslant |\mathbf{NF}_\ell(\mathbf{ab})|$ for all pairs $(\mathbf{a}, \mathbf{b}) \in \mathcal{S} \times \mathbf{G}^+$. Indeed, consider the word $b_1 \cdot b_2 \cdot \ldots \cdot b_k := \mathbf{NF}_\ell(\mathbf{b})$. Lemma 2.37 proves that $\mathbf{a} \leqslant_\ell \alpha_\ell(\mathbf{ab}) \leqslant_\ell \mathbf{a}\alpha_\ell(\mathbf{b}) = \mathbf{a}b_1$: let $\mathbf{c}$ and $\mathbf{d}$ be elements of $\mathbf{G}^+$ such that $\alpha_\ell(\mathbf{ab}) = \mathbf{ac}$ and $\mathbf{a}b_1 = \alpha_\ell(\mathbf{ab})\mathbf{d}$. Since $\mathbf{a}b_1 = \mathbf{acd}$, it follows that $\mathbf{d}$ is a right-divisor of b_1, hence is simple. The induction hypothesis indicates then that

$$|\mathbf{NF}_\ell(\mathbf{ab})| = |\alpha_\ell(\mathbf{ab}) \cdot \mathbf{NF}_\ell(\mathbf{d}b_2b_3 \ldots b_k)| = 1 + |\mathbf{NF}_\ell(\mathbf{d}b_2b_3 \ldots b_k)| \geqslant |\mathbf{NF}_\ell(\mathbf{b})|,$$

which completes the induction.

It follows that $\mathbf{b} \mapsto |\mathbf{NF}_\ell(\mathbf{b})|$ is non-decreasing for $\geqslant_r$, which proves that $3 \Rightarrow 4$. One shows analogously that $2 \Rightarrow 5 \Rightarrow 1 \Rightarrow 3$.

In addition, the implications $8 \Rightarrow 6$ and $9 \Rightarrow 7$ are immediate, while the converse implications $6 \Rightarrow 8$ and $7 \Rightarrow 9$ follow from a straightforward induction on u. Finally, the equivalence $6 \Leftrightarrow 7$ is due to the fact that $\Delta^u \mathbf{c} = \phi_\Delta^{-u}(\mathbf{c})\Delta^u$ and $\mathbf{c}\Delta^u = \Delta^u \phi_\Delta^u(\mathbf{c})$ for all elements $\mathbf{c} \in \mathbf{G}^+$. $\square$

Corollary 2.50.
Let $\mathbf{G}^+$ be a Garside monoid, and let $\mathbf{b}$ be an element of the monoid $\mathbf{G}^+$. The Garside words $\mathbf{NF}_\ell(\mathbf{b})$ and $\mathbf{NF}_r(\mathbf{b})$ have the same length, i.e. $|\mathbf{NF}_\ell(\mathbf{b})| = |\mathbf{NF}_r(\mathbf{b})|$.

The equivalence between 2 and 3 in Proposition 2.49 shows that each element Δ^u is itself a Garside element of the monoid $\mathbf{G}^+$. Hence, we will use below the short-cut notation $\succcurlyeq$, which we define as follows: $\Delta^u \succcurlyeq \mathbf{b}$ if and only if $\Delta^u \geqslant_r \mathbf{b}$ or, equivalently, if and only if $\mathbf{b} \leqslant_\ell \Delta^u$; and $\Delta^u \preccurlyeq \mathbf{b}$ if and only if $\Delta^u \leqslant_\ell \mathbf{b}$ or, equivalently, if and only if $\mathbf{b} \geqslant_r \Delta^u$. We call *infimum* of $\mathbf{b}$, and denote by $\inf(\mathbf{b})$, the largest integer u such that $\Delta^u \preccurlyeq \mathbf{b}$.

In addition, since the lengths $|\mathbf{NF}_\ell(\mathbf{b})|$ and $|\mathbf{NF}_r(\mathbf{b})|$ are equal, we henceforth denote by $\|\mathbf{b}\|$ this common length. An additional consequence of Proposition 2.49 is the following one.

Corollary 2.51.
Let $\mathbf{G}^+$ be a Garside monoid, with Garside element Δ. Let u be a non-negative integer and let $\mathbf{b}$ be an element of the monoid $\mathbf{G}^+$. In addition, let $b_1 \cdot \ldots \cdot b_k$ be the left Garside normal form of $\mathbf{b}$, and let $c_{-k} \cdot \ldots \cdot c_1$ be the right Garside normal form of $\mathbf{b}$, with $k := \|\mathbf{b}\|$. We have $\mathbf{GCD}_{\leqslant_\ell}(\mathbf{b}, \Delta^u) = b_1 b_2 \ldots b_{\min\{k,u\}}$, and $\mathbf{GCD}_{\geqslant_r}(\mathbf{b}, \Delta^u) = c_{-\min\{k,u\}} \ldots c_{-2}c_{-1}$.

Proof. If $u \geqslant k$, then $\mathbf{b} \preccurlyeq \Delta^u$, and therefore Corollary 2.51 is immediate. Hence, we assume that $u < k$.

First, Proposition 2.49 proves that $b_1 b_2 \ldots b_u \leqslant_\ell \mathbf{GCD}_{\leqslant_\ell}(\mathbf{b}, \Delta^u)$. Now, let m be some generator of $\mathbf{G}^+$. If $m \leqslant_\ell b_{u+1} \ldots b_k$, then $b_u \leqslant_\ell \alpha_\ell(b_u m) \leqslant_\ell \alpha_\ell(b_u \ldots b_k) = b_u$, hence $b_u \cdot m$ is a left Garside word, and Corollary 2.38 proves that $b_1 \cdot \ldots \cdot b_u \cdot m$ is also a left Garside word. Therefore, Proposition 2.49 proves that $b_1 b_2 \ldots b_u m$ does not divide Δ^u. Likewise, if m is not a left-divisor of $b_{u+1} \ldots b_k$, then $b_1 b_2 \ldots b_u m$ does not divide $\mathbf{b}$.

Hence, in both cases, we know that $b_1 b_2 \ldots b_u m$ cannot divide $\mathbf{GCD}_{\leqslant_\ell}(\mathbf{b}, \Delta^u)$, and therefore that $b_1 b_2 \ldots b_u = \mathbf{GCD}_{\leqslant_\ell}(\mathbf{b}, \Delta^u)$. We show similarly that $c_{-u} \ldots c_{-2}c_{-1} = \mathbf{GCD}_{\geqslant_r}(\mathbf{b}, \Delta^u)$, which completes the proof. $\square$

We also say that a word $\underline{\mathbf{w}}$ whose letters belong to the set $\mathcal{S}^\circ := \mathcal{S} \backslash \{\mathbf{1}, \Delta\}$ is a Δ-*free word*, and that an element $\mathbf{b}$ of $\mathbf{G}^+$ that is not divisible by Δ is Δ-free. Proposition 2.49

proves that the elements whose left Garside normal form (or, equivalently, right Garside normal form) is Δ-free are exactly the Δ-free elements of $\mathbf{G}^+$.

Pushing further in that direction, we wish to work only with Δ-free Garside words, which amounts to cancelling the Δ factors that occur in the factorisation of any element of $\mathbf{G}^+$. This is of course infeasible *per se* (since not all elements of $\mathbf{G}^+$ are Δ-free), but we already can work on assigning to the Δ factors a specific place, either to the left of to the right of the factorisation (by considering left and right Garside normal forms), or "in the middle", as shown by the following result.

Lemma 2.52.
Let $\mathbf{G}^+$ be a Garside monoid, with Garside element Δ, and let u be a non-negative integer. Consider elements $\mathbf{a}$ and $\mathbf{b}$ of the monoid $\mathbf{G}^+$ such that $\mathbf{ab} \geqslant \Delta^u$. There exists elements $\mathbf{a}'$, $\mathbf{a}''$, $\mathbf{b}'$ and $\mathbf{b}''$ of $\mathbf{G}^+$ such that $\mathbf{a} = \mathbf{a}'\mathbf{a}''$, $\mathbf{b} = \mathbf{b}'\mathbf{b}''$ and $\mathbf{a}''\mathbf{b}' = \Delta^u$.

Proof. The claim is obvious for $u = 0$. Hence, we first prove it for $u = 1$. Since $\mathbf{ab} \geqslant \Delta$, it follows that $\Delta = \alpha_r(\mathbf{ab}) = \alpha_r(\alpha_r(\mathbf{a})\mathbf{b})$, i.e. that $\alpha_r(\mathbf{a})\mathbf{b} \geqslant_r \Delta$, and therefore that $\Delta \leqslant_\ell \alpha_r(\mathbf{a})\mathbf{b}$. This implies that

$$\Delta = \alpha_\ell(\alpha_r(\mathbf{a})\mathbf{b}) = \alpha_\ell(\alpha_r(\mathbf{a})\alpha_\ell(\mathbf{b})) \leqslant_\ell \alpha_r(\mathbf{a})\alpha_\ell(\mathbf{b}).$$

This means that $\alpha_r(\mathbf{a})^{-1}\Delta \leqslant_\ell \alpha_\ell(\mathbf{b}) \leqslant_\ell \mathbf{b}$. Defining $\mathbf{a}'' = \alpha_r(\mathbf{a})$ and $\mathbf{b}' = \alpha_r(\mathbf{a})^{-1}\Delta$, then $\mathbf{a}' = \mathbf{a}(\mathbf{a}'')^{-1}$ and $\mathbf{b}'' = (\mathbf{b}')^{-1}\mathbf{b}$ completes the proof.

We proceed now by induction on $u \geqslant 2$. Consider elements $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ of $\mathbf{G}^+$ such that $\mathbf{a} = \mathbf{a}_1\mathbf{a}_2$, $\mathbf{b} = \mathbf{b}_1\mathbf{b}_2$ and $\mathbf{a}_2\mathbf{b}_1 = \Delta$. Since $\phi_\Delta(\mathbf{a}_1)\mathbf{b}_2 \geqslant \Delta^{u-1}$, there exists elements $\mathbf{a}_3, \mathbf{a}_4, \mathbf{b}_3, \mathbf{b}_4$ of $\mathbf{G}^+$ such that $\phi_\Delta(\mathbf{a}_1) = \mathbf{a}_3\mathbf{a}_4$, $\mathbf{b}_2 = \mathbf{b}_3\mathbf{b}_4$ and $\mathbf{a}_4\mathbf{b}_3 = \Delta^{u-1}$. Hence, let us define $\mathbf{a}' = \phi_\Delta^{-1}(\mathbf{a}_3)$, $\mathbf{a}'' = \phi_\Delta^{-1}(\mathbf{a}_4)\mathbf{a}_2$, $\mathbf{b}' = \mathbf{b}_1\mathbf{b}_3$ and $\mathbf{b}'' = \mathbf{b}_4$: we have $\mathbf{a}'\mathbf{a}'' = \phi_\Delta^{-1}(\mathbf{a}_3\mathbf{a}_4)\mathbf{a}_2 = \mathbf{a}_1\mathbf{a}_2 = \mathbf{a}$ and $\mathbf{b}'\mathbf{b}'' = \mathbf{b}_1(\mathbf{b}_3\mathbf{b}_4) = \mathbf{b}_1\mathbf{b}_2 = \mathbf{b}$, as well as $\mathbf{a}''\mathbf{b}' = \phi_\Delta^{-1}(\mathbf{a}_4)\mathbf{a}_2\mathbf{b}_1\mathbf{b}_3 = \phi_\Delta^{-1}(\mathbf{a}_4)\Delta\mathbf{b}_3 = \Delta\mathbf{a}_4\mathbf{b}_3 = \Delta^u$. □

In addition, the Garside monoids can be embedded in their group of fractions, as follows.

Definition & Proposition 2.53 (Garside group)**.**
Let $\mathbf{G}^+$ be a Garside monoid, and consider the group morphism

$$\begin{array}{rcl} \Phi : \mathbb{Z} & \mapsto & \mathrm{Aut}(\mathbf{G}^+). \\ k & \mapsto & \phi_\Delta^k \end{array}$$

The morphism Φ gives rise to the semidirect product $\mathbf{G}^+ \rtimes_\Phi \mathbb{Z}$, such that $(\mathbf{a}, k) \cdot (\mathbf{b}, \ell) := (\mathbf{a}\Phi_\Delta^k(\mathbf{b}), k + \ell)$.

For all integers $k \geqslant 0$, the pair (Δ^k, k) is central in $\mathbf{G}^+ \rtimes_\Phi \mathbb{Z}$, so that the submonoid $\mathbf{N} := \{(\Delta^k, k) : k \geqslant 0\}$ is normal in $\mathbf{G}^+ \rtimes_\Phi \mathbb{Z}$. The quotient monoid $\mathbf{G} := (\mathbf{G}^+ \rtimes_\Phi \mathbb{Z})/\mathbf{N}$ is the group of fractions *of $\mathbf{G}^+$. Moreover, each element $\mathbf{x}$ of $\mathbf{G}$ can be written as a product $\mathbf{x} = \mathbf{y}^{-1}\mathbf{z}$, with $\mathbf{y}, \mathbf{z} \in \mathbf{G}^+$. We say that $\mathbf{G}$ is the* Garside group *associated with $\mathbf{G}^+$.*

Proof. First, recall that $\phi_\Delta(\Delta) = \Delta$. Hence, a direct computation shows that, for all $(\mathbf{a}, k, \ell) \in \mathbf{G}^+ \times \mathbb{Z} \times \mathbb{Z}$, we have $(\mathbf{a}, k) \cdot (\Delta^\ell, \ell) = (\mathbf{a}\Delta^\ell, k + \ell) = (\Delta^\ell \phi_\Delta^\ell(\mathbf{a}), k + \ell) = (\Delta^\ell, \ell) \cdot (\mathbf{a}, k)$, which proves that (Δ^ℓ, ℓ) is central in $\mathbf{G}^+ \rtimes_\Phi \mathbb{Z}$ and therefore that $\mathbf{N} := \{(\Delta^k, k) : k \geqslant 0\}$ is normal in $\mathbf{G}^+ \rtimes_\Phi \mathbb{Z}$.

Second, for all $(\mathbf{a}, k) \in \mathbf{G}^+ \times \mathbb{Z}$, since $\mathbf{a} \leqslant_\ell \Delta^{\lambda(\mathbf{a})}$, there exists an element $\mathbf{b} \in \mathbf{G}^+$ such that $\mathbf{ab} = \Delta^{\lambda(\mathbf{a})}$. It follows that

$$(\mathbf{a}, k) \cdot (\phi_\Delta^{-k}(\mathbf{b}), \lambda(\mathbf{a}) - k) = (\mathbf{ab}, k + \lambda(\mathbf{a}) - k) = (\Delta^{\lambda(\mathbf{a})}, \lambda(\mathbf{a})) \in \mathbf{N},$$

hence each element $\mathbf{x}$ of $\mathbf{G}$ has a right inverse, which we denote by $\mathbf{x}^{-1}$. Since $\mathbf{x}^{-1}\mathbf{x} = \mathbf{x}^{-1}\mathbf{x}\mathbf{x}^{-1}(\mathbf{x}^{-1})^{-1} = \mathbf{x}^{-1}(\mathbf{x}^{-1})^{-1} = \mathbf{1}$, it follows that $\mathbf{x}^{-1}$ is also a left inverse of $\mathbf{x}$, i.e. that $\mathbf{G}$ is a group.

Third, identifying the monoid $\mathbf{G}^+$ with the subset $\{(\mathbf{a}, 0) : \mathbf{a} \in \mathbf{G}^+\}$ of the quotient $\mathbf{G}$, we observe that

$$(\mathbf{a}, k) \cdot (\phi_\Delta^{-k}(\mathbf{b})\Delta^{|k|}, 0) = (\mathbf{ab}\Delta^{|k|}, k) = (\mathbf{ab}\Delta^{|k|-k}, 0),$$

where the last equality holds only in the quotient $(\mathbf{G}^+ \rtimes_\Phi \mathbb{Z})/\mathbf{N}$. This proves that each element $\mathbf{x}$ of $\mathbf{G}$ is of the form $\mathbf{x} = \mathbf{y}^{-1}\mathbf{z}$, where $\mathbf{y}$ and $\mathbf{z}$ belong to $\{(\mathbf{a}, 0) : \mathbf{a} \in \mathbf{G}^+\}$, which completes the proof. □

From now on, we will just identify the Garside group $\mathbf{G}$ with the group of fractions of $\mathbf{G}^+$, and assume that $\mathbf{G}^+$ is a submonoid of $\mathbf{G}$. Observe that, analogously, the Artin–Tits group $\mathbf{A}$ is exactly (by definition) the group of fractions of the monoid $\mathbf{A}^+$, hence that $\mathbf{A}$ is the Garside group associated with the Garside monoid $\mathbf{A}^+$.

Each kind of Garside normal forms on $\mathbf{G}^+$ can be generalised to a normal form on the Garside group $\mathbf{G}$.

Definition 2.54 (Garside normal forms in the group $\mathbf{G}$)**.**
Let $\mathbf{a}$ be an element of the Garside group $\mathbf{G}$, and let u be the largest integer such that $\Delta^{-u}\mathbf{a}$ belongs to the monoid $\mathbf{G}^+$. Let us also define the integer $\zeta := 1$ if $u \geqslant 0$, or $\zeta := -1$ if $u < 0$. Let $\mathbf{a}_1 \cdot \ldots \cdot \mathbf{a}_k$ be the left Garside normal form of $\Delta^{-u}\mathbf{a}$, and let $\mathbf{a}'_1 \cdot \ldots \cdot \mathbf{a}'_k$ be the right Garside normal form of $\Delta^{-u}\mathbf{a}$ (in the sense of Definition 2.34).

The left Garside normal form *of $\mathbf{a}$ in the group $\mathbf{G}$ is defined as the word $\mathbf{NF}_\ell(\mathbf{a}) := \Delta^\zeta \cdot \ldots \cdot \Delta^\zeta \cdot \mathbf{a}_1 \cdot \ldots \cdot \mathbf{a}_k$, with $|u|$ occurrences of the letter Δ^ζ. The* right Garside normal form *of $\mathbf{a}$ in the group $\mathbf{G}$ is defined as the word $\mathbf{NF}_r(\mathbf{a}) := \mathbf{a}'_1 \cdot \ldots \cdot \mathbf{a}'_k \cdot \Delta^\zeta \cdot \ldots \cdot \Delta^\zeta$, with $|u|$ occurrences of the letter Δ^ζ.*

Like in the case of Garside monoids, the Garside normal forms are synchronously automatic in Garside groups [47].

Proposition 2.55.
Let $\mathbf{G}$ be a Garside monoid. The left and right Garside normal forms of $\mathbf{G}$ are synchronously automatic.

Proof. Like in the monoid case, we consider the generating set $\Gamma := \{\Delta^{-1}\} \cup \mathcal{S} \backslash \{\mathbf{1}\}$ of $\mathbf{G}$, to which we may add a supplementary letter $\mathbf{1}$, thereby obtaining the set $\Gamma_{\mathbf{1}} := \{\Delta^{-1}\} \cup \mathcal{S}$, and first prove that the left Garside normal form $\mathbf{NF}_\ell : \mathbf{G} \mapsto \Gamma$ is synchronously left automatic.

For all simple elements $\beta \in \mathcal{S}$ and all Δ-free elements γ of the monoid $\mathbf{G}^+$, we have $0 \leqslant \inf(\beta\gamma) \leqslant 1$. It follows that the set

$$\{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_{\mathbf{1}}^* \times \Gamma_{\mathbf{1}}^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists \gamma \in \mathbf{G} \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}_\ell(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}_\ell(\beta\gamma)\}$$

is equal to the union of the sets

$$\begin{aligned}
&\{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_{\mathbf{1}}^* \times \Gamma_{\mathbf{1}}^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists \gamma \in \mathbf{G}^+ \text{ s.t. } (\underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}_\ell(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}_\ell(\beta\gamma))\};\\
&\{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_{\mathbf{1}}^* \times \Gamma_{\mathbf{1}}^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists k \geqslant 1, \exists \gamma \in \mathbf{G}^+ \text{ s.t. } \inf(\phi_\Delta^{-k}(\beta)\gamma) = 0,\\
&\qquad \underline{\mathbf{w}} \twoheadrightarrow (\Delta^{-1})^k \cdot \mathbf{NF}_\ell(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow (\Delta^{-1})^k \cdot \mathbf{NF}_\ell(\phi_\Delta^{-k}(\beta)\gamma)\};\\
&\{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_{\mathbf{1}}^* \times \Gamma_{\mathbf{1}}^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists k \geqslant 0, \exists \gamma \in \mathbf{G}^+ \text{ s.t. } \inf(\Delta\phi_\Delta^{-k}(\beta)^{-1}\gamma) = 0,\\
&\qquad \underline{\mathbf{w}} \twoheadrightarrow (\Delta^{-1})^{k+1} \cdot \mathbf{NF}_\ell(\Delta\phi_\Delta^{-k}(\beta)^{-1}\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow (\Delta^{-1})^k \cdot \mathbf{NF}_\ell(\gamma)\},
\end{aligned}$$

which, as a consequence of Proposition 2.46, are regular.

We prove in a similar way that $\mathbf{NF}_\ell$ is synchronously right automatic and that $\mathbf{NF}_r$ is synchronously automatic. □

Note that we might also have proved Proposition 2.55 by referring to the following standard result [47].

Definition 2.56 (Incremental difference sets)**.**
Let G be a finitely generated group, with generating set $\Gamma := \{g_1, \ldots, g_n\}$, and let $\mathbf{NF} : G \mapsto \Gamma^*$ be a normal form. Consider two elements γ and γ' of G, with respective normal forms $\mathbf{NF}(\gamma) = a_1 \cdot \ldots \cdot a_k$ and $\mathbf{NF}(\gamma') = a'_1 \cdot \ldots \cdot a'_\ell$.

The left incremental difference set *of γ and γ' for the normal form $\mathbf{NF}$ is defined as the set*

$$\Delta_{\mathbf{NF}}^{\mathbf{left}}(\gamma, \gamma') := \{(a_{\min\{k+1,i\}} \ldots a_k)(a'_{\min\{\ell+1,i\}} \ldots a'_\ell)^{-1} : i \geqslant 1\}.$$

In addition, the left incremental difference set *of the normal form $\mathbf{NF}$ is defined as the set*

$$\Delta_{\mathbf{NF}}^{\mathbf{left}} := \bigcup_{\gamma \in G} \bigcup_{i=1}^{n} \Delta_{\mathbf{NF}}^{\mathbf{left}}(g_i\gamma, \gamma).$$

Similarly, the right incremental difference set *of γ and γ' for the normal form $\mathbf{NF}$ is defined as the set*

$$\Delta_{\mathbf{NF}}^{\mathbf{right}}(\gamma, \gamma') := \{(a_1 \ldots a_{\min\{k,i\}})^{-1}(a'_1 \ldots a'_{\min\{\ell,i\}}) : i \geqslant 1\}$$

and the right incremental difference set *of the normal form $\mathbf{NF}$ is defined as the set*

$$\Delta_{\mathbf{NF}}^{\mathbf{right}} := \bigcup_{\gamma \in G} \bigcup_{i=1}^{n} \Delta_{\mathbf{NF}}^{\mathbf{right}}(\gamma, \gamma g_i).$$

Theorem 2.57.
Let G be a finitely generated group, with generating set $\Gamma := \{g_1, \ldots, g_n\}$, and let $\mathbf{NF} : G \mapsto \Gamma^*$ be a regular normal form. The normal form $\mathbf{NF}$ is synchronously left-automatic if and only if the left incremental difference set $\Delta^{\mathbf{left}}_{\mathbf{NF}}$ is finite, and $\mathbf{NF}$ is synchronously right-automatic if and only if the right incremental difference set $\Delta^{\mathbf{right}}_{\mathbf{NF}}$ is finite.

Proof. Let ε be a copy of the trivial element of G that does not belong to Γ. We first prove that, for each generator $g_i \in \Gamma$, the set

$$\mathbf{S}^{\mathbf{left}}_i := \{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma^*_\varepsilon \times \Gamma^*_\varepsilon : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists \gamma \in G \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}(g_i\gamma)\}$$

is regular if and only if the set $\Delta^{\mathbf{left}}_i := \bigcup_{\gamma \in G} \Delta^{\mathbf{left}}_{\mathbf{NF}}(g_i\gamma, \gamma)$ is finite.

Indeed, if $\mathbf{S}^{\mathbf{left}}_i$ is recognised by some minimal deterministic automaton $\mathcal{A} = (\Gamma_\varepsilon \times \Gamma_\varepsilon, V, \delta, \mathbf{s}, F)$, consider some element γ of G, and let $\underline{\mathbf{w}}$ and $\underline{\mathbf{x}}$ be two elements of Γ^*_ε such that $|\underline{\mathbf{w}}| = |\underline{\mathbf{x}}|$, $\underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}(\gamma)$ and $\underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}(g_i\gamma)$. In addition, for all non-negative integers $k \leqslant |\underline{\mathbf{x}}|$, let $s_k \in V$ be the state of $\mathcal{A}$ obtained after having read the k (pairs of) letters $(\underline{\mathbf{w}}_1, \underline{\mathbf{x}}_1), \ldots, (\underline{\mathbf{w}}_k, \underline{\mathbf{x}}_k)$. There exists a path of length $\ell \leqslant |V|$ from the initial state i to the state s_k, i.e. there exists ℓ pairs of letters $(\underline{\mathbf{w}}'_1, \underline{\mathbf{x}}'_1), \ldots, (\underline{\mathbf{w}}'_\ell, \underline{\mathbf{x}}'_\ell)$ such that the pair $(\underline{\mathbf{w}}'_1 \cdot \ldots \cdot \underline{\mathbf{w}}'_\ell \cdot \underline{\mathbf{w}}_{k+1} \cdot \ldots \cdot \underline{\mathbf{w}}_{|\underline{\mathbf{w}}|}, \underline{\mathbf{x}}'_1 \cdot \ldots \cdot \underline{\mathbf{x}}'_\ell \cdot \underline{\mathbf{x}}_{k+1} \cdot \ldots \cdot \underline{\mathbf{x}}_{|\underline{\mathbf{w}}|})$ belongs to $\mathbf{S}^{\mathbf{left}}_i$. It follows that

$$\begin{aligned} g_i &= (\underline{\mathbf{x}}'_1 \ldots \underline{\mathbf{x}}'_\ell \underline{\mathbf{x}}_{k+1} \ldots \underline{\mathbf{x}}_{|\underline{\mathbf{w}}|})(\underline{\mathbf{w}}'_1 \ldots \underline{\mathbf{w}}'_\ell \underline{\mathbf{w}}_{k+1} \ldots \underline{\mathbf{w}}_{|\underline{\mathbf{w}}|})^{-1} \\ &= (\underline{\mathbf{x}}'_1 \ldots \underline{\mathbf{x}}'_\ell)(\underline{\mathbf{x}}_{k+1} \ldots \underline{\mathbf{x}}_{|\underline{\mathbf{w}}|})(\underline{\mathbf{w}}_{k+1} \ldots \underline{\mathbf{w}}_{|\underline{\mathbf{w}}|})^{-1}(\underline{\mathbf{w}}'_1 \ldots \underline{\mathbf{w}}'_\ell)^{-1} \end{aligned}$$

and therefore that $(\underline{\mathbf{x}}_{k+1} \ldots \underline{\mathbf{x}}_{|\underline{\mathbf{w}}|})(\underline{\mathbf{w}}_{k+1} \ldots \underline{\mathbf{w}}_{|\underline{\mathbf{w}}|})^{-1} \in \bigcup_{a=0}^{|V|} \bigcup_{b=0}^{|V|} \Gamma^{-a} g_i \Gamma^b$. This proves that the set $\Delta^{\mathbf{left}}_i = \bigcup_{\gamma \in G} \Delta^{\mathbf{left}}_{\mathbf{NF}}(g_i\gamma, \gamma)$ is finite.

Conversely, if the set $\Delta^{\mathbf{left}}_i$ is finite, let $\mathbb{A} = (\Gamma_\varepsilon, W, \epsilon, \mathbf{t}, H)$ be the minimal deterministic automaton that accepts the set $\{\underline{\mathbf{w}} \in \Gamma_\varepsilon : \exists \gamma \in G \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}(\gamma)\}$. Then, consider the automaton $\mathcal{A}_i = (A, V, \delta, \mathbf{s}, F)$, with

- alphabet $A = \Gamma_\varepsilon \times \Gamma_\varepsilon$;
- set of states $V = W \times W \times \Delta^{\mathbf{left}}_i$;
- transition function δ with domain $\{((\mathbf{x}, \mathbf{y}, \zeta), (\mathbf{a}, \mathbf{b})) \in V \times A : (\mathbf{x}, \mathbf{a})$ and $(\mathbf{y}, \mathbf{b})$ belong to the domain of ϵ and $\mathbf{b}^{-1}\zeta\mathbf{a} \in \Delta^{\mathbf{left}}_i\}$ and such that $\delta : ((\mathbf{x}, \mathbf{y}, \zeta), (\mathbf{a}, \mathbf{b})) \mapsto (\epsilon(\mathbf{x}, \mathbf{a}), \epsilon(\mathbf{y}, \mathbf{b}), \mathbf{b}^{-1}\zeta\mathbf{a})$ if $((\mathbf{x}, \mathbf{y}, \zeta), (\mathbf{a}, \mathbf{b}))$ belongs to the domain of δ;
- initial state $\mathbf{s} = (\mathbf{t}, \mathbf{t}, g_i)$;
- set of accepting states $F = H \times H \times \{\mathbf{1}\}$.

One shows easily that $\mathcal{A}_i$ recognises the set $\mathbf{S}^{\mathbf{left}}_i$.

Overall, it follows that $\mathbf{NF}$ is synchronously left-automatic if and only if each set $\Delta^{\mathbf{left}}_i$ is finite, i.e. if and only if the set $\Delta^{\mathbf{left}}_{\mathbf{NF}} = \bigcup_{i=1}^n \Delta^{\mathbf{left}}_i$ is finite. One shows similarly that $\mathbf{NF}$ is synchronously right-automatic if and only if the set $\Delta^{\mathbf{right}}_{\mathbf{NF}}$ is finite. $\square$

Corollary 2.58.
Let G be a finitely generated group, with generating set $\Gamma := \{g_1, \ldots, g_n\}$, and let $\mathbf{NF} : G \mapsto \Gamma^*$ be a computable, regular normal form, whose language is accepted by some fixed automaton $\mathcal{A}$.

If $\mathbf{NF}$ is synchronously left-automatic, then, for all integers $i \in \{1, \ldots, n\}$, the minimal deterministic automaton that accepts the language

$$\mathbf{S}_i^{\mathbf{left}} := \{(\underline{\mathbf{w}}, \underline{\mathbf{x}}) \in \Gamma_\varepsilon^* \times \Gamma_\varepsilon^* : |\underline{\mathbf{w}}| = |\underline{\mathbf{x}}| \text{ and } \exists \gamma \in G \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}(\gamma) \text{ and } \underline{\mathbf{x}} \twoheadrightarrow \mathbf{NF}(g_i\gamma)\}$$

is computable. Similarly, if $\mathbf{NF}$ is right-automatic, then, for all integers $i \in \{1, \ldots, n\}$, the minimal deterministic automaton that accepts the language $\mathbf{S}_i^{\mathbf{right}}$ is computable.

Proof. We prove that, if the set $\Delta_i^{\mathbf{left}}$ is finite, then we can compute the automaton $\mathcal{A}_i$ mentioned in the above proof of Theorem 2.57. From the knowledge of $\mathcal{A}$, we can compute a (minimal deterministic) automaton $\mathbb{A} = (\Gamma_\varepsilon, W, \epsilon, \mathbf{t}, H)$ that accepts the set $\mathcal{L} = \{\underline{\mathbf{w}} \in \Gamma_\varepsilon : \exists \gamma \in G \text{ s.t. } \underline{\mathbf{w}} \twoheadrightarrow \mathbf{NF}(\gamma)\}$.

We compute now increasingly large subsets $S^0, S^1, \ldots$ of the incremental difference set $\Delta_i^{\mathbf{left}}$ and automata $\mathbb{A}^0, \mathbb{A}^1, \ldots$ accepting increasingly large subsets $\mathcal{L}^0, \mathcal{L}^1, \ldots$ of $\mathcal{L}$. Note that, since $\Delta_i^{\mathbf{left}}$ is finite, the sequence $S^0, S^1, \ldots$ can contain only finitely many terms. In addition, let D_ϵ be the domain of the transition function ϵ.

We first define $S^0 = \{g_i\}$. Then, for all integers j such that S^j exists, we define the non-deterministic automaton $\mathbb{A}^j = (\Gamma_\varepsilon, V^j, \delta^j, \mathbf{s}, F)$ with

- state set $V^j = W \times W \times S^j$;
- non-deterministic transition function δ^i with domain $V^j \times \Gamma_\varepsilon$ and such that

$$\begin{aligned} \delta^i : ((\mathbf{x}, \mathbf{y}, \zeta), \mathbf{a}) \quad \mapsto \quad & \{(\epsilon(\mathbf{x}, \mathbf{a}), \epsilon(\mathbf{y}, \mathbf{b}), \mathbf{b}^{-1}\zeta\mathbf{a}) : \mathbf{b} \in \Gamma_\varepsilon, \\ & (\mathbf{x}, \mathbf{a}) \in D_\epsilon, (\mathbf{y}, \mathbf{b}) \in D_\epsilon \text{ and } \mathbf{b}^{-1}\zeta\mathbf{a} \in S^j\}; \end{aligned}$$

- initial state $\mathbf{s} = (\mathbf{t}, \mathbf{t}, g_i)$;
- set of accepting states $F = H \times H \times \{\mathbf{1}\}$.

By construction, we have $\mathcal{L}^j \cdot \varepsilon^* \subseteq \mathcal{L}$, with equality if and only if $S^j = \Delta_i^{\mathbf{left}}$. Hence, as long as $\mathcal{L}^j \cdot \varepsilon^* \subsetneq \mathcal{L}$, we select some element γ of G such that $\mathbf{NF}(\gamma) \notin \mathcal{L}^j \cdot \varepsilon^*$, then we define S^{j+1} as $S^{j+1} = S^j \cup \Delta_{\mathbf{NF}}^{\mathbf{left}}(g_i\gamma, \gamma)$. By definition, the set $\Delta_{\mathbf{NF}}^{\mathbf{left}}(g_i\gamma, \gamma)$ was not a subset of S^j, which proves that the sequence $S^0, S^1, \ldots$ is strictly increasing, and proves that our process terminates with the computation of $\Delta_i^{\mathbf{left}}$ itself. It is then straightforward to compute the automaton $\mathcal{A}_i$ and to minimise it.

Applying similar constructions for the sets $\Delta_i^{\mathbf{right}}$ completes the proof. $\square$

2.3.3 Normal Forms in Artin–Tits Monoids and Groups of Spherical Type

We come back now to the case where $\mathbf{G}^+$ is an Artin–Tits monoid of spherical type $\mathbf{A}^+$, not only a generic Garside monoid; we will work under that assumption until the end of this section, and will always asume that Δ is the *smallest* Garside element of $\mathbf{A}^+$, i.e. the lowest upper bound of its generators. In that narrower context, there exist alternative criteria for characterising Garside words, which will lead to a criterion for concatenating directly Garside words.

Proposition 2.59.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type, let $\mathcal{S}$ be the set of simple elements of $\mathbf{A}^+$, and let $\underline{\mathbf{b}} := b_1 \cdot b_2 \cdot \ldots \cdot b_k$ be a word with letters in $\mathcal{S} \backslash \{\mathbf{1}\}$. The word $\underline{\mathbf{b}}$ is a left Garside word if and only if $\mathbf{left}(b_{i+1}) \subseteq \mathbf{right}(b_i)$ for all $i \in \{1, \ldots, k-1\}$. Analogously, the word $\underline{\mathbf{b}}$ is a right Garside word if and only if $\mathbf{right}(b_i) \subseteq \mathbf{left}(b_{i+1})$ for all $i \in \{1, \ldots, k-1\}$.

Proof. Following Lemma 2.40, it suffices to prove that $\overline{\mathbf{left}}(\mathbf{a}) = \mathbf{left}(\mathbf{a})$ and $\overline{\mathbf{right}}(\mathbf{a}) = \mathbf{right}(\mathbf{a})$ for all braids $\mathbf{a} \in \mathcal{S} \backslash \{\mathbf{1}\}$. Both equalities are straightforward consequences of Lemma 2.16. □

In addition, Theorem 2.26 provides us with a classification of the finite Coxeter groups, from which it follows that Δ^2 belongs to the centre of the monoid $\mathbf{A}^+$.

Proposition 2.60.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type with generators $\sigma_1, \ldots, \sigma_n$, and consider its Garside element $\Delta := \mathbf{LCM}_{\leqslant_\ell}(\sigma_1, \ldots, \sigma_n)$. The morphism of monoids $\phi_\Delta : \mathbf{A}^+ \mapsto \mathbf{A}^+$, such that $\Delta \phi_\Delta(\mathbf{a}) = \mathbf{a}\Delta$ for all $\mathbf{a} \in \mathbf{A}^+$, is an involution.

Proof. The monoid $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, which means that its Coxeter group $\mathbf{W}$ is finite. Therefore, Theorem 2.26 states that there exists a partition $J_1, \ldots, J_k$ of $\{\sigma_1, \ldots, \sigma_n\}$ such that each Coxeter group generated by J_ℓ (for $\ell \in \{1, \ldots, k\}$) is presented by a finite irreducible Coxeter system.

Hence, let $\mathbf{A}_1^+, \ldots, \mathbf{A}_\ell^+$ be the associated Artin–Tits monoids of spherical type, and let $\Delta_1, \ldots, \Delta_\ell$ be the associated Garside elements. The monoid $\mathbf{A}^+$ is the direct product $\prod_{i=1}^{\ell} \mathbf{A}_i^+$, and its Garside element Δ is the (commutative) product $\prod_{i=1}^{\ell} \Delta_i$. Therefore, it remains to treat the case where $\mathbf{W}$ is itself presented by a finite irreducible Coxeter system.

Now, let Γ be the Coxeter diagram associated with $\mathbf{W}$. Proposition 2.47 proves that ϕ_Δ is a morphism of monoids that maps the set $\{\sigma_1, \ldots, \sigma_n\}$ to itself. It follows that ϕ_Δ induces an isomorphism of labelled graphs from Γ to itself. Let κ be the smallest positive integer such that $\phi_\Delta^\kappa = \mathbf{Id}_\Gamma$.

It comes immediately that

- $\kappa = 1$ if $\mathbf{W} = B_n$, $\mathbf{W} = E_7$, $\mathbf{W} = E_8$, $\mathbf{W} = H_3$ or $\mathbf{W} = H_4$;
- $\kappa \in \{1, 2\}$ if $\mathbf{W} = A_n$, $\mathbf{W} = D_n$ (with $n \neq 4$), $\mathbf{W} = E_6$, $\mathbf{W} = F_4$ or $\mathbf{W} = I_2(a)$.[3]

Finally, if $\mathbf{W} = D_4$, then $\phi_\Delta(\sigma_3) = \sigma_3$, and the generators σ_1, σ_2 and σ_4 play symmetric roles. Consequently, if the morphism ϕ_Δ were to map σ_1 to σ_2, then it should also map σ_1 to σ_4. This proves that $\phi_\Delta(\sigma_1) = \sigma_1$ and, similarly, that $\phi_\Delta(\sigma_2) = \sigma_2$ and $\phi_\Delta(\sigma_4) = \sigma_4$, which proves that $\kappa = 1$.[4]

Overall, we have proved that κ must divide 2, which proves that $\phi_\Delta^2 = \mathbf{Id}_\Gamma$, i.e. that ϕ_Δ^2 maps each generator σ_i to itself, and therefore that $\phi_\Delta^2 = \mathbf{Id}_{\mathbf{A}^+}$. □

If Δ^2 is always central in $\mathbf{A}^+$, this is not necessarily the case of the element Δ. For instance, if $\mathbf{A}^+$ is a dihedral monoid (i.e. $\mathbf{W}$ is of type $I_2(a)$), then Δ belongs to the centre of $\mathbf{A}^+$ if and only if a is an even integer. Therefore, it is natural to project the monoid $\mathbf{A}^+$ onto the quotient $\mathbf{A}^+/\Delta^2$. We describe now such projections, which we will use extensively throughout Chapter 5, as well as the (immediate) result that follows.

Definition 2.61 (Projection on $\mathbf{A}^+/\Delta^2$).
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type, with let $\mathbf{b}$ be an element of $\mathbf{A}^+$, and let u be the largest integer such that $\mathbf{b} \geqslant \Delta^{2u}$, where $\Delta := \mathbf{LCM}_{\leqslant_\ell}(\sigma_1, \ldots, \sigma_n)$.. We denote by $\delta(\mathbf{b})$ the element $\Delta^{-2u}\mathbf{b}$ of $\mathbf{A}^+$, which we identify with the projection of $\mathbf{b}$ onto $\mathbf{A}^+/\Delta^2$.

Lemma 2.62.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type. The equality $\delta(\mathbf{ab}) = \delta(\mathbf{a}\delta(\mathbf{b})) = \delta(\delta(\mathbf{a})\mathbf{b})$ holds for all elements $\mathbf{a}$ and $\mathbf{b}$ of the monoid $\mathbf{A}^+$.

Moreover, from Proposition 2.60 follows a notion of symmetric Garside normal form in Artin–Tits groups of spherical type.

Definition & Proposition 2.63 (Symmetric Garside normal form in the group $\mathbf{A}$).
Let $\mathbf{A}$ be an Artin–Tits group of spherical type. The symmetric Garside normal form *of a braid $\mathbf{a} \in \mathbf{A}$ is defined as the (unique) word $\mathbf{NF}_{\mathrm{sym}}(\mathbf{a}) := a_k^{-1} \cdot \ldots \cdot a_1^{-1} \cdot b_1 \cdot \ldots \cdot b_\ell$ such that*

- *$\mathbf{a} = a_k^{-1} \ldots a_1^{-1} b_1 \ldots b_\ell$;*
- *$a_1 \cdot \ldots \cdot a_k$ and $b_1 \cdot \ldots \cdot b_\ell$ are left Garside words (in the sense of Definition 2.34);*
- *either $k = 0$ or $\ell = 0$ or the elements a_1 and b_1 have no common non-trivial divisor in the monoid $\mathbf{A}^+$.*

[3]We may even prove as follows that $\kappa = 1$ if $\mathbf{W} = F_4$ or $\mathbf{W} = I_2(a)$ with $a \in 2\mathbb{Z}$. Indeed, the number of occurrences of letters σ_1 or σ_2 (if $\mathbf{W} = F_4$), or σ_1 (if $\mathbf{W} = I_2(a)$) is invariant among all the words that represent a given braid. Hence, the generator σ_1 is not conjugate with σ_4 (if $\mathbf{W} = F_4$) or σ_2 (if $\mathbf{W} = I_2(a)$), and therefore we must have $\kappa = 1$.

[4]Alternatively, Proposition 2.33 proves that the group isomorphism $\iota : D_n \mapsto \mathfrak{S}_n^\pm$ maps Δ to the permutation $-\mathbf{Id}_{\{-n,\ldots,n\}}$, which commutes with each element of $\mathfrak{S}_n^\pm$, and therefore proves that $\kappa = 1$.

The subword $\mathbf{NF}^{-}_{\mathrm{sym}}(\mathbf{a}) := a_k^{-1} \cdot \ldots \cdot a_1^{-1}$ *is called the* negative part *of the braid* $\mathbf{a}$*, and the subword* $\mathbf{NF}^{+}_{\mathrm{sym}}(\mathbf{a}) := b_1 \cdot \ldots \cdot b_\ell$ *is called the* positive part *of* $\mathbf{a}$*. The words of the form* $\mathbf{NF}_{\mathrm{sym}}(\mathbf{a})$ *are called* symmetric Garside words.

Proof. Let $\mathbf{a}$ be an element of $\mathbf{A}$. We first prove that a word such as $\mathbf{NF}_{\mathrm{sym}}(\mathbf{a}) = a_k^{-1} \cdot \ldots \cdot a_1^{-1} \cdot b_1 \cdot \ldots \cdot b_\ell$ exists. According to Proposition 2.53, there exists elements $\overline{\alpha}$ and $\overline{\beta}$ of $\mathbf{A}^+$ such that $\mathbf{a} = \overline{\alpha}^{-1}\overline{\beta}$. Consider the elements $\delta := \mathbf{GCD}_{\leqslant_\ell}(\overline{\alpha}, \overline{\beta})$, $\alpha := \delta^{-1}\overline{\alpha}$ and $\beta := \delta^{-1}\overline{\beta}$, and consider the words $a_1 \cdot \ldots \cdot a_k = \mathbf{NF}_\ell(\alpha)$ and that $b_1 \cdot \ldots \cdot b_\ell = \mathbf{NF}_\ell(\gamma)$.

First, we have $a_k^{-1} \ldots a_1^{-1} b_1 \ldots b_\ell = \alpha^{-1}\beta = \overline{\alpha}^{-1}\delta\delta^{-1}\overline{\beta} = \overline{\alpha}^{-1}\overline{\beta} = \mathbf{a}$. Second, by construction, both $a_1 \cdot \ldots \cdot a_k$ and $b_1 \cdot \ldots \cdot b_\ell$ are left Garside words. Third, we have $\delta = \mathbf{GCD}_{\leqslant_\ell}(\overline{\alpha}, \overline{\beta}) = \mathbf{GCD}_{\leqslant_\ell}(\delta\alpha, \delta\beta) = \delta\mathbf{GCD}_{\leqslant_\ell}(\alpha, \beta)$, hence $\mathbf{GCD}_{\leqslant_\ell}(\alpha, \beta) = \mathbf{1}$, and therefore $\mathbf{GCD}_{\leqslant_\ell}(a_1, b_1) \leqslant_\ell \mathbf{GCD}_{\leqslant_\ell}(\alpha, \beta) = \mathbf{1}$.

It remains to prove that the word $\mathbf{NF}_{\mathrm{sym}}(\mathbf{a}) = a_k^{-1} \cdot \ldots \cdot a_1^{-1} \cdot b_1 \cdot \ldots \cdot b_\ell$ is unique. Consider the elements $\alpha = a_1 \ldots a_k$ and $\beta = b_1 \ldots b_\ell$ of $\mathbf{A}^+$. If $\mathbf{a} \in \mathbf{A}^+$ and if $k \geqslant 1$, then $\beta = \alpha\mathbf{a}$, and therefore $\ell \geqslant 1$ and $\mathbf{1} = \mathbf{GCD}_{\leqslant_\ell}(a_1, b_1) = \mathbf{GCD}_{\leqslant_\ell}(\alpha, \beta, \Delta) = \mathbf{GCD}_{\leqslant_\ell}(\alpha, \alpha\mathbf{a}, \Delta) = \mathbf{GCD}_{\leqslant_\ell}(\alpha, \Delta) = a_1$, which is impossible. Hence, if $\mathbf{a} \in \mathbf{A}^+$, we have $k = 0$ and $b_1 \cdot \ldots \cdot b_\ell$ is the left Garside normal form of $\mathbf{a}$. Similarly, if $\mathbf{a}^{-1} \in \mathbf{A}^+$, then $\ell = 0$ and $a_1 \cdot \ldots \cdot a_k$ is the left Garside normal form of $\mathbf{a}^{-1}$.

Now, we assume that neither $\mathbf{a}$ nor $\mathbf{a}^{-1}$ belongs to $\mathbf{A}^+$. Hence, both k and ℓ are positive integers, and both $a_1 \cdot \ldots \cdot a_k$ and $b_1 \cdot \ldots \cdot b_\ell$ are Δ-free words. Indeed, a_1 and b_1 must be non-trivial, i.e. $\mathbf{left}(a_1) \neq \varnothing$ and $\mathbf{left}(b_1) \neq \varnothing$, and since a_1 and b_1 have no common non-trivial divisor in $\mathbf{A}^+$ it follows that $\mathbf{left}(a_1) \cap \mathbf{left}(b_1) = \varnothing$, whence $\mathbf{left}(a_1) \subseteq \{\sigma_1, \ldots, \sigma_n\} \backslash \mathbf{left}(b_1) \subsetneq \{\sigma_1, \ldots, \sigma_n\} = \mathbf{left}(\Delta)$ and, similarly, $\mathbf{left}(b_1) \subsetneq \mathbf{left}(\Delta)$. Let $u \geqslant 1$ be the smallest integer such that $\Delta^u\mathbf{a} \in \mathbf{A}^+$. The integer u is the unique integer such that $\Delta^u\mathbf{a}$ is a Δ-free element of $\mathbf{A}^+$. For all integers $i \in \{1, \ldots, k\}$, consider the simple element $a_i^* := \partial_\Delta^{2i-1}(a_i)$ of $\mathbf{A}^+$. We prove now that $a_k^* \cdot \ldots \cdot a_1^* \cdot b_1 \cdot \ldots \cdot b_\ell$ is a left Garside word.

According to Corollary 2.38, it is enough to prove that the words $a_{i+1}^* \cdot a_i^*$ (when $1 \leqslant i \leqslant k-1$) and $a_1^* \cdot b_1$ are left Garside words. Recall that $u \cdot v$ is a left Garside word if and only if $u = \alpha_\ell(uv)$. In addition, recall the function $\partial_\Delta : \mathcal{S} \mapsto \mathcal{S}$ introduced in the proof of Proposition 2.47, such that $\mathbf{a}\partial_\Delta(\mathbf{a}) = \Delta$ and such that $\phi_\Delta = \partial_\Delta^2$. Since $\phi_\Delta^2 = \partial_\Delta^4 = \mathbf{Id}_{\mathcal{S}}$, it follows that

$$\begin{aligned} u = \alpha_\ell(uv) &\Leftrightarrow u = \mathbf{GCD}_{\leqslant_\ell}(\Delta, uv) = \mathbf{GCD}_{\leqslant_\ell}(u\partial_\Delta(u), uv) \\ &\Leftrightarrow \mathbf{GCD}_{\leqslant_\ell}(\partial_\Delta(u), v) = \mathbf{1} \Leftrightarrow \mathbf{GCD}_{\leqslant_\ell}(\partial_\Delta^4(v), \partial_\Delta(u)) = \mathbf{1} \\ &\Leftrightarrow \partial_\Delta^3(v) \cdot \partial_\Delta(u) \text{ is a left Garside word.} \end{aligned}$$

Hence, for all $i \in \{1, \ldots, k-1\}$, and since $a_i \cdot a_{i+1}$ is a left Garside word, so is $\partial_\Delta^3(a_{i+1}) \cdot \partial_\Delta(a_i)$. Then, since ϕ_Δ is an isomorphism of monoids, the word $a_{i+1}^* \cdot a_i^* = \phi_\Delta^{i-3}\partial_\Delta^3(a_{i+1}) \cdot \phi_\Delta^{i-3}\partial_\Delta(a_i)$ is a left Garside word too.

Finally, a direct computation shows that $a_k^* \ldots a_1^* b_1 \ldots b_\ell = \Delta^k\mathbf{a}$. Moreover, since

$a_k \neq \mathbf{1}$, we know that $a_k^* \neq \Delta$. Hence, $\Delta^k \mathbf{a}$ is a Δ-free element of $\mathbf{A}^+$, and therefore $k = u$. This proves that the braids $a_1^*, \ldots, a_k^*, b_1, \ldots, b_\ell$ are uniquely defined, and that so is the word $\mathbf{NF}_{\text{sym}}(\mathbf{a})$. □

Example 2.64.
Consider the braid group $\mathbf{B}_3 := \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. The symmetric Garside words (i.e. the words of the form $\mathbf{NF}_{\text{sym}}(\mathbf{a})$ for $\mathbf{a} \in \mathbf{B}_3$) is the language accepted by the (non-deterministic) automaton represented in Fig. 2.65, in which each state is both initial and final. Moreover, for the sake of readability of Fig. 2.65, we chose to denote by $\overline{\sigma_1}$ and $\overline{\sigma_2}$ the generators σ_1^{-1} and σ_2^{-1}.

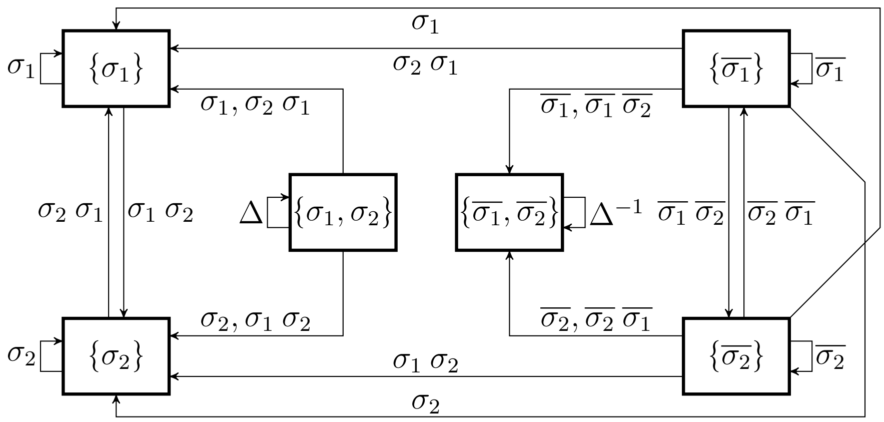


Figure 2.65 – Automaton accepting the language $\{\mathbf{NF}_{\text{sym}}(\mathbf{a}) : \mathbf{a} \in \mathbf{B}_3\}$

2.3.4 Heap Monoids

We focus here on *heap monoids* and *heap groups*, which are combinatorial objects similar to braid monoids and braid groups. Along with braid monoids, trace monoids form a very popular class of Artin–Tits monoids, and have been studied in depth [24, 43, 44, 91].

Definition 2.66 (Heap monoid and heap group)**.**
Let $\{\sigma_1, \ldots, \sigma_n\}$ be a finite alphabet, let $D \subseteq \{\sigma_1, \ldots, \sigma_n\}^2$ be a symmetric reflexive relation, which we call dependence *or* incompatibility *relation, and let $I = \{\sigma_1, \ldots, \sigma_n\}^2 \backslash D$ be the associated* independence *relation.*

The heap monoid *(also called* trace monoid, right-angled Artin–Tits monoid*, or* free partially commutative monoid*) generated by the pair $(\{\sigma_1, \ldots, \sigma_n\}, I)$ is the monoid $\mathcal{M}^+$ presented as follows:*

$$\mathcal{M}^+ := \langle \sigma_1, \ldots, \sigma_n \mid \forall (a, b) \in I, ab = ba \rangle^+ .$$

The heap group *generated by the pair (Σ, I) is the group $\mathcal{M}$ presented as follows:*

$$\mathcal{M} := \langle \sigma_1, \ldots, \sigma_n \mid \forall (a, b) \in I, ab = ba \rangle .$$

Note that heap monoids and groups are particular instances of Artin–Tits monoids and groups, in which every exponent $m_{i,j}$ belongs to the set $\{0, 2\}$. However, unless the monoid is the abelian monoid $\mathbb{Z}_{\geqslant 0}^{\Sigma}$, its Coxeter group is never finite. A popular example of heap monoids is the *dimer model*, defined by

$$\mathcal{M}_n^+ := \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geqslant 2 \rangle^+ .$$

The dimer model is analogous to braid monoids, from which it is obtained by removing the braid relations $\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}$. This model is well-known in combinatorics [18] and has recently attracted interest in statistical physics as a model of random growth [68, 91].

This analogy reflects on the Coxeter diagram of the dimer model. Each generator σ_i is represented by the vertex with label i. Vertices i and j are linked by an unlabelled, white edge if $m_{i,j} = \infty$, i.e. if $i \neq j$ and $(\sigma_i, \sigma_j) \notin I$.

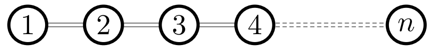


Figure 2.67 – Coxeter diagram of the dimer model $\mathcal{M}_n^+$

Like braid monoids, each heap monoid has a graphical representation, in terms of the *heap diagrams* introduced by Viennot [92]. Each generating element σ_i is represented by a horizontal (*not necessarily connected*) brick that may move vertically but not horizontally. Two elements are independent if and only if the bricks that represent them can be placed on the same vertical layer without overlapping each other.

Figure 2.69 shows the diagrams of two words, $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_3\sigma_1$ and $\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_1$, that belong to the same heap, in the heap monoid $\mathcal{M}^+$ generated by the pair

$$(\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}, \{(\sigma_2, \sigma_3), (\sigma_3, \sigma_2), (\sigma_2, \sigma_5), (\sigma_5, \sigma_2), (\sigma_3, \sigma_4), (\sigma_4, \sigma_3), (\sigma_3, \sigma_5), (\sigma_5, \sigma_3)\}).$$

A canonical representation of the heap is obtained by letting *gravity* act on the blocks that form the diagrams, so that several blocks fall onto the same vertical layer. This representation is unique, and is linked to the Cartier-Foata normal form introduced below.

Note that, unlike braid diagrams, heap diagrams shall be read *from bottom to top.*

Like all Artin–Tits monoids, the heap monoid $\mathcal{M}^+$ is invariant under word reversal, homogeneous and cancellative, and come with a notion of left and right sets. Aside from simple elements, some elements of $\mathcal{M}^+$, called *cliques*, give rise to Garside normal forms analogous to the case of Garside monoids.

Definition 2.68 (Cliques of a heap monoid)**.**
Let $\mathcal{M}^+$ be a heap monoid, and let $a_1, \ldots, a_k$ be pairwise independent generators of $\mathcal{M}^+$, i.e. such that $\forall i, j \in \{1, \ldots, k\}, i \neq j \Rightarrow (a_i, a_j) \in I$. We say that the (commutative) product $a_1 \ldots a_k$ is a clique *of $\mathcal{M}^+$, and we denote by $\mathcal{C}$ the set of cliques of $\mathcal{M}^+$.*

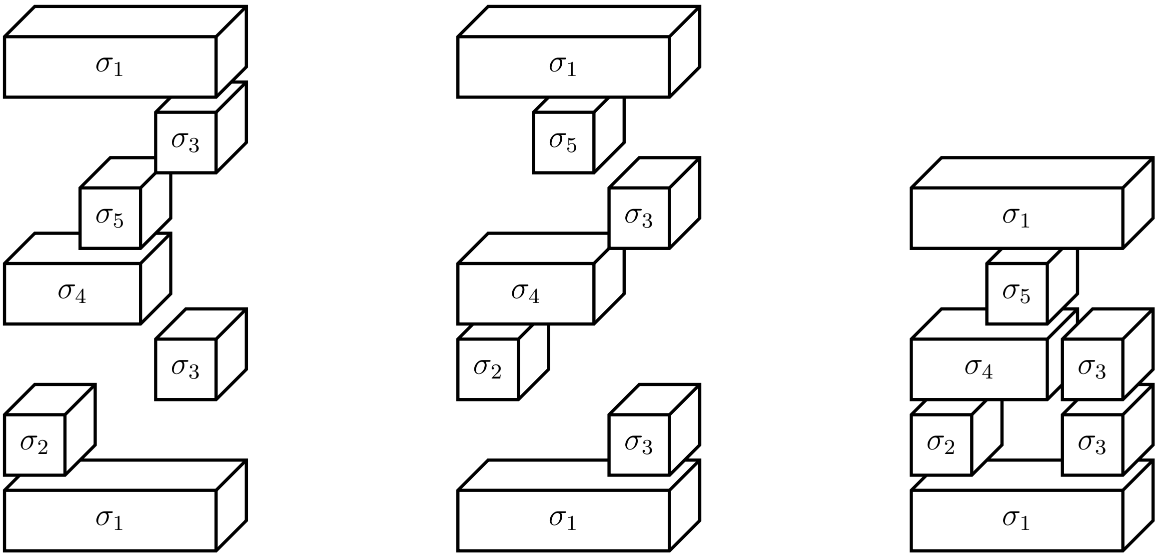


Figure 2.69 – Heap diagrams associated with the heap $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_3\sigma_1$

Lemma 2.70.
Let $\mathcal{M}^+$ be a heap monoid with generators $\sigma_1, \ldots, \sigma_n$ and let $\mathbf{a}$ *be an element of $\mathcal{M}^+$. The elements of the set* $\mathbf{left}(\mathbf{a}) := \{\sigma_i : \sigma_i \leqslant_\ell \mathbf{a}\}$ *are pairwise independent. In addition, let $\alpha_\ell(\mathbf{a})$ be the (commutative) product $\prod_{\sigma \in \mathbf{left}(\mathbf{a})} \sigma$. The heap $\alpha_\ell(\mathbf{a})$ is the* maximal left-dividing clique *of* $\mathbf{a}$*, i.e. $\alpha_\ell(\mathbf{a}) \in \mathcal{C}$, $\alpha_\ell(\mathbf{a}) \leqslant_\ell \mathbf{a}$ and $\forall \mathbf{b} \in \mathcal{C}, \mathbf{b} \leqslant_\ell \mathbf{a} \Leftrightarrow \mathbf{b} \leqslant_\ell \alpha_\ell(\mathbf{a})$.*

Similarly, the elements of the set $\mathbf{right}(\mathbf{a}) := \{\sigma_i : \mathbf{a} \geqslant_r \sigma_i\}$ *are pairwise independent, and the heap $\alpha_r(\mathbf{a}) := \prod_{\sigma \in \mathbf{right}(\mathbf{a})} \sigma$ is the* maximal right-dividing clique *of* $\mathbf{a}$.

Proof. Due to word reversal, we only focus on the part that concerns the left set and left-divisors of $\mathbf{a}$. Let $a_1 \ldots a_k$ be a factorisation of $\mathbf{a}$ into generators or $\mathcal{M}^+$. One shows easily that $\mathbf{left}(\mathbf{a}) = \{\sigma_i : \exists u \in \{1, \ldots, k\}, a_u = \sigma_i \text{ and } \forall v \in \{1, \ldots, u-1\}, (a_u, a_v) \in I\}$. Hence, the elements of $\mathbf{left}(\mathbf{a})$ must be pairwise independent, and therefore commute with each other.

Then, let $\overline{a}_1 \ldots \overline{a}_m$ be a factorisation of $\alpha_\ell(\mathbf{a})$, with $\mathbf{left}(\mathbf{a}) = \{\overline{a}_1, \ldots, \overline{a}_m\}$. An immediate induction on i shows that $\overline{a}_1 \ldots \overline{a}_i \leqslant_\ell \mathbf{a}$ for all $i \in \{1, \ldots, m\}$, which proves that $\alpha_\ell(\mathbf{a}) \leqslant_\ell \mathbf{a}$. Finally, if $\mathbf{b}$ is a left-dividing clique of $\mathbf{a}$, then we have $\mathbf{left}(\mathbf{b}) \subseteq \mathbf{left}(\mathbf{a})$, hence $\mathbf{b} = \alpha_\ell(\mathbf{b}) \leqslant_\ell \alpha_\ell(\mathbf{a})$. □

Definition 2.71 (Garside normal forms in the monoid $\mathcal{M}^+$)**.**
Let $\mathcal{M}^+$ be a heap monoid, and let $\mathbf{a}$ *be an element of $\mathcal{M}^+$. The* left Garside normal form*, or* Cartier-Foata normal form*, of* $\mathbf{a}$ *is defined as the word* $\mathbf{NF}_\ell(\mathbf{a}) := \mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k$ *such that:*

- $\mathbf{a} = \mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_k$*,*
- $\mathbf{a}_1, \ldots, \mathbf{a}_k$ *are cliques of $\mathcal{M}^+$,*
- *either $k = 0$ or $\mathbf{a}_k \neq \mathbf{1}$, and*

- $\mathbf{a}_i = \alpha_\ell(\mathbf{a}_i \mathbf{a}_{i+1} \ldots \mathbf{a}_k)$ *for all* $i \in \{1, \ldots, k\}$.

The right Garside normal form *of* $\mathbf{a}$ *is defined as the word* $\mathbf{NF}_r(\mathbf{a}) := \mathbf{a}'_1 \cdot \mathbf{a}'_2 \cdot \ldots \cdot \mathbf{a}'_\ell$ *such that:*

- $\mathbf{a} = \mathbf{a}'_1 \mathbf{a}'_2 \ldots \mathbf{a}'_\ell$,
- $\mathbf{a}'_1, \ldots, \mathbf{a}'_\ell$ *are cliques of* $\mathcal{M}^+$,
- *either* $\ell = 0$ *or* $\mathbf{a}'_1 \neq \mathbf{1}$, *and*
- $\mathbf{a}'_i = \alpha_r(\mathbf{a}'_1 \mathbf{a}'_2 \ldots \mathbf{a}'_i)$ *for all* $i \in \{1, \ldots, \ell\}$.

For instance, the left Garside normal form of the word represented in Fig. 2.69 is $\sigma_1 \cdot \sigma_2\sigma_3 \cdot \sigma_3\sigma_4 \cdot \sigma_5 \cdot \sigma_1$.

Moreover, like when defining the Garside normal forms on Garside monoids in Definition 2.34, requiring that $\mathbf{a}_k \neq \mathbf{1}$, in the definition of the left Garside normal form, amounts to requiring that $\mathbf{a}_i \neq \mathbf{1}$ for all $i \in \{1, \ldots, k\}$, and requiring that $\mathbf{a}'_1 \neq \mathbf{1}$, in the definition of the right Garside normal form, amounts to requiring that $\mathbf{a}'_i \neq \mathbf{1}$ for all $i \in \{1, \ldots, \ell\}$.

However, unlike in Garside groups, some elements of the heap group $\mathcal{M}$, such as the heap $\sigma_2\sigma_1^{-1}$, cannot be factored as a product $\mathbf{y}^{-1}\mathbf{z}$ such that $\mathbf{y}$ and $\mathbf{z}$ belong to the heap monoid $\mathcal{M}^+$. Hence, there is no direct way to derive a normal form on heap groups from the Cartier-Foata normal form on heap monoids. Nevertheless, there still exists a notion of Cartier-Foata normal form on heap groups.

Definition 2.72 (Cliques of a heap group)**.**
Let $\mathcal{M}$ *be a heap group, and let* $a_1, \ldots, a_k$ *be pairwise independent generators of the heap monoid* $\mathcal{M}^+$, *i.e. such that* $\forall i, j \in \{1, \ldots, k\}, i \neq j \Rightarrow (a_i, a_j) \in I$. *In addition, consider a tuple* $(\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k$. *We say that the (commutative) product* $a_1^{\epsilon_1} \ldots a_k^{\epsilon_k}$ *is a* clique *of the group* $\mathcal{M}$, *and we denote by* $\mathcal{C}$ *the set of all cliques of* $\mathcal{M}$.

In particular, the cliques of the heap monoid $\mathcal{M}^+$ are the cliques of $\mathcal{M}$ that belong to the monoid $\mathcal{M}^+$. Moreover, the restriction of the Artin length to the monoid $\mathcal{M}^+$ coincides with a variant of the product length that we introduced in Definition 2.35 for Garside monoids.

Definition 2.73 (Product length on heap groups)**.**
Let $\mathcal{M}$ *be a heap group with generators* $\sigma_1, \ldots, \sigma_n$, *and let* $\mathbf{a}$ *be an element of* $\mathcal{M}$. *The* product length *of* $\mathbf{a}$, *which we denote by* $\chi(\mathbf{a})$, *is the smallest integer such that* $\mathbf{a}$ *can be written as a product of* $\chi(\mathbf{a})$ *generators* $\sigma_i^{\pm 1}$.

The product length leads to a generalisation of the left and right divisibility relations, as well as of left and right sets of a heap.

Definition 2.74 (Divisibility relations and left and right (outgoing) sets)**.**
Let $\mathcal{M}$ *be a heap group, and let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two elements of* $\mathcal{M}$. *We define the partial divisibility orderings* $\leqslant_\ell$ *and* $\geqslant_r$ *as follows:*

$$\mathbf{a} \leqslant_\ell \mathbf{ab} \Leftrightarrow \mathbf{ab} \geqslant_r \mathbf{b} \Leftrightarrow \chi(\mathbf{a}) + \chi(\mathbf{b}) = \chi(\mathbf{ab}).$$

We call left set *of* $\mathbf{a}$*, and denote by* $\mathbf{left}(\mathbf{a})$*, the set* $\{\sigma_i^{\epsilon_i} : \epsilon_i = \pm 1, \sigma_i^{\epsilon_i} \leqslant_\ell \mathbf{a}\}$*. We also call* right set *of* $\mathbf{a}$*, and denote by* $\mathbf{right}(\mathbf{a})$*, the set* $\{\sigma_i^{\epsilon_i} : \epsilon_i = \pm 1, \mathbf{a} \geqslant_r \sigma_i\}$*.*

In addition, if $\mathbf{a}$ *is a clique, we call* left outgoing set *of* $\mathbf{a}$*, and denote by* $\overline{\mathbf{left}}(\mathbf{a})$*, the set* $\{\sigma_i^{\epsilon_i} : \epsilon_i = \pm 1, \sigma_i^{\epsilon_i}\mathbf{a} \notin \mathcal{C}\}$*. We also call* right outgoing set *of* $\mathbf{a}$*, and denote by* $\overline{\mathbf{right}}(\mathbf{a})$*, the set* $\{\sigma_i^{\epsilon_i} : \epsilon_i = \pm 1, \mathbf{a}\sigma_i^{\epsilon_i} \notin \mathcal{C}\}$*.*

Note that, since $\lambda(\mathbf{a}) = \chi(\mathbf{a})$ for all positive heaps $\mathbf{a} \in \mathcal{M}^+$, Definition 2.74 is consistent with the usual definitions of $\leqslant_\ell$, $\geqslant_r$, $\mathbf{left}$, $\mathbf{right}$, $\overline{\mathbf{left}}$ and $\overline{\mathbf{right}}$. In particular, for all heaps $\mathbf{a} \in \mathcal{M}^+$, Definitions 2.39 and 2.74 lead to the same sets $\mathbf{left}(\mathbf{a})$, $\mathbf{right}(\mathbf{a})$, $\overline{\mathbf{left}}(\mathbf{a})$ and $\overline{\mathbf{right}}(\mathbf{a})$, which legitimates using the same notations.

Moreover, the left and right sets are crucial for characterising the relations $\leqslant_\ell$ and $\geqslant_r$, due to the following immediate result.

Lemma 2.75.
Let $\mathcal{M}$ *be a heap group, and let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two elements of* $\mathcal{M}$*. We have* $\chi(\mathbf{a}) + \chi(\mathbf{b}) = \chi(\mathbf{ab})$ *if and only if* $\forall \sigma^\epsilon \in \mathbf{right}(\mathbf{a}), \forall \tau^\eta \in \mathbf{left}(\mathbf{b}), \sigma^\epsilon \tau^\eta \neq \mathbf{1}$*.*

Lemma 2.76.
Let $\mathbf{a}$ *be an element of the heap group* $\mathcal{M}$*. The elements of the set* $\mathbf{left}(\mathbf{a}) := \{\sigma_i^{\epsilon_i} : \sigma_i^{\epsilon_i} \leqslant_\ell \mathbf{a}\}$ *are pairwise independent. In addition, let* $\alpha_\ell(\mathbf{a})$ *be the (commutative) product* $\prod_{\sigma^\epsilon \in \mathbf{left}(\mathbf{a})} \sigma^\epsilon$*. The heap* $\alpha_\ell(\mathbf{a})$ *is the* maximal left-dividing clique *of* $\mathbf{a}$*, i.e.* $\alpha_\ell(\mathbf{a}) \in \mathcal{C}$*,* $\alpha_\ell(\mathbf{a}) \leqslant_\ell \mathbf{a}$ *and* $\forall \mathbf{b} \in \mathcal{C}, \mathbf{b} \leqslant_\ell \mathbf{a} \Leftrightarrow \mathbf{b} \leqslant_\ell \alpha_\ell(\mathbf{a})$*.*

Similarly, the elements of the set $\mathbf{right}(\mathbf{a}) := \{\sigma_i^{\epsilon_i} : \mathbf{a} \geqslant_r \sigma_i^{\epsilon_i}\}$ *are pairwise independent, and the heap* $\alpha_r(\mathbf{a}) := \prod_{\sigma^\epsilon \in \mathbf{right}(\mathbf{a})} \sigma^\epsilon$ *is the* maximal right-dividing clique *of* $\mathbf{a}$*.*

Proof. The proof is analogous to that of Lemma 2.70. Due to word reversal, we only focus on the part that concerns the left set and left-divisors of $\mathbf{a}$. Let $a_1^{\theta_1} \ldots a_k^{\theta_k}$ be a minimal-length factorisation of $\mathbf{a}$ into generators or $\mathcal{M}$, i.e. such that $k = \chi(\mathbf{a})$. One shows easily that $\mathbf{left}(\mathbf{a}) = \{\sigma_i^{\epsilon_i} : \exists u \in \{1, \ldots, k\}, a_u^{\theta_u} = \sigma_i^{\epsilon_i}$ and $\forall v \in \{1, \ldots, u-1\}, (a_u, a_v) \in I\}$. Hence, the elements of $\mathbf{left}(\mathbf{a})$ must be pairwise independent, and therefore commute with each other.

Then, let $\overline{a}_1^{\overline{\theta}_1} \ldots \overline{a}_m^{\overline{\theta}_m}$ be a factorisation of $\alpha_\ell(\mathbf{a})$, with $\mathbf{left}(\mathbf{a}) = \{\overline{a}_1^{\overline{\theta}_1}, \ldots, \overline{a}_m^{\overline{\theta}_m}\}$. An immediate induction on i shows that $\overline{a}_1^{\overline{\theta}_1} \ldots \overline{a}_i^{\overline{\theta}_i} \leqslant_\ell \mathbf{a}$ for all $i \in \{1, \ldots, m\}$, which proves that $\alpha_\ell(\mathbf{a}) \leqslant_\ell \mathbf{a}$. Finally, if $\mathbf{b}$ is a left-dividing clique of $\mathbf{a}$, then we have $\mathbf{left}(\mathbf{b}) \subseteq \mathbf{left}(\mathbf{a})$, hence $\mathbf{b} = \alpha_\ell(\mathbf{b}) \leqslant_\ell \alpha_\ell(\mathbf{a})$. □

Consequently, we extend the Garside normal forms to heap groups.

Definition 2.77 (Garside normal forms in the heap group $\mathcal{M}$)**.**
Let $\mathcal{M}$ *be a heap group, and let* $\mathbf{a}$ *be an element of* $\mathcal{M}$*. The* left Garside normal form *of* $\mathbf{a}$ *is defined as the word* $\mathbf{NF}_\ell(\mathbf{a}) := \mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k$ *such that:*

- $\mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_k$*,*

- $\mathbf{a}_1, \ldots, \mathbf{a}_k$ *are cliques of* $\mathcal{M}$*,*
- *either* $k = 0$ *or* $\mathbf{a}_k \neq \mathbf{1}$*, and*
- $\mathbf{a}_i = \alpha_\ell(\mathbf{a}_i \mathbf{a}_{i+1} \ldots \mathbf{a}_k)$ *for all* $i \in \{1, \ldots, k\}$*.*

The right Garside normal form *of* $\mathbf{a}$ *is defined as the word* $\mathbf{NF}_r(\mathbf{a}) := \mathbf{a}'_1 \cdot \mathbf{a}'_2 \cdot \ldots \cdot \mathbf{a}'_\ell$ *such that:*

- $\mathbf{a} = \mathbf{a}'_1 \mathbf{a}'_2 \ldots \mathbf{a}'_\ell$*,*
- $\mathbf{a}'_1, \ldots, \mathbf{a}'_\ell$ *are cliques of* $\mathcal{M}$*,*
- *either* $\ell = 0$ *or* $\mathbf{a}_1 \neq \mathbf{1}$*, and*
- $\mathbf{a}'_i = \alpha_r(\mathbf{a}'_1 \mathbf{a}'_2 \ldots \mathbf{a}'_i)$ *for all* $i \in \{1, \ldots, \ell\}$*.*

Again, requiring that $\mathbf{a}_k \neq \mathbf{1}$ amounts to requiring that $\mathbf{a}_i \neq \mathbf{1}$ for all $i \in \{1, \ldots, k\}$, and requiring that $\mathbf{a}'_1 \neq \mathbf{1}$ amounts to requiring that $\mathbf{a}'_i \neq \mathbf{1}$ for all $i \in \{1, \ldots, \ell\}$.

2.3.5 Artin–Tits Monoids of FC Type

The left Garside normal form, for braid monoids, and the Cartier-Foata normal form, for heap monoids, seem very similar. Hence, we look for a common framework to study both Artin–Tits monoids of spherical type and heap monoids, that would lead to generalisations of both the left Garside normal form and the Cartier-Foata normal form, as well as of simple elements and of cliques. It is possible to do so by considering variants of *Garside families* (see [37]), and we settle for the following framework.

Definition 2.78 (Garside family and two-way Garside family)**.**
Let $\mathbf{A}^+$ *be an Artin–Tits monoid, with generators* $\sigma_1, \ldots, \sigma_n$*. A* Garside family *of the monoid* $\mathbf{A}^+$ *is a set* $\mathbf{S}$ *such that:*

1. $\sigma_1, \ldots, \sigma_n$ *all belong to* $\mathbf{S}$*;*
2. $\mathbf{S}$ *is closed under* $\leqslant_\ell$*-LCM in* $\mathbf{A}^+$*, i.e. every pair of elements of* $\mathbf{S}$ *with a common* $\leqslant_\ell$*-multiple (in* $\mathbf{A}^+$*) has a least common* $\leqslant_\ell$*-multiple, which belongs to* $\mathbf{S}$*;*
3. $\mathbf{S}$ *is closed under* $\geqslant_r$*-division, i.e. for all* $\mathbf{a} \in \mathbf{S}$*, the* $\geqslant_r$*-divisors of* $\mathbf{a}$ *(in* $\mathbf{A}^+$*) belong to* $\mathbf{S}$*.*

If, furthermore, $\mathbf{S}$ *is closed under* $\leqslant_\ell$*-division and under* $\geqslant_r$*-LCM in* $\mathbf{A}^+$*, then we say that* $\mathbf{S}$ *is a* two-way *Garside family.*

While all Artin–Tits monoids admit a finite (one-way) Garside family [38], we aim now at proving Theorem 2.85, which states that an Artin–Tits monoid $\mathbf{A}^+$ admits a finite two-way Garside family if and only if $\mathbf{A}^+$ is an *Artin–Tits monoid of FC type*. Artin–Tits monoids of FC type were introduced by Charney and Davis [29] as a natural extension of both braids and heaps [6, 7, 41, 57, 58].

Definition 2.79 (Artin–Tits monoid of FC type)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid, with generators $\sigma_1, \ldots, \sigma_n$. We say that $\mathbf{A}^+$ is a monoid of FC type *if, for all sets $I \subseteq \{1, \ldots, n\}$, either we have $m_{i,j} = \infty$ for some $i, j \in I$, or the submonoid of $\mathbf{A}^+$ generated by $\{\sigma_i : i \in I\}$ is an Artin–Tits monoid of spherical type.*

First, it comes immediately that any intersection of two-way Garside families is also a two-way Garside family. Consequently, there always exists a *smallest* two-way Garside family, which may be finite or infinite, depending on $\mathbf{A}^+$. Below, we call *strong elements* of $\mathbf{A}^+$ the elements of the smallest two-way Garside family of $\mathbf{A}^+$.

Lemma 2.80.
Artin–Tits monoids of spherical type and heap monoids are Artin–Tits monoids of FC type.

In fact, Artin–Tits monoids of spherical type and heap monoids are prototypical examples of monoids that admit a finite two-way Garside family, as shown below.

Lemma 2.81.
Let $\mathbf{A}^+$ be an Artin–Tits monoid, with generators $\sigma_1, \ldots, \sigma_n$. Let $\mathbf{S}$ be the smallest set containing the generators $\sigma_1, \ldots, \sigma_n$ and such that:

- $\mathbf{S}$ *is closed under $\leqslant_\ell$-division;*
- $\mathbf{S}$ *is closed under* incremental *$\leqslant_\ell$-LCM, i.e.* $\forall \mathbf{a} \in \mathbf{S}, \forall i, j \in \{1, \ldots, n\}, (\{\mathbf{a}\sigma_i, \mathbf{a}\sigma_j\} \subseteq \mathbf{S}$ *and* $i \neq j) \Rightarrow \mathbf{a}[\sigma_i \sigma_j]^{m_{i,j}} \in \mathbf{S}$*;*
- $\mathbf{S}$ *is closed under $\geqslant_r$-division and under* incremental *$\geqslant_r$-LCM.*

The set $\mathbf{S}$ is the smallest two-way Garside family of $\mathbf{A}^+$.

Proof. Each two-way Garside family of $\mathbf{A}^+$ is closed under $\leqslant_\ell$-division, incremental $\leqslant_\ell$-LCM, $\geqslant_r$-division and incremental $\geqslant_r$-LCM. Hence, it remains to prove that $\mathbf{S}$ is closed under $\leqslant_\ell$-LCM and $\geqslant_r$-LCM. The proof is very similar to the proof of Lemma 2.19.

Let $\mathbf{z}$ be an element of $\mathbf{A}^+$, let $\mathbf{a}$ and $\mathbf{b}$ be two $\leqslant_\ell$-divisors of $\mathbf{z}$ that belong to $\mathbf{S}$, and let $\mathbf{c} := \mathbf{GCD}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b})$. We prove by induction on $\lambda(\mathbf{z}) - \lambda(\mathbf{c})$ that $\mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b}) \in \mathbf{S}$. If $\lambda(\mathbf{c}) = \lambda(\mathbf{z})$, then $\lambda(\mathbf{c}) \geqslant \max\{\lambda(\mathbf{a}), \lambda(\mathbf{b})\}$, hence $\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b})$. Hence, let us assume that $\lambda(\mathbf{c}) < \lambda(\mathbf{z})$.

If $\mathbf{a} = \mathbf{c}$, then $\mathbf{b} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b})$. Similarly, if $\mathbf{b} = \mathbf{c}$, then $\mathbf{a} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b})$. Hence, we focus on the case where $\mathbf{c} <_\ell \mathbf{a}$ and $\mathbf{c} <_\ell \mathbf{b}$. Consider two generators σ_i and σ_j of $\mathbf{A}^+$ such that $\mathbf{c}\sigma_i \leqslant_\ell \mathbf{a}$ and $\mathbf{c}\sigma_j \leqslant_\ell \mathbf{b}$. Then, let $\mathbf{d} := \mathbf{c}[\sigma_i \sigma_j]^{m_{i,j}}$ and $\mathbf{e} := \mathbf{GCD}_{\leqslant_\ell}(\mathbf{a}, \mathbf{d})$. Both $\mathbf{c}\sigma_i$ and $\mathbf{c}\sigma_j$ belong to $\mathbf{S}$ and $\leqslant_\ell$-divide $\mathbf{z}$, hence so does $\mathbf{d} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{c}\sigma_i, \mathbf{c}\sigma_j)$. Since $\mathbf{c}\sigma_i \leqslant_\ell \mathbf{e}$, it follows that $\lambda(\mathbf{c}) < \lambda(\mathbf{c}\sigma_i) \leqslant \lambda(\mathbf{e})$, and the induction hypothesis states that the element $\mathbf{z_a} := \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{d})$ belongs to $\mathbf{S}$.

We show similarly that the element $\mathbf{z_b} := \mathbf{LCM}_{\leqslant_\ell}(\mathbf{b}, \mathbf{d})$ belongs to $\mathbf{S}$. Finally observe that $\mathbf{d} \leqslant_\ell \mathbf{GCD}_{\leqslant_\ell}(\mathbf{z_a}, \mathbf{z_b})$, and therefore the induction hypothesis states that the element

$\mathbf{y} := \mathbf{LCM}_{\leqslant_\ell}(\mathbf{z_a}, \mathbf{z_b})$ belongs to $\mathbf{S}$. This shows that

$$\mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b}) = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b}, \mathbf{c}\sigma_i, \mathbf{c}\sigma_j) = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{a}, \mathbf{b}, \mathbf{d}) = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{z_a}, \mathbf{z_b}) = \mathbf{y}$$

belongs to $\mathbf{S}$, which completes the induction. We show similarly that $\mathbf{S}$ is closed under $\geqslant_r$-LCM. □

The proof of Lemma 2.81 is illustrated in Fig. 2.82.

$$\begin{tikzcd}
 & & \mathbf{c}\sigma_i \arrow[r] & \mathbf{e} \arrow[r] \arrow[dr] & \mathbf{a} \arrow[r] & \mathbf{z_a} \arrow[dr] & & \\
\mathbf{1} \arrow[r] & \mathbf{c} \arrow[ur] \arrow[dr] & & & \mathbf{d} \arrow[ur] \arrow[dr] & & \mathbf{y} \arrow[r] & \mathbf{z} \\
 & & \mathbf{c}\sigma_j \arrow[urr] \arrow[rr] & & \mathbf{b} \arrow[r] & \mathbf{z_b} \arrow[ur] & &
\end{tikzcd}$$

Figure 2.82 – $\mathbf{S}$ is closed under $\leqslant_\ell$-LCM

Definition 2.83 (Self-independent element)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid, with generators $\sigma_1, \ldots, \sigma_n$, and let $\mathbf{a}$ be an element of $\mathbf{A}^+$. If there exists two generators σ_i and σ_j of $\mathbf{A}^+$ and three elements $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ of $\mathbf{A}^+$ such that $\sigma_i \neq \sigma_j$, that $m_{i,j} = \infty$ and that $\mathbf{a} = \mathbf{u}\sigma_i\mathbf{v}\sigma_j\mathbf{w}$, then we say that $\mathbf{a}$ is self-dependent*. Otherwise, we say that $\mathbf{a}$ is* self-independent*.*

Proposition 2.84.
Let $\mathbf{A}^+$ be an Artin–Tits monoid. Let $\mathbf{S}$ be the smallest two-way Garside family of $\mathbf{A}^+$, let $\mathcal{S}$ be the set of simple elements of $\mathbf{A}^+$, and let $\mathbb{I}$ be the set of self-independent elements of $\mathbf{A}^+$. We have $\mathbf{S} = \mathcal{S} \cap \mathbb{I}$.

Proof. We first prove by induction on $\lambda(\mathbf{x})$ that $\mathbf{x} \in \mathbf{S}$ for all $\mathbf{x} \in \mathcal{S} \cap \mathbb{I}$. Indeed, if $\lambda(\mathbf{x}) = 0$ or $\lambda(\mathbf{x}) = 1$, then by construction we have $\mathbf{x} \in \mathbf{S}$. Then, if $|\mathbf{right}(\mathbf{x})| \geqslant 2$, consider two generators σ_i and σ_j of $\mathbf{A}^+$ that belong to $\mathbf{right}(\mathbf{x})$. Lemma 2.19 proves that $\mathcal{S}$ is closed under $\leqslant_\ell$-division, and it comes immediately that $\mathbb{I}$ is closed under $\leqslant_\ell$-division too. Consequently, we know that both $\mathbf{x}\sigma_i^{-1}$ and $\mathbf{x}\sigma_j^{-1}$ belong to $\mathcal{S} \cap \mathbb{I}$ and therefore, by induction hypothesis, to $\mathbf{S}$. It follows that $\mathbf{x} = \mathbf{LCM}_{\leqslant_\ell}(\mathbf{x}\sigma_i^{-1}, \mathbf{x}\sigma_j^{-1})$ also belongs to $\mathbf{S}$.

Hence, we treat the case where $\lambda(\mathbf{x}) \geqslant 2$ and $|\mathbf{right}(\mathbf{x})| = 1$. Consider two generators σ_i and σ_j of $\mathbf{A}^+$ such that $\mathbf{x} \geqslant_r \sigma_i\sigma_j$. Since $\mathbf{x} \in \mathcal{S}$, Corollary 2.18 states that $\mathbf{x}$ is σ^2-free, whence $\sigma_i \neq \sigma_j$. Then, let k be the largest integer such that $\mathbf{x} \geqslant [\sigma_i\sigma_j]^{-k}$, where the notation $[\sigma_i\sigma_j]^{-k}$ was introduced in Definition 2.8, and let $\mathbf{y}$ be the element of $\mathbf{A}^+$ such that $\mathbf{x} = \mathbf{y}[\sigma_i\sigma_j]^{-k}$. Since $\mathbf{right}(\mathbf{x}) = \{\sigma_j\}$, it follows that $2 \leqslant k < m_{i,j}$.

Since $\mathbf{x}$ is σ^2-free and since k is maximal, we know that neither σ_i nor σ_j belongs to $\mathbf{right}(\mathbf{y})$, hence that both $\mathbf{y}\sigma_i$ and $\mathbf{y}\sigma_j$ belong to $\mathcal{S}$. Then, since $\mathbf{x}$ is self-independent, we know that so are $\mathbf{y}\sigma_i$ and $\mathbf{y}\sigma_j$. Since $k \geqslant 2$, the induction hypothesis proves that both $\mathbf{y}\sigma_i$

and $\mathbf{y}\sigma_j$ belong to $\mathbf{S}$, and therefore that so does $\mathbf{yLCM}_{\leqslant_\ell}(\sigma_i, \sigma_j)$, which is $\leqslant_\ell$-divisible by $\mathbf{x}$. This means that $\mathbf{x} \in \mathbf{S}$, which completes the induction and proves that $\mathcal{S} \cap \mathbb{I} \subseteq \mathbf{S}$.

Conversely, we prove that $\mathbf{S} \subseteq \mathcal{S} \cap \mathbb{I}$. Theorem 2.14 already proves that $\mathcal{S}$ is a two-way Garside family of $\mathbf{A}^+$, whence $\mathbf{S} \subseteq \mathcal{S}$. Hence, we shall prove that $\mathbf{S} \subseteq \mathbb{I}$. Let $\mathbf{x}$ be some element of $\mathbf{S}$. Lemma 2.81 proves that there exists an integer $k \geqslant 0$ elements $\mathbf{x}_0, \ldots, \mathbf{x}_k$ of $\mathbf{S}$ such that $\mathbf{x}_k = \mathbf{x}$ and, for all $i \in \{0, \ldots, k\}$, either

- $\mathbf{x}_i \in \{\mathbf{1}, \sigma_1, \ldots, \sigma_n\}$;
- $\mathbf{x}_i \leqslant_\ell \mathbf{x}_j$ or $\mathbf{x}_j \geqslant_r \mathbf{x}_i$ for some integer $j \in \{0, \ldots, i-1\}$;
- there exists generators σ_u and σ_v, an element $\mathbf{y}$ of $\mathbf{A}^+$ and integers $j, k \in \{0, \ldots, i-1\}$ such that $\mathbf{x}_j = \mathbf{y}\sigma_u$, $\mathbf{x}_k = \mathbf{y}\sigma_v$ and $\mathbf{x}_i = \mathbf{yLCM}_{\leqslant_\ell}(\sigma_u, \sigma_v)$;
- there exists generators σ_u and σ_v, an element $\mathbf{y}$ of $\mathbf{A}^+$ and integers $j, k \in \{0, \ldots, i-1\}$ such that $\mathbf{x}_j = \sigma_u\mathbf{y}$, $\mathbf{x}_k = \sigma_v\mathbf{y}$ and $\mathbf{x}_i = \mathbf{LCM}_{\geqslant_r}(\sigma_u, \sigma_v)\mathbf{y}$.

We prove by induction on i that $\mathbf{x}_i \in \mathbb{I}$ for all $i \in \{0, \ldots, k\}$:

- If $\mathbf{x}_i \in \{\mathbf{1}, \sigma_1, \ldots, \sigma_n\}$, then of course $\mathbf{x}_i \in \mathbb{I}$.
- If $\mathbf{x}_i \leqslant_\ell \mathbf{x}_j$ or $\mathbf{x}_j \geqslant_r \mathbf{x}_i$ for some integer $j \in \{0, \ldots, i-1\}$, then the induction hypothesis states that $\mathbf{x}_j \in \mathbb{I}$, and therefore $\mathbf{x}_i \in \mathbb{I}$ too.
- If $\mathbf{x}_i = \mathbf{yLCM}_{\leqslant_\ell}(\sigma_u, \sigma_v)$ for some $\mathbf{y} \in \mathbf{A}^+$ and some generators σ_u and σ_v such that $\{\mathbf{y}\sigma_u, \mathbf{y}\sigma_v\} \subseteq \{\mathbf{x}_0, \ldots, \mathbf{x}_{i-1}\}$, then the induction hypothesis states that both $\mathbf{y}\sigma_u$ and $\mathbf{y}\sigma_v$ belong to $\mathbb{I}$. Hence, let $\mathbb{L}$ be the set $\{u, v\} \cup \{k : \sigma_k$ is a factor of $\mathbf{y}\}$. Since both $\mathbf{y}\sigma_u$ and $\mathbf{y}\sigma_v$ belong to $\mathbb{I}$, it follows that $m_{y,z} \neq 0$ for each pair (y, z) of elements of $\mathbb{L}$ such that $y \neq z$. This proves that $\mathbf{x}_i = \mathbf{y}[\sigma_u\sigma_v]^{m_{u,v}}$ also belongs to $\mathbb{I}$.
- Likewise, if $\mathbf{x}_i = \mathbf{LCM}_{\geqslant_r}(\sigma_u, \sigma_v)\mathbf{y}$ for some $\mathbf{y} \in \mathbf{A}^+$ and some generators σ_u and σ_v such that $\{\sigma_u\mathbf{y}, \sigma_v\mathbf{y}\} \subseteq \{\mathbf{x}_0, \ldots, \mathbf{x}_{i-1}\}$, then $\mathbf{x}_i \in \mathbb{I}$.

This completes the induction and the proof of Proposition 2.84. □

Theorem 2.85.
Let $\mathbf{A}^+$ be an Artin–Tits monoid. The monoid $\mathbf{A}^+$ admits a finite two-way Garside family if and only if $\mathbf{A}^+$ is of FC type.

Proof. Let $\mathbf{S}$ be the smallest two-way Garside family of $\mathbf{A}^+$. Let us first assume that, for some set $I \subseteq \{1, \ldots, n\}$, we have $m_{i,j} < +\infty$ for all $i, j \in I$, and that the submonoid of $\mathbf{A}^+$ generated by $\{\sigma_i : i \in I\}$ is not an Artin–Tits monoid of spherical type. Let $\mathbf{B}^+$ be that submonoid, and let $\mathcal{S}_\mathbf{B}$ be the set of simple elements of $\mathbf{B}^+$. Theorem 2.14 states that $\mathcal{S}_\mathbf{B}$ is infinite. Since $\mathcal{S}_\mathbf{B} \subseteq \mathcal{S}$ and $\mathcal{S}_\mathbf{B} \subseteq \mathbf{B}^+ \subseteq \mathbb{I}$, it follows that $\mathcal{S}_\mathbf{B} \subseteq \mathbf{S}$, which proves that $\mathbf{S}$ is infinite.

Conversely, let us assume that $\mathbf{S}$ is infinite. This means that $\mathbf{S}$ contains elements of arbitrarily large length. Hence, for all $k \geqslant 0$, let $\mathbf{a}_k$ be an element of $\mathbf{S}$ such that $\lambda(\mathbf{a}_k) = k$, and let L_k be the set of letters that appear in $\mathbf{a}_k$. Then, let $\mathcal{L}$ be a subset of $\{1, \ldots, n\}$ such that the set $\Omega := \{k \geqslant 0 : L_k = \{\sigma_i : i \in \mathcal{L}\}\}$ is infinite. In addition, let $\mathbf{C}^+$ be the submonoid of $\mathbf{A}^+$ generated by $\{\sigma_i : i \in \mathcal{L}\}$, and let $\mathcal{S}_\mathbf{C}$ be the set of simple elements of $\mathbf{C}^+$. It necessarily follows that $m_{i,j} < +\infty$ for all $i, j \in \mathcal{L}$, since each word $\mathbf{a}_k$ belongs to $\mathbb{I}$ when $k \in \Omega$. Moreover, observe that each $\mathbf{a}_k$ is a σ^2-free element of $\mathbf{C}^+$ when $k \in \Omega$, whence $\mathbf{a}_k \in \mathcal{S}_{\mathbf{C}}$. It follows that $\mathcal{S}_{\mathbf{C}}$ is infinite, i.e. that $\mathbf{C}^+$ is not an Artin–Tits monoid of spherical type. This completes the proof. □

Let us restate Theorem 2.85 in terms of Coxeter diagrams. Recall that the Coxeter diagram of the monoid $\mathbf{A}^+$ is the graph $\mathbf{G}$ with vertices $1, \ldots, n$, and in which two vertices i and j are linked by:

- an unlabelled, white edge if $m_{i,j} = \infty$;
- no edge if $m_{i,j} = 2$;
- an unlabelled, black edge if $m_{i,j} = 3$;
- a black edge with label $m_{i,j}$ if $m_{i,j} \geqslant 4$.

An Artin–Tits monoid $\mathbf{A}^+$ is strong if the induced subgraphs of its Coxeter diagram either contain a ○═○ edge (i.e. a white edge) or are Coxeter diagrams of Artin–Tits monoids of spherical type. This amounts to saying that each maximal ○═○-independent induced subgraph of the Coxeter diagram of $\mathbf{A}^+$ can be split into connected components which are isomorphic to the diagrams classified in Fig. 2.27.

Example 2.86.
Consider the matrix $M := (m_{i,j})_{1 \leqslant i,j \leqslant 6}$ such that

$$M = \begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 3 \\ 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 \\ 2 & 2 & 3 & 1 & 4 & \infty \\ 2 & 2 & 2 & 4 & 1 & 3 \\ 3 & 2 & 2 & \infty & 3 & 1 \end{pmatrix},$$

and let $\mathbf{A}_M^+$ be the Artin–Tits monoid defined by

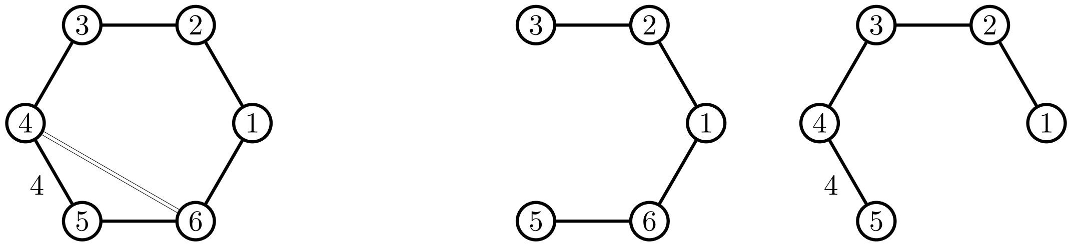
$$\mathbf{A}_M^+ := \langle \sigma_1, \ldots, \sigma_6 \mid [\sigma_i \sigma_j]^{m_{i,j}} = [\sigma_j \sigma_i]^{m_{i,j}} \rangle^+,$$

whose Coxeter diagram is presented in Fig. 2.88.

Its maximal ○═○-independent induced subgraphs are outlined in Fig. 2.88, and are isomorphic to the Coxeter diagrams of type A_5 and B_5. Hence, $\mathbf{A}_M^+$ is an Artin–Tits monoid of FC type.

It follows from Theorem 2.85 that the smallest (one-way) Garside family of an Artin–Tits monoid $\mathbf{A}^+$ may be a strict subset of the smallest two-way Garside family of $\mathbf{A}^+$, as shown below.

Proposition 2.87.
Let $\mathbf{A}^+$ be an Artin–Tits monoid, let $\mathbf{S}_1$ be the smallest Garside family of $\mathbf{A}^+$, and let $\mathbf{S}_2$ be the smallest two-way Garside family of $\mathbf{A}^+$. The inclusion $\mathbf{S}_1 \subseteq \mathbf{S}_2$ holds, with equality if and only if $\mathbf{A}^+$ has FC type.



Coxeter diagram — Maximal $\circ\!\!=\!\!\circ$-free induced subgraphs

Figure 2.88 – A Coxeter diagram and its maximal $\circ\!\!=\!\!\circ$-free induced subgraphs

Proof. Since each two-way Garside family is also a (one-way) Garside family, the inclusion relation $\mathbf{S}_1 \subseteq \mathbf{S}_2$ is straightforward. Then, if $\mathbf{A}^+$ has FC type, let $\mathbf{x}$ be some element of $\mathbf{S}_2$, and let $\mathbf{L}(\mathbf{x})$ denote the set of Artin generators $\{\sigma_i : \exists \mathbf{u}, \mathbf{v} \in \mathbf{A}^+, \mathbf{x} = \mathbf{u}\sigma_i\mathbf{v}\}$. It follows from Proposition 2.84 that $\mathbf{x} \in \mathcal{S} \cap \mathbb{I} \cap \langle \mathbf{L}(\mathbf{x})\rangle^+$ and that the monoid $\langle \mathbf{L}(\mathbf{x})\rangle^+$ has spherical type. Therefore, $\mathbf{x}$ is a simple element of the monoid generated by $\mathbf{L}(\mathbf{x})$, i.e. a $\geqslant_r$-divisor of the element $\Delta_{\mathbf{L}(\mathbf{x})} := \mathbf{LCM}(\mathbf{L}(\mathbf{x}))$. By construction, the element $\Delta_{\mathbf{L}(\mathbf{x})}$ belongs to $\mathbf{S}_1$, and therefore so does $\mathbf{x}$ belong to $\mathbf{S}_1$, which proves that $\mathbf{S}_1 = \mathbf{S}_2$.

Conversely if $\mathbf{A}^+$ does not have FC type, then we mentioned above that $\mathbf{S}_1$ is finite, whereas $\mathbf{S}_2$ is infinite, whence $\mathbf{S}_1 \subsetneq \mathbf{S}_2$. □

Proposition 2.87 is illustrated by Example 2.89 (when $\mathbf{A}^+$ does *not* have FC type) and by Example 2.91 (when $\mathbf{A}^+$ has FC type).

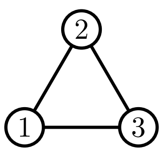
Example 2.89.
Consider the matrices $M := (m_{i,j})_{1 \leqslant i,j \leqslant 3}$ and $N := (n_{i,j})_{1 \leqslant i,j \leqslant 3}$ such that

$$M = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}.$$

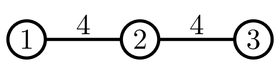
Let $\tilde{\mathbf{A}}_2^+$ and $\tilde{\mathbf{C}}_2^+$ be the Artin–Tits monoids defined by

$$\tilde{\mathbf{A}}_2^+ := \langle \sigma_1, \sigma_2, \sigma_3 \mid [\sigma_i\sigma_j]^{m_{i,j}} = [\sigma_j\sigma_i]^{m_{i,j}}\rangle^+ ; \tilde{\mathbf{C}}_2^+ := \langle \sigma_1, \sigma_2, \sigma_3 \mid [\sigma_i\sigma_j]^{n_{i,j}} = [\sigma_j\sigma_i]^{n_{i,j}}\rangle^+.$$

According to Theorem 2.26, these monoids are not of spherical type, and therefore are not of FC type either, as indicated by their respective Coxeter diagrams:



$\tilde{\mathbf{A}}_2^+$



$\tilde{\mathbf{C}}_2^+$

Hence, their smallest (finite) Garside families, which are represented in Fig. 2.90, are strict subsets of their smallest (infinite) two-way Garside families. In Figure 2.90, we represent the relation $u \leqslant_\ell v$ by (the reflexive transitive closure of) arrows that go from u to v, and we omit representing the relation $u \geqslant_r v$.

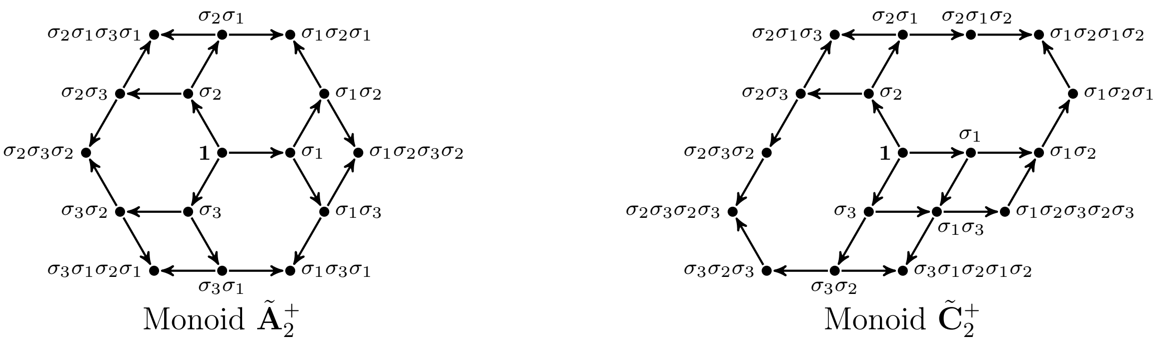


Figure 2.90 – Smallest Garside families in the monoids $\tilde{\mathbf{A}}_2^+$ and $\tilde{\mathbf{C}}_2^+$

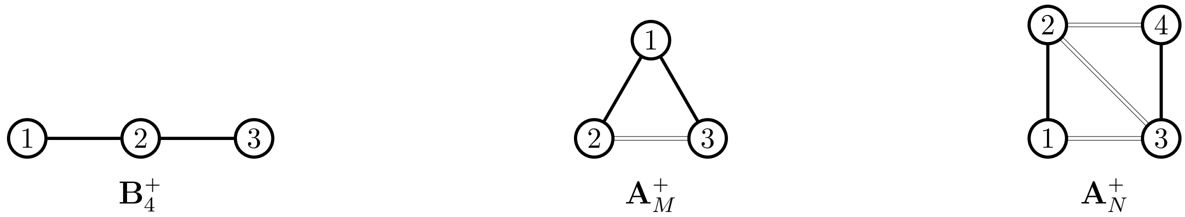
Example 2.91.
Consider the matrices $M := (m_{i,j})_{1 \leqslant i,j \leqslant 3}$ and $N := (n_{i,j})_{1 \leqslant i,j \leqslant 4}$ such that

$$M = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & \infty \\ 3 & \infty & 1 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 3 & \infty & \infty \\ 3 & 1 & \infty & \infty \\ \infty & \infty & 1 & 3 \\ \infty & \infty & 3 & 1 \end{pmatrix}.$$

Let $\mathbf{A}_M^+$ and $\mathbf{A}_N^+$ be the Artin–Tits monoids defined by

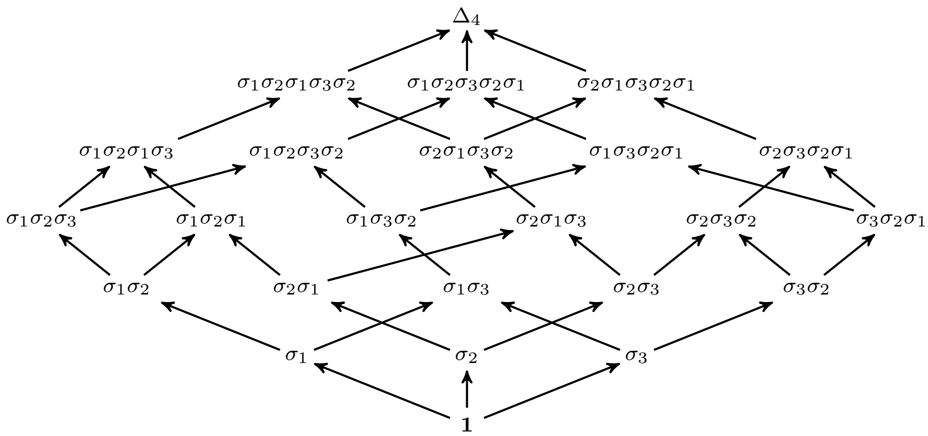
$$\mathbf{A}_M^+ := \langle \sigma_1, \sigma_2, \sigma_3 \mid [\sigma_i \sigma_j]^{m_{i,j}} = [\sigma_j \sigma_i]^{m_{i,j}} \rangle^+; \mathbf{A}_N^+ := \langle \sigma_1, \ldots, \sigma_4 \mid [\sigma_i \sigma_j]^{n_{i,j}} = [\sigma_j \sigma_i]^{n_{i,j}} \rangle^+.$$

The braid monoid $\mathbf{B}_4^+$ and the monoids $\mathbf{A}_M^+$ and $\mathbf{A}_N^+$ have FC type, as indicated by their respective Coxeter diagrams:

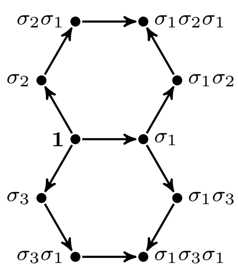


Hence, their smallest one-way and two-way Garside families, which are represented in Fig. 2.92, are equal to each other.

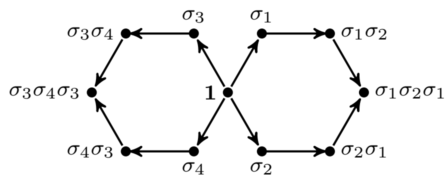
We generalise Garside normal forms to Artin–Tits monoids of FC type, using the functions $\alpha_\ell : \mathbf{a} \mapsto \mathbf{GCD}_{\leqslant_\ell}(\{\mathbf{x} \in \mathbf{S} : \mathbf{x} \leqslant_\ell \mathbf{a}\})$ and $\alpha_r : \mathbf{a} \mapsto \mathbf{GCD}_{\geqslant_r}(\{\mathbf{x} \in \mathbf{S} : \mathbf{a} \geqslant_r \mathbf{x}\})$.



Monoid $\mathbf{B}_4^+$



Monoid $\mathbf{A}_M^+$



Monoid $\mathbf{A}_N^+$

Figure 2.92 – Smallest (two-way) Garside families in the monoids $\mathbf{B}_4^+$, $\mathbf{A}_M^+$ and $\mathbf{A}_N^+$

Definition 2.93 (Garside normal forms in the monoid $\mathbf{A}^+$)**.**
Let $\mathbf{A}^+$ *be an Artin–Tits monoid of FC type and let* $\mathbf{a}$ *be an element of* $\mathbf{A}^+$*. The* left Garside normal form *of* $\mathbf{a}$ *is defined as the word* $\mathbf{NF}_\ell(\mathbf{a}) := \mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k$ *such that:*

- $\mathbf{a} = \mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_k$,
- $\mathbf{a}_1, \ldots, \mathbf{a}_k$ *are strong elements of* $\mathbf{A}^+$,
- *either* $k = 0$ *or* $\mathbf{a}_k \neq \mathbf{1}$, *and*
- $\mathbf{a}_i = \alpha_\ell(\mathbf{a}_i\mathbf{a}_{i+1} \ldots \mathbf{a}_k)$ *for all* $i \in \{1, \ldots, k\}$.

The right Garside normal form *of* $\mathbf{a}$ *is defined as the word* $\mathbf{NF}_r(\mathbf{a}) := \mathbf{a}'_1 \cdot \mathbf{a}'_2 \cdot \ldots \cdot \mathbf{a}'_\ell$ *such that:*

- $\mathbf{a} = \mathbf{a}'_1\mathbf{a}'_2 \ldots \mathbf{a}'_\ell$,
- $\mathbf{a}'_1, \ldots, \mathbf{a}'_\ell$ *are strong elements of* $\mathbf{A}^+$,
- *either* $\ell = 0$ *or* $\mathbf{a}_1 \neq \mathbf{1}$, *and*
- $\mathbf{a}'_i = \alpha_r(\mathbf{a}'_1\mathbf{a}'_2 \ldots \mathbf{a}'_i)$ *for all* $i \in \{1, \ldots, \ell\}$.

Like in Garside monoids and in heap monoids, requiring that $\mathbf{a}_k \neq \mathbf{1}$ amounts to requiring that $\mathbf{a}_i \neq \mathbf{1}$ for all $i \in \{1, \ldots, k\}$, and requiring that $\mathbf{a}'_1 \neq \mathbf{1}$ amounts to requiring that $\mathbf{a}'_i \neq \mathbf{1}$ for all $i \in \{1, \ldots, \ell\}$.

Note that the Cartier-Foata normal form for heap monoids is in fact a specific instance of left Garside normal forms for heap monoids. Hence, replacing the *set of simple elements* $\mathcal{S}$ of a Garside monoid $\mathbf{G}^+$ by the *set of strong elements* $\mathbf{S}$ of an Artin–Tits monoid of FC type $\mathbf{A}^+$, we generalise immediately the Definitions, Lemmas, Corollaries and Propositions 2.37 to 2.42, as well as the notations $\mathbf{a} \longrightarrow \mathbf{b}$ and $\mathbf{a} \longleftarrow \mathbf{b}$.

In particular, we redefine the notions of left and right outgoing set of a strong element. These notions have simple characterisations.

Definition & Proposition 2.94 (Incompatibility set)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type, with smallest two-way Garside family $\mathbf{S}$, and let $\mathbf{x}$ be an element of $\mathbf{S}$. We call set of letters *of $\mathbf{x}$, and denote by $\mathbb{L}(\mathbf{x})$, the set $\{\sigma_i : \exists \mathbf{u}, \mathbf{v} \in \mathbf{A}^+, \mathbf{x} = \mathbf{u}\sigma_i\mathbf{v}\}$. We also call* incompatibility set *of $\mathbf{x}$, and denote by $\mathbb{I}(\mathbf{x})$, the set $\{\sigma_i : \exists \sigma_j \in \mathbb{L}(\mathbf{x}), m_{i,j} = \infty\}$. The left and right outgoing sets of $\mathbf{x}$ satisfy the relations*

$$\overline{\mathbf{left}}(\mathbf{x}) = \mathbf{left}(\mathbf{x}) \cup \mathbb{I}(\mathbf{x}) \text{ and } \overline{\mathbf{right}}(\mathbf{x}) = \mathbf{right}(\mathbf{x}) \cup \mathbb{I}(\mathbf{x}).$$

Proof. Both relations follow immediately from the set equality $\mathbf{S} = \mathbb{I} \cap \mathcal{S}$. □

Furthermore, a result analogous to Proposition 2.49 also holds in the framework of Artin–Tits monoids of FC type.

Proposition 2.95.
Let u be a non-negative integer, let $\mathbf{b}$ be an element of the monoid $\mathbf{A}^+$, and let $\mathbf{S}^u$ be the product set $\{b_1 \dots b_u : b_1, \dots, b_u \in \mathbf{S}\}$. The following statements are equivalent:

1. $\mathbf{b} \in \mathbf{S}^u$;
2. $|\mathbf{NF}_\ell(\mathbf{b})| \leqslant u$;
3. $|\mathbf{NF}_r(\mathbf{b})| \leqslant u$.

Moreover, if $u \geqslant 1$, then $\mathbf{S}^u$ is a two-way Garside family.

Proof. First, the implication $2 \Rightarrow 1$ is immediate. Second, we show by induction on $|\mathbf{NF}_\ell(\mathbf{b})|$ that $|\mathbf{NF}_\ell(\mathbf{b})| \leqslant |\mathbf{NF}_\ell(\mathbf{ab})|$ for all pairs $(\mathbf{a}, \mathbf{b}) \in \mathbf{S} \times \mathbf{A}^+$. This induction is exactly the same as the analogous one used for proving Proposition 2.49 in the case of standard Garside groups.

Third, we prove by induction on u that a $1 \Rightarrow 2$. First, the statement is immediate if $u \leqslant 1$. Then, if $u \geqslant 1$, let $b_1, \dots, b_u$ be strong elements and let $\mathbf{b} := b_1 \dots b_u$. Since $b_1 \leqslant_\ell \alpha_\ell(\mathbf{b})$, let $\mathbf{c}$ and $\mathbf{d}$ be the elements of $\mathbf{A}^+$ such that $b_1\mathbf{c} = \alpha_\ell(\mathbf{b})$ and $\mathbf{cd} = b_2 \dots b_u$. Since $\alpha_\ell(\mathbf{b}) \geqslant_r \mathbf{c}$, we know that $\mathbf{c} \in \mathbf{S}$, hence that $|\mathbf{NF}_\ell(\mathbf{d})| \leqslant |\mathbf{NF}_\ell(\mathbf{cd})|$, and the induction hypothesis states that $|\mathbf{NF}_\ell(\mathbf{cd})| \leqslant u - 1$. Since $\mathbf{NF}_\ell(\mathbf{b}) = \alpha_\ell(\mathbf{b}) \cdot \mathbf{NF}_\ell(\mathbf{d})$, it follows that $|\mathbf{NF}_\ell(\mathbf{b})| \leqslant u$. This completes our second induction, and proves that $1 \Rightarrow 2$. One shows analogously that $1 \Leftrightarrow 3$.

It remains to show that $\mathbf{S}^u$ is a two-way Garside family. First, each generator σ_i of $\mathbf{A}^+$ belongs to $\mathbf{S}^u$ and $\mathbf{S}^u$ is closed under $\leqslant_\ell$-division and $\geqslant_r$-division. Now, we show by induction on u that $\mathbf{S}^u$ is closed under $\leqslant_\ell$-LCM. First, the statement is immediate if $u = 1$. Then, if $u \geqslant 1$, let a_1, b_1 be elements of $\mathbf{S}$, and let a_2, b_2 be elements of $\mathbf{S}^u$, such that $a_1 a_2$ and $b_1 b_2$ have some $\leqslant_\ell$-multiple $\mathbf{d}$. Since $\mathbf{d}$ is a $\leqslant_\ell$-multiple of a_1 and b_1, we know that $c_1 := \mathbf{LCM}_{\leqslant_\ell}(a_1, b_1)$ belongs to $\mathbf{S}$.

Let $\overline{a}_1$ and $\overline{b}_1$ be strong elements such that $c_1 = a_1\overline{a}_1 = b_1\overline{b}_1$, and let d_2 be an element of $\mathbf{A}^+$ such that $c_1 d_2 = \mathbf{d}$. Since both $a_1 a_2$ and $a_1\overline{a}_1 = \mathbf{LCM}_{\leqslant_\ell}(a_1, b_1)$ divide $\mathbf{d}$, we know that a_2 and $\overline{a}_1$ have a common $\leqslant_\ell$-multiple. Hence, consider the element $\overline{a}_2$ of $\mathbf{A}^+$ such that $\mathbf{LCM}_{\leqslant_\ell}(a_2, \overline{a}_1) = \overline{a}_1\overline{a}_2$. By induction hypothesis, we know that both $\overline{a}_1\overline{a}_2$ and $\overline{a}_2$ belong to $\mathbf{S}^{u-1}$. Moreover, observe that

$$c_1\overline{a}_2 = a_1\overline{a}_1\overline{a}_2 = a_1\mathbf{LCM}_{\leqslant_\ell}(a_2, \overline{a}_1) = \mathbf{LCM}_{\leqslant_\ell}(a_1 a_2, a_1\overline{a}_1) \leqslant_\ell \mathbf{d} = c_1 d_2,$$

whence $\overline{a}_2 \leqslant d_2$.

Similarly, there exists an element $\overline{b}_2$ of $\mathbf{A}^+$ such that $\mathbf{LCM}_{\leqslant_\ell}(b_2, \overline{b}_1) = \overline{b}_1\overline{b}_2$, and we know both that $\overline{b}_2 \in \mathbf{S}^{u-1}$ and that $\overline{b}_2 \leqslant d_2$. Hence, the induction hypothesis proves that $c_2 := \mathbf{LCM}_{\leqslant_\ell}(\overline{a}_2, \overline{b}_2)$ belongs to $\mathbf{S}^{u-1}$. Then, observe that $a_1 a_2 \leqslant_\ell c_1\overline{a}_2 \leqslant_\ell c_1 c_2$ and that, similarly, $b_1 b_2 \leqslant c_1 c_2$. This proves that $a_1 a_2$ and $b_1 b_2$ have a common $\leqslant_\ell$-multiple in $\mathbf{S}^u$, and since $\mathbf{S}^u$ is closed under $\leqslant_\ell$-division, it follows that $\mathbf{LCM}_{\leqslant_\ell}(a_1 a_2, b_1 b_2) \in \mathbf{S}^u$. This completes the proof that $\mathbf{S}^u$ is closed under $\leqslant_\ell$-LCM, and one shows analogously that $\mathbf{S}$ is closed under $\geqslant_r$-LCM. □

The proof of Proposition 2.95 is illustrated in Fig. 2.96. Plain black lines with double arrows indicate multiplications to the right by some element of $\mathbf{S}$, plain black lines with simple arrows indicate multiplications by some element of $\mathbf{S}^{u-1}$, and the unique dotted line (with a simple arrow) indicates a multiplication by some element of $\mathbf{A}^+$.

$$\begin{tikzcd}
& a_1 \arrow[r] \arrow[dr, two heads] & b_1b_2 \arrow[r] & c_1\overline{b}_2 \arrow[dr] & & \\
\mathbf{1} \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dr, two heads] & & c_1 \arrow[ur, two heads] \arrow[rr] \arrow[dr, two heads] & & c_1c_2 \arrow[r, dotted] & c_1d_2 \\
& a_2 \arrow[r] \arrow[ur, two heads] & a_1a_2 \arrow[r] & c_1\overline{a}_2 \arrow[ur] & &
\end{tikzcd}$$

Figure 2.96 – $\mathbf{S}^u$ is closed under $\leqslant_\ell$-LCM

Corollary 2.97.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type and let $\mathbf{b}$ be an element of $\mathbf{A}^+$. The Garside words $\mathbf{NF}_\ell(\mathbf{b})$ and $\mathbf{NF}_r(\mathbf{b})$ have the same length, i.e. $|\mathbf{NF}_\ell(\mathbf{b})| = |\mathbf{NF}_r(\mathbf{b})|$.

Hence, we also denote by $\|\mathbf{b}\|$ both lengths $|\mathbf{NF}_\ell(\mathbf{b})|$ and $|\mathbf{NF}_r(\mathbf{b})|$.

Note the duality between the Garside element Δ^u, in the case of a (standard) Garside monoid, and the two-way Garside family $\mathbf{S}^u = \{b_1 \ldots b_u : b_1, \ldots b_u \in \mathbf{S}\}$. In particular, if $\mathbf{A}^+$ is a Garside monoid, then we have $\mathbf{S} = \mathcal{S}$ and $\mathbf{S}^u = \mathcal{S}^u = \{\mathbf{a} \in \mathbf{A}^+ : \mathbf{a} \leqslant \Delta^u\}$.

Furthermore, for all integers $u \geqslant 1$, it makes sense to consider the functions $\alpha_\ell^u : \mathbf{a} \mapsto \mathbf{LCM}_{\leqslant_\ell}(\{\mathbf{x} \in \mathbf{S}^u : \mathbf{x} \leqslant_\ell \mathbf{a}\})$ and $\alpha_r^u : \mathbf{a} \mapsto \mathbf{LCM}_{\geqslant_r}(\{\mathbf{x} \in \mathbf{S}^u : \mathbf{a} \geqslant_r \mathbf{x}\})$ and the associated Garside normal forms.

Proposition 2.98.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type, with two-way Garside family $\mathbf{S}$, let $u \geqslant 1$ be a positive integer, and let $\mathbf{x}$ be an element of $\mathbf{A}^+$. Let $x_1 \cdot \ldots \cdot x_k$ be the left Garside normal form of $\mathbf{x}$ with respect to the two-way Garside family $\mathbf{S}$, and let $y_1 \cdot \ldots \cdot y_\ell$ be the left Garside normal form of $\mathbf{x}$ with respect to the two-way Garside family $\mathbf{S}^u$. We have $\ell = \lceil k/u \rceil$, and $y_i = x_{u(i-1)+1} x_{u(i-1)+2} \ldots x_{\max\{k,ui\}}$ for all $i \in \{1, \ldots, \ell\}$.

In addition, from Corollary 2.38 and Proposition 2.95 follow *dual* statements.

Lemma 2.99.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type, with two-way Garside family $\mathbf{S}$. Let I be a subset of $\{\sigma_1, \ldots, \sigma_n\}$ that has some common multiple, and let $\Delta_I := \mathbf{LCM}_{\leqslant_\ell}(I) = \mathbf{LCM}_{\geqslant_r}(I)$. Then, let $\mathbf{x}$ be a non-trivial element of $\mathbf{A}^+$, and let $x_1 \cdot \cdots \cdot \ldots \cdot x_k$ be the left Garside normal form of $\mathbf{x}$:

- *if $I \cap \overline{\mathbf{right}}(x_k) = \varnothing$, then $\|\mathbf{x}\Delta_I\| = k$;*
- *if $I \subseteq \overline{\mathbf{right}}(x_k)$, then $\mathbf{NF}_\ell(\mathbf{x}\Delta_I) = \mathbf{NF}_\ell(\mathbf{x}) \cdot \Delta_I$;*
- *in all other cases, we have $\|\mathbf{x}\Delta_I\| = k+1$, and $\mathbf{x} <_\ell \alpha_\ell^k(\mathbf{x}\Delta_I) <_\ell \mathbf{x}\Delta_I$.*

Proof. First, if $I \cap \overline{\mathbf{right}}(x_k) = \varnothing$, then $x_k\sigma_i \in \mathbf{S}$ for all $\sigma_i \in I$, and therefore $x_k\Delta_I = \mathbf{LCM}_{\leqslant_\ell}(\{x_k\sigma_i : \sigma_i \in I\})$ also belongs to $\mathbf{S}$. Hence, Proposition 2.95 states that $\mathbf{xy} = x_1 x_2 \ldots x_{k-1}(x_k\Delta_I)$ belongs to $\mathbf{S}^k$, i.e. that $\|\mathbf{x}\Delta_I\| \leqslant k$. Since $k = \|\mathbf{x}\| \leqslant \|\mathbf{x}\Delta_I\|$, it follows that $\|\mathbf{x}\Delta_I\| = k$.

Second, Corollary 2.38 already proves that $\mathbf{NF}_\ell(\mathbf{x}\Delta_I) = \mathbf{NF}_\ell(\mathbf{x}) \cdot \Delta_I$ if and only if $I \subseteq \overline{\mathbf{right}}(x_k)$.

Hence, if none of the relations $I \cap \overline{\mathbf{right}}(x_k) = \varnothing$ or $I \subseteq \overline{\mathbf{right}}(x_k)$ holds, we know both that $\mathbf{NF}_\ell(\mathbf{x}\Delta_I) \neq \mathbf{NF}_\ell(\mathbf{x}) \cdot \Delta_I$ and that there exists a generator $\sigma_i \in I \cap \overline{\mathbf{right}}(x_k)$. Since σ_i is atomic, we have $\mathbf{left}(\sigma_i) = \{\sigma_i\}$, whence Corollary 2.38 proves that $x_1 \cdot \ldots \cdot x_k \cdot \sigma_i$ is in left Garside normal form. It follows that $k+1 \leqslant \|\mathbf{x}\sigma_i\| \leqslant \|\mathbf{x}\Delta_I\|$ and $\alpha_\ell^k(\mathbf{x}\Delta_I) < \mathbf{x}\Delta_I$. Moreover, $\mathbf{x}\Delta_I$ belongs to $\mathbf{S}^{k+1}$, hence Proposition 2.95 states that $k+1 = \|\mathbf{x}\Delta_I\|$, and therefore that $\mathbf{x} <_\ell \alpha_\ell^k(\mathbf{x}\Delta_I) <_\ell \mathbf{x}\Delta_I$. □

2.4 A Geometric Approach to Braids

Having covered briefly some standard results about the algebraic and combinatorial nature of braid groups and monoids, let us now focus on the topological nature of braids. Theorems 2.3 and 2.6 were first milestones in that respect, outlining the connection between braids and topological structures such as isotopy groups of braid diagrams and fundamental groups of configuration spaces.

Like the algebraic approach that we treated above, the geometric approach to braids has been abundantly treated in the literature [13, 39, 48]. Hence, we review here some standard results that concern the geometric nature of braids, and that we will use in Chapters 3 and 4.

2.4.1 Braids, Laminations and Curve Diagrams

Let $D^2 \subseteq \mathbb{C}$ be the closed unit disk, let ∂D^2 be the unit circle (i.e. the boundary of D^2), and let $P_n \subseteq (-1, 1)$ be a set of cardinality n. Let H_n be the group of orientation-preserving homeomorphisms $h : \mathbb{C} \to \mathbb{C}$ such that $h(P_n) = P_n$, $h(\partial D^2) = \partial D^2$ and $h(1) = 1$, $h(-1) = -1$, i.e. the homeomorphisms fixing ∂D^2 and P_n setwise, and ± 1 pointwise. The following result is equivalent to Theorem 2.3.

Theorem 2.100.
The group $\mathbf{B}_n$ of braids on n strands is isomorphic to the mapping class group of the punctured disk $D^2 \backslash P_n$, i.e. isomorphic to the quotient group of H_n by the isotopy relation.

Remarkably, this definition does *not* depend on which set P_n we consider. We will refer below to the elements of P_n as being *mobile punctures* in the disk D^2, and number them from left to right: $P_n = \{p_i : 1 \leqslant i \leqslant n\}$, with $p_1 < \ldots < p_n$. We also call *left point*, or *fixed puncture*, the point -1, which we also denote by p_0; we call *right point* the point $+1$, which we also denote by p_{n+1}.

There exist many variants of Theorem 2.100, which offer flexibility when considering braids as isotopy classes of homeomorphisms. We present below such two such variants.

Let H_n^* be the group of orientation-preserving homeomorphisms $h : \mathbb{C} \to \mathbb{C}$ such that $h(P_n) = P_n$, $h(-1) = -1$ and $h(1) = 1$, i.e. fixing $\{-1\}$, $\{1\}$ and P_n setwise.

Theorem 2.101.
The group $\mathbf{B}_n$ of braids on n strands is isomorphic to the quotient group of H_n^* by the isotopy relation.

Let $H_n^\diamond$ be the group of orientation-preserving homeomorphisms $h : \mathbb{C} \mapsto \mathbb{C}$ such that $h(P_n) = P_n$, $h(\mathbb{R}_{<-1}) = \mathbb{R}_{<-1}$ and $h(\partial D^2) = \partial D^2$, where $\mathbb{R}_{<-1}$ denotes the open real interval $(-\infty, -1)$.

Theorem 2.102.
The group $\mathbf{B}_n$ of n-strand braids is isomorphic to the quotient group of $H_n^\diamond$ by the isotopy relation.

Hence, a braid is the isotopy class $[h]$ of some homeomorphism h. According to standard notations for braids, we will let braids act on the complex plane *on the right*, i.e. denote by $[g][h]$ the isotopy class of the homeomorphism $[h \circ g]$: composition to the left gives rise to a braid multiplication to the right, and vice-versa.

Theorems 2.100, 2.101 and 2.102 identify braids with isotopy classes of homeomorphisms of $\mathbb{C}$. More precisely, let $\mathbf{S}$ be a subset of the complex plane and let β be a braid, and consider the isotopy class $\mathbf{S}\cdot\beta = \{h(\mathbf{S}) : h \text{ is an homeomorphism that represents } \beta\}$. The group of braids $\mathbf{B}_n$ acts transitively on the set $\{\mathbf{S}\cdot\beta : \beta \in \mathbf{B}_n\}$, which induces an equivalence relation on the group $\mathbf{B}_n$ itself.

We focus below on two subsets of $\mathbb{C}$. We call respectively *trivial closed lamination*, *trivial open lamination* and *trivial curve diagram* the sets

$$\begin{aligned}
&\mathbf{L}^c := \{z : |2z + 1 - p_j| = |1 + p_{j+1}| : 0 \leqslant j \leqslant n\},\\
&\mathbf{L}^o := \left\{\tfrac{1}{2}(p_j + p_{j+1}) + iz : z \in \mathbb{R}, 1 \leqslant j \leqslant n-1\right\}, \text{ and}\\
&\mathbf{D} := [-1, 1].
\end{aligned}$$

From a geometric point of view, the set $\mathbf{L}^c$ is a collection of circles, $\mathbf{L}^o$ is a collection of vertical lines, and $\mathbf{D}$ is a horizontal segment.

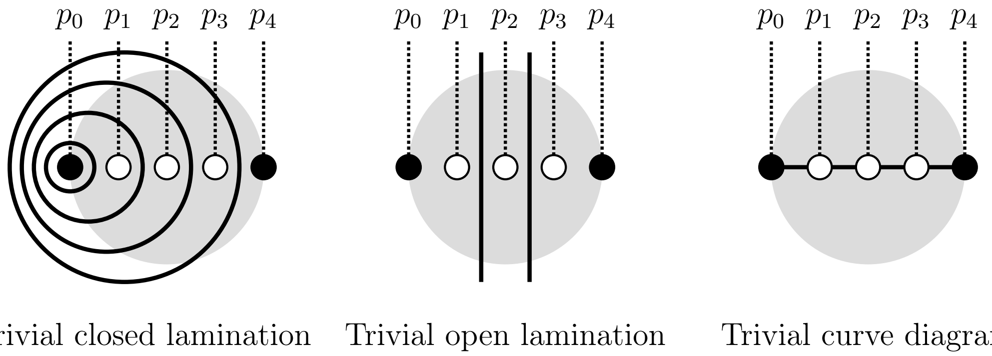


Figure 2.103 – Trivial closed lamination, open lamination and curve diagram

If $\mathbf{S} = \mathbf{L}^c$, $\mathbf{S} = \mathbf{L}^o$ or $\mathbf{S} = \mathbf{D}$, then the action of $\mathbf{B}_n$ on $\{\mathbf{S}\cdot\beta : \beta \in \mathbf{B}_n\}$ is free (see [13] for details). This means that the isotopy classes $\mathbf{S}\cdot\beta$ and $\mathbf{S}\cdot\gamma$ are disjoint as soon as $\beta \neq \gamma$. Hence, each set $h(\mathbf{S})$ belongs to a class $\mathbf{S}\cdot\beta$ for one unique braid β; we say that $h(\mathbf{S})$ *represents* the braid β. We focus below on such sets $h(\mathbf{S})$.

Closed laminations will be studied in Chapter 3, while curve diagrams will be studied in Chapter 4, in connection with the trivial open lamination. We introduce below closed laminations, then open laminations, and finally curve diagrams.

Definition 2.104 (Closed lamination).
Consider the set P_n of n mobile punctures inside the disk D^2. We call closed lamination, *and denote by $\mathcal{L}^c$, the union of $n+1$ non-intersecting closed curves $\mathcal{L}^c_0, \ldots, \mathcal{L}^c_n$ such that each curve $\mathcal{L}^c_j$ splits the plane $\mathbb{C}$ into one* inner *region that contains the left point and j mobile punctures, and one* outer *region that contains the right point and $n - j$ mobile punctures.*

Figure 2.105 represents two closed laminations, including $\mathbf{L}^c$, the trivial one. From this point on, closed laminations will always be represented as follows. Mobile punctures

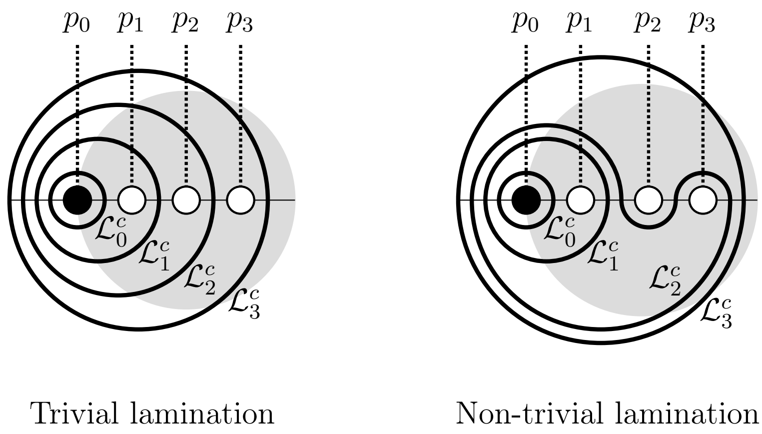


Figure 2.105 – Closed laminations

are indicated by white dots, the fixed puncture (i.e. the left point) is indicated by a black dot, and the right point is omitted. In addition, the unit disk D^2 is represented by a gray area, the curves of the lamination are drawn in black, and the real axis $\mathbb{R}$ is drawn with a thin horizontal line. Hereafter, and depending on the context, we may omit to represent the unit disk D^2, as well as the names $p_0, \ldots, p_n$.

Geometrically, a braid $\beta \in \mathbf{B}_n$, represented by some braid diagram $\mathfrak{D}$, acts on the trivial closed lamination $\mathbf{L}^c$ as follows. We place the n mobile punctures on the top of the n strands of $\mathfrak{D}$, then let these punctures "slide" along the strands of $\mathfrak{D}$, until we reach the bottom of $\mathfrak{D}$. At the same time, we force the $n+1$ curves to follow the motion prescribed by the punctures. Doing so, we obtain the lamination $\mathbf{L}^c \cdot \beta$, which will henceforth represent the braid β itself. Figure 2.107 illustrates the action of the braid σ_2 on $\mathbf{L}^c$.

When a braid β acts on some closed lamination $\mathcal{L}^c$, it moves both the mobile punctures and the curves of $\mathcal{L}^c$. The names of the punctures $p_0, \ldots, p_n$ depend uniquely of the order of the punctures. Hence, when applying the braid $\sigma_i^{\pm 1}$ on a lamination, both punctures p_i and p_{i+1} move, and they are respectively renamed p_{i+1} and p_i. On the contrary, and although the curves $\mathcal{L}_j^c$ may move, they are not renamed, as shown in Fig. 2.105.

Definition 2.106 (Open lamination).
Consider the set P_n of n mobile punctures inside the disk D^2. We call open lamination*, and denote by $\mathcal{L}^o$, the union of $n-1$ non-intersecting open curves $\mathcal{L}_1^o, \ldots, \mathcal{L}_{n-1}^o$ such that each curve $\mathcal{L}_j^o$*

- *contains two vertical half-lines with opposite directions (i.e. sets $z_j + i\mathbb{R}_{\leqslant 0}$ and $z'_j + i\mathbb{R}_{\geqslant 0}$);*
- *splits the plane $\mathbb{C}$ into one* left *region that contains the left point and j punctures, and one* right *region that contains the right point and $n-j$ punctures.*

Figure 2.108 represents two open laminations, including $\mathbf{L}^o$, the trivial one. From this point on, open laminations will always be represented as follows. Mobile punctures are indicated by white dots, and both the left point and the right point are indicated by

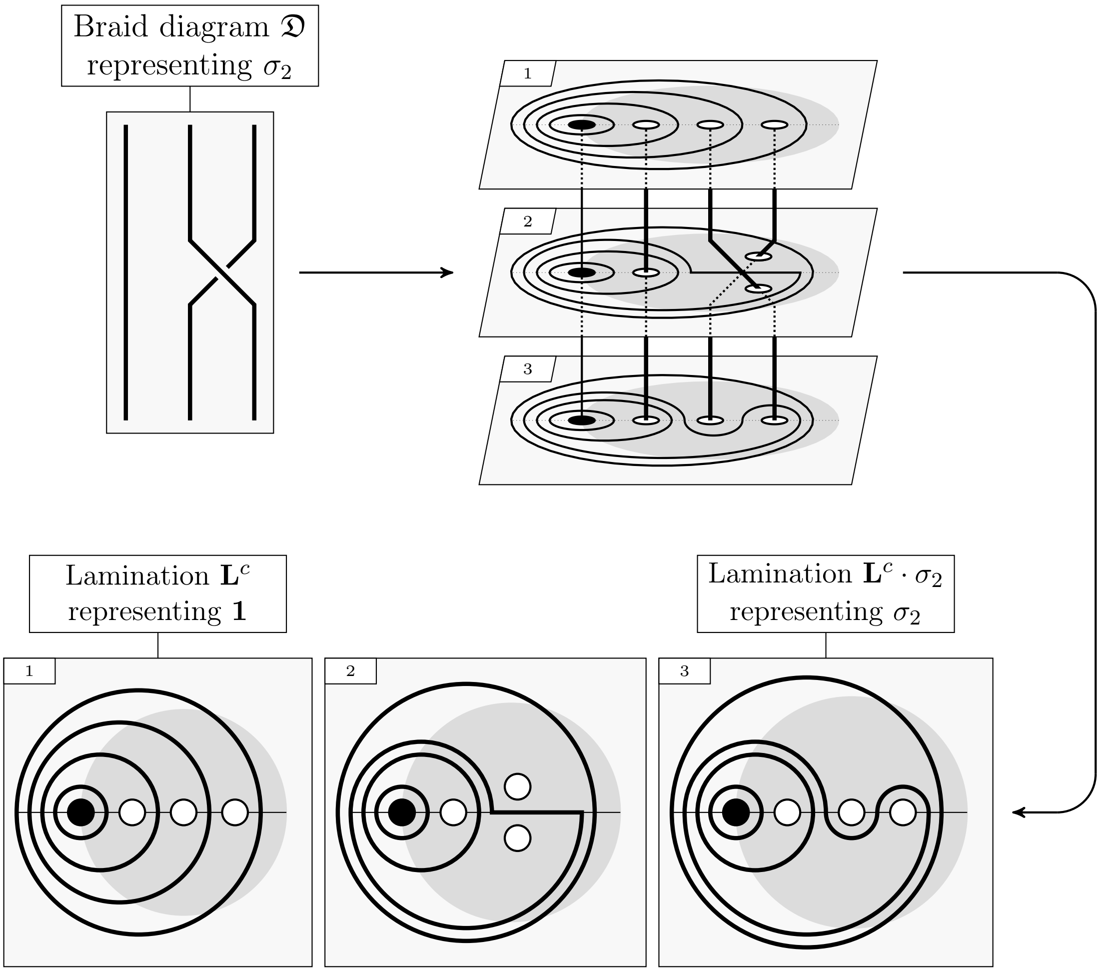
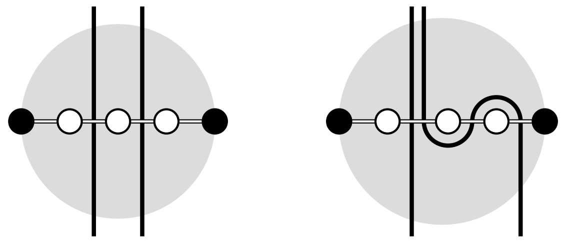


Figure 2.107 – Braid acting on a closed lamination

black dots. The unit disk D^2 is represented by a gray area, the curves of the lamination are drawn in black, and the segment $[-1, 1]$ (i.e. the trivial curve diagram $\mathbf{D}$) is drawn in white.

The action of braids on both types of laminations is very similar. A braid $\beta \in \mathbf{B}_n$, represented by some braid diagram $\mathfrak{D}$, acts on the trivial open lamination $\mathbf{L}^o$ as follows. We place the n mobile punctures on the top of the n strands of $\mathfrak{D}$, then let these punctures "slide" along the strands of $\mathfrak{D}$, until we reach the bottom of $\mathfrak{D}$. At the same time, we force the $n-1$ curves to follow the motion prescribed by the punctures. Doing so, we obtain the lamination $\mathbf{L}^o \cdot \beta$, which will henceforth represent the braid β itself. This is analogous to the process illustrated in Fig. 2.107.

Closed laminations and open laminations are in fact small variants of each other. More precisely, an open lamination $\mathcal{L}^o$ gives rise to a closed lamination $\mathcal{L}^c$ as follows. For each integer $i \leqslant n-1$, bend both half-lines of the curve $\mathcal{L}^o_i$ to the left, in order to transform $\mathcal{L}^o_i$ into a closed line $\mathcal{L}^c_i$. Then, add a closed curve $\mathcal{L}^c_n$ that encloses all (fixed or mobile) punctures but does not enclose the right point, and add a closed curve $\mathcal{L}^c_0$ that encloses



Trivial lamination Non-trivial lamination

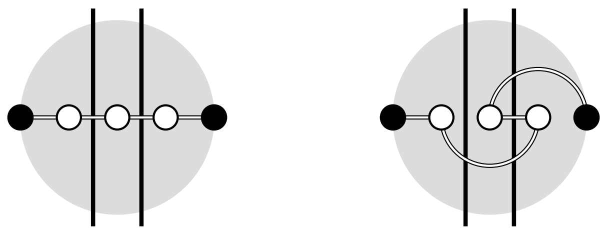
Figure 2.108 – Open laminations

the fixed puncture but does not enclose any mobile puncture nor the right point. We thereby obtain a closed lamination $\mathcal{L}^c$ that represents the same braid as $\mathcal{L}^o$.

For instance, the closed laminations presented in Fig. 2.105 are obtained from the open laminations of Fig. 2.108 by following this process.

Definition 2.109 (Curve diagram)**.**
Consider the set P_n of n mobile punctures inside the disk D^2. We call curve diagram*, and denote by $\mathcal{D}$, each non-intersecting open curve, with endpoints -1 and $+1$, that contains each puncture of the disk.*

Figure 2.110 represents two curve diagrams, including $\mathbf{D}$, the trivial one. From this point on, curve diagrams will always be represented in the same way as open laminations, as follows. Mobile punctures are indicated by white dots, and both the left point and the right point are indicated by black dots. In addition, the unit disk D^2 is represented by a gray area, the curve $\mathbf{D}$ is drawn in white, and the trivial open lamination $\mathbf{L}^o$ is drawn in black.



Trivial curve diagram Non-trivial curve diagram

Figure 2.110 – Curve diagrams

Once again, the action of braids on curve diagram has the flavour of the action of braids on laminations. A braid $\beta \in \mathbf{B}_n$, represented by some braid diagram $\mathfrak{D}$, acts on the trivial curve diagram $\mathbf{D}$ as follows. We place the n mobile punctures on the top of the n strands of $\mathfrak{D}$, then let these punctures "slide" along the strands of $\mathfrak{D}$, until we reach the bottom of $\mathfrak{D}$. At the same time, we force the curve of $\mathbf{D}$ to follow the motion prescribed by the punctures. Doing so, we obtain the curve diagram $\mathbf{D} \cdot \beta$, which will henceforth represent the braid β itself. This is analogous to the process illustrated in Fig. 2.107.

2.4.2 Norms of Laminations, of Curve Diagrams and of Braids

Following Dynnikov and Wiest [46], we define the norm of a (closed or open) lamination or of a curve diagram, and the associated norm of a braid.

Definition 2.111 (Closed laminated norm and tight closed lamination).
Let β be a braid on n strands, and let $\mathcal{L}^c$ be a closed lamination representing β.

The norm *of $\mathcal{L}^c$, which we denote by $\|\mathcal{L}^c\|_\ell$, is the cardinality of the set $\mathcal{L}^c \cap \mathbb{R}$, i.e. the number of intersection points between the real axis $\mathbb{R}$ and the $n+1$ curves of the lamination $\mathcal{L}^c$.*

Moreover, if, among all the closed laminations that represent β, the lamination $\mathcal{L}^c$ has a minimal norm, then we say that $\mathcal{L}^c$ is a tight *lamination. In this case, we also define the* closed laminated norm *of the braid β, which we denote by $\|\beta\|^c_\ell$, as the norm $\|\mathcal{L}^c\|_\ell$.*

Although we call the mapping $\beta \mapsto \|\beta\|^c_\ell$ a *norm*, following the seminal paper of Dynnikov and Wiest [46], this mapping does not satisfy standard properties of norms on metric spaces, such as separation axioms (i.e. that $\beta = \mathbf{1}$ if and only if $\|\beta\|^c_\ell = 0$) or sub-additivity axioms (i.e. that $\|\beta \cdot \gamma\|^c_\ell \leqslant \|\beta\|^c_\ell + \|\gamma\|^c_\ell$ for all $\beta, \gamma \in \mathbf{B}_n$). Counterexamples to those properties are provided by the fact that $\|\mathbf{1}\|^c_\ell = 2(n+1)$ and that $\|(\sigma_1\sigma_2)^k\|^c_\ell = 2F_{2k+3} + 2(n-1)$, where F_k denotes the k-th Fibonacci number.

However, Dynnikov and Wiest prove in [46] that the mapping $\beta \mapsto \log \|\beta\|^c_\ell$ is *comparable* to a norm, i.e. that there exists positive constants m_n and M_n and a norm $\mathcal{N}$ of $\mathbf{B}_n$ such that $m_n(\mathcal{N}(\beta) - 1) \leqslant \log \|\beta\|^c_\ell \leqslant M_n(\mathcal{N}(\beta) + 1)$ for all $\beta \in \mathbf{B}_n$.

Tight closed laminations are important, due to the following classical result (see [39, 46, 48] for details).

Theorem 2.112.
Two tight closed laminations represent the same braid if and only if they are related by an isotopy that preserves the real axis $\mathbb{R}$ setwise and the point -1.

From this point on, we will refer to *the* tight closed lamination of a braid, as illustrated in Fig. 2.114.

Definition 2.113 (Open laminated norm and tight open lamination).
Let β be a braid on n strands, and let $\mathcal{L}^o$ be an open lamination representing β.

The norm *of $\mathcal{L}^o$, which we denote by $\|\mathcal{L}^o\|_\ell$, is the cardinality of the set $\mathcal{L}^o \cap (-1, 1)$, i.e. the number of intersection points between the real interval $(-1, 1)$ and the $n-1$ curves of the lamination $\mathcal{L}^o$.*

Moreover, if, among all the open laminations that represent β, the lamination $\mathcal{L}^o$ has a minimal norm, then we say that $\mathcal{L}^o$ is a tight *lamination. In this case, we also define the* open laminated norm *of the braid β, which we denote by $\|\beta\|^o_\ell$, as the norm $\|\mathcal{L}^o\|_\ell$.*

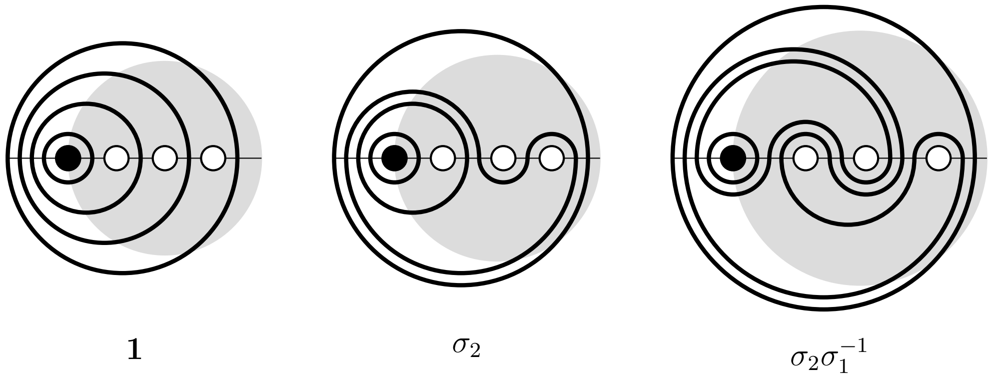


Figure 2.114 – Identifying braids with tight closed laminations

Due to the similarity between closed and open laminations, the open and closed laminated norm are also very similar.

Lemma 2.115.
Let $\mathcal{L}^o$ be an open lamination with n mobile punctures, and let $\mathcal{L}^c$ be the closed lamination obtained by "bending" the curves of $\mathcal{L}^o$ and adding the curves $\mathcal{L}_0^c$ and $\mathcal{L}_n^c$. We have $\|\mathcal{L}^c\|_\ell = \|\mathcal{L}^o\|_\ell + n + 3$.

Corollary 2.116.
Let β be a braid on n strands. We have $\|\beta\|_\ell^c = \|\beta\|_\ell^o + n + 3$.

In addition, tight open laminations are the variants of tight closed laminations, which explains the following result, which is equivalent to Theorem 2.112.

Theorem 2.117.
Two tight open laminations represent the same braid if and only if they are related by an isotopy that preserves the real axis $\mathbb{R}$ setwise and the point -1.

Hence, we refer to *the* tight open lamination of a braid, as illustrated in Fig. 2.118.

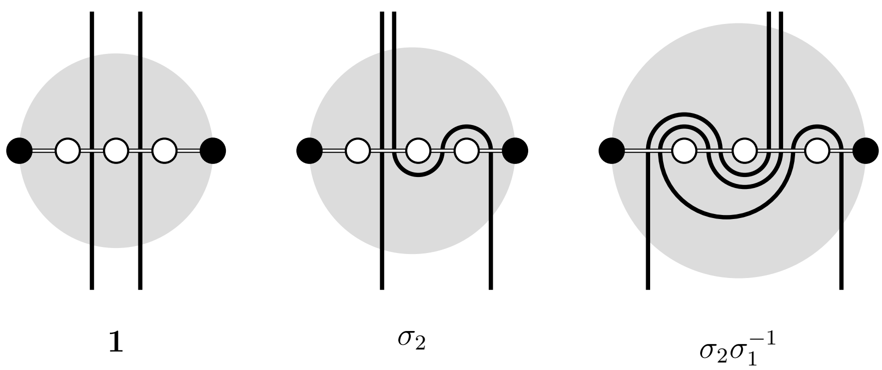


Figure 2.118 – Identifying braids with tight open laminations

Definition 2.119 (Diagrammatic norm and tight curve diagram).
Let β be a braid on n strands, and let $\mathcal{D}$ be a curve diagram representing β.

The norm *of $\mathcal{D}$, which we denote by $\|\mathcal{D}\|_d$, is the cardinality of the set $\mathcal{D} \cap \mathbf{L}^o$, i.e. the number of intersection points between the curve diagram $\mathcal{D}$ and the $n-1$ curves of the trivial open lamination $\mathbf{L}^o$.*

Moreover, if, among all the curve diagrams that represent β, the diagram $\mathcal{D}$ has a minimal norm, then we say that $\mathcal{D}$ is a tight curve diagram*. In this case, we also define the* diagrammatic norm *of the braid β, which we denote by $\|\beta\|_d$, as the norm $\|\mathcal{D}\|_d$.*

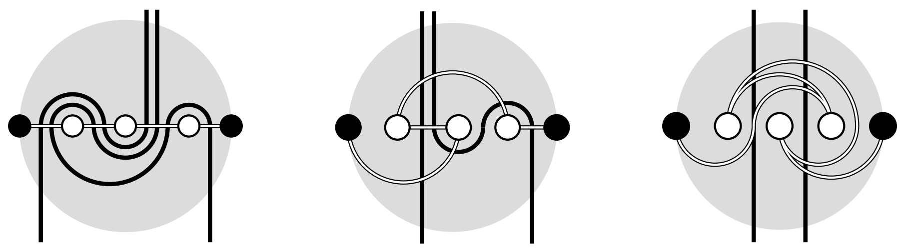
The connection between the laminated norms and the diagrammatic of a braid is not as immediate as the connection between the two kinds of laminated norms. Nevertheless, this connection remains quite simple, as illustrated in Fig. 2.121 and shown below.

Proposition 2.120.
Let β be a braid on n strands. We have $\|\beta\|_\ell^o = \|\beta^{-1}\|_d$, i.e. the open laminated norm of the braid β is equal to the diagrammatic norm of the braid β^{-1}.

Proof. Let $\mathbf{L}^o$ be the trivial open lamination and let $h \in H_n^*$ be a representative of the braid β such that $h(\mathbf{L}^o)$ is a tight open lamination. Since the curve $h^{-1}(\mathbf{D})$ is a curve diagram of the braid β^{-1}, it follows that

$$\|\beta\|_\ell = |h(\mathbf{L}^o) \cap \mathbf{D}| = |h^{-1}(h(\mathbf{L}^o) \cap \mathbf{D})| = |\mathbf{L}^o \cap h^{-1}(\mathbf{D})| \geqslant \|\beta^{-1}\|_d.$$

One proves similarly that $\|\beta^{-1}\|_d \geqslant \|\beta\|_\ell^o$, which completes the proof. □

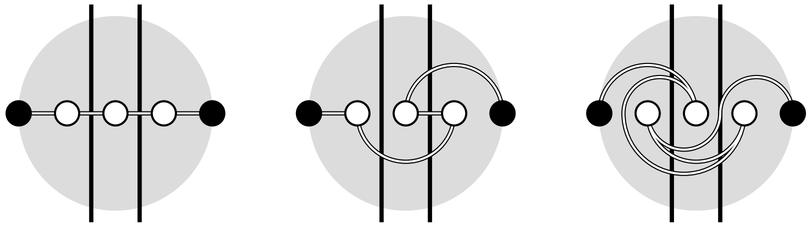


$|\mathbf{L}^o \cdot \sigma_2\sigma_1^{-1} \cap \mathbf{D}| = 8$ $|\mathbf{L}^o \cdot \sigma_2 \cap \mathbf{D} \cdot \sigma_1| = 8$ $|\mathbf{L}^o \cap \mathbf{D} \cdot \sigma_1\sigma_2^{-1}| = 8$

Figure 2.121 – From $\|\sigma_2\sigma_1^{-1}\|_\ell^o$ to $\|\sigma_1\sigma_2^{-1}\|_d$

This connection leads to a new variant of Theorem 2.112, in terms of curve diagrams.

Theorem 2.122.
Two tight curve diagrams represent the same braid if and only if they are related by an isotopy that preserves the trivial open lamination $\mathbf{L}^o$ and the points -1 and $+1$.



$\mathbf{1}$ σ_2^{-1} $\sigma_2^{-1}\sigma_1$

Figure 2.123 – Identifying braids with tight curve diagrams

Hence, we refer to *the* tight curve diagram of a braid, as illustrated in Fig. 2.123.

Geometrical symmetries induce some additional invariance properties of the above-mentioned norms. Consider the group morphisms $\mathbf{S}_v$, $\mathbf{S}_h$ and $\mathbf{S}_c$ such that $\mathbf{S}_v : \sigma_i \mapsto \sigma_{n-i}^{-1}$, $\mathbf{S}_h : \sigma_i \mapsto \sigma_i^{-1}$ and $\mathbf{S}_c : \sigma_i \mapsto \sigma_{n-i}$. Observe that $\mathbf{S}_h \circ \mathbf{S}_v = \mathbf{S}_v \circ \mathbf{S}_h = \mathbf{S}_c$ is the conjugation morphism ϕ_Δ, i.e. the morphism $\beta \mapsto \Delta^{-1}\beta\Delta$.

Lemma 2.124.
Let $\mathbf{B}_n$ be the group of s-strand braids. For all braids $\beta \in \mathbf{B}_n$, we have

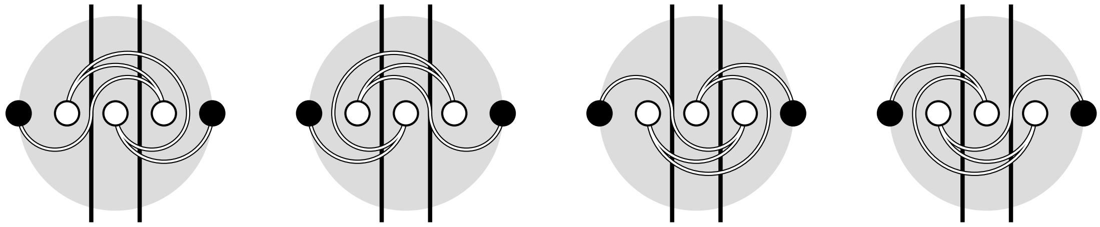
$$\begin{aligned}
\|\beta\|_\ell^c &= \|\mathbf{S}_v(\beta)\|_\ell^c = \|\mathbf{S}_h(\beta)\|_\ell^c = \|\mathbf{S}_c(\beta)\|_\ell^c,\\
\|\beta\|_\ell^o &= \|\mathbf{S}_v(\beta)\|_\ell^o = \|\mathbf{S}_h(\beta)\|_\ell^o = \|\mathbf{S}_c(\beta)\|_\ell^o, \text{ and}\\
\|\beta\|_d &= \|\mathbf{S}_v(\beta)\|_d = \|\mathbf{S}_h(\beta)\|_d = \|\mathbf{S}_c(\beta)\|_d,
\end{aligned}$$

i.e. the laminated and diagrammatic norms are invariant under $\mathbf{S}_v$, $\mathbf{S}_h$ and $\mathbf{S}_c$.

Proof. From a geometric point of view, the braid morphisms $\mathbf{S}_v$, $\mathbf{S}_h$ and $\mathbf{S}_c$ respectively induce vertical, horizontal and central symmetries on the laminations and the curve diagrams. More precisely, if $\mathcal{L}^c$, $\mathcal{L}^o$ and $\mathcal{D}$ are a closed lamination, an open lamination and a curve diagram representing some braid $\beta \in \mathbf{B}_n$, then:

- their vertically symmetric laminations $\mathcal{L}_v^c$, $\mathcal{L}_v^o$ and curve diagram $\mathcal{D}_v$ represent the braid $\mathbf{S}_v(\beta)$;
- their horizontally symmetric laminations $\mathcal{L}_h^c$, $\mathcal{L}_h^o$ and curve diagram $\mathcal{D}_h$ represent the braid $\mathbf{S}_h(\beta)$;
- their centrally symmetric laminations $\mathcal{L}_c^c$, $\mathcal{L}_c^o$ and curve diagram $\mathcal{D}_c$ represent the braid $\mathbf{S}_c(\beta)$.

□



Initial diagram	Vertical symmetry	Horizontal symmetry	Central symmetry
$\beta = \sigma_1^{-1}\sigma_2$	$\mathbf{S}_v(\beta) = \sigma_2\sigma_1^{-1}$	$\mathbf{S}_h(\beta) = \sigma_1\sigma_2^{-1}$	$\phi_\Delta(\beta) = \sigma_2^{-1}\sigma_1$

Figure 2.125 – Vertical, horizontal and central symmetries of a curve diagram

2.4.3 Arcs, Bigons and Tightness

The notion of tightness of a lamination or of a curve diagram, such as introduced in Definitions 2.111, 2.113 and 2.119, are in fact specific instances of a more general notion of tightness with respect to a union of curves.

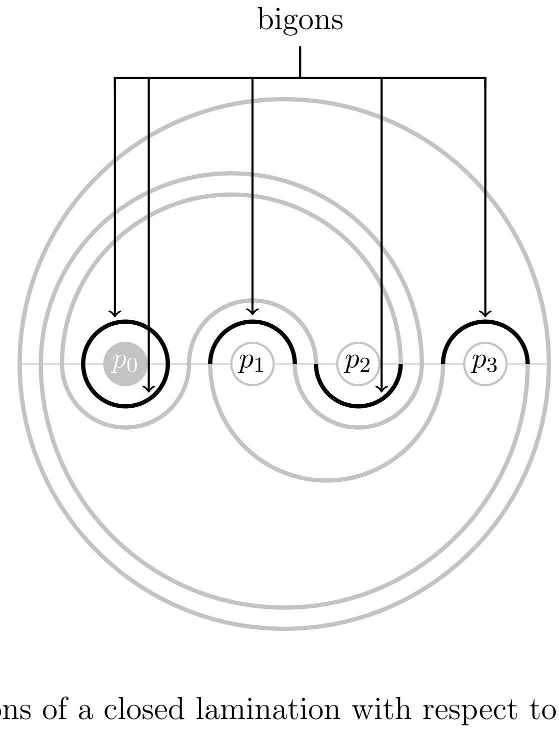


Figure 2.126 – Bigons of a closed lamination with respect to $\mathbb{R}$ and to punctures

Definition 2.127 (Arcs, adjacent endpoints and bigons).
Let $\mathcal{F}$ and $\mathcal{G}$ be two unions of curves, such that the intersection $\mathcal{F} \cap \mathcal{G}$ is finite. If the curves of $\mathcal{F}$ and $\mathcal{G}$ actually cross *each other at each point of $\mathcal{F} \cap \mathcal{G}$, then we say that $\mathcal{F}$ and $\mathcal{G}$ are* transverse *to each other.*

We call arc *of $\mathcal{F}$ with respect to $\mathcal{G}$, or $(\mathcal{F}, \mathcal{G})$-arc for short, a connected component of $\mathcal{F} \backslash \mathcal{G}$. If an arc A arc is bounded, then A must have two endpoints, which must lie on $\mathcal{G}$.*

We call these endpoints adjacent endpoints *in $\mathcal{F}$ with respect to $\mathcal{G}$, or $(\mathcal{F},\mathcal{G})$-adjacent endpoints.*

Finally, if these endpoints are also $(\mathcal{G},\mathcal{F})$-adjacent endpoints, then we say that A is a bigon *of $\mathcal{F}$ with respect to $\mathcal{G}$, or $(\mathcal{F},\mathcal{G})$-bigon. In addition, if B is a $(\mathcal{G},\mathcal{F})$-bigon that whose endpoints are the endpoints of A, then we say that the closed curve $A \cup B$ is a* bigon complex *of $\mathcal{F}$ and $\mathcal{G}$.*

Figure 2.126 shows a closed lamination and the bigons of this lamination with respect to $\mathbb{R}$ and to a set of punctures (here, $\{p_0, p_1, p_2, p_3\}$).

Bigons and bigon pairs play a crucial role in providing an intrinsic and easy to handle characterisation of tightness.

Definition 2.128 (Tightness)**.**
Let $\mathcal{F}$ and $\mathcal{G}$ be two unions of curves that are transverse to each other. In addition, let $\mathcal{P}$ be a set of points. We say that $\mathcal{F}$ and $\mathcal{G}$ are tight *with respect to each other and to the set $\mathcal{P}$ if, for all bigon complexes C of $\mathcal{F}$ and $\mathcal{G}$, there exists a point $p \in \mathcal{P}$ that lies either on C or inside the finite area delimited by C.*

Theorem 2.129.
A closed lamination $\mathcal{L}^c$ is tight in the sense of Definition 2.111 if and only if $\mathcal{L}^c$ and the real axis $\mathbb{R}$ are tight with respect to each other and to the set $\{p_0, \ldots, p_n\}$ of all *(fixed or mobile) punctures.*

An open lamination $\mathcal{L}^o$ is tight in the sense of Definition 2.113 if and only if $\mathcal{L}^o$ and the trivial curve diagram $\mathbf{D}$ are tight with respect to each other and to the set $\{p_1, \ldots, p_n\}$ of mobile punctures.

A curve diagram $\mathcal{D}$ is tight in the sense of Definition 2.119 if and only if $\mathcal{D}$ and the trivial open lamination $\mathbf{L}^o$ are tight with respect to each other and to the set $\{p_1, \ldots, p_n\}$ of mobile punctures.

Theorem 2.129 is of utmost importance, since it provides us with a direct way of checking that a lamination or a curve diagram is tight. In addition, it also paves the way to constructing tight laminations or tight curve diagrams, by removing iteratively unnecessary bigon complexes.

Chapter 3

The Relaxation Normal Form of Braids is Regular

Résumé

Les tresses peuvent être représentées de manière géométrique, en tant que laminations de disques épointés. La complexité géométrique d'une tresse est la plus petite complexité d'une lamination représentant cette tresse, et les laminations minimales d'une tresse en sont les représentants de complexité minimale. Les laminations minimales sont à l'origine d'une forme normale pour les tresses, via un algorithme de relaxation. Nous étudions ici cet algorithme de relaxation et la forme normale associée. Nous prouvons que cette forme normale est rationnelle et close par passage au préfixe. Nous construisons un automate déterministe qui reconnaît cette forme normale, dont nous comparons la taille à la taille de l'automate minimal reconnaissant la forme normale de relaxation. Enfin, nous mettons en évidence des liens entre la forme normale de relaxation et la notion de σ-positivité.

La majeure partie du contenu de ce chapitre est apparue dans [63] et a été soumise pour publication.

Abstract

Braids can be represented geometrically as laminations of punctured disks. The geometric complexity of a braid is the minimal complexity of a lamination that represents it, and tight laminations are representatives of minimal complexity. These laminations give rise to a normal form for braids, via a relaxation algorithm. We study here this relaxation algorithm and the associated normal form. We prove that this normal form is regular and prefix-closed. We provide an effective construction of a deterministic automaton that recognises this normal form, and we compare the size of this automaton to the size of the minimal automaton that recognises the relaxation normal form. We also relate the relaxation normal form and the notion of σ-positivity.

Most of the content of this chapter appeared in [63], and was submitted for publication.

Chapter 3 is devoted to the study of the relaxation normal form studied in [25], and which belongs to a larger class of relaxation normal forms [46]. Our main result consists in proving that the relaxation normal form is regular. Relaxation normal forms are based on relaxation algorithms, which consist in decreasing step-by-step the geometric complexity of a braid β, by applying *relaxation moves* to the tight (closed) lamination that represents β, i.e. multiplying β by another braid γ chosen from a finite set and so that $\|\beta\gamma\|_\ell < \|\beta\|_\ell$.

To our knowledge, we provide here the first known example of a regular normal form stemming from geometric representations of braids as tight (closed) laminations. Our proof relies heavily on the choice of the relaxation moves used in the relaxation algorithm, and therefore it does not provide any insights on whether other relaxation normal forms might be regular. In fact, we suspect that the relaxation normal forms studied in [46] are not regular, although suitable choices of relaxation moves might lead to other regular relaxation normal forms.

A similar result holds for the Bressaud normal form [19], which has a geometric nature and is regular. However, the Bressaud normal form comes from another geometric representation of braids, and therefore the tools used for proving that the relaxation normal form is regular are distinct from those used to prove that the Bressaud normal form is regular.

In particular, the Bressaud normal form and the relaxation normal form differ in several respects. First, they do not use the same alphabet, i.e. consist in factorisations where the factors are chosen in two distinct sets. While the Bressaud normal form of the group $\mathbf{B}_n$ uses the alphabet $\{\sigma_i\sigma_{i+1}\cdots\sigma_j : 1 \leqslant i \leqslant j < n\} \cup \{\sigma_i^{-1}\sigma_{i-1}^{-1}\cdots\sigma_1^{-1} : 1 \leqslant i < n\}$, the relaxation normal form uses the alphabet $\{\sigma_i\sigma_{i+1}\cdots\sigma_j : 1 \leqslant i \leqslant j < n\} \cup \{\sigma_i^{-1}\sigma_{i+1}^{-1}\cdots\sigma_j^{-1} : 1 \leqslant i \leqslant j < n\}$.

Second, the Bressaud normal form is accepted by some deterministic automaton of size $n(n-1)$, while Proposition 3.57 will prove below that each deterministic automaton that accepts the relaxation normal form is of size at least $2^{n/2-1}$. Third, one may observe directly that some braids have distinct Bressaud and relaxation normal forms, as shown in Fig. 3.1.

Bressaud normal form	Relaxation normal form
$(\sigma_2^{-1}\sigma_1^{-1}) \cdot \sigma_2 \cdot \sigma_1$	$\sigma_1 \cdot \sigma_2^{-1}$
$(\sigma_2\sigma_3) \cdot \sigma_1^{-1}$	$(\sigma_1^{-1}\sigma_2^{-1}) \cdot (\sigma_1\sigma_2\sigma_3)$

Figure 3.1 – Bressaud and relaxation normal forms

Sections 3.1.1, 3.2.1 and 3.4 present standard notions, which can be found in [13, 39, 46, 48]. On the contrary, the notions and objects introduced in Sections 3.1.2, 3.1.3, 3.2.2 and 3.3 are original, and were designed and developed in order to derive the results of Chapter 3.

3.1 Closed Lamination, Cell Map and Lamination Tree

In Chapter 3, we will consider only *closed* laminations, and cast both open laminations and curve diagrams aside. Therefore, we henceforth omit all mentions of the word *closed*, and refer only to *laminations* and to the *laminated norm* of a braid. Accordingly, we will only denote by $\|\beta\|_\ell$ the (closed) laminated norm of the braid β.

3.1.1 Arcs and Bigons of a Closed Lamination

First of all, let us introduce here on the standard notions of *arcs* and of *bigons* of a closed lamination [13, 46].

Definition 3.2 (Arcs, real projection and shadow).
Let $\mathcal{L}$ be a (closed) lamination. We say that $\mathcal{L}$ is a $\mathbb{R}$-lamination *if $\mathcal{L}$ is transverse to $\mathbb{R}$. Let $\mathcal{C}$ be an arc of $\mathcal{L}$ with respect to $\mathbb{R}$, as introduced in Definition 2.127. Henceforth, we will just say that $\mathcal{C}$ is an* arc *of $\mathcal{L}$.*

We say that $\mathcal{C}$ is an upper *arc if $\mathcal{C}$ is contained in the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant 0\}$, and that $\mathcal{C}$ is a* lower *arc if $\mathcal{C}$ is contained in the lower half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) \leqslant 0\}$.*

Since each curve of $\mathcal{L}$ is closed, the arc $\mathcal{C}$ must have two endpoints on the real axis $\mathbb{R}$. Let e and E be these endpoints, with $e < E$. We say that e is the left endpoint *of $\mathcal{C}$, and that E is the* right endpoint *of $\mathcal{C}$. We also call* real projection *of $\mathcal{C}$, and denote by $\pi_{\mathbb{R}}(\mathcal{C})$, the open interval $(e, E) \subseteq \mathbb{R}$.*

Finally, if $\{p_0, \ldots, p_n\}$ are the punctures of $\mathcal{L}$, we call shadow *of $\mathcal{C}$ in the lamination $\mathcal{L}$, and denote by $\pi_{\mathcal{L}}(\mathcal{C})$, the set $\{i \in \{0, \ldots, n\} : p_i \in \pi_{\mathbb{R}}(\mathcal{C})\}$. By extension, for each point $p_i \in \{p_0, \ldots, p_n\}$, we also denote by $\pi_{\mathcal{L}}(p_i)$ the singleton set $\{i\}$.*

Definition 3.3 (Blinding ordering).
Let $\mathcal{L}$ be a $\mathbb{R}$-lamination, and let A and A' be two arcs of $\mathcal{L}$. We say that A blinds *A', which we denote by $A' \overset{\pi}{\subseteq} A$, if the relation $\pi_{\mathbb{R}}(A') \subseteq \pi_{\mathbb{R}}(A)$ holds.*

If $\pi_{\mathbb{R}}(A') \subsetneq \pi_{\mathbb{R}}(A)$, then we say that A strictly *blinds A', which we denote by $A' \overset{\pi}{\subsetneq} A$.*

Finally, if a real point p belongs to $\pi_{\mathbb{R}}(A)$, then we also say that A blinds *p, which we denote by $p \overset{\pi}{\in} A$.*

The blinding ordering provides us with an alternative characterisation of bigons and of tightness.

Proposition 3.4.
Let $\mathcal{L}$ be a $\mathbb{R}$-lamination. The bigons of $\mathcal{L}$ are the arcs of $\mathcal{L}$ whose real projection is minimal for the inclusion ordering.

Proposition 3.5.
A lamination $\mathcal{L}$ that is transverse to $\mathbb{R}$ is tight if and only if each of its bigons (or, equivalently, each of its arcs) blinds at least one puncture of $\mathcal{L}$.

In what follows, we write $A \overset{\pi}{\circ} A'$ when two arcs A and A' have non-intersecting real projections, i.e. when $\pi_{\mathbb{R}}(A) \cap \pi_{\mathbb{R}}(A') = \varnothing$. Lemma 3.6 follows immediately from the fact that the arcs of a lamination do not cross each other.

Lemma 3.6.
Let $\mathcal{L}$ be a tight lamination, and let A and A' be upper (respectively, lower) arcs of $\mathcal{L}$. Either $A \overset{\pi}{\subseteq} A'$ or $A \overset{\pi}{\supseteq} A'$ or $A \overset{\pi}{\circ} A'$.

Corollary 3.7.
Let $\mathcal{L}$ be a tight lamination, and let A be an upper arc of $\mathcal{L}$. The following propositions are equivalent:

1. *the arc A is a bigon;*
2. *no upper arc of $\mathcal{L}$ is strictly blinded by A;*
3. *for each upper arc A' of $\mathcal{L}$, either $A \overset{\pi}{\subseteq} A'$ or $A \overset{\pi}{\circ} A'$;*
4. *for each lower arc A' of $\mathcal{L}$, either $A \overset{\pi}{\subseteq} A'$ or $A \overset{\pi}{\circ} A'$;*
5. *no endpoint of any upper arc of $\mathcal{L}$ is blinded by A;*
6. *no endpoint of any lower arc of $\mathcal{L}$ is blinded by A;*
7. *no endpoint of any arc of $\mathcal{L}$ is blinded by A.*

Similar statements hold when A is a lower arc.

Proof. The implications $7 \Rightarrow 1$, $1 \Rightarrow 2$, $3 \Leftrightarrow 5$ and $4 \Leftrightarrow 6$ are immediate. Moreover, Lemma 3.6 shows that $2 \Rightarrow 3$. Finally, since $\mathcal{L}$ is tight, being an endpoint of some upper arc of $\mathcal{L}$ is equivalent to being an endpoint of some lower arc of $\mathcal{L}$, i.e. to being a real point of $\mathcal{L}$, which proves that $5 \Leftrightarrow 6 \Leftrightarrow 7$. □

In addition, and although Proposition 3.5 gives a necessary and sufficient condition for a lamination to be tight, it does not provide us with any algorithm for computing such a tight lamination. We define below such an algorithm, based on the notion of *extensions*.

Definition 3.8 (Extensions of an arc)**.**
Let $\mathcal{L}$ be a $\mathbb{R}$-lamination and let A be an arc of $\mathcal{L}$. We call left extension *of A, and denote by $\overleftarrow{\mathbf{e}}_{\mathcal{L}}(A)$, the arc of $\mathcal{L}$ with which A shares its left endpoint. We call* right extension *of A, and denote by $\overrightarrow{\mathbf{e}}_{\mathcal{L}}(A)$, the arc of $\mathcal{L}$ with which A shares its right endpoint.*

If there is no ambiguity about the lamination $\mathcal{L}$, we may also write $\overleftarrow{\mathbf{e}}(A)$ and $\overrightarrow{\mathbf{e}}(A)$ instead of $\overleftarrow{\mathbf{e}}_{\mathcal{L}}(A)$ and $\overrightarrow{\mathbf{e}}_{\mathcal{L}}(A)$.

For each real interval I, we denote by $\mathcal{C}^{\uparrow}(I)$ the upper semi-circle with diameter I and by $\mathcal{C}^{\downarrow}(I)$ the lower semi-circle with diameter I. In addition, in what follows, if e and E are real points such that $e < E$, we may denote both by (e, E) and (E, e) the real interval $\{x \in \mathbb{R} : e < x < E\}$.

```
input : Lamination 𝓛 with punctures {p_0, ..., p_n}
for each upper arc A ⊆ 𝓛 do
    replace the arc A by C↑(π_ℝ(A))
end
for each lower arc A ⊆ 𝓛 do
    replace the arc A by C↓(π_ℝ(A))
end
while 𝓛 contains a bigon B that blinds no point p ∈ {p_0, ..., p_n} do
    let e_ℓ, e'_ℓ, e_r and e'_r be such that π_ℝ(B) = (e_ℓ, e_r), π_ℝ(←e(B)) = (e_ℓ, e'_ℓ) and
      π_ℝ(→e(B)) = (e_r, e'_r)
    if B is an upper arc then
        replace the arcs B, ←e(B) and →e(B) by C↓(e'_ℓ, e'_r)
    else
        replace the arcs B, ←e(B) and →e(B) by C↑(e'_ℓ, e'_r)
    end
end
output: Tight lamination isotopic to 𝓛
```

Algorithm 3.9: Tightening a lamination

Algorithm 3.9 transforms a lamination into a tight lamination representing the same braid. It uses two steps. First, we "normalise" the arcs of the lamination, making sure that we will not create unnecessary intersections between arcs. Second, we remove incrementally all the useless bigons. Note that, for implementation purposes, actually drawing each arc is useless. The only relevant information is the vertical (upper or lower) orientation of each arc and the relative locations of its endpoints on the real axis.

3.1.2 Cells, Boundaries and Cell Map

Having recalled some state-of-the-art results, we focus now on two original notions, which are the concepts of *cell* and of *cell map*.

Definition 3.10 (Cells and boundaries).
Let $\mathcal{L}$ be a $\mathbb{R}$-lamination. We call cell *of $\mathcal{L}$ each finite connected component of the set $\mathbb{C}\backslash(\mathcal{L} \cup \mathbb{R})$.*

We also call arc boundaries *of a cell $\mathcal{C}$ the arcs of $\mathcal{L}$ that belong to the boundary $\partial\mathcal{C}$, and* real boundaries *of $\mathcal{C}$ the connected segments of the set $\mathbb{R} \cap \partial\mathcal{C}$. Observe that one arc boundary of $\mathcal{C}$ blinds all the other boundaries of $\mathcal{C}$. This boundary is called the* parent boundary *of $\mathcal{C}$, and the other arc boundaries of $\mathcal{C}$ are called the* children boundaries *of $\mathcal{C}$.*

Finally, we say that $\mathcal{C}$ *is an* upper *cell* $\mathcal{C}$ *is contained in the half-plane* $\{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant 0\}$*, and we say that* $\mathcal{C}$ lower *cell if* $\mathcal{C}$ *is contained in the half-plane* $\{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant 0\}$*.*

Figure 3.11 shows a cell of some lamination, as well as its (arc and real) boundaries.

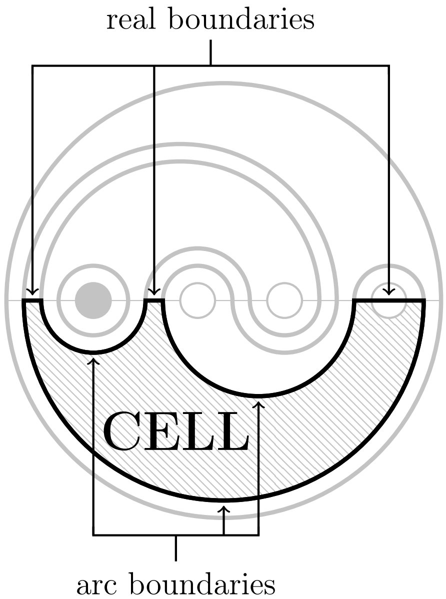


Figure 3.11 – A cell and its boundaries

Definition & Proposition 3.12 (Cell map)**.**
Let $\mathcal{L}$ *be a tight lamination. The* cell map *of the lamination* $\mathcal{L}$*, which we denote by* $\mathcal{M}(\mathcal{L})$*, is the bipartite planar map (i.e. an embedding of a planar graph into the plane) obtained as follows:*

- *inside each cell* $\mathcal{C}$ *of* $\mathcal{L}$*, we draw a vertex* $v_{\mathcal{C}}$ *of the map* $\mathcal{M}(\mathcal{L})$*;*
- *for each cell* $\mathcal{C}$ *of* $\mathcal{L}$ *and each real boundary* B *of* $\mathcal{C}$*, we draw, inside the cell* $\mathcal{C}$ *itself, one half-edge between the vertex* $v_{\mathcal{C}}$ *and the midpoint of the real boundary* B*, so that the half-edges drawn inside of* $\mathcal{C}$ *do not cross each other;*
- *each real boundary* B *belongs to one upper cell* $\mathcal{C}$ *and one lower cell* $\mathcal{C}'$*: we merge the half-edges that link the midpoint of* B *to the vertices* $v_{\mathcal{C}}$ *and* $v_{\mathcal{C}'}$*, thereby obtaining one edge of the map* $\mathcal{M}(\mathcal{L})$ *between* $v_{\mathcal{C}}$ *and* $v_{\mathcal{C}'}$*.*

Note that the cell map is not supposed to be connected nor simple (and, actually, is *never* connected nor simple), although it does not contain loops. Nevertheless, the relative positions of its connected components is important. Indeed, the topological properties of tight laminations are reflected on their cell maps and arc trees.

Theorem 3.13.
Let $\mathcal{L}$ *be a tight lamination and let* $\mathcal{M}(\mathcal{L})$ *be the cell map of* $\mathcal{L}$*. The map* $\mathcal{M}(\mathcal{L})$ *consists of* $n+1$ *connected components* $\mathbf{C}_0, \ldots, \mathbf{C}_n$*, which we can order so that*

- *the component* $\mathbf{C}_0$ *consists of two vertices and one edge;*

- *each component* $\mathbf{C}_i$ *(with* $1 \leqslant i \leqslant n$*) has a "lasso" shape, i.e. it contains one unique cycle, and has at most one vertex of degree* 3*; moreover, this unique cycle has an even length and encloses the components* $\mathbf{C}_0, \ldots, \mathbf{C}_{i-1}$.

Proof. Let $\mathcal{L}_0, \ldots, \mathcal{L}_n$ be the $n+1$ curves of the lamination $\mathcal{L}$. Then, let $\mathcal{Z}_0$ be the inner area defined by the curve $\mathcal{L}_0$ and, for $1 \leqslant i \leqslant n$, let $\mathcal{Z}_i$ be the area enclosed between the curves $\mathcal{L}_{i-1}$ and $\mathcal{L}_i$. First, each area $\mathcal{Z}_i$ is connected, hence the set $\mathbf{C}_i := \{v_\mathcal{C} : \mathcal{C} \subseteq \mathcal{Z}_i\}$ is a connected subset of the cell map $\mathcal{M}(\mathcal{L})$. Second, two cells $\mathcal{C}$ and $\mathcal{C}'$ that belong to two distinct areas $\mathcal{Z}_i$ and $\mathcal{Z}_j$ cannot have any common real boundary, i.e. the vertices $v_\mathcal{C}$ and $v_{\mathcal{C}'}$ cannot be neighbours in $\mathcal{M}(\mathcal{L})$. Therefore, the sets $\mathbf{C}_i$ are the connected components of $\mathcal{M}(\mathcal{L})$.

Since $\|\mathcal{L}\|_\ell$ is minimal, the curve $\mathcal{L}_0$ must consist of two bigons with the same real projection, hence $\mathbf{C}_0$ consists of two vertices and one edge. In addition, each component $\mathbf{C}_i$ with $1 \leqslant i \leqslant n$ must contain one unique cycle, which encloses the components $\mathbf{C}_0, \ldots, \mathbf{C}_{i-1}$: indeed, by construction of the areas $\mathcal{Z}_j$, the set $\bigcup_{j \leqslant i-1} \mathcal{Z}_j$ is the unique "hole" in the area $\mathcal{Z}_i$.

Moreover, the map $\mathcal{M}(\mathcal{L})$ is bipartite, since each edge of $\mathcal{M}(\mathcal{L})$ links an upper cell and a lower cell. Therefore, each cycle must have an even length.

Finally, for $1 \leqslant i \leqslant n$ and for each vertex $v_\mathcal{C} \in \mathbf{C}_i$ of degree 1, the cell $\mathcal{C}$ has exactly one arc boundary, which must be a bigon, and exactly one real boundary, which must contain the unique puncture of $\mathcal{Z}_i$. If $v_{\mathcal{C}'}$ is another vertex of $\mathbf{C}_i$ with degree 1, then $\mathcal{C}$ and $\mathcal{C}'$ must therefore share the same real boundary. Hence, $\mathbf{C}_i = \{v_\mathcal{C}, v_{\mathcal{C}'}\}$, which contradicts the fact that $\mathbf{C}_i$ must contain one cycle. This proves that $\mathbf{C}_i$ may have at most one vertex with degree 1: since $\mathbf{C}_i$ already has one unique cycle, $\mathbf{C}_i$ must have a lasso shape. □

Note that each component $\mathbf{C}_i$ (with $0 \leqslant i \leqslant n$) either contains only simple edges or contains exactly one double edge, which then forms the cycle (of length 2) of the component. In particular, the component $\mathbf{C}_1$ necessarily contains one double edge.

Figure 3.15 shows the cell map of some tight lamination $\mathcal{L}$. The arcs of $\mathcal{L}$ are gray lines, the edges of $\mathcal{M}(\mathcal{L})$ are black lines, and the vertices of $\mathcal{M}(\mathcal{L})$ are white circles. In addition, each of the connected components $\mathbf{C}_0$, $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$ is labelled.

3.1.3 Lamination Trees

We pursue here uncovering original combinatorial objects related to tight closed laminations, and introduce and investigate the notion of *lamination tree.*

Definition 3.14 (Left-right order in $\mathcal{L}$)**.**
Let $\mathcal{L}$ *be a tight lamination, and let* A *and* B *be two arcs or punctures of* $\mathcal{L}$.

We say that A lies to the left of B in $\mathcal{L}$, *which we denote by* $A \prec_\mathcal{L} B$, *if and only if* $\max \pi_\mathcal{L}(A) < \min \pi_\mathcal{L}(B)$. *The order* $\prec_\mathcal{L}$ *is called* left-right order in $\mathcal{L}$.

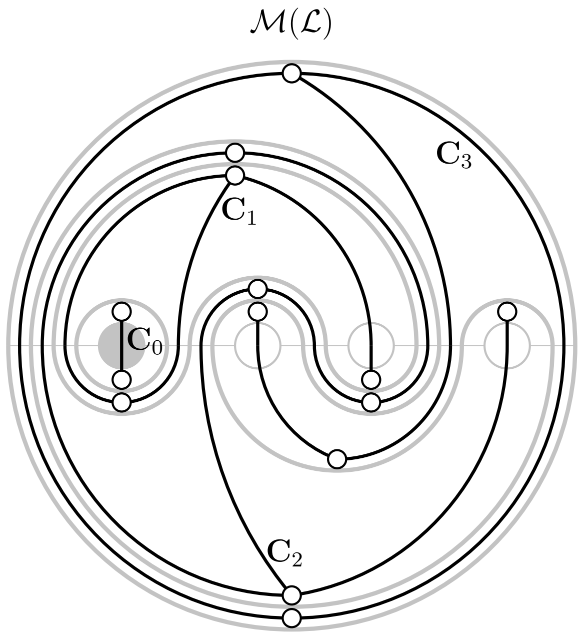


Figure 3.15 – Cell map of a tight lamination $\mathcal{L}$

Definition & Proposition 3.16 (Lamination trees and arc trees)**.**
Let $\mathcal{L}$ be a tight lamination. Consider the directed graph $\mathcal{T}^{\uparrow}(\mathcal{L})$ that we define as follows. The vertices of $\mathcal{T}^{\uparrow}(\mathcal{L})$ are either of the form v_A, where A is an upper arc of $\mathcal{L}$, or of the form v_p, where p is a puncture of $\mathcal{L}$. In addition, each upper arc A of $\mathcal{L}$ is the parent boundary of some upper cell $\mathcal{C}$:

- *for each child boundary B of $\mathcal{C}$, one edge of $\mathcal{T}^{\uparrow}(\mathcal{L})$ goes from v_A to v_B;*
- *for each puncture p that belongs to a real boundary of $\mathcal{C}$, one edge of $\mathcal{T}^{\uparrow}(\mathcal{L})$ goes from v_A to v_p.*

The graph $\mathcal{T}^{\uparrow}(\mathcal{L})$ is a rooted tree. In addition, we order $\mathcal{T}^{\uparrow}(\mathcal{L})$ as follows:

- *if $v_{A_1}, \ldots, v_{A_k}$ are the children of some vertex v_A (where A_i may be an arc or a puncture of $\mathcal{L}$) such that $A_1 \prec_{\mathcal{L}} \ldots \prec_{\mathcal{L}} A_k$, then v_{A_i} is the i-th child of v_A.*

We say that the tree $\mathcal{T}^{\uparrow}(\mathcal{L})$ is the upper lamination tree *of $\mathcal{L}$. We define similarly the* lower lamination tree *of $\mathcal{L}$, which we denote by $\mathcal{T}^{\downarrow}(\mathcal{L})$. Finally, when we refer to* some *arc tree, regardless of whether this is the upper or the lower lamination tree, we simply write $\mathcal{T}^{\updownarrow}(\mathcal{L})$.*

In addition, we define the (upper and lower) arc trees *of $\mathcal{L}$, which we respectively denote by $\mathcal{T}^{\uparrow}_{\mathrm{arc}}(\mathcal{L})$ and $\mathcal{T}^{\downarrow}_{\mathrm{arc}}(\mathcal{L})$, by removing the vertices v_p (where p is a puncture of $\mathcal{L}$) from the lamination trees $\mathcal{T}^{\updownarrow}(\mathcal{L})$.*

Proof. We need to prove that the directed graph $\mathcal{T}^{\uparrow}(\mathcal{L})$ is a tree, then that it can be ordered.

First, consider two nodes A and B of $\mathcal{T}^{\uparrow}(\mathcal{L})$. It comes immediately that some directed path goes from A to B if and only if A blinds B. Hence, Lemma 3.6 proves that $\mathcal{T}^{\uparrow}(\mathcal{L})$ is a forest. Second, the (outermost) curve $\mathcal{L}_n$ contains exactly one upper arc, which is a maximal element for the relation $\overset{\pi}{\subseteq}$. This arc is therefore the unique root of $\mathcal{T}^{\uparrow}(\mathcal{L})$, which must then be a tree.

Now, consider some upper cell $\mathcal{C}$. Let A be the parent boundary of $\mathcal{C}$. Since the children of v_A are associated either with children boundaries of $\mathcal{C}$ or to points belonging to the real boundaries of $\mathcal{C}$, it comes immediately that these children boundaries and points can be ordered according to $\prec_{\mathcal{L}}$. □

Figure 3.17 shows the two arc trees $\mathcal{T}^{\uparrow}(\mathcal{L})$ and $\mathcal{T}^{\downarrow}(\mathcal{L})$. Each vertex v_A is represented by a white circle lying on A. Each vertex v_p is represented by a black circle that lies on the puncture p.

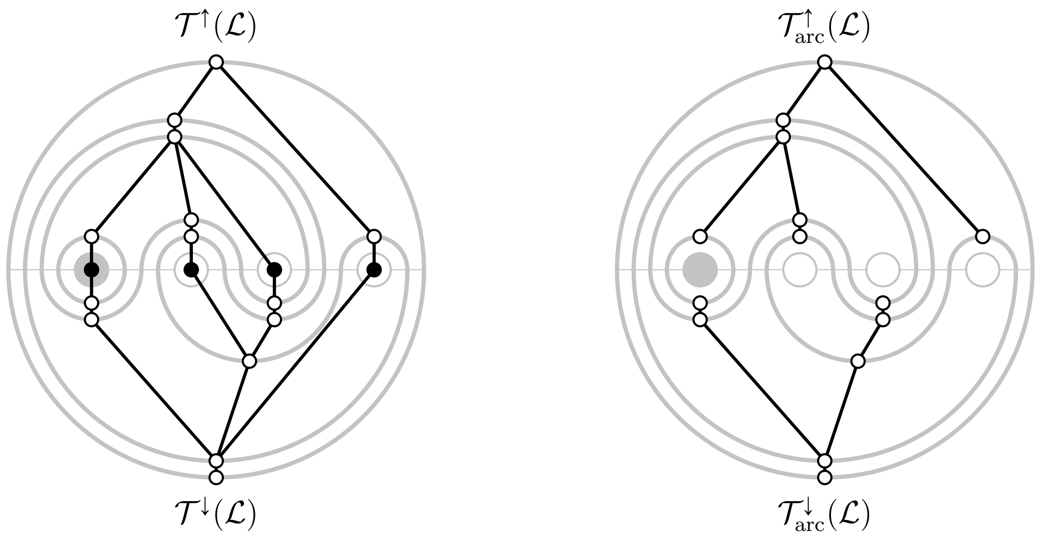


Figure 3.17 – Lamination trees and arc trees of a tight lamination

In what follows, we will frequently identify the vertex v_A with the arc A, and the vertex v_p with the puncture p. In particular, when A is an arc of $\mathcal{L}$, we may unambiguously refer to *the parent* and to *the children* of A. However, each puncture p_i of $\mathcal{L}$ belongs to both trees, hence has two parents (one in each tree).

Moreover, note that two distinct punctures of $\mathcal{L}$ cannot belong to real boundaries of the same cell, since they have to be separated by some arc of $\mathcal{L}$. Hence, no arc of $\mathcal{L}$ has more than one puncture among its children.

Proposition 3.18.
Let $\mathcal{L}$ be a tight lamination. The leaves of the trees $\mathcal{T}^{\updownarrow}(\mathcal{L})$ are the punctures of $\mathcal{L}$, and the leaves of the trees $\mathcal{T}^{\updownarrow}_{\mathrm{arc}}(\mathcal{L})$ are the bigons of $\mathcal{L}$.

Proof. First, each point p must be a leaf of $\mathcal{T}^{\updownarrow}(\mathcal{L})$, and the converse holds due to Proposition 3.5. The equivalence between 1 and 2 in Corollary 3.7 proves the second part of Proposition 3.18. □

Corollary 3.19.
Let $\mathcal{L}$ be a tight lamination. Let A be an arc of $\mathcal{L}$ and let $C_1, \ldots, C_k$ be the children of A. The family $(\pi_{\mathcal{L}}(C_i))_{1 \leqslant i \leqslant k}$ forms a partition of $\pi_{\mathcal{L}}(A)$ into integer intervals, i.e sets of consecutive integers.

Definition & Corollary 3.20 (Index of a bigon)**.**
If B is a bigon, then $\pi_{\mathcal{L}}(B)$ is a singleton set: we call index *of B in $\mathcal{L}$ the integer b such that $\pi_{\mathcal{L}}(B) = \{b\}$.*

Proposition 3.21.
Let $\mathcal{L}$ be a tight lamination, and let B and B' be two distinct bigons of $\mathcal{L}$. Either $B \prec_{\mathcal{L}} B'$ or $B' \prec_{\mathcal{L}} B$ or $\pi_{\mathbb{R}}(B) = \pi_{\mathbb{R}}(B')$; in the latter case, the union $B \cup B'$ is the (innermost) curve $\mathcal{L}_0$.

Proof. If $\pi_{\mathbb{R}}(B) \cap \pi_{\mathbb{R}}(B') \neq \varnothing$, then Corollary 3.7 proves that $\pi_{\mathbb{R}}(B) \subseteq \pi_{\mathbb{R}}(B') \subseteq \pi_{\mathbb{R}}(B)$. Hence, we have $\pi_{\mathbb{R}}(B) = \pi_{\mathbb{R}}(B')$, and therefore $B \cup B'$ is a closed curve, i.e. one curve $\mathcal{L}_i$ of $\mathcal{L}$. The internal area of $\mathcal{L}_i$ contains no other curve of $\mathcal{L}$, hence $i = 0$. Finally, if $\pi_{\mathbb{R}}(B) \cap \pi_{\mathbb{R}}(B') = \varnothing$, then of course either $B \prec_{\mathcal{L}} B'$ or $B' \prec_{\mathcal{L}} B$. □

In particular, the left-right order in $\mathcal{L}$ induces a total order on the set $\mathcal{S}$ of bigons that are not contained in $\mathcal{L}_0$. Moreover, if $\mathcal{L}$ is non-trivial, then the set $\mathcal{S}$ is non-empty, and its elements are greater than any of the two bigons that belong to $\mathcal{L}_0$. This leads to the notion of *rightmost bigon*, which is already an important notion in [25].

Definition & Proposition 3.22 (Rightmost bigon and rightmost index)**.**
Let $\mathcal{L}$ be a tight lamination. If $\mathcal{L}$ is non-trivial, then there exists a unique $\prec_{\mathcal{L}}$-maximal bigon of $\mathcal{L}$, which we call the rightmost bigon *of $\mathcal{L}$. We also call* rightmost index *of $\mathcal{L}$ is the index of this bigon in $\mathcal{L}$.*

Proposition 3.23.
Let $\mathcal{L}$ be a non-trivial, tight lamination. Let B be the rightmost bigon of $\mathcal{L}$, let k be the rightmost index of $\mathcal{L}$, and let A be some arc of $\mathcal{L}$. Then, $\min(\pi_{\mathcal{L}}(A)) \leqslant k$. Moreover, if A and B have opposite vertical orientations (i.e. one is an upper arc and the other one a lower arc), then $\min(\pi_{\mathcal{L}}(A)) \leqslant k - 1$.

Proof. Without loss of generality, we assume here that B is an upper bigon. Let v_C be the leftmost leaf of $\mathcal{T}^{\updownarrow}_{\mathrm{arc}}(\mathcal{L})$ for which v_A is an ancestor. Proposition 3.18 proves that C is a bigon of $\mathcal{L}$. Let c be the index of C in $\mathcal{L}$.

Since $C \preccurlyeq_{\mathcal{L}} B$, then either $B = C$, in which case A is an upper arc and $\min(\pi_{\mathcal{L}}(A)) \leqslant c = k$, or $C \prec_{\mathcal{L}} B$, in which case $\min(\pi_{\mathcal{L}}(A)) \leqslant c < k$. □

Finally, from Theorem 3.13 follow topological results about arc trees.

Corollary 3.24.
Let $\mathcal{L}$ be a tight lamination. The trees $\mathcal{T}^{\uparrow}_{\mathrm{arc}}(\mathcal{L})$ and $\mathcal{T}^{\downarrow}_{\mathrm{arc}}(\mathcal{L})$ are unary-binary, i.e. each node in those trees has at most two children.

Proof. Each vertex A of $\mathcal{L}$ is the parent arc boundary of some cell $\mathcal{C}$. Let $A_1, \ldots, A_k$ be the children boundaries of $\mathcal{C}$. The vertices $v_{A_1}, \ldots, v_{A_k}$ are the k children of the vertex v_A in $\mathcal{T}^{\updownarrow}_{\mathrm{arc}}(\mathcal{L})$. Hence, the cell $\mathcal{C}$ has $k+1$ real boundaries, i.e. degree $k+1$ in the map $\mathcal{M}(\mathcal{L})$. Since no vertex of $\mathcal{M}(\mathcal{L})$ has a degree more than 3, the result follows. □

Lemma 3.25.
Let $\mathcal{L}$ be a non-trivial tight lamination, let k be the rightmost index in $\mathcal{L}$, and let A be an arc of $\mathcal{L}$ with shadow $\{k\}$. We have $\overleftarrow{\mathbf{e}}(A) \overset{\pi}{\subseteq} \overrightarrow{\mathbf{e}}(A)$, and there exists integers $i \leqslant j \leqslant k-1$ such that $\pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(A)) = \{i, \ldots, k\}$ and $\pi_{\mathcal{L}}(\overleftarrow{\mathbf{e}}(A)) = \{j, \ldots, k-1\}$.

Proof. Without loss of generality, we assume that A is a lower arc. Let $i \leqslant I$ and $j \leqslant J$ be integers such that $\pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(A)) = \{i, \ldots, I\}$ and $\pi_{\mathcal{L}}(\overleftarrow{\mathbf{e}}(A)) = \{j, \ldots, J\}$. The right endpoint of A is an endpoint of $\overrightarrow{\mathbf{e}}(A)$ and belongs to the interval (p_k, p_{k+1}), whence $i = k+1$ or $I = k$. Similarly, we have $j = k$ or $J = k-1$. Proposition 3.23 proves that $i \leqslant k-1$ and that $j \leqslant k-1$, which shows that $I = k$ and $J = k-1$.

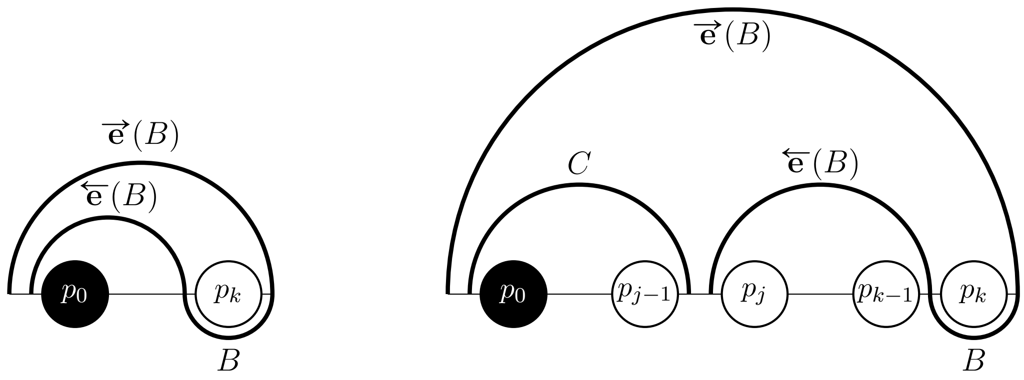
In addition, note that both $\overrightarrow{\mathbf{e}}(A)$ and $\overleftarrow{\mathbf{e}}(A)$ blind the puncture p_{k-1}, but that only $\overrightarrow{\mathbf{e}}(A)$ blinds p_k. This proves that the relations $\overrightarrow{\mathbf{e}}(A) \circ \overleftarrow{\mathbf{e}}(A)$ and $\overrightarrow{\mathbf{e}}(A) \overset{\pi}{\subseteq} \overleftarrow{\mathbf{e}}(A)$ are impossible, and Lemma 3.6 then proves that $\overleftarrow{\mathbf{e}}(A) \overset{\pi}{\subseteq} \overrightarrow{\mathbf{e}}(A)$, thereby showing that $i \leqslant j$ and completing the proof. □

Corollary 3.26.
Let $\mathcal{L}$ be a non-trivial tight lamination, let B be the rightmost bigon of $\mathcal{L}$, and let k be the index of B in $\mathcal{L}$. If $\pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(B)) = \{0, \ldots, k\}$, then the arc $\overrightarrow{\mathbf{e}}(B)$ has three children in $\mathcal{T}^{\updownarrow}(\mathcal{L})$, which are one arc with shadow $\{0, 1, \ldots, j-1\}$, the arc $\overleftarrow{\mathbf{e}}(B)$, with shadow $\{j, \ldots, k-1\}$, and the point p_k.

Proof. Without loss of generality, we assume that B is a lower arc.

The arcs $\overleftarrow{\mathbf{e}}(B)$, B and $\overrightarrow{\mathbf{e}}(B)$ all belong to the same curve $\mathcal{L}_u$ of the lamination $\mathcal{L}$. Since $\mathcal{L}_u$ intersects once the interval $(-\infty, -1)$, it follows that the case $j = 0$, illustrated in the left part of Fig. 3.27, is impossible.

Hence, we know that $j \geqslant 1$. The vertex $v_{\overleftarrow{\mathbf{e}}(B)}$ is a left sibling of v_{p_k} in $\mathcal{T}^{\uparrow}_{\mathrm{arc}}(\mathcal{L})$, hence $\overleftarrow{\mathbf{e}}(B)$ is a child of $\overrightarrow{\mathbf{e}}(B)$. Therefore, the arc $\overrightarrow{\mathbf{e}}(B)$ must have additional child, which we call C, as shown in the right part of Fig. 3.27. Since $\pi_{\mathcal{L}}(C)$, $\pi_{\mathcal{L}}(\overleftarrow{\mathbf{e}}(B))$ and $\{k\}$ form a partition of $\pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(B))$ into integer intervals (i.e. sets of consecutive integers), Corollary 3.26 follows. □



Case #1:$j = 0$ (impossible) Case #2: $1 \leqslant j$

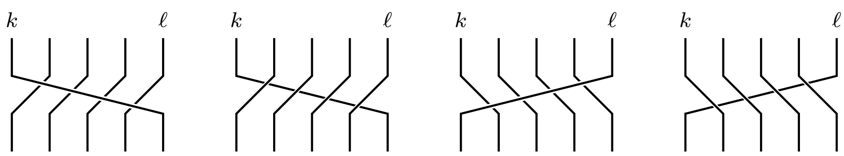
Figure 3.27 – Proof of Corollary 3.26

3.2 The Relaxation Normal Form is Regular

3.2.1 A Prefix-Closed Normal Form

We present now the *right-relaxation algorithm*, introduced by Caruso [25] and which is a particular case of the transmission-relaxation algorithms of Dynnikov and Wiest [46]. This algorithm consists in applying a sequence of elementary homeomorphisms to a given tight lamination, in order to obtain a tight lamination of the trivial braid **1**. This algorithm gives rise to a *relaxation normal form*, which is the main concern of this paper.

We first introduce some notions, which are central for the right-relaxation algorithm: the family of *sliding braids*, whose braid diagrams are given in Fig. 3.28, and the *right-relaxation move.*



$[k \hookrightarrow \ell]$ $[k \curvearrowright \ell]$ $[k \hookleftarrow \ell]$ $[k \curvearrowleft \ell]$

Figure 3.28 – Braid diagrams of sliding braids

Definition 3.29 (Sliding braid)**.**
The family of sliding braids *(with n strands) is the family that contains the following braids, for some integers k and ℓ such that $1 \leqslant k < \ell \leqslant n$:*

- $[k \hookrightarrow \ell] = \sigma_k \sigma_{k+1} \ldots \sigma_{\ell-1}$;
- $[k \curvearrowright \ell] = \sigma_k^{-1} \sigma_{k+1}^{-1} \ldots \sigma_{\ell-1}^{-1}$;
- $[k \hookleftarrow \ell] = \sigma_{\ell-1}^{-1} \sigma_{\ell-1}^{-1} \ldots \sigma_k^{-1}$;
- $[k \curvearrowleft \ell] = \sigma_{\ell-1} \sigma_{\ell-1} \ldots \sigma_k$.

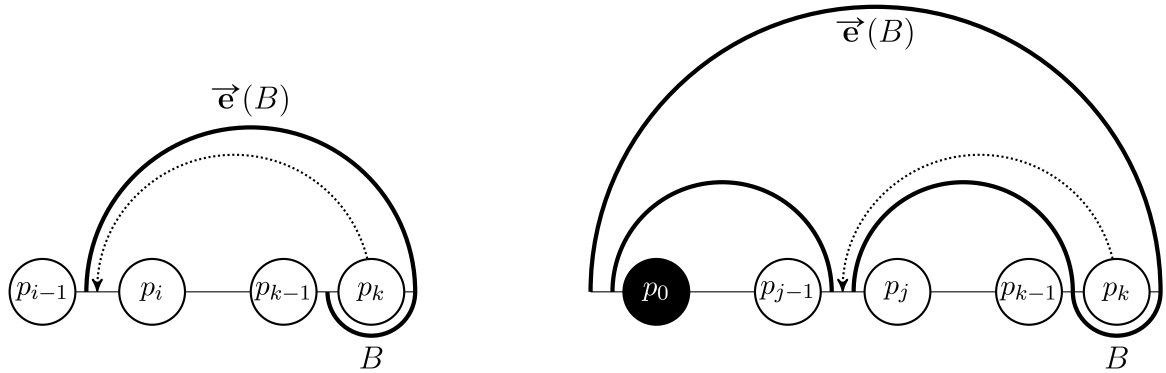
We call right-oriented sliding braids *the braids $[k \curvearrowright \ell]$ and $[k \hookrightarrow \ell]$, and we denote by Σ the set of right-oriented sliding braids.*

Observe that $[k \hookrightarrow \ell] = [k \hookleftarrow \ell]^{-1}$ and that $[k \curvearrowright \ell] = [k \curvearrowleft \ell]^{-1}$.

Definition 3.30 (Right-relaxation move)**.**
Let β be a non-trivial braid, let $\mathcal{L}$ be the tight lamination that represents β, let B be the rightmost bigon of $\mathcal{L}$ and let k be the index of B in $\mathcal{L}$. We define the right-relaxation move *of β, which we denote by $\mathbf{R}(\beta)$, as follows, and as illustrated in Fig. 3.31.*

According to Lemma 3.25 and to Corollary 3.26, two cases are possible:

- *if $\pi_{\mathcal{L}}(B) = \{i, \ldots, k\}$ for some $i \geqslant 1$, then $\mathbf{R}(\beta) := [i \hookleftarrow k]$ if B is an upper bigon, and $\mathbf{R}(\beta) := [i \curvearrowleft k]$ if B is a lower bigon;*
- *if $\pi_{\mathcal{L}}(B) = \{0, \ldots, k\}$, then $\pi_{\mathcal{L}}(\overleftarrow{\mathbf{e}}(B)) = \{j, \ldots, k-1\}$ for some $j \geqslant 1$; then, $\mathbf{R}(\beta) := [j \hookleftarrow k]$ if B is an upper bigon, and $\mathbf{R}(\beta) := [j \curvearrowleft k]$ if B is a lower bigon.*



Case #1: $0 \notin \pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(B))$ Case #2: $0 \in \pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(B))$

Figure 3.31 – Relaxation move (when B is a lower bigon)

Right-relaxation moves are at the core of the relaxation normal form, as they allow incremental simplifications of laminations.

Lemma 3.32.
Let β be a non-trivial braid. We have $\|\beta\mathbf{R}(\beta)\|_\ell \leqslant \|\beta\|_\ell - 1$.

Proof. Let $\mathcal{L}$ be the tight lamination, with set of punctures $\{p_0, \ldots, p_n\}$ and with rightmost bigon B, that represents β. Let $\mathcal{L}'$ be the lamination obtained from $\mathcal{L}$ as follows:

- if $0 \notin \pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(B))$, the puncture p_k is slid along the arc $\overrightarrow{\mathbf{e}}(B)$ until it reaches the real axis;
- if $0 \in \pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(B))$, then the puncture p_k is slid along the arc $\overleftarrow{\mathbf{e}}(B)$ until it reaches the real axis.

Let ρ denote the real point to which the puncture p_k was slid: $\mathcal{L}'$ is a lamination with the same arcs as $\mathcal{L}$, with set of punctures $\{p_0, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n, \rho\}$, and $\mathcal{L}'$ represents the braid $\beta\mathbf{R}(\beta)$. Moreover, the arc A blinds none of the new punctures, hence $\mathcal{L}'$ is not tight. This proves that $\|\beta\mathbf{R}(\beta)\|_\ell < \|\mathcal{L}'\|_\ell = \|\mathcal{L}\|_\ell = \|\beta\|_\ell$, i.e. that $\|\beta\mathbf{R}(\beta)\|_\ell \leqslant \|\beta\|_\ell - 1$. □

Lemma 3.32 proves that, starting from a non-trivial braid β and performing successive right-relaxation moves, we end with the trivial braid.

Definition & Proposition 3.33 (Relaxation normal form)**.**
Let β be a braid, and consider the sequence of braids inductively defined by $\beta_0 = \beta$, and $\beta_{i+1} = \beta_i \mathbf{R}(\beta_i)$ whenever $\beta_i \neq \mathbf{1}$. There exists some integer k such that $\beta_k = \mathbf{1}$, and $\beta = \mathbf{R}(\beta_{k-1})^{-1}\mathbf{R}(\beta_{k-2})^{-1} \dots \mathbf{R}(\beta_0)^{-1}$.

We call relaxation normal form *of the braid β, and denote by $\mathbf{RNF}(\beta)$, the word*

$$\mathbf{R}(\beta_{k-1})^{-1} \cdot \mathbf{R}(\beta_{k-2})^{-1} \cdot \ldots \cdot \mathbf{R}(\beta_0)^{-1},$$

where $\cdot$ is the concatenation symbol.

Using Algorithm 3.9 makes this definition constructive, since it provides us with a way of computing the relaxation normal form of any braid. Indeed, for each braid $\beta \in \mathbf{B}_n$, we can compute a (non necessarily tight) lamination of β. Then, using repeatedly Algorithm 3.9 and applying right-relaxation moves, we can indeed compute the relaxation normal form of β.

One key feature of the relaxation normal form is that the set of words $\{\mathbf{RNF}(\beta) : \beta \in \mathbf{B}_n\}$ is prefix-closed. This follows from the equality $\mathbf{RNF}(\beta) = \mathbf{RNF}(\beta\mathbf{R}(\beta)) \cdot \mathbf{R}(\beta)^{-1}$, which holds for each non-trivial braid β.

This prefix-closure property offers numerous possibilities. The relaxation normal form induces a tree, whose nodes are the words in relaxation normal form, and where the children of a word $\mathbf{w}$ are the words of the type $\mathbf{w} \cdot \lambda$ (for some sliding braid λ) that are in normal form. Hence, this tree is a sub-graph of the oriented Cayley graph of $\mathbf{B}_n$ for the right-oriented sliding braids, i.e. the set of generators

$$\Sigma = \{[k \rightharpoondown \ell] : 1 \leqslant k < \ell \leqslant n\} \cup \{[k \rightharpoonup \ell] : 1 \leqslant k < \ell \leqslant n\}.$$

This is useful for studying random processes: for instance, we may define a random walk by jumping from one word in relaxation normal form to one of its children that we choose at random. Another example is testing if a word is in relaxation normal form: it is possible to proceed by induction, checking only whether, for some relaxation normal word $\mathbf{w}$ and some sliding braid λ, the sequence $\mathbf{w} \cdot \lambda$ is in relaxation. The latter property will be useful when proving that the set $\{\mathbf{RNF}(\beta) : \beta \in \mathbf{B}_n\}$ is regular.

3.2.2 One Letter Further

Having defined precisely the relaxation form on which Chapter 3 is focused, we introduce once again some original notions, which we call *neighbour arcs*, *extended shadow* and *λ-relaxed lamination*, that will be crucial for proving that the relaxation normal form is regular.

Definition 3.34 (Neighbour points and arcs).
Let $\mathcal{L}$ be a tight lamination. Recall that $\mathcal{L}_{\mathbb{R}}$ denotes the set of all intersection points between the real axis and the curves of $\mathcal{L}$. Let p be a point on the open real interval $(\min \mathcal{L}_{\mathbb{R}}, \max \mathcal{L}_{\mathbb{R}})$. We call left neighbour *of p in $\mathcal{L}$ the point $p^-_{\mathcal{L}} := \max\{z \in \mathcal{L}_{\mathbb{R}} : z < p\}$ and* right neighbour *of p in $\mathcal{L}$ the point $p^+_{\mathcal{L}} := \min\{z \in \mathcal{L}_{\mathbb{R}} : z > p\}$.*

The point $p^-_{\mathcal{L}}$ belongs to two arcs of $\mathcal{L}$. We denote the upper one by $\mathcal{A}^{\uparrow}_{-}(p, \mathcal{L})$, and we call it left upper arc *of p in $\mathcal{L}$. We denote the lower one by $\mathcal{A}^{\downarrow}_{-}(p, \mathcal{L})$, and we call it* left lower arc *of p in $\mathcal{L}$. Similarly, the point $p^+_{\mathcal{L}}$ belongs to two arcs of $\mathcal{L}$. We denote the upper one by $\mathcal{A}^{\uparrow}_{+}(p, \mathcal{L})$, and we call it* right upper arc *of p in $\mathcal{L}$. We denote the lower one by $\mathcal{A}^{\downarrow}_{+}(p, \mathcal{L})$, and we call it* right lower arc *of p in $\mathcal{L}$. These four arcs are called* neighbour arcs *of p in $\mathcal{L}$.*

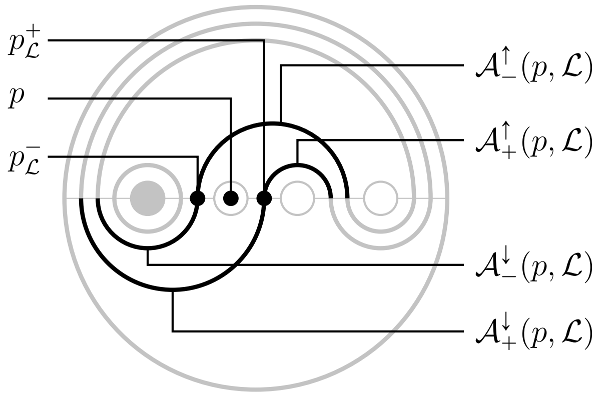


Figure 3.35 – A puncture, its neighbour points and its neighbour arcs

Figure 3.35 shows some tight lamination, in which a puncture p, the neighbour points of p and the neighbour arcs of p have been highlighted.

Definition 3.36 (Shadow and extended shadow).
Let $\mathcal{L}$ be a tight lamination that represents some braid β, and let k be the rightmost index of $\mathcal{L}$. Let A be an arc of $\mathcal{L}$. We define the extended shadow *of A in $\mathcal{L}$ as the pair $(\pi_{\mathcal{L}}(A), \pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(A)))$ if $k \in \pi_{\mathcal{L}}(A)$, or $(\pi_{\mathcal{L}}(A), \varnothing)$ if $k \notin \pi_{\mathcal{L}}(A)$. We denote this pair by $\pi^2_{\mathcal{L}}(A)$.*

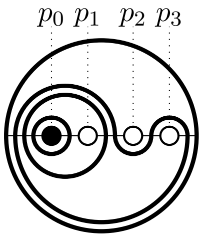
Then, for $1 \leqslant i \leqslant n$, $\diamond \in \{+,-\}$ and $\vartheta \in \{\downarrow, \uparrow\}$, we denote by $\pi_\beta(i, \diamond, \vartheta)$ the shadow of the arc $\mathcal{A}^{\vartheta}_{\diamond}(p_i, \mathcal{L})$ in $\mathcal{L}$, and we denote by $\pi^2_\beta(i, \diamond, \vartheta)$ the extended shadow of the arc $\mathcal{A}^{\vartheta}_{\diamond}(p_i, \mathcal{L})$ in $\mathcal{L}$.

By abuse of notation, we define the shadow *of β as the mapping*

$$\begin{array}{rcll} \pi_\beta & : & \{1, \ldots, n\} \times \{+,-\} \times \{\downarrow, \uparrow\} & \mapsto & 2^{\{0,\ldots,n\}} \\ & & (i, \diamond, \vartheta) & \mapsto & \pi_\beta(i, \diamond, \vartheta) \end{array}$$

and the extended shadow *of β is defined as the mapping*

$$\begin{array}{rcll} \pi^2_\beta & : & \{1,\ldots,n\}\times\{+,-\}\times\{\downarrow,\uparrow\} & \mapsto \quad 2^{\{0,\ldots,n\}}\times 2^{\{0,\ldots,n\}}. \\ & & (i,\diamond,\vartheta) & \mapsto \quad \pi^2_\beta(i,\diamond,\vartheta) \end{array}$$



$\pi^2_\beta(i,\diamond,\vartheta)$

$\diamond$	ϑ	$i = 1$	$i = 2$	$i = 3$
$-$	$\uparrow$	$(\{0\},\varnothing)$	$(\{0,1\},\varnothing)$	$(\{3\},\{0,1,2,3\})$
$+$	$\uparrow$	$(\{0,1\},\varnothing)$	$(\{3\},\{0,1,2,3\})$	$(\{3\},\{0,1,2,3\})$
$-$	$\downarrow$	$(\{0\},\varnothing)$	$(\{2\},\varnothing)$	$(\{2\},\varnothing)$
$+$	$\downarrow$	$(\{0,1\},\varnothing)$	$(\{2\},\varnothing)$	$(\{0,1,2,3\},\{3\})$

Figure 3.37 – A tight lamination and its extended shadow

According to Proposition 3.23, saying that $k \in \pi_{\mathcal{L}}(A)$ is equivalent to saying that $k \leqslant \max(\pi_{\mathcal{L}}(A))$. Consequently, if $k \in \pi_{\mathcal{L}}(A)$, then $k \in \pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(A))$ as well, and the right endpoint of A is also the right endpoint of $\overrightarrow{\mathbf{e}}(A)$.

For example, Fig. 3.37 represents the tight lamination $\mathcal{L}$ associated with the braid $\beta = \sigma_2$ and the extended shadow of β. Here, $\pi^2_\beta(3,+,\downarrow) = (\{0,1,2,3\},\{3\})$. Indeed, the rightmost index of $\mathcal{L}$ is 3, the shadow of $\mathcal{A}^{\downarrow}_{+}(p_3)$ in $\mathcal{L}$ is $\{0,1,2,3\}$, and $\mathcal{A}^{\downarrow}_{+}(p_3)$ shares its right endpoint with an arc of shadow $\{3\}$ in $\mathcal{L}$.

Let λ be the right-oriented sliding braid of the form $[k \rightharpoonup \ell]$, where $1 \leqslant k < \ell \leqslant n$. We characterise which braids β are such that the word $\mathbf{RNF}(\beta)\cdot\lambda$ is in relaxation normal form. An analogous characterisation holds when λ is a braid $[k \rightharpoondown \ell]$.

Definition 3.38 (λ-relaxed lamination)**.**
Let β be an n-strand braid. We say that β is λ-relaxed if all of the following requirements are fulfilled:

1. $\pi_\beta(k,+,\downarrow) \neq \{k\}$;
2. *either* $\pi_\beta(k,+,\uparrow) = \{0,\ldots,k\}$ *or* $\pi_\beta(k,-,\uparrow) \subseteq \{k,\ldots,\ell-1\}$;
3. *for all* $i \in \{\ell+2,\ldots,n\}$, $\ell+1 \in \pi_\beta(i,-,\uparrow) \cap \pi_\beta(i,-,\downarrow)$;
4. *if* $\ell < n$, *then* $k \in \pi_\beta(\ell+1,+,\uparrow)$;
5. *if* $\ell < n$, *then either* $\pi_\beta(\ell+1,+,\downarrow) \neq \{\ell+1\}$ *or* $\pi_\beta(\ell+1,-,\uparrow) \subseteq \{k+1,\ldots,\ell\}$.

The five requirements of Definition 3.38 are illustrated in Fig. 3.40.

Proposition 3.39.
Let β be some n-strand braid, and let λ be a right-oriented sliding braid. If the equality $\mathbf{RNF}(\beta\lambda) = \mathbf{RNF}(\beta)\cdot\lambda$ *holds, then β is λ-relaxed.*

Proof. We prove Proposition 3.39 when $\lambda = [k \rightharpoonup \ell]$. The proof is analogous when $\lambda = [k \rightharpoondown \ell]$. Let β be a braid such that $\mathbf{RNF}(\beta\lambda) = \mathbf{RNF}(\beta)\cdot\lambda$, i.e. such that

$\mathbf{R}(\beta\lambda) = \lambda^{-1}$. Let $\mathcal{L}$ be a tight lamination of β, and let $\overline{\mathcal{L}}$ be a tight lamination of $\beta\lambda$. Until the end of the proof, we denote by $p_0, \ldots, p_n$ the punctures of $\mathcal{L}$, and we denote by $\overline{p}_0, \ldots, \overline{p}_n$ the punctures of $\overline{\mathcal{L}}$.

We first show how to draw $\mathcal{L}$ by modifying $\overline{\mathcal{L}}$. Observe that ℓ is the rightmost index of $\overline{\mathcal{L}}$ and that $\overline{p}_\ell$ belongs to a lower bigon of $\overline{\mathcal{L}}$. Since $\mathbf{R}(\beta\lambda) = \lambda^{-1}$, the puncture $\overline{p}_\ell$ of $\overline{\mathcal{L}}$ is slid along one upper neighbouring arc of $\overline{p}_\ell$ in $\overline{\mathcal{L}}$, and arrives at some position $p_k \in (\overline{p}_{k-1}, \overline{p}_k)$. Then, applying Algorithm 3.9 amounts to doing what follows.

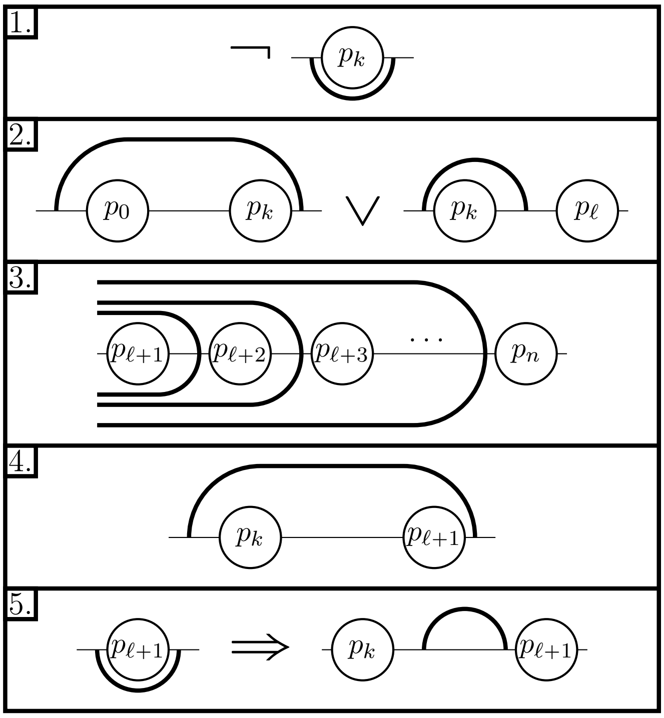


Figure 3.40 – Requirements in Definition 3.38

1. For each (lower) arc $\overline{A}$ of $\overline{\mathcal{L}}$ with shadow $\{\ell\}$ in $\overline{\mathcal{L}}$, consider the real projections $(e, E) := \pi_{\mathbb{R}}(\overleftarrow{\mathbf{e}}_{\overline{\mathcal{L}}}(\overline{A}))$ and $(e', E') := \pi_{\mathbb{R}}(\overrightarrow{\mathbf{e}}_{\overline{\mathcal{L}}}(\overline{A}))$. Lemma 3.25 shows that $e' < e < \overline{p}_{\ell-1} < E < \overline{p}_\ell < E' < p_{\ell+1}$. We merge the arcs $\overline{A}$, $\overleftarrow{\mathbf{e}}_{\overline{\mathcal{L}}}(\overline{A})$ and $\overrightarrow{\mathbf{e}}_{\overline{\mathcal{L}}}(\overline{A})$ into one upper arc with real projection (e', e).
2. We do not modify other arcs of $\overline{\mathcal{L}}$.
3. We replace the puncture $\overline{p}_\ell$ by the puncture p_k.

Note that this process does not depend of the order in which the arcs $\overline{A}$ are treated in the first step. Doing so, we obtain a lamination $\mathcal{L}$ with set of punctures

$$\{p_0, \ldots, p_n\} = \{\overline{p}_0, \ldots, \overline{p}_{k-1}, p_k, \overline{p}_k, \ldots, \overline{p}_{\ell-1}, \overline{p}_{\ell+1}, \ldots, \overline{p}_n\},$$

which represents the braid β.

In addition, consider some (lower) arc $\overline{A}$ of $\overline{\mathcal{L}}$ that was erased. Let B be the (lower) rightmost bigon of $\overline{\mathcal{L}}$, and consider the real projections $(e_B, E_B) := \pi_{\mathbb{R}}(\overleftarrow{\mathbf{e}}_{\overline{\mathcal{L}}}(B))$ and

$(e'_B, E'_B) := \pi_{\mathbb{R}}(\overrightarrow{\mathbf{e}}_{\overline{\mathcal{L}}}(B))$. It follows from the relation $\mathbf{R}(\beta\lambda) = \lambda^{-1}$ that $e'_B < p_k < e_B$. Since $B \overset{\pi}{\subseteq} \overline{A}$ this proves that

$$e' \leqslant e'_B < p_k < e_B \leqslant e < \overline{p}_{\ell-1} < E \leqslant E_B < \overline{p}_\ell < E'_B \leqslant E' < \overline{p}_{\ell+1}.$$

The arc $\overline{A}$ was therefore merged into an upper arc that blinds p_k but not $\overline{p}_{\ell-1}$. Conversely, each arc of $\mathcal{L}$ that either is a lower arc, or blinds $\overline{p}_{\ell-1} = p_\ell$, or does not blind p_k, is also an arc of $\overline{\mathcal{L}}$.

Lemma 3.25 and Corollary 3.26 prove that the intervals $\pi_{\beta\lambda}(\ell, -, \uparrow)$ and $\pi_{\beta\lambda}(\ell, +, \uparrow)$ are respectively of the form $\{i, \ldots, \ell-1\}$ and $\{j, \ldots, \ell\}$, with either $j = 0 < i = k \leqslant \ell-1$ or $1 \leqslant j = k \leqslant i \leqslant \ell - 1$.

We prove now that β satisfies each of the five requirements that characterise λ-relaxed braids, and thereby complete the proof of Proposition 3.39.

1. Let us assume here that $\pi_\beta(k, +, \downarrow) = \{k\}$. Then, the arc $\mathcal{A}^\downarrow_+(p_k, \mathcal{L})$ is a lower bigon of $\mathcal{L}$ and, like all lower arcs of $\mathcal{L}$, it is also an arc of $\overline{\mathcal{L}}$. Moreover, note that $\pi_{\mathbb{R}}(\mathcal{A}^\downarrow_+(p_k, \mathcal{L})) \subseteq (p_{k-1}, p_{k+1}) = (\overline{p}_{k-1}, \overline{p}_k)$. Hence, the arc $\mathcal{A}^\downarrow_+(p_k, \mathcal{L})$ blinds no puncture of $\overline{\mathcal{L}}$, contradicting the tightness of $\overline{\mathcal{L}}$. This means that our assumption was false, i.e. that $\pi_\beta(k, +, \downarrow) \neq \{k\}$.
2. The arcs $\mathcal{A}^\uparrow_+(\overline{p}_\ell, \overline{\mathcal{L}})$, $\mathcal{A}^\uparrow_-(\overline{p}_\ell, \overline{\mathcal{L}})$ and $\mathcal{A}^\downarrow_-(\overline{p}_\ell, \overline{\mathcal{L}}) = \mathcal{A}^\downarrow_+(\overline{p}_\ell, \overline{\mathcal{L}})$ were merged into one arc A of $\mathcal{L}$:
   - if $j \geqslant 1$, then $\overline{p}_\ell$ was slid along the arc $\mathcal{A}^\uparrow_+(\overline{p}_\ell, \overline{\mathcal{L}})$, hence $A = \mathcal{A}^\uparrow_-(p_k, \mathcal{L})$, and $\pi_{\mathcal{L}}(A) = \pi_\beta(k, -, \uparrow) = \{k, \ldots, i\} \subseteq \{k, \ldots, \ell-1\}$;
   - if $j = 0$, then $\overline{p}_\ell$ was slid along the arc $\mathcal{A}^\uparrow_-(\overline{p}_\ell, \overline{\mathcal{L}})$, hence $A = \mathcal{A}^\uparrow_+(p_k, \mathcal{L})$, and $\pi_{\mathcal{L}}(A) = \pi_\beta(k, +, \uparrow) = \{0, \ldots, k\}$.
3. Consider some integer $i \geqslant \ell + 2$. Proposition 3.23 implies that $\overline{p}_{\ell+1} \overset{\pi}{\in} \mathcal{A}^\uparrow_-(\overline{p}_i, \overline{\mathcal{L}})$. Due to the equalities $p_{\ell+1} = \overline{p}_{\ell+1}$ and $\mathcal{A}^\uparrow_-(\overline{p}_i, \overline{\mathcal{L}}) = \mathcal{A}^\uparrow_-(p_i, \mathcal{L})$, it follows that $\ell + 1 \in \pi_\beta(i, -, \uparrow)$. We prove similarly that $\ell + 1 \in \pi_\beta(i, -, \downarrow)$.
4. Let us assume that $\ell < n$. Corollary 3.26 proves that the upper parent of $\overline{p}_\ell$ in $\overline{\mathcal{L}}$ is the arc $\mathcal{A}^\uparrow_+(\overline{p}_\ell, \overline{\mathcal{L}})$. Moreover, Proposition 3.23 shows that $\overline{p}_\ell \overset{\pi}{\in} \mathcal{A}^\uparrow_+(\overline{p}_{\ell+1}, \overline{\mathcal{L}})$, which implies that $\overline{p}_\ell \overset{\pi}{\in} \mathcal{A}^\uparrow_+(\overline{p}_\ell, \overline{\mathcal{L}}) \overset{\pi}{\subseteq} \mathcal{A}^\uparrow_+(\overline{p}_{\ell+1}, \overline{\mathcal{L}})$. In addition, since $\mathbf{R}(\beta\lambda) = \lambda^{-1}$, the arc $\mathcal{A}^\uparrow_+(\overline{p}_\ell, \overline{\mathcal{L}})$ blinds the point p_k. This shows that $p_k \overset{\pi}{\in} \mathcal{A}^\uparrow_+(\overline{p}_\ell, \overline{\mathcal{L}}) \overset{\pi}{\subseteq} \mathcal{A}^\uparrow_+(\overline{p}_{\ell+1}, \overline{\mathcal{L}}) = \mathcal{A}^\uparrow_+(p_{\ell+1}, \mathcal{L})$, hence that $k \in \pi_\beta(\ell + 1, +, \uparrow)$.
5. Let us assume that $\ell < n$ and that $\pi_\beta(\ell + 1, +, \downarrow) = \{\ell + 1\}$. This means that $p_{\ell+1}$ belongs to a lower bigon in $\mathcal{L}$: let A be this bigon. Since A is a lower arc of $\mathcal{L}$, A is also an arc of $\overline{\mathcal{L}}$, which blinds the point $p_{\ell+1} = \overline{p}_{\ell+1}$ and no other common puncture of $\mathcal{L}$ and $\overline{\mathcal{L}}$. Proposition 3.23 shows that A also blinds some point $\overline{p}_i$ with $i \leqslant \ell$. Hence, A blinds $\overline{p}_\ell$, and $\pi_{\overline{\mathcal{L}}}(A) = \{\ell, \ell + 1\}$.
   Let B be the (lower) rightmost bigon of $\overline{\mathcal{L}}$, and let C and D denote respectively the arcs $\overleftarrow{\mathbf{e}}_{\mathcal{L}}(A)$ and $\overleftarrow{\mathbf{e}}_{\overline{\mathcal{L}}}(B)$. Let (e_C, E_C) and (e_D, E_D) be the real projections of C and D. Lemma 3.25 proves that $e_C < p_\ell < E_C < p_{\ell+1}$. This shows that C blinds p_ℓ, and thus that C is also an arc of $\overline{\mathcal{L}}$, as illustrated in Fig. 3.41.

Lemma 3.25 shows that $e_D < \overline{p}_{\ell-1} < E_D < \overline{p}_\ell$. Since $\overline{p}_\ell \in \pi_{\mathbb{R}}(B) \subseteq \pi_{\mathbb{R}}(A) \subseteq (\overline{p}_{\ell-1}, \overline{p}_{\ell+2})$, it follows that $\overline{p}_{\ell-1} < E_C < E_D < \overline{p}_\ell$. This proves that $C \overset{\pi}{\subsetneq} D$. Moreover, note that $D = \mathcal{A}^{\uparrow}_{-}(\overline{p}_\ell, \overline{\mathcal{L}})$, so that $\pi_{\mathbb{R}}(D) \subseteq (p_k, \overline{p}_\ell)$, which proves that $\pi_{\mathbb{R}}(C) \subseteq (p_k, \overline{p}_\ell) \subseteq (p_k, p_{\ell+1})$. Hence, the arc $C = \mathcal{A}^{\uparrow}_{-}(p_{\ell+1}, \mathcal{L})$ has shadow $\pi_\beta(\ell + 1, -, \uparrow) \subseteq \{k+1, \ldots, \ell\}$ in $\mathcal{L}$.

□

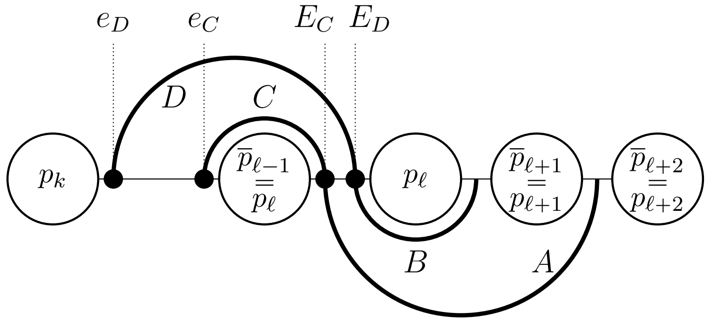


Figure 3.41 – A fragment of the lamination $\overline{\mathcal{L}}$

We also prove the converse implication of Proposition 3.39.

Proposition 3.42.
Let β be some n-strand braid, and let λ be a right-oriented sliding braid. If β is λ-relaxed, then the equality $\mathbf{RNF}(\beta\lambda) = \mathbf{RNF}(\beta) \cdot \lambda$ *holds.*

Proof. Like in the proof of Proposition 3.39, we assume hereafter that $\lambda = [k \rightharpoondown \ell]$. Let β be a λ-relaxed braid. Let $\mathcal{L}$ be a tight lamination of β, and let $\overline{\mathcal{L}}$ be a tight lamination of $\beta\lambda$. We denote by $p_0, \ldots, p_n$ the punctures of $\mathcal{L}$, and we denote by $\overline{p}_0, \ldots, \overline{p}_n$ the punctures of $\overline{\mathcal{L}}$.

We proceed by first showing how to draw $\overline{\mathcal{L}}$ by modifying $\mathcal{L}$. Let $\mathbf{A}^\uparrow$ denote the set of the upper arcs of $\mathcal{L}$.

We partition $\mathbf{A}^\uparrow$ in several subsets, and as illustrated in Fig. 3.43:

- $\Omega_1 = \{A \in \mathbf{A}^\uparrow : k \in \pi_{\mathcal{L}}(A) \text{ and } \ell, \ell+1 \notin \pi_{\mathcal{L}}(A)\}$,
- $\Omega_2 = \{A \in \mathbf{A}^\uparrow : \ell \in \pi_{\mathcal{L}}(A) \text{ and } k, \ell+1 \notin \pi_{\mathcal{L}}(A)\}$,
- $\Omega_3 = \{A \in \mathbf{A}^\uparrow : \ell+1 \in \pi_{\mathcal{L}}(A) \text{ and } k, \ell \notin \pi_{\mathcal{L}}(A)\}$,
- $\Omega_4 = \{A \in \mathbf{A}^\uparrow : k, \ell \in \pi_{\mathcal{L}}(A) \text{ and } \ell+1 \notin \pi_{\mathcal{L}}(A)\}$,
- $\Omega_5 = \{A \in \mathbf{A}^\uparrow : \ell, \ell+1 \in \pi_{\mathcal{L}}(A) \text{ and } k \notin \pi_{\mathcal{L}}(A)\}$,
- $\Omega_6 = \{A \in \mathbf{A}^\uparrow : k, \ell, \ell+1 \in \pi_{\mathcal{L}}(A)\}$,
- $\Omega_7 = \{A \in \mathbf{A}^\uparrow : \pi_{\mathcal{L}}(A) \subseteq \{0, \ldots, k-1\}\}$,
- $\Omega_8 = \{A \in \mathbf{A}^\uparrow : \pi_{\mathcal{L}}(A) \subseteq \{k+1, \ldots, \ell-1\}\}$, and

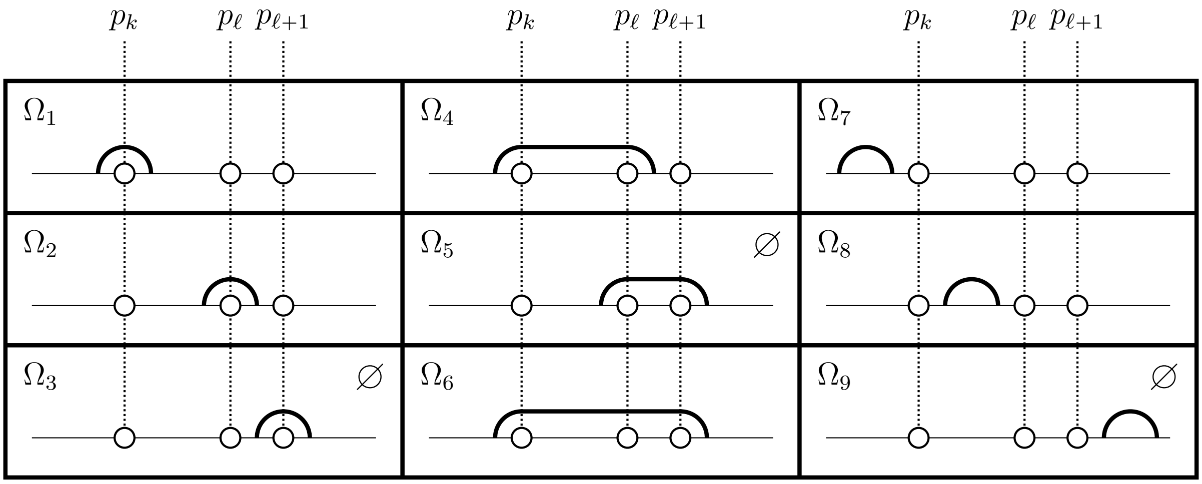


Figure 3.43 – Nine classes of upper arcs

- $\Omega_9 = \{A \in \mathbf{A}^\uparrow : \pi_{\mathcal{L}}(A) \subseteq \{\ell+2, \ldots, n\}\}$.

First, suppose that some arc A belongs to Ω_9. Let v_B be a leaf of $\mathcal{T}^{\updownarrow}_{\text{arc}}(\mathcal{L})$ such that v_A is an ancestor of v_B: Proposition 3.18 proves that B is a bigon of $\mathcal{L}$. Let j be the index of B in $\mathcal{L}$. Since $\{j\} = \pi_{\mathcal{L}}(B) \subseteq \pi_{\mathcal{L}}(A) \subseteq \{\ell+2, \ldots, n\}$, we know that $j \geqslant \ell + 2$, and since $B = \mathcal{A}^{\uparrow}_{-}(p_j)$, the requirement 3 is falsified. This contradiction proves that $\Omega_9 = \varnothing$.

Then, if $\ell = n$, then of course $\Omega_5 = \Omega_3 = \varnothing$. If $\ell < n$, then the requirement 4 indicates that $k \in \pi_{\mathcal{L}}(\mathcal{A}^{\uparrow}_{+}(p_{\ell+1}, \mathcal{L}))$. Consequently, p_k and $p_{\ell+1}$ are respectively a descendant and a child of $\mathcal{A}^{\uparrow}_{+}(p_{\ell+1}, \mathcal{L})$. Therefore, each upper arc A that blinds $p_{\ell+1}$ also blinds p_k, whence $\Omega_5 = \Omega_3 = \varnothing$. This is why we added the symbol $\varnothing$ when depicting in Fig. 3.43 the elements arcs in Ω_3, Ω_5 and Ω_9.

Moreover, let us extend the notations $\overset{\pi}{\subseteq}$ and $\overset{\pi}{\in}$ as follows: if X and Y are two subsets of $\mathbf{A}^\uparrow$ such that $A \overset{\pi}{\subsetneq} B$ for each ordered pair $(A, B) \in X \times Y$, then we write $X \overset{\pi}{\subseteq} Y$. Similarly, if $p \in \mathbb{R}$ is such that $p \overset{\pi}{\in} A$ for each arc $A \in X$, then we write $p \overset{\pi}{\in} X$. Using these notations, the relations shown in Fig. 3.44 are clear.

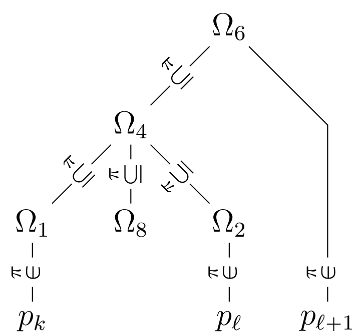


Figure 3.44 – Blinding relations between sets Ω_i

In addition, note that, for each point $p \in \mathbb{R}$, the relation $\overset{\pi}{\subseteq}$ induces a total order on the set of the upper arcs of $\mathcal{L}$ that blind the point p. Consequently, each of the sets Ω_1, Ω_2, Ω_4 and Ω_6 is totally ordered by $\overset{\pi}{\subseteq}$: we denote by $A_1^i \overset{\pi}{\subsetneq} \ldots \overset{\pi}{\subsetneq} A_{\omega_i}^i$ the elements of Ω_i, where $\omega_i = |\Omega_i|$, and we denote by (e_j^i, E_j^i) the real projection of the arc A_j^i.

It is then straightforward to see that the elements of $(p_\ell, p_{\ell+1}) \cap \mathcal{L}$ are

$$p_\ell < E_1^2 < \ldots < E_{\omega_2}^2 < E_1^4 < \ldots < E_{\omega_4}^4 < p_{\ell+1}.$$

Hence, let a_j^- and a_j^+ (for $1 \leqslant j \leqslant \omega_1$) and $\overline{p}_\ell$ be real numbers such that

$$p_\ell < E_1^2 < \ldots < E_{\omega_2}^2 < a_{\omega_1}^- < \ldots < a_1^- < \overline{p}_\ell < a_1^+ < \ldots < a_{\omega_1}^+ < E_1^4 < \ldots < E_{\omega_4}^4 < p_{\ell+1},$$

as illustrated in the top picture of Fig. 3.45.

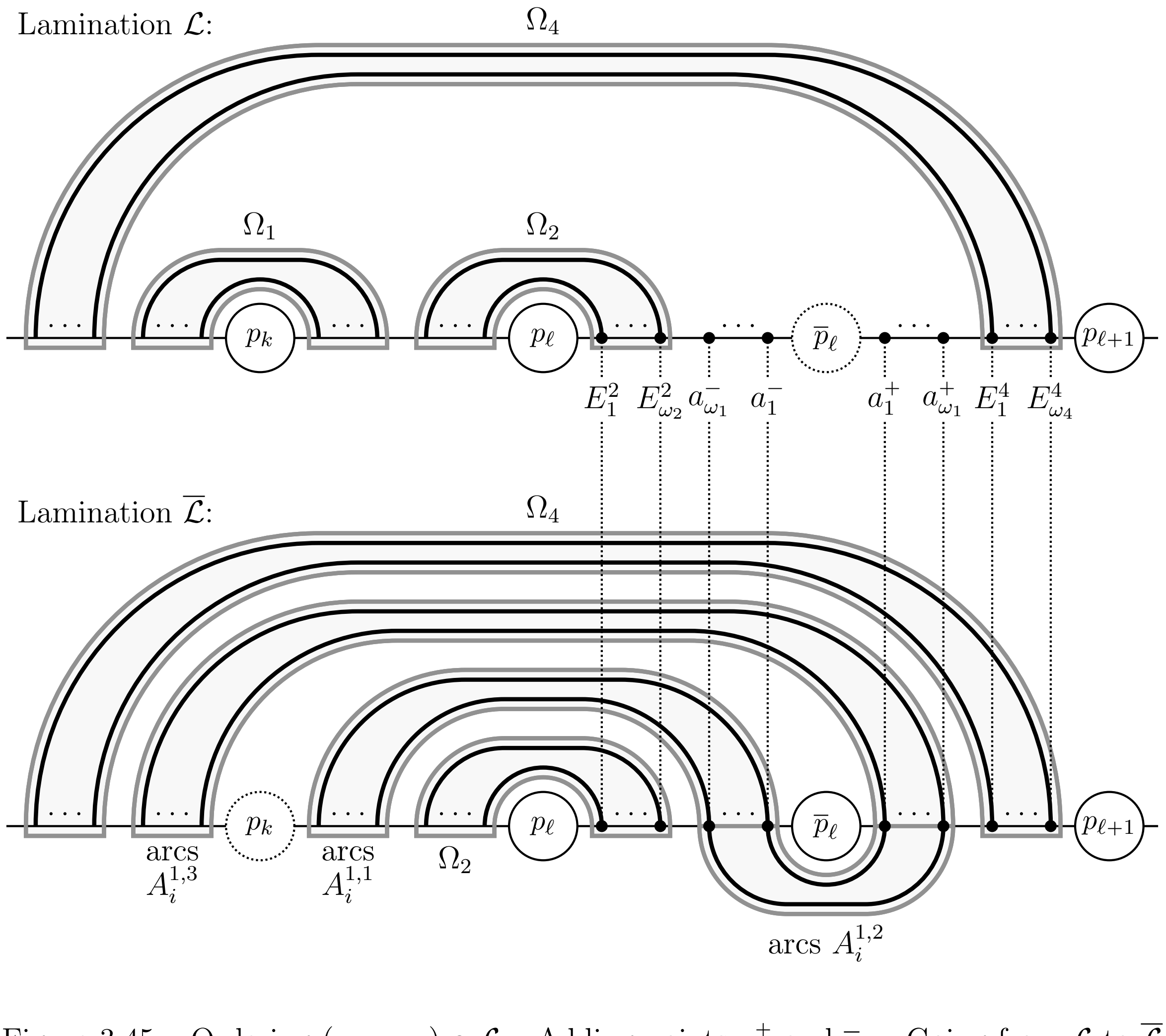


Figure 3.45 – Ordering $(p_\ell, p_{\ell+1}) \cap \mathcal{L}$ – Adding points $a_i^\pm$ and $\overline{p}_\ell$ – Going from $\mathcal{L}$ to $\overline{\mathcal{L}}$

Aiming to emulate a "backward" version of Algorithm 3.9, as illustrated in the bottom picture of Fig. 3.45, we draw the lamination $\overline{\mathcal{L}}$ as follows.

1. We replace each arc $A_j^1 \in \Omega_1$ by three arcs: one upper arc $A_j^{1,1}$ with real projection (E_j^1, a_j^-), one lower arc $A_j^{1,2}$ with real projection (a_j^-, a_j^+), and one upper arc $A_j^{1,3}$ with real projection (e_j^1, a_j^+).

2. We do not modify other arcs of $\mathcal{L}$.
3. We replace the puncture p_k by the puncture $\overline{p}_\ell$.

Note that this process does not depend in which the arcs A^i_j are treated in the first step.

Let us prove that, doing so, we obtain a lamination $\overline{\mathcal{L}}$ with set of punctures

$$\{\overline{p}_0, \ldots, \overline{p}_n\} = \{p_0, \ldots, p_{k-1}, p_{k+1}, \ldots, p_\ell, \overline{p}_\ell, p_{\ell+1}, \ldots, p_n\}.$$

First, checking that $\overline{\mathcal{L}}$ is a lamination amounts to checking that, if X and Y are both upper (or lower) arcs of $\overline{\mathcal{L}}$, then either $\pi_\mathbb{R}(X) \subseteq \pi_\mathbb{R}(Y)$ or $\pi_\mathbb{R}(X) \supseteq \pi_\mathbb{R}(Y)$ or $\pi_\mathbb{R}(X) \cap \pi_\mathbb{R}(Y) = \emptyset$. The case where either both or none of X and Y are arcs of $\mathcal{L}$ is immediate, and so is the case where X is of the type $A^{1,2}_j$. Finally, if X is of the type $A^{1,1}_j$ or $A^{1,3}_j$, it suffices to treat separately the cases where $Y \in \Omega_i$ for $i \in \{2, 4, 6, 7, 8\}$, and each of these cases is immediate too.

In addition, proving that $\overline{\mathcal{L}}$ is a tight lamination amounts to checking that each arc of $\overline{\mathcal{L}}$ blinds some puncture of $\overline{\mathcal{L}}$. There are two cases to check. First, each of the arcs $A^{1,1}_j$, $A^{1,2}_j$ and $A^{1,3}_j$ blinds either $p_\ell = \overline{p}_{\ell-1}$ or $\overline{p}_\ell$. Second, consider some arc A that belongs to both $\mathcal{L}$ and $\overline{\mathcal{L}}$. Since $\mathcal{L}$ is tight, A blinds some point p_i.

Let us assume that A blinds the point p_k. If A is an upper arc, then $A \notin \Omega_1$, hence $A \in \Omega_4 \cup \Omega_6$ and A blinds p_ℓ. If A is a lower arc, then the requirement 1 implies that $\pi_\mathcal{L}(A) \neq \{k\}$, and therefore A also blinds some point p_i with $i \neq k$. Hence, in all cases, A blinds some point p_i with $i \neq k$. Since p_i must be a puncture of both $\mathcal{L}$ and $\overline{\mathcal{L}}$, this proves that $\overline{\mathcal{L}}$ is tight.

Then, let us prove that ℓ is the rightmost index of $\overline{\mathcal{L}}$. Let B be the rightmost bigon of $\overline{\mathcal{L}}$, and let i be the index of B in $\overline{\mathcal{L}}$. Since $\overline{p}_\ell$ belongs to a bigon of $\overline{\mathcal{L}}$, we know that $i \geqslant \ell$. If $i \geqslant \ell + 1$, then B is also a bigon of $\mathcal{L}$, and $\pi_\mathcal{L}(B) = \pi_{\overline{\mathcal{L}}}(B) = \{i\}$. However, we shall prove now that no bigon of $\mathcal{L}$ with index $\ell + 1$ or more can be a bigon of $\overline{\mathcal{L}}$.

Indeed, consider some bigon C of $\mathcal{L}$ with index $c \geqslant \ell + 1$, if such a bigon exists. Note that $C = \mathcal{A}^\uparrow_-(p_c, \mathcal{L}) = \mathcal{A}^\uparrow_+(p_c, \mathcal{L})$ or that $C = \mathcal{A}^\downarrow_-(p_c, \mathcal{L}) = \mathcal{A}^\downarrow_+(p_c, \mathcal{L})$, depending on whether C is an upper or a lower bigon. Hence, we are in one of the following cases.

- If $c \geqslant \ell + 2$, then the requirement 3 states that $\ell + 1 \in \pi_\mathcal{L}(C)$, which is false.
- If $c = \ell + 1$ and if C is an upper bigon, then the requirement 4 states that $k \in \pi_\mathcal{L}(C)$, which is false.
- If $c = \ell + 1$ and if C is a lower bigon, then $\overleftarrow{\mathbf{e}}(C) = \mathcal{A}^\uparrow_-(p_{\ell+1}, \mathcal{L})$. Therefore, the requirement 5 shows that $\overleftarrow{\mathbf{e}}(C) \in \Omega_2$. This proves that $\Omega_4 = \emptyset$ and, since some curve of $\mathcal{L}$ must separate the punctures p_k and $p_{\ell+1}$, that $\Omega_1 \neq \emptyset$. Consequently, the new arc $A^{1,2}_1$ was "inserted" inside the former bigon C, which is therefore not a bigon of $\overline{\mathcal{L}}$. This situation is represented in Fig. 3.46.

It follows that $i \leqslant \ell$, i.e. that $i = \ell$, and therefore that $A^{1,2}_1$ is necessarily the rightmost bigon of $\overline{\mathcal{L}}$.

Finally, let γ be the braid that $\overline{\mathcal{L}}$ represents. Let us prove that $\mathbf{R}(\gamma) = \lambda^{-1}$ and that $\gamma\lambda^{-1} = \beta$. According to the requirement 2, two cases are possible.

- If $\pi_\beta(k, +, \uparrow) = \{0, \ldots, k\}$, then $A_1^1 = \mathcal{A}_+^\uparrow(p_k, \mathcal{L})$, and therefore both A_1^1 and $A_1^{1,3}$ blind $p_0 = \overline{p}_0$. Moreover, the interval (p_k, E_1^1) contains no endpoint of any arc of $\mathcal{L}$ nor of $\overline{\mathcal{L}}$. The lamination $\mathcal{L}$ is therefore obtained from $\overline{\mathcal{L}}$ by sliding the puncture $\overline{p}_\ell$ along the arc $A_1^{1,1} = \mathcal{A}_-^\uparrow(\overline{p}_\ell, \overline{\mathcal{L}})$, then merging arcs of $\overline{\mathcal{L}}$ like in Algorithm 3.9.
- If $\pi_\beta(k, -, \uparrow) \subseteq \{k, \ldots, \ell - 1\}$, then $p_k \overset{\pi}{\in} \mathcal{A}_-^\uparrow(p_k)$ and $A_1^1 = \mathcal{A}_-^\uparrow(p_k, \mathcal{L})$, and therefore $A_{1,3}$ does not blind $p_0 = \overline{p}_0$. Moreover, the interval (e_1^1, p_k) contains no endpoint of any arc of $\mathcal{L}$ nor of $\overline{\mathcal{L}}$. The lamination $\mathcal{L}$ is therefore obtained from $\overline{\mathcal{L}}$ by sliding the puncture $\overline{p}_\ell$ along the arc $A_1^{1,3} = \mathcal{A}_+^\uparrow(\overline{p}_\ell, \overline{\mathcal{L}})$, then merging arcs of $\overline{\mathcal{L}}$.

In both cases, it follows that $\mathbf{R}(\gamma) = \lambda^{-1}$ and that $\gamma\lambda^{-1} = \beta$.

This means that $\overline{\mathcal{L}}$ was indeed the tight lamination of $\beta\lambda$, and that $\mathbf{RNF}(\beta\lambda) = \mathbf{RNF}(\beta) \cdot \lambda$, which completes the proof of Proposition 3.42. □

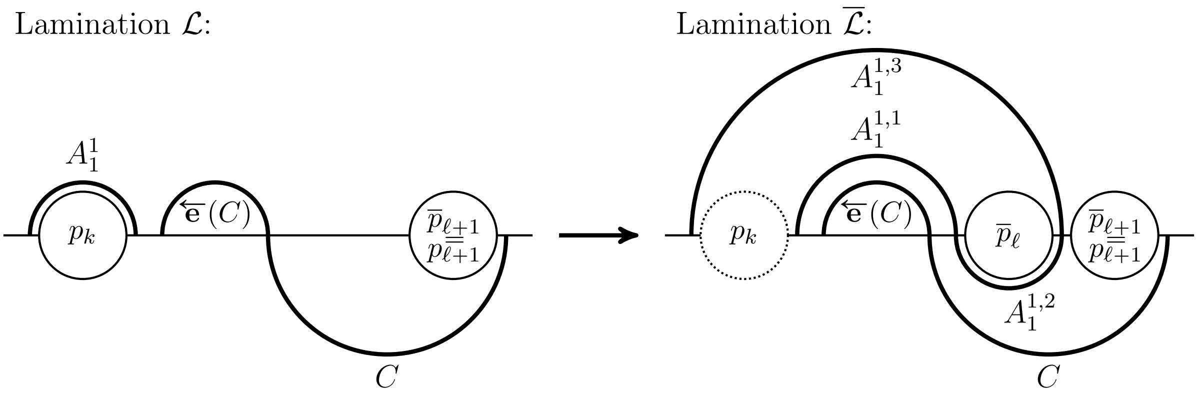


Figure 3.46 – From $\mathcal{L}$ to $\overline{\mathcal{L}}$ when $p_{\ell+1}$ belongs to a lower bigon of $\mathcal{L}$

Corollary 3.47.
Let β be some braid, and let λ be a right-oriented sliding braid. The equality $\mathbf{RNF}(\beta\lambda) = \mathbf{RNF}(\beta) \cdot \lambda$ *holds if and only if β is λ-relaxed.*

3.2.3 An Automaton for the Relaxation Normal Form

Corollary 3.47 paves the way for proving that the relaxation normal form is regular.

Theorem 3.48.
Let Σ be the set of all right-oriented sliding n-strand braids. There exists two functions **decide** *and* **compute** *that take as inputs the extended shadow π_β^2 of some braid β and a right-oriented sliding braid $\lambda \in \Sigma$, and such that*

1. $\mathbf{decide}(\pi_\beta^2, \lambda) = \mathbf{true}$ *if* $\mathbf{RNF}(\beta) \cdot \lambda = \mathbf{RNF}(\beta\lambda)$*, and* **false** *otherwise;*

2. $\mathbf{compute}(\pi^2_\beta, \lambda) = \pi^2_{\beta\lambda}$ *if* $\mathbf{RNF}(\beta) \cdot \lambda = \mathbf{RNF}(\beta\lambda)$.

Proof. Corollary 3.47 implies that knowing π_β is sufficient to check whether $\mathbf{R}(\beta\lambda) = \lambda^{-1}$. Hence, knowing π^2_β is also sufficient, which proves the first part of Theorem 3.48.

We prove now the second part of Theorem 3.48 when $\lambda = [k \rightharpoondown \ell]$. First, if $\beta \neq \mathbf{1}$, note that $\pi^2_\beta \neq \pi^2_{\mathbf{1}}$. Indeed, the rightmost index of β is some positive integer i, which means that either $\pi_\beta(i, +, \uparrow) = \{i\}$ or $\pi_\beta(i, +, \downarrow) = \{i\}$, whereas $\pi_{\mathbf{1}}(i, +, \updownarrow) = \{0, \ldots, i\}$. Therefore, we may already define the partial function $\mathbf{compute}(\pi^2_{\mathbf{1}}, \cdot) : \lambda \mapsto \pi^2_\lambda$, and there remains to build an appropriate function **compute** on pairs (π^2_β, λ) such that β is a non-trivial λ-relaxed braid.

Since $\mathbf{RNF}(\beta\lambda) = \mathbf{RNF}(\beta) \cdot \lambda$, we know that neither β nor $\beta\lambda$ is trivial. Let $\mathcal{L}$ and $\overline{\mathcal{L}}$ be the (non-trivial) tight laminations that represent respectively the braids β and $\beta\lambda$. Once again, we denote by $p_0, \ldots, p_n$ the punctures of $\mathcal{L}$, and we denote by $\overline{p}_0, \ldots, \overline{p}_n$ the punctures of $\overline{\mathcal{L}}$. In addition, consider the functions $\psi : i \mapsto i - \mathbf{1}_{k<i}$, $\overline{\psi} : i \mapsto i - \mathbf{1}_{k<i\leqslant\ell}$ and

$$\begin{array}{llll}
\Psi & : & I & \mapsto & \{x : \psi(\min I) \leqslant x \leqslant \max I\} \text{ if } I \neq \varnothing, \text{ or } \varnothing \text{ if } I = \varnothing; \\
\Psi^* & : & I & \mapsto & \{x : \psi(\min I) \leqslant x \leqslant \psi(\max I)\} \text{ if } I \neq \varnothing, \text{ or } \varnothing \text{ if } I = \varnothing; \\
\Theta^\uparrow & : & (I, J) & \mapsto & (\Psi(I), \Psi(J)) \text{ if } \{k, \ell\} \subseteq I, \text{ or } (\Psi^*(I), \varnothing) \text{ if } \{k, \ell\} \nsubseteq I; \\
\Theta^\downarrow & : & (I, J) & \mapsto & (\Psi(I), \Psi(J)) \text{ if } \{k, \ell\} \subseteq J, \text{ or } (\Psi^*(I), \varnothing) \text{ if } \{k, \ell\} \nsubseteq J,
\end{array}$$

where I and J are subintervals of $\{0, \ldots, n\}$. The functions ψ, $\overline{\psi}$, Ψ, Ψ^*, $\Theta^\uparrow$ and $\Theta^\downarrow$ will play a crucial role in computing $\pi^2_{\beta\lambda}$.

Intuitively, the functions ψ and $\overline{\psi}$ are meant to reflect the fact that some punctures of $\mathcal{L}$ and $\overline{\mathcal{L}}$ have different names: $p_i = \overline{p}_{\psi(i)} = \overline{p}_{\overline{\psi}(i)}$ if $i < k$ or if $k < i \leqslant \ell$, and $p_i = \overline{p}_i = \overline{p}_{\overline{\psi}(i)}$ if $\ell < i$. In addition, let A be an arc of both $\mathcal{L}$ and $\overline{\mathcal{L}}$, i.e. an arc of $\mathcal{L}$ that does not belong to Ω_1. We prove below that $\pi_{\overline{\mathcal{L}}}(A) = \Psi(\pi_{\mathcal{L}}(A))$ if A blinds $\overline{p}_\ell$, and that $\pi_{\overline{\mathcal{L}}}(A) = \Psi^*(\pi_{\mathcal{L}}(A))$ otherwise. We also prove that $\pi^2_{\overline{\mathcal{L}}}(A) = \Theta^\uparrow(\pi^2_{\mathcal{L}}(A))$ if A is an upper arc, and that $\pi^2_{\overline{\mathcal{L}}}(A) = \Theta^\downarrow(\pi^2_{\mathcal{L}}(A))$ if A is a lower arc.

Indeed, remember how the lamination $\overline{\mathcal{L}}$ was drawn in the proof of Proposition 3.42. We split the set $\mathbf{A}^\uparrow$ of upper arcs of $\mathcal{L}$ into six subsets Ω_1, Ω_2, Ω_4, Ω_6, Ω_7 and Ω_8, and replaced each arc A^1_j by three arcs $A^{1,1}_j$, $A^{1,2}_j$ and $A^{1,3}_j$, then replaced the puncture p_k by a new puncture $\overline{p}_\ell$, and did not modify any other arc or puncture.

Furthermore, let (e, E) be the real projection of A. Proposition 3.23 shows that $e < \overline{p}_\ell$, hence one shows easily that $\pi^2_{\overline{\mathcal{L}}}(A) = (\Psi^*(\pi_{\mathcal{L}}(A)), \varnothing)$ if $E < \overline{p}_\ell$, and that $\pi_{\overline{\mathcal{L}}}(A) = \Psi(\pi_{\mathcal{L}}(A))$ if $E > \overline{p}_\ell$.

In addition, if $E > \overline{p}_\ell$, let I, J be subintervals of $\{0, \ldots, n\}$ such that $\pi^2_{\mathcal{L}}(A) = (I, J)$, and let m be the rightmost index of $\mathcal{L}$. Let us prove that J is non-empty. Due to the requirement 3 in Definition 3.38, we know that $m \leqslant \ell + 1$. If $m = \ell + 1$, then the interval $(\overline{p}_\ell, p_{\ell+1})$ contains no endpoint of any arc of $\mathcal{L}$, and therefore $E > p_{\ell+1}$, so that $J \neq \varnothing$;

if $m \leqslant \ell$, then of course $J \neq \varnothing$ as well. Hence, regardless of the value of m, we have $J \neq \varnothing$, i.e. $J = \pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(A))$.

If A is an upper arc, then $E > \overline{p}_\ell$ if and only if $A \in \Omega_4 \cup \Omega_6$, i.e. if and only if $\{k, \ell\} \subseteq \pi_{\mathcal{L}}(A) = I$. Consequently, if A is a lower arc, then $E > \overline{p}_\ell$ if and only if $\{k, \ell\} \subseteq \pi_{\mathcal{L}}(\overrightarrow{\mathbf{e}}(A)) = J$. This proves that $\pi^2_{\overline{\mathcal{L}}}(A) = \Theta^{\uparrow}(\pi^2_{\mathcal{L}}(A))$ for all upper arcs $A \notin \Omega_1$ of $\mathcal{L}$, and that $\pi^2_{\overline{\mathcal{L}}}(A) = \Theta^{\downarrow}(\pi^2_{\mathcal{L}}(A))$ for all lower arcs A of $\mathcal{L}$.

Finally, if A^1_i is an element of Ω_1 and has shadow $\{u, \ldots, v\}$ in $\mathcal{L}$, then the arcs $A^{1,1}_i$, $A^{1,2}_i$ and $A^{1,3}_i$ of $\overline{\mathcal{L}}$ have respective shadows $\{v, \ldots, \ell{-}1\}$, $\{\ell\}$ and $\{u, \ldots, \ell\}$ in $\overline{\mathcal{L}}$. Therefore, the last remaining challenge is to identify the neighbouring arcs of the punctures of $\overline{\mathcal{L}}$. We do it, and thereby we compute $\pi^2_{\beta\lambda}$ as a function of π^2_β and of λ, as follows.

1. Let $(\diamond, u, v)$ be the unique triple in $\{(+, 0, k)\} \cup \{(-, k, z) : k \leqslant z < \ell\}$ such that $\pi_\beta(k, \diamond, \uparrow) = \{u, \ldots, v\}$. In $\mathcal{L}$, the upper parent of p_k has a shadow $\{u, \ldots, v\}$, whence
$$\begin{aligned} \pi^2_{\beta\lambda}(\ell, +, \uparrow) &= (\{u, \ldots, \ell\}, \{\ell\}); \\ \pi^2_{\beta\lambda}(\ell, -, \uparrow) &= (\{v, \ldots, \ell - 1\}, \varnothing); \\ \pi^2_{\beta\lambda}(\ell, \downarrow, \pm) &= (\{\ell\}, \{u, \ldots, \ell\}). \end{aligned}$$

2. First, recall that $\mathcal{A}^{\uparrow}_{+}(p_{\ell+1}, \mathcal{L}) \in \Omega_6$. Let x be the integer such that $\pi_\beta(\ell + 1, +, \uparrow) = \{x, \ldots, \ell + 1\}$: we have $x \leqslant k$.
If $k \in \pi_\beta(\ell + 1, -, \uparrow)$, then $p_{\ell+1} = \overline{p}_{\ell+1}$ has the same neighbour arcs in $\mathcal{L}$ and in $\overline{\mathcal{L}}$.
If $k \notin \pi_\beta(\ell + 1, -, \uparrow)$, then Fig. 3.49 illustrates the case where $x < k$ (the case where $x = k$ is analogous). First, observe that $\Omega_4 = \varnothing$. Since some arcs must separate the punctures p_k, p_ℓ and $p_{\ell+1}$ in $\mathcal{L}$, it follows that Ω_1 and Ω_2 are non-empty. Consequently, the neighbour arcs of the puncture $\overline{p}_{\ell+1} = p_{\ell+1}$ in $\overline{\mathcal{L}}$ are
$$\begin{aligned} &\mathcal{A}^{\uparrow}_{+}(\overline{p}_{\ell+1}, \overline{\mathcal{L}}) = \mathcal{A}^{\uparrow}_{+}(p_{\ell+1}, \mathcal{L}) = A^6_1, && \mathcal{A}^{\downarrow}_{+}(\overline{p}_{\ell+1}, \overline{\mathcal{L}}) = \mathcal{A}^{\downarrow}_{+}(p_{\ell+1}, \mathcal{L}), \\ &\mathcal{A}^{\uparrow}_{-}(\overline{p}_{\ell+1}, \overline{\mathcal{L}}) = A^{1,3}_{\omega_1}, && \mathcal{A}^{\downarrow}_{-}(\overline{p}_{\ell+1}, \overline{\mathcal{L}}) = A^{1,2}_{\omega_1}, \end{aligned}$$
Moreover, since $\Omega_4 = \varnothing$, the arcs $A^1_{\omega_1}$ and $A^2_{\omega_2}$ must be (the first two) children of A^6_1 in $\mathcal{L}$, whose rightmost child is then $p_{\ell+1}$. Since $x \leqslant k$, it follows that $x = \min \pi_{\mathcal{L}}(A^6_1) = \min \pi_{\mathcal{L}}(A^1_{\omega_1}) = \min \pi_{\overline{\mathcal{L}}}(A^{1,3}_{\omega_1})$.

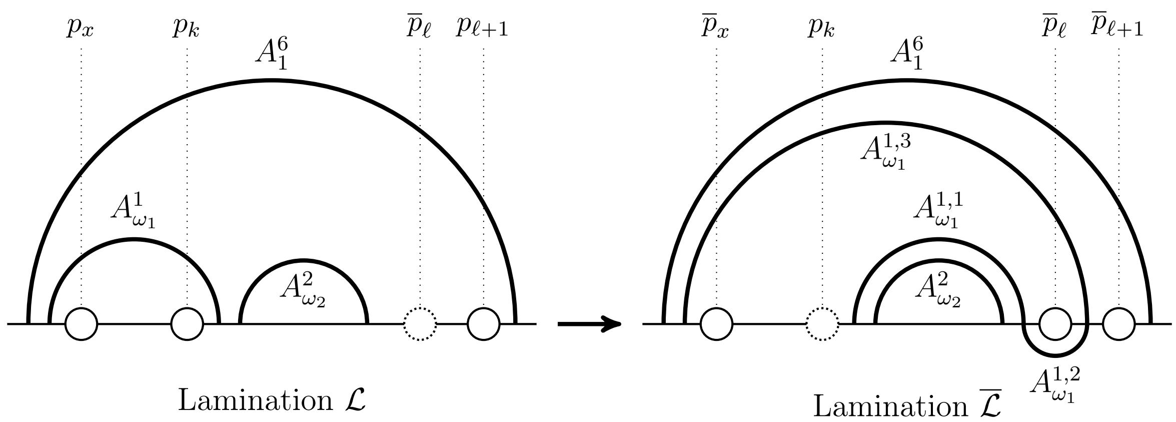


Figure 3.49 – Computing $\pi^2_{\beta\lambda}(\ell + 1, \pm, \updownarrow)$ when $k \notin \pi_\beta(\ell + 1, -, \uparrow)$ — assuming $x < k$

Adding these two cases, we obtain

$$\begin{array}{rcl}
\pi^2_{\beta\lambda}(\ell+1,+,\uparrow) & = & \Theta^{\uparrow}(\pi^2_\beta(\ell+1,+,\uparrow)); \\
\pi^2_{\beta\lambda}(\ell+1,+,\downarrow) & = & \Theta^{\downarrow}(\pi^2_\beta(\ell+1,+,\downarrow)); \\
\pi^2_{\beta\lambda}(\ell+1,-,\uparrow) & = & \Theta^{\uparrow}(\pi^2_\beta(\ell+1,-,\uparrow)) \text{ if } k \in \pi_\beta(\ell+1,-,\uparrow) \\
 & & (\{x,\ldots,\ell\},\{\ell\}) \text{ if } k \notin \pi_\beta(\ell+1,-,\uparrow); \\
\pi^2_{\beta\lambda}(\ell+1,-,\downarrow) & = & \Theta^{\downarrow}(\pi^2_\beta(\ell+1,-,\downarrow)) \text{ if } k \in \pi_\beta(\ell+1,-,\uparrow) \\
 & & (\{\ell\},\{x,\ldots,\ell\}) \text{ if } k \notin \pi_\beta(\ell+1,-,\uparrow).
\end{array}$$

3. Observe that $\mathcal{A}^{\uparrow}_{+}(p_\ell,\mathcal{L}) \in \Omega_2 \cup \Omega_4$ and that $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}) \in \Omega_1 \cup \Omega_2 \cup \Omega_8$. In addition, either $\mathcal{A}^{\uparrow}_{+}(p_\ell,\mathcal{L}) = \mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L})$, or $\mathcal{A}^{\uparrow}_{+}(p_\ell,\mathcal{L})$ is the parent of $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L})$. Note that the former case arises if and only if p_ℓ belongs to an upper bigon of $\mathcal{L}$ or, equivalently, if $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}) \in \Omega_2$. Hence, let y be the integer such that $\pi_\beta(\ell,+,\uparrow) = \{y,\ldots,\ell\}$. In addition, if $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}) \in \Omega_1 \cup \Omega_8$, let z be the integer such that $\pi_\beta(\ell,-,\uparrow) = \{z,\ldots,\ell-1\}$.
If $k \notin \pi_\beta(\ell,+,\uparrow)$, then $\mathcal{A}^{\uparrow}_{+}(p_\ell,\mathcal{L}) \in \Omega_2$ and $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}) \in \Omega_2 \cup \Omega_8$. In this case, the puncture $p_\ell = \overline{p}_{\ell-1}$ has the same neighbour arcs in $\mathcal{L}$ and in $\overline{\mathcal{L}}$.
If $k \in \pi_\beta(\ell,-,\uparrow)$, then $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}) \in \Omega_1$ and $\mathcal{A}^{\uparrow}_{+}(p_\ell,\mathcal{L}) \in \Omega_4$. Hence, $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}) = A^1_{\omega_1}$ and $\Omega_2 = \varnothing$, which shows that $\overline{p}_{\ell-1}$ belongs to an upper bigon of $\overline{\mathcal{L}}$. Consequently, the neighbour arcs of the puncture $\overline{p}_{\ell-1} = p_\ell$ in $\overline{\mathcal{L}}$ are

$$\begin{array}{ll}
\mathcal{A}^{\uparrow}_{+}(\overline{p}_{\ell-1},\overline{\mathcal{L}}) = A^{1,1}_{\omega_1}, & \mathcal{A}^{\downarrow}_{+}(\overline{p}_{\ell-1},\overline{\mathcal{L}}) = A^{1,2}_{\omega_1}, \\
\mathcal{A}^{\uparrow}_{-}(\overline{p}_{\ell-1},\overline{\mathcal{L}}) = A^{1,1}_{\omega_1}, & \mathcal{A}^{\downarrow}_{-}(\overline{p}_{\ell-1},\overline{\mathcal{L}}) = \mathcal{A}^{\downarrow}_{-}(p_\ell,\mathcal{L}).
\end{array}$$

Moreover, since $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}) \in \Omega_1$, the integer z is well-defined, and satisfies the inequality $z \leqslant k$: Figure 3.50a illustrates the case where $z < k$. It follows that $z = \min \pi_{\mathcal{L}}(A^1_{\omega_1}) = \min \pi_{\overline{\mathcal{L}}}(A^{1,3}_{\omega_1})$.
Finally, if $k \in \pi_\beta(\ell,+,\uparrow)$ and $k \notin \pi_\beta(\ell,-,\uparrow)$, then $\mathcal{A}^{\uparrow}_{+}(p_\ell,\mathcal{L}) \in \Omega_4$ and $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}) \in \Omega_8$. It follows that $\Omega_2 = \varnothing$. Consequently, the neighbour arcs of the puncture $\overline{p}_{\ell-1} = p_\ell$ in $\overline{\mathcal{L}}$ are

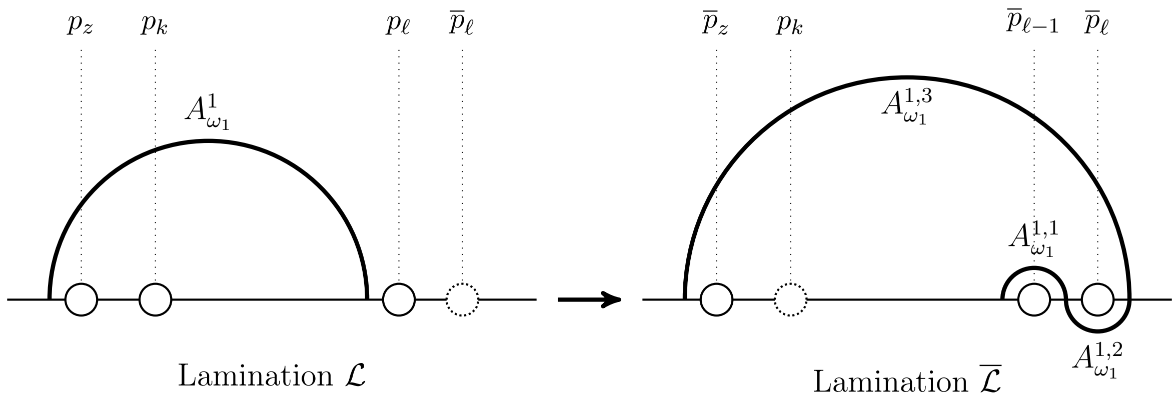
$$\begin{array}{ll}
\mathcal{A}^{\uparrow}_{+}(\overline{p}_{\ell-1},\overline{\mathcal{L}}) = A^{1,1}_{\omega_1}, & \mathcal{A}^{\downarrow}_{+}(\overline{p}_{\ell-1},\overline{\mathcal{L}}) = A^{1,2}_{\omega_1}, \\
\mathcal{A}^{\uparrow}_{-}(\overline{p}_{\ell-1},\overline{\mathcal{L}}) = \mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L}), & \mathcal{A}^{\downarrow}_{-}(\overline{p}_{\ell-1},\overline{\mathcal{L}}) = \mathcal{A}^{\downarrow}_{-}(p_\ell,\mathcal{L}),
\end{array}$$

Moreover, the arcs $A^1_{\omega_1}$ and $\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L})$ are therefore (the first two) children of $\mathcal{A}^{\uparrow}_{+}(p_\ell,\mathcal{L}) = A^4_1$ in $\mathcal{L}$, whose rightmost child is then p_ℓ. Hence, both integers y and z are well-defined, and they satisfy the inequalities $y \leqslant k < z$: Figure 3.50b illustrates the case where $y < k$. It follows that

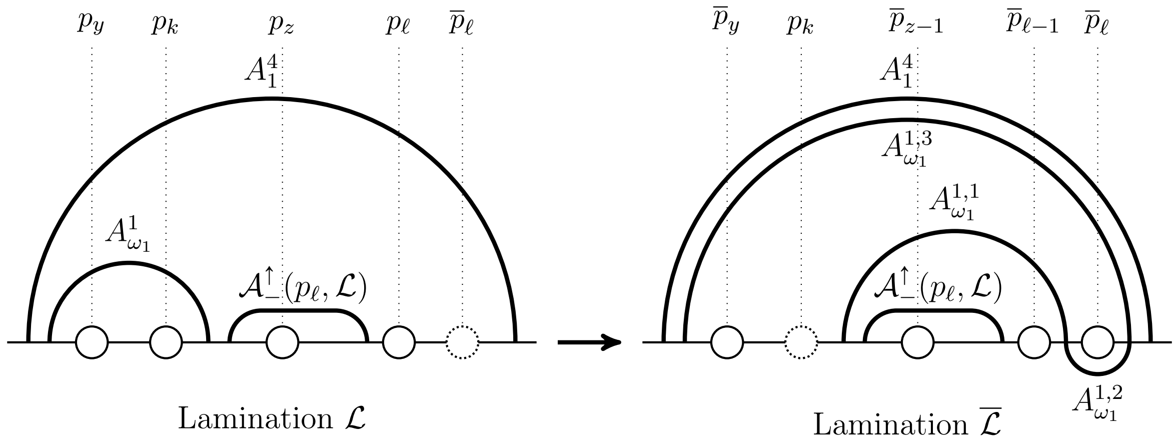
$$y = \min \pi_{\mathcal{L}}(A^4_1) = \min \pi_{\mathcal{L}}(A^1_{\omega_1}) = \min \pi_{\overline{\mathcal{L}}}(A^{1,3}_{\omega_1}) \text{ and that}$$
$$z - 1 = \min \pi_{\mathcal{L}}(\mathcal{A}^{\uparrow}_{-}(p_\ell,\mathcal{L})) - 1 = \max \pi_{\mathcal{L}}(A^1_{\omega_1}) = \min \pi_{\overline{\mathcal{L}}}(A^{1,1}_{\omega_1}).$$

Adding these three cases, we obtain

$$
\begin{aligned}
\pi^2_{\beta\lambda}(\ell-1,+,\uparrow) &= \Theta^{\uparrow}(\pi^2_{\beta}(\ell,+,\uparrow)) \text{ if } k \notin \pi_\beta(\ell,+,\uparrow)\\
& (\{\ell-1\},\varnothing) \text{ if } k \in \pi_\beta(\ell,-,\uparrow)\\
& (\{z-1,\ldots,\ell\},\varnothing) \text{ if } k \in \pi_\beta(\ell,+,\uparrow) \text{ and } k \notin \pi_\beta(\ell,-,\uparrow);\\
\pi^2_{\beta\lambda}(\ell-1,-,\uparrow) &= \Theta^{\uparrow}(\pi^2_{\beta}(\ell,-,\uparrow)) \text{ if } k \notin \pi_\beta(\ell,-,\uparrow)\\
& (\{\ell-1\},\varnothing) \text{ if } k \in \pi_\beta(\ell,-,\uparrow);\\
\pi^2_{\beta\lambda}(\ell-1,+,\downarrow) &= \Theta^{\downarrow}(\pi^2_{\beta}(\ell,+,\downarrow)) \text{ if } k \notin \pi_\beta(\ell,+,\uparrow)\\
& (\{\ell\},\{z,\ldots,\ell\}) \text{ if } k \in \pi_\beta(\ell,-,\uparrow)\\
& (\{\ell\},\{y,\ldots,\ell\}) \text{ if } k \in \pi_\beta(\ell,+,\uparrow) \text{ and } k \notin \pi_\beta(\ell,-,\uparrow);\\
\pi^2_{\beta\lambda}(\ell-1,-,\downarrow) &= \Theta^{\downarrow}(\pi^2_{\beta}(\ell,-,\downarrow)).
\end{aligned}
$$



(a) Case #1: $k \in \pi_\beta(\ell,-,\uparrow)$ — assuming $z < k$



(b) Case #2: $k \in \pi_\beta(\ell,+,\uparrow)$ and $k \notin \pi_\beta(\ell,-,\uparrow)$ — assuming $y < k$

Figure 3.50 – Computing $\pi^2_{\beta\lambda}(\ell-1,\pm,\updownarrow)$ when $k \in \pi_\beta(\ell,+,\uparrow)$

4. Let i be an integer such that $i \notin \{k,\ell,\ell+1\}$. The neighbour arcs of p_i in $\mathcal{L}$ either are some arc $A^1_j \in \Omega_1$ (which will be replaced by $A^{1,1}_j$ if $i > k$, or $A^{1,3}_j$ if $i < k$, when transforming $\mathcal{L}$ into $\overline{\mathcal{L}}$) or are also neighbour arcs of p_i in $\overline{\mathcal{L}}$. It follows that

$$
\begin{aligned}
\pi^2_{\beta\lambda}(\overline{\psi}(i),+,\uparrow) &= (\{i+1,\ldots,\ell\},\{\ell\}) \text{ if } i<k,\ k\in\pi_\beta(i,+,\uparrow) \text{ and } \\
&\qquad\qquad\qquad\qquad \ell\notin\pi_\beta(i,+,\uparrow) \\
& (\{i,\ldots,\ell-1\},\varnothing) \text{ if } k<i<\ell \text{ and } k\in\pi_\beta(i,+,\uparrow) \\
& \Theta^\uparrow(\pi_\beta(i,+,\uparrow)) \text{ otherwise;} \\
\pi^2_{\beta\lambda}(\overline{\psi}(i),-,\uparrow) &= (\{i,\ldots,\ell\},\{\ell\}) \text{ if } i<k,\ k\in\pi_\beta(i,-,\uparrow) \text{ and } \ell\notin\pi_\beta(i,-,\uparrow) \\
& (\{i-1,\ldots,\ell-1\},\varnothing) \text{ if } k<i<\ell \text{ and } k\in\pi_\beta(i,-,\uparrow) \\
& \Theta^\uparrow(\pi_\beta(i,-,\uparrow)) \text{ otherwise;} \\
\pi^2_{\beta\lambda}(\overline{\psi}(i),+,\downarrow) &= \Theta^\downarrow(\pi^2_\beta(i,+,\downarrow)); \\
\pi^2_{\beta\lambda}(\overline{\psi}(i),-,\downarrow) &= \Theta^\downarrow(\pi^2_\beta(i,-,\downarrow)).
\end{aligned}
$$

This disjunction of cases provides us with a complete characterisation of $\pi^2_{\beta\lambda}$ as a function depending only on π_β and of λ, which completes the proof of Theorem 3.48. □

Corollary 3.51.
Let n be a positive integer. The language of all relaxation normal words in the braid group $\mathbf{B}_n$ is regular, and is recognised by the deterministic automaton $\mathcal{A} = (\Sigma, Q, i, \delta, Q)$, with

- *alphabet $\Sigma = \{[k \frown \ell] : 1 \leqslant k < \ell \leqslant n\} \cup \{[k \smile \ell] : 1 \leqslant k < \ell \leqslant n\}$;*
- *state set $Q = \{\pi^2_\beta : \beta \in \mathbf{B}_n\}$;*
- *initial state $i = \pi^2_{\mathbf{1}}$;*
- *transition function $\delta = \{(\pi^2_\beta, \lambda, \pi^2_{\beta\lambda}) : \mathbf{R}(\beta\lambda) = \lambda^{-1}\}$;*
- *set of accepting states Q.*

Figure 3.52 presents the minimal automaton accepting the language $\mathbf{RNF}(\mathbf{B}_3)$.

This minimal automaton is obtained by merging states of the above defined automaton $\mathcal{A}$. Hence, each state $\mathbf{s}$ of the minimal automaton is a subset of Q, and is represented in Fig. 3.52 by some braid β such that $\pi^2_\beta \in \mathbf{s}$. The initial state is the state $\{\pi^2_{\mathbf{1}}\}$, and each state is accepting. Moreover, for the sake of readability of Fig. 3.52, we chose to denote by $\overline{\beta}$ the braid β^{-1}.

3.3 Is This Automaton Really Efficient?

Corollary 3.51 provides us with a deterministic automaton that accepts the relaxation normal form. A natural question is that of the size of the deterministic automaton for that normal form. Is the automaton provided in Corollary 3.51 minimal or close to minimal? Before answering this question, we first introduce some combinatorial objects that arise from the concepts of lamination trees and of neighbour arcs, as illustrated in Fig. 3.53.

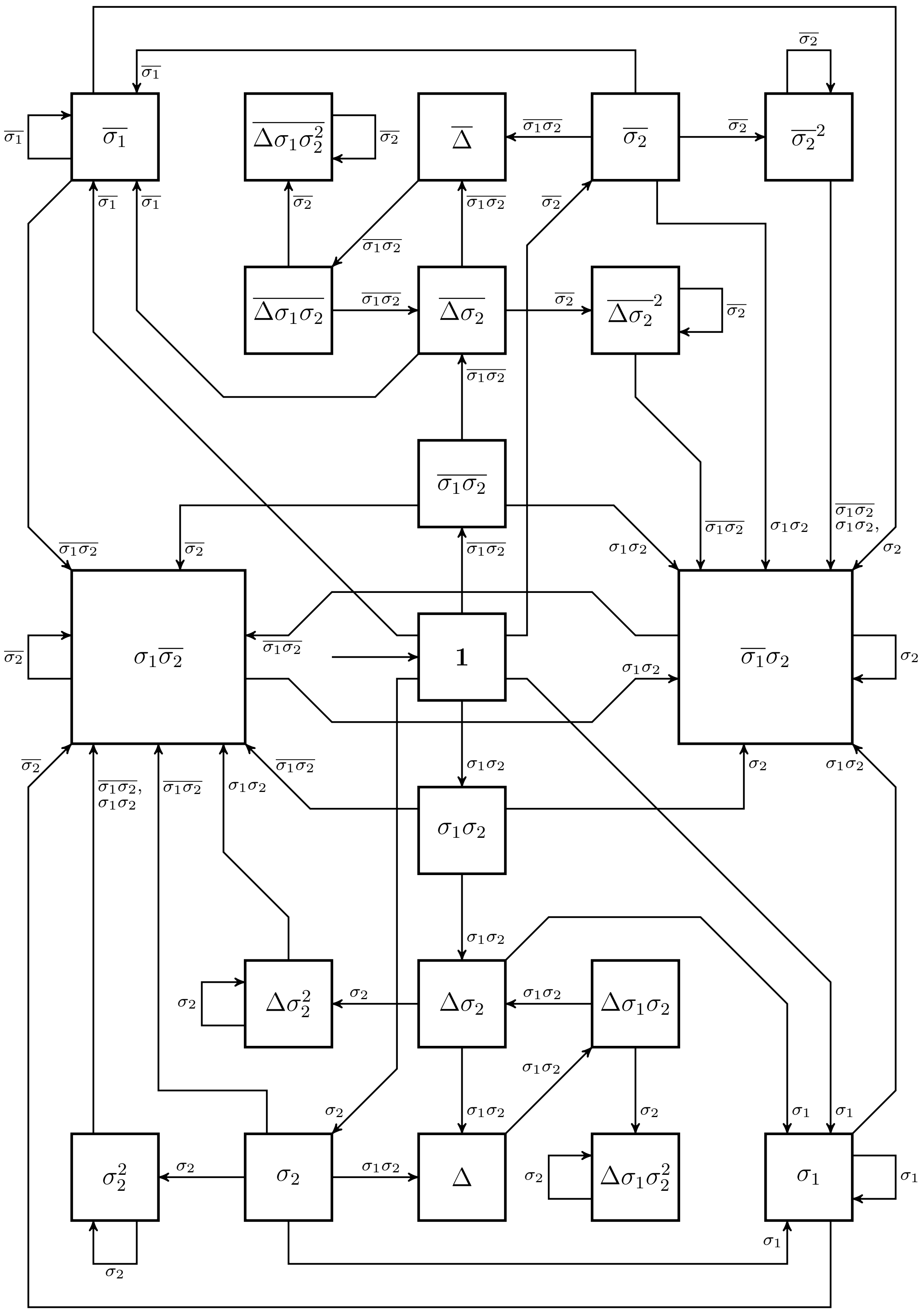


Figure 3.52 – Minimal automaton accepting the language $\mathbf{RNF}(\mathbf{B}_3)$

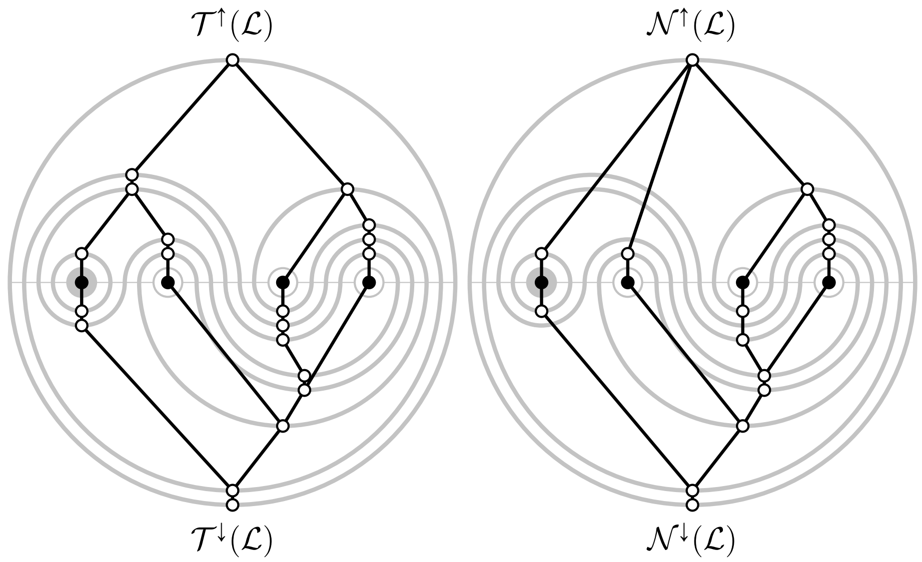


Figure 3.53 – Lamination trees and neighbour trees of a tight lamination

Definition & Proposition 3.54 (Neighbour trees).
Let $\mathcal{L}$ be tight lamination, with set of punctures $\{p_0, \ldots, p_n\}$ and with rightmost index k. Let $\mathbf{A}_n^{\top}$ be the upper arc contained in the curve $\mathcal{L}_n$, i.e. the root of the tree $\mathcal{T}^{\uparrow}(\mathcal{L})$. In addition, let $\Lambda^{\uparrow}$ be the subset of upper arcs of $\mathcal{L}$ defined by

$$\Lambda^{\uparrow} := \{\mathcal{A}_{\pm}^{\uparrow}(p_i) : 0 \leqslant i \leqslant n\} \cup \{\mathbf{A}_n^{\top}\} \cup \{A : p_k \overset{\pi}{\in} A \text{ and } \exists i \leqslant n, \overrightarrow{\mathbf{e}}(A) = \mathcal{A}_{\pm}^{\downarrow}(p_i)\}.$$

The upper neighbour tree *of the lamination $\mathcal{L}$ which we denote by $\mathcal{N}^{\uparrow}(\mathcal{L})$, and defined as follows. The vertices of $\mathcal{N}^{\uparrow}(\mathcal{L})$ are of the form v_A, where A is an upper arc of $\mathcal{L}$ such that $A \in \Lambda^{\uparrow}$, or of the form v_p, where p is a puncture of $\mathcal{L}$. A vertex v_A is an ancestor of v_B in $\mathcal{N}^{\uparrow}(\mathcal{L})$ if and only if A blinds B. Finally, if v_A is a vertex whose children are vertices $v_{A_1}, \ldots, v_{A_k}$ such that $A_1 \prec_{\mathcal{L}} \ldots \prec_{\mathcal{L}} A_k$, then v_{A_i} is the i-th child of v_A.*

We define similarly the lower neighbour tree *of $\mathcal{L}$, which we denote by $\mathcal{N}^{\downarrow}(\mathcal{L})$.*

Proof. We only need to prove that, if $v_{A_1}, \ldots, v_{A_k}$ are the children of v_A in $\mathcal{N}^{\uparrow}(\mathcal{L})$, then $A_1, \ldots, A_k$ are ordered by the relation $\prec_{\mathcal{L}}$. This statement follows from Corollary 3.19 and from the fact that$\{A_1, \ldots, A_k\}$ is a transversal section of the vertices of $\mathcal{T}^{\uparrow}(\mathcal{L})$ that descend from v_A. □

Following Proposition 3.18, the leaves of $\mathcal{N}^{\uparrow}(\mathcal{L})$ and $\mathcal{N}^{\downarrow}(\mathcal{L})$ are the punctures of $\mathcal{L}$. Furthermore, a puncture p_i belongs to an upper (respectively, lower) bigon if and only if it has no sibling in $\mathcal{N}^{\uparrow}(\mathcal{L})$ (respectively, in $\mathcal{N}^{\downarrow}(\mathcal{L})$).

Lemma 3.55.
Let β and β' be n-strand braids, with respective tight laminations $\mathcal{L}$ and $\mathcal{L}'$. If $\mathcal{N}^{\uparrow}(\mathcal{L}) = \mathcal{N}^{\uparrow}(\mathcal{L}')$ and $\mathcal{N}^{\downarrow}(\mathcal{L}) = \mathcal{N}^{\downarrow}(\mathcal{L}')$, then $\pi_{\beta}^2 = \pi_{\beta'}^2$.

Proof. Let $v_{\mathbf{A}_n^\top}$ and $v_{\mathbf{A}_n^\perp}$ be the respective roots of $\mathcal{N}^\uparrow(\mathcal{L})$ and of $\mathcal{N}^\downarrow(\mathcal{L})$, and consider the sets

$$\begin{aligned}\Lambda^\uparrow &:= \{\mathcal{A}_\pm^\uparrow(p_i) : 0 \leqslant i \leqslant n\} \cup \{\mathbf{A}_n^\top\} \cup \{A : p_k \overset{\pi}{\in} A \text{ and } \exists i \leqslant n, \overrightarrow{\mathbf{e}}(A) = \mathcal{A}_\pm^\downarrow(p_i)\},\\ \Lambda^\downarrow &:= \{\mathcal{A}_\pm^\downarrow(p_i) : 0 \leqslant i \leqslant n\} \cup \{\mathbf{A}_n^\perp\} \cup \{A : p_k \overset{\pi}{\in} A \text{ and } \exists i \leqslant n, \overrightarrow{\mathbf{e}}(A) = \mathcal{A}_\pm^\uparrow(p_i)\}.\end{aligned}$$

Let p be some puncture of $\mathcal{L}$. The arc $v_{\mathcal{A}_-^\uparrow(p,\mathcal{L})}$ is either the left sibling of p in $\mathcal{N}^\uparrow(\mathcal{L})$, if such a left sibling exists, or the parent of p in $\mathcal{N}^\uparrow(\mathcal{L})$, if p has no left sibling in $\mathcal{N}^\uparrow(\mathcal{L})$. We identify similarly the vertices $v_{\mathcal{A}_-^\downarrow(p,\mathcal{L})}$ and $v_{\mathcal{A}_+^\updownarrow(p,\mathcal{L})}$ among the nodes of $\mathcal{N}^\uparrow(\mathcal{L})$ and $\mathcal{N}^\downarrow(\mathcal{L})$.

Moreover, let k be the rightmost index of $\mathcal{L}$. We identify k since p_k is the rightmost puncture that does not have siblings in both $\mathcal{N}^\uparrow(\mathcal{L})$ and $\mathcal{N}^\downarrow(\mathcal{L})$. Let $A_1 \overset{\pi}{\subsetneq} \ldots \overset{\pi}{\subsetneq} A_u$ be the arcs blinding p_k and belonging to $\Lambda^\uparrow$. Let $B_1 \overset{\pi}{\subsetneq} \ldots \overset{\pi}{\subsetneq} B_v$ be the arcs blinding p_k and belonging to $\Lambda^\downarrow$. It comes immediately that $u = v$ and that $A_j = \overrightarrow{\mathbf{e}}(B_j)$ whenever $1 \leqslant j \leqslant u$. Hence, we identify each arc of the set $\{A : p_k \overset{\pi}{\in} A \text{ and } \exists i \leqslant n, \overrightarrow{\mathbf{e}}(A) = \mathcal{A}_\pm^\downarrow(p_i)\}$ among the nodes of $\mathcal{N}^\uparrow(\mathcal{L})$. Similarly, we identify each arc of the set $\{A : p_k \overset{\pi}{\in} A \text{ and } \exists i \leqslant n, \overrightarrow{\mathbf{e}}(A) = \mathcal{A}_\pm^\uparrow(p_i)\}$ among the nodes of $\mathcal{N}^\downarrow(\mathcal{L})$.

Hence, we can compute $\pi_\beta^2(i, \diamond, \vartheta)$ for each triple $(i, \diamond, \vartheta) \in \{0, \ldots, n\} \times \{+, -\} \times \{\uparrow, \downarrow\}$, which means that the trees $\mathcal{N}^\uparrow(\mathcal{L})$ and $\mathcal{N}^\downarrow(\mathcal{L})$ uniquely determine π_β^2. This completes the proof. □

Corollary 3.56.
Let $\mathcal{A} = (\Sigma, Q, i, \delta, Q)$ be the automaton provided in Corollary 3.51. Its state set Q is of cardinality $|Q| \leqslant 2^{20(n+1)}$.

Proof. Let $\mathbf{N}$ be the set $\{(\mathcal{N}^\uparrow(\mathcal{L}), \mathcal{N}^\downarrow(\mathcal{L})) : \mathcal{L} \text{ is a tight lamination}\}$ and let Π be the set $\{\pi_\beta^2 : \beta \in \mathbf{B}_n\}$. Lemma 3.55 states that there exists some surjective projection $\mathbf{N} \mapsto \Pi$, hence that $|\Pi| \leqslant |\mathbf{N}|$. Since $Q = \Pi$, it remains to show that $|\mathbf{N}| \leqslant 2^{20(n+1)}$.

Let $\mathcal{L}$ be some tight lamination. The tree $\mathcal{N}^\uparrow(\mathcal{L})$ contains at most $n + 1$ nodes of the type v_{p_i}, $2(n + 1)$ nodes of the type $v_{\mathcal{A}_\pm^\uparrow(p_i)}$, 1 node of the type $\mathbf{A}_n^\top$ and $2(n + 1)$ nodes of the type v_A, where $p_k \overset{\pi}{\in} A$ and $\overrightarrow{\mathbf{e}}(A) = \mathcal{A}_\pm^\downarrow(p_i)$ for some i. This proves that $\mathcal{N}^\uparrow(\mathcal{L})$ has at most $5n + 6$ nodes. Similarly, $\mathcal{N}^\downarrow(\mathcal{L})$ has at most $5n + 6$ nodes.

Moreover, both $\mathcal{N}^\uparrow(\mathcal{L})$ and $\mathcal{N}^\downarrow(\mathcal{L})$ are rooted ordered trees. For each integer k, there exists C_{k-1} rooted ordered trees with k nodes, where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number (see [49, p. 35]). Moreover, the relations

$$\sum_{i=0}^{k} C_i \leqslant (k+1)C_k = \binom{2k}{k} = \prod_{i=1}^{k} \frac{2i}{i} \cdot \frac{2i-1}{i} \leqslant 2^{2k}$$

show that there exist at most $2^{10(n+1)}$ rooted ordered trees with at most $5n + 6$ nodes. It follows that $|Q| = |\Pi| \leqslant |\mathbf{N}| \leqslant 2^{20(n+1)}$. □

We can then prove that the size of the automaton $\mathcal{A}$ has the same order of magnitude as the size of the minimal automaton.

Proposition 3.57.
Let $\mathcal{A}_{\min} = (\Sigma, Q_{\min}, i_{\min}, \delta_{\min}, F_{\min})$ be the minimal deterministic automaton that accepts the set of relaxation normal words for the braid group $\mathbf{B}_n$. The sets $F_{\min}$ and $Q_{\min}$ are equal, with cardinality $|Q_{\min}| \geqslant 2^{n/2-1}$.

Proof. Since $\mathcal{A}_{\min}$ is minimal, each of its states is co-accessible: from each state $s \in Q_{\min}$, one can reach a state $s' \in F_{\min}$. Since the relaxation normal form is prefix-closed, it follows that $Q_{\min} \subseteq F_{\min}$, i.e. that $F_{\min} = Q_{\min}$.

To each braid α corresponds a unique relaxation normal word in Σ^*, hence one unique state in $Q_{\min}$. We denote this state by $\delta^*(\alpha)$. Now, let $m = \lfloor \frac{n-1}{2} \rfloor$. To each tuple $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m) \in \{-1, 1\}^m$, we associate the braid $\beta_\epsilon = \sigma_1^{\epsilon_1} \sigma_3^{\epsilon_2} \ldots \sigma_{2m-1}^{\epsilon_m} \in \mathbf{B}_n$. An immediate induction on m shows that $\sigma_1^{\epsilon_1} \cdot \sigma_3^{\epsilon_2} \cdot \ldots \cdot \sigma_{2m-1}^{\epsilon_m}$ is a relaxation normal word.

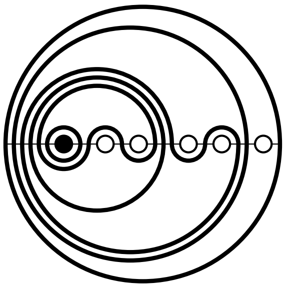


Figure 3.58 – The braid $\beta_{(-1,1)}$ (for $n = 5$)

Then, if ϵ and η are distinct tuples in $\{-1, 1\}^m$, consider some integer $i \leqslant m$ such that $\epsilon_i \neq \eta_i$. Without loss of generality, we assume that $\epsilon_i = 1$ and that $\eta_i = -1$. One shows easily that $\mathbf{RNF}(\beta_\epsilon) \cdot [2i \rightharpoondown n]$ is a relaxation normal word, whereas $\mathbf{RNF}(\beta_\eta) \cdot [2i \rightharpoondown n]$ is not. This shows that $\delta^*(\beta_\epsilon) \neq \delta^*(\beta_\eta)$ and, consequently, that $|Q_{\min}| \geqslant 2^m \geqslant 2^{n/2-1}$. $\square$

For example, Fig. 3.58 shows the 5-strand braid $\beta_{(-1,1)} = [1 \rightharpoondown 2][3 \rightharpoonup 4]$: we have $\mathbf{RNF}(\beta_{(-1,1)}) = [1 \rightharpoondown 2] \cdot [3 \rightharpoonup 4]$, $\mathbf{RNF}(\beta_{(-1,1)}[2 \rightharpoondown 5]) = [1 \rightharpoondown 5] \cdot [3 \rightharpoonup 4]$ and $\mathbf{RNF}(\beta_{(-1,1)}[4 \rightharpoondown 5]) = [1 \rightharpoondown 2] \cdot [3 \rightharpoonup 4] \cdot [4 \rightharpoondown 5]$.

Corollary 3.59.
Both automata $\mathcal{A}$ and $\mathcal{A}_{\min}$ introduced in Corollary 3.51 and Proposition 3.57 have state sets with cardinalities $2^{\Omega(n)}$.

3.4 Relaxation Normal Form and Braid Positivity

One of the main features of the braid group is that it is *left-orderable*, meaning that there exists a total order $\lessdot$ on $\mathbf{B}_n$ such that, if α, β and γ are braids such that $\alpha \lessdot \beta$, then $\gamma\alpha \lessdot \gamma\beta$. Such a property allows us to characterise the order $\lessdot$ just by knowing its positive elements, i.e. the set $\{\alpha \in \mathbf{B}_n : \mathbf{1} \lessdot \alpha\}$.

One such left-order is called the *Dehornoy order*. This order has been thoroughly studied [34, 35, 39], and its set of positive elements can be represented simply in terms of *σ-positive* braids.

Definition 3.60 (σ_i-positivity and σ-positivity)**.**
Let $\beta \in \mathbf{B}_n$ be a braid on n strands, and $\sigma_i \in \mathbf{B}_n$ be an Artin generator, where $i \leqslant n$. We say that β is σ_i-neutral *if it belongs to the subgroup of $\mathbf{B}_n$ generated by the set $\{\sigma_j : i+1 \leqslant j \leqslant n\}$.*

We also say that β is σ_i-positive *(respectively,* σ_i-negative*) if it can be expressed as a product $\beta = \gamma_0\sigma_i^\epsilon\gamma_1\sigma_i^\epsilon \ldots \sigma_i^\epsilon\gamma_k$ such that $k \geqslant 1$, each braid γ_j is σ_i-neutral, and $\epsilon = 1$ (respectively, $\epsilon = -1$).*

Finally, we say that β is σ-positive *(respectively,* σ-negative*) if it is σ_i-positive (respectively, σ_i-negative) for some $i \leqslant n$.*

This notion of σ-positivity and σ-negativity comes with a wealth of properties, including the fact that every non-trivial braid is either σ-positive or σ-negative, but not both (a proof of this result can be found in [39]). It immediately follows that the Dehornoy order, for which a braid α is smaller than a braid β if and only if $\alpha^{-1}\beta$ is σ-positive, has the property of being a total left-order.

Several normal form, stemming both from geometric and algebraic frameworks, are related to the Dehornoy order. For instance, up to a conjugation by the involutive mapping $\sigma_i \leftrightarrow \sigma_{n-i}$ (which transforms the normal form such as decribed in [19] into the variant mentioned in [40]), the Bressaud normal form maps each σ-positive braid β to a σ-positive braid word that represents β, although its does *not* map each σ-negative braid to a σ-negative braid word.

Similarly, the cycling normal form introduced by Fromentin [51, 52], which has an algebraic flavour, identifies each braid β with a σ-consistent braid word that represent β. Furthermore, this word is, up to a multiplicative constant, a shortest representative of β.

Hence, both these normal forms lead to computing σ-consistent braid words, which allows checking whether a braid is σ-positive, and therefore allows computing the Dehornoy order. We prove below that the relaxation normal form of a braid β, although it is not σ-consistent in general, helps checking whether the braid β itself is σ-positive.

Indeed, σ-positivity and σ-negativity is directly expressible in terms of tight laminations.

Definition & Proposition 3.61 (Second right arcs)**.**
Let $\mathcal{L}$ be a tight lamination and let p be some puncture of $\mathcal{L}$, except the rightmost one. Since $\mathcal{L}_\mathbb{R}$ intersects both intervals (p, p_n) and (p_n, ∞), the point $p_\mathcal{L}^{++} = \left(p_\mathcal{L}^+\right)_\mathcal{L}^+ = \min\{z \in \mathcal{L}_\mathbb{R} : z > p_\mathcal{L}^+\}$ is well-defined. We call this point the second right neighbour point *of p in $\mathcal{L}$.*

The point $p_{\mathcal{L}}^{++}$ *belongs to two arcs of* $\mathcal{L}$*. We call these arcs the* second right upper arc *and the* second right lower arc *of* p *in* $\mathcal{L}$*, and denote them respectively by* $\mathcal{A}_{++}^{\uparrow}(p, \mathcal{L})$ *and by* $\mathcal{A}_{++}^{\downarrow}(p, \mathcal{L})$*. The two arcs* $\mathcal{A}_{++}^{\uparrow}(p, \mathcal{L})$ *and* $\mathcal{A}_{++}^{\downarrow}(p, \mathcal{L})$ *are called* second right arcs *of* p *in* $\mathcal{L}$*.*

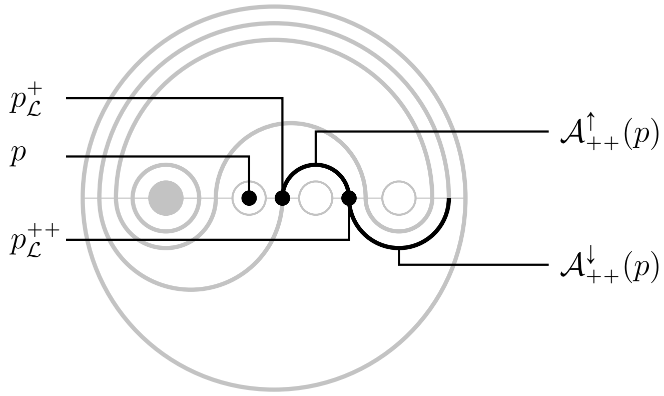


Figure 3.62 – A puncture and its second right neighbour and arcs

Figure 3.62 shows some tight lamination, in which a puncture p, the right neighbour point and the second right neighbour of p, and the second right arcs of p have been highlighted. Second right arcs provide us with a geometrical characterisation of σ_i-positive and σ_i-negative braids (see [39] for details), which we reformulate here.

Proposition 3.63.
Let $\mathcal{L}$ *be the tight lamination of a braid* $\beta \in \mathbf{B}_n$ *and let* σ_i *be an Artin generator of the braid group* $\mathbf{B}_n$*. The braid* β *is* σ_i*-neutral if and only if* $0 \in \pi_\beta(j, +, \uparrow) \cap \pi_\beta(j, +, \downarrow)$ *for all* $j \leqslant i$*. In addition,* β *is*

- σ_i*-positive if and only if* β *is* σ_{i-1}*-neutral and* $i \in \pi_{\mathcal{L}}(\mathcal{A}_{++}^{\downarrow}(p_{i-1}, \mathcal{L})) \subseteq \{i, \ldots, n\}$*;*
- σ_i*-negative if and only if* β *is* σ_{i-1}*-neutral and* $i \in \pi_{\mathcal{L}}(\mathcal{A}_{++}^{\uparrow}(p_{i-1}, \mathcal{L})) \subseteq \{i, \ldots, n\}$*.*

From Proposition 3.63 follows a characterisation of the σ_i-positive and σ_i-negative braids according to their relaxation normal forms. Indeed, for each integer $j \in \{1, \ldots, n\}$, let $\mathcal{S}_j^{\uparrow}$, $\mathcal{S}_j^{\downarrow}$ and Σ_j be respectively the subsets $\{[j \rightharpoonup v] : j < v\}$, $\{[j \rightharpoondown v] : j < v\}$ and $\bigcup_{k \geqslant j}(\mathcal{S}_k^{\uparrow} \cup \mathcal{S}_k^{\downarrow})$ of the set Σ of all right-oriented sliding braids.

Theorem 3.64.
Let $\beta \in \mathbf{B}_n$ *be a braid and let* σ_i *be an Artin generator of the braid group* $\mathbf{B}_n$*. The braid* β *is* σ_i*-positive (respectively,* σ_i*-negative) if and only if* $\mathbf{RNF}(\beta) \in \Sigma_{i+1}^* \cdot \mathcal{S}_i^{\downarrow} \cdot \Sigma_i^*$ *(respectively,* $\mathbf{RNF}(\beta) \in \Sigma_{i+1}^* \cdot \mathcal{S}_i^{\uparrow} \cdot \Sigma_i^*$*).*

Proof. The sets $\{\mathbf{1}\}$, $\Sigma_{i+1}^* \cdot \mathcal{S}_i^{\downarrow} \cdot \Sigma_i^*$ and $\Sigma_{i+1}^* \cdot \mathcal{S}_i^{\uparrow} \cdot \Sigma_i^*$ for $(1 \leqslant i \leqslant n-1)$ form a partition of the free monoid Σ^*. Moreover, a braid β is clearly σ_i-positive if $\mathbf{RNF}(\beta) \in \Sigma_{i+1}^* \cdot \mathcal{S}_i^{\downarrow}$, or σ_i-negative if $\mathbf{RNF}(\beta) \in \Sigma_{i+1}^* \cdot \mathcal{S}_i^{\uparrow}$. Hence, and without loss of generality, it suffices to prove that if β is a σ_i-positive, $[i \rightharpoonup j]$-relaxed braid for some $j \geqslant i+1$, then $\beta[i \rightharpoonup j]$ is σ_i-positive.

Then, let $\mathcal{L}$ and $\overline{\mathcal{L}}$ be tight laminations of β and $\beta[i \rightharpoonup j]$. Since β is σ_i-positive, Proposition 3.63 shows that

$$0 \in \pi_\beta(u,+,\uparrow) \cap \pi_\beta(u,+,\downarrow) = \pi_{\beta[i\rightharpoonup j]}(u,+,\uparrow) \cap \pi_{\beta[i\rightharpoonup j]}(u,+,\downarrow)$$

for all $u < i$. Furthermore, consider the arc $A := \mathcal{A}^{\downarrow}_{++}(p_{i-1},\mathcal{L}) = \mathcal{A}^{\downarrow}_{++}(p_{i-1},\overline{\mathcal{L}})$. Proposition 3.63 also states $i \in \pi_{\mathcal{L}}(A)$, hence that $\pi_{\mathcal{L}}(A) \subseteq \{i,\ldots,n\}$, and the requirement 1 of Definition 3.38 proves that $\pi_{\mathcal{L}}(A) \neq \{i\}$. It follows that $\{i,i+1\} \subseteq \pi_{\mathcal{L}}(A)$ and therefore $i \in \pi_{\overline{\mathcal{L}}}(A) \subseteq \{i,\ldots,n\}$, which completes the proof. □

If follows that the sets $\{\mathbf{RNF}(\beta) : \beta \text{ is } \sigma_i\text{-positive}\}$ and $\{\mathbf{RNF}(\beta) : \beta \text{ is } \sigma_i\text{-negative}\}$ are regular, and that each prefix of a σ_i-positive word must be σ_i-positive or σ_i-neutral. In particular, the sets $\{\mathbf{RNF}(\beta) : \beta \text{ is } \sigma\text{-positive}\}$ and $\{\mathbf{RNF}(\beta) : \beta \text{ is } \sigma\text{-negative}\}$ are also regular.

3.5 Experimental Data, Conjectures and Open Questions

A natural question that follows Section 3.3 is whether our results can be refined. Can we build a smaller deterministic automaton that would recognise the relaxation normal form? At which cost? A first way or proceeding is to modify slightly the notion of extended shadow of a braid.

Theorem 3.48 consists in saying that, in order to check whether a word $w_1 \cdot \ldots \cdot w_k$ belongs to the relaxation normal form, it is enough to read the word from left to right and to remember the extended shadows π^2 of the braids $w_1 \ldots w_i$, for $i \in \{1,\ldots,k\}$. The construction of the automaton $\mathcal{A}$ in Corollary 3.51 follows from this result.

However, the extended shadows may contain unnecessary information. In particular, let β be a braid, let $\mathcal{L}$ be the tight lamination associated with β, and let k be the rightmost index of $\mathcal{L}$. Consider the sets $B^\uparrow := \{j \in \{0,\ldots,k-1\} : p_j \text{ belongs to an upper bigon}\}$ and $B^\downarrow := \{j \in \{0,\ldots,k-1\} : p_j \text{ belongs to a lower bigon}\}$. For all pairs $(j,\diamond,\vartheta) \in \{0,\ldots,n\} \times \{-,+\} \times \{\downarrow,\uparrow\}$, consider the two intervals $(I,J) := \pi^2_\beta(j,\diamond,\vartheta)$, and let us denote by $\overline{\pi}^2_\beta$ the pair of sets $(I, J \backslash B^\vartheta)$.

We may prove a variant of Theorem 3.48, where the function π^2_β is replaced by the function $\overline{\pi}^2_\beta$. Observe that $\pi^2_{\sigma_1^2} \neq \pi^2_{\sigma_1^3}$, while $\overline{\pi}^2_{\sigma_1^2} = \overline{\pi}^2_{\sigma_1^3}$. Therefore, since $\overline{\pi}^2_\beta$ depends only on π^2_β, the set $\{\overline{\pi}^2_\beta(\beta) : \beta \in \mathbf{B}_n\}$ has fewer elements than the set $\{\pi^2_\beta(\beta) : \beta \in \mathbf{B}_n\}$. In particular, we might build an automaton $\overline{\mathcal{A}}$ whose states are the elements of the set $\{\overline{\pi}^2_\beta(\beta) : \beta \in \mathbf{B}_n\}$, which means that $\overline{\mathcal{A}}$ is smaller than as $\mathcal{A}$.

Of course, we might directly compute the minimal deterministic automaton $\mathcal{A}_{\min}$ of the relaxation normal form by computing the automaton $\mathcal{A}$ and minimising it. However, since $\mathcal{A}$ is larger than $\mathcal{A}_{\min}$, such a way of proceeding may appear as uselessly costly. In that context, being able to build from scratch the automaton $\overline{\mathcal{A}}$ provides us with a better way of obtaining small automata recognising the relaxation normal form.

On the contrary, we may look for tighter lower bounds on the number of states of the minimal deterministic automaton $\mathcal{A}_{\min}$ of the relaxation normal form. Here is such a lower bound, which is nevertheless not optimal. For all braids $\alpha \in \mathbf{B}_n$, let $\delta^*(\alpha)$ be the corresponding state in $\mathcal{A}_{\min}$. For each integer $i \in \{1, \ldots, n-1\}$, consider the braid $\beta_i := [i \hookrightarrow n][i \hookleftarrow n]$. In addition, for each tuple $\epsilon := (\epsilon_1, \ldots, \epsilon_{n-1}) \in \{-1, +1\}^{n-1}$, consider the braid $\gamma_\epsilon := \beta_1^{\epsilon_1} \beta_2^{2\epsilon_2} \ldots \beta_{n-1}^{(n-1)\epsilon_{n-1}}$.

Like in the proof of Proposition 3.57, we observe that $\mathbf{RNF}(\gamma_\epsilon) \cdot [i \hookrightarrow n]$ is a relaxation normal word if and only if $\epsilon_i = 1$, and therefore that the states $\delta^*(\gamma_\epsilon)$ are pairwise distinct. This proves that the automaton $\mathcal{A}_{\min}$ must have at least 2^{n-1} states, which is quadratically better than the lower bound found in Proposition 3.57.

Figure 3.65 presents, for $n \in \{2, 3, 4, 5\}$, the size $|\mathcal{A}_{\min}|$ of the minimal deterministic automaton of the relaxation normal form in the braid group $\mathbf{B}_n$, the size $|\overline{\mathcal{A}}|$ of the above-constructed automaton $\overline{\mathcal{A}}$, as well as the lower bound 2^{n-1} and the upper bound $2^{20(n+1)}$ on $|\mathcal{A}_{\min}|$ that we obtained above. It suggests that the lower and upper bounds 2^{n-1} and $2^{20(n+1)}$ are far from optimal but, at the same time, that the construction of $|\mathcal{A}_{\min}|$ is not so inefficient in practice.

n	2^{n-1}	$\vert\mathcal{A}_{\min}\vert$	$\vert\overline{\mathcal{A}}\vert$	$2^{20(n+1)}$
2	2	3	3	$\approx 10^{18}$
3	4	21	29	$\approx 10^{24}$
4	8	207	307	$\approx 10^{30}$
5	16	2261	3549	$\approx 10^{36}$

Figure 3.65 – Size of the automata accepting the relaxation normal form in $\mathbf{B}_n$

Another question is that of the automaticity of the relaxation normal form. A complete book about automatic groups and automatic normal forms is [47], which we invite the reader to look at.

Proposition 3.66.
Consider some integer $n \in \{2, 3\}$. Let $\mathbf{RS}$ *denote the set of right-oriented sliding braids in $\mathbf{B}_n$, i.e.* $\mathbf{RS} := \{[k \rightharpoondown \ell] : 1 \leqslant k < \ell \leqslant n\} \cup \{[k \hookrightarrow \ell] : 1 \leqslant k < \ell \leqslant n\}$. *The relaxation normal form* $\mathbf{RNF} : \mathbf{B}_n \mapsto \mathbf{RS}$ *is synchronously automatic.*

Proof. Proposition 3.66 is self-evident if $n = 2$, hence we focus on the case $n = 3$. We might have built directly automata recognising the sets $\mathbf{RNF}_i^{\mathbf{left}}$ and $\mathbf{RNF}_i^{\mathbf{right}}$ for $i = 1$

and $i = 2$. However, we had to face a state space explosion, and could not cope with the large size of the involved automata. Consequently, we had to find an alternative proof.

Let $\psi : \mathbf{RS}^* \mapsto \{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}\}$ be the morphism of monoids such that $\psi : \sigma_1\sigma_2 \mapsto \sigma_1 \cdot \sigma_2$, $\psi : \sigma_1^{-1}\sigma_2^{-1} \mapsto \sigma_1^{-1} \cdot \sigma_2^{-1}$ and $\psi : \mathbf{x} \mapsto \mathbf{x}$ if $\mathbf{x} \in \{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}\}$. The normal form $\overline{\mathbf{RNF}} = \psi \circ \mathbf{RNF}$, which maps each braid $\beta \in \mathbf{B}_3$ to the word $\psi(\mathbf{RNF}(\beta))$, is not prefix-closed but remains regular. In particular, it is easy to compute the minimal automaton, denoted by $\mathcal{A}_0$, recognising the set $\{\overline{\mathbf{RNF}}(\beta) : \beta \in \mathbf{B}_3\} \cdot \{\varepsilon\}^*$. This automaton is of size $|\mathcal{A}_0| = 34$ (i.e. $\mathcal{A}_0$ has 34 states), which is small enough for carrying subsequent computations.

Then, using the recipe provided by Corollary 2.58, we compute directly (minimal deterministic) automata, which we denote by $\overline{\mathcal{A}}_i^{\mathbf{left}}$ and $\overline{\mathcal{A}}_i^{\mathbf{right}}$, that recognise the sets $\overline{\mathbf{RNF}}_i^{\mathbf{left}}$ and $\overline{\mathbf{RNF}}_i^{\mathbf{right}}$ for $i = 1$ and $i = 2$. These automata are of respective sizes $\left|\overline{\mathcal{A}}_1^{\mathbf{left}}\right| = 212$, $\left|\overline{\mathcal{A}}_1^{\mathbf{right}}\right| = 518$, $\left|\overline{\mathcal{A}}_2^{\mathbf{left}}\right| = 214$ and $\left|\overline{\mathcal{A}}_2^{\mathbf{right}}\right| = 56$, which makes handling them a tractable task.

Tables presenting the automaton $\mathcal{A}$ and the automata $\overline{\mathcal{A}}_1^{\mathbf{left}}$, $\overline{\mathcal{A}}_2^{\mathbf{left}}$, $\overline{\mathcal{A}}_1^{\mathbf{right}}$ and $\overline{\mathcal{A}}_2^{\mathbf{right}}$ (which are too large to be represented graphically) are available on

http://www.irif.univ-paris-diderot.fr/~vjuge/papers/Automates.pdf.

We wish to derive the synchronous automaticity of $\mathbf{RNF}$ from that of $\overline{\mathbf{RNF}}$. However, reading synchronously letters of words in $\overline{\mathbf{RNF}}(\mathbf{B}_3)$ may result in reading asynchronously letter of words in $\mathbf{RNF}(\mathbf{B}_3)$. For instance, consider the words

$$\begin{aligned}
\underline{\mathbf{a}} &:= \sigma_1\sigma_2 \cdot \sigma_1\sigma_2 \cdot \sigma_1 \cdot \sigma_1 \cdot \sigma_1\sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \text{ and} \\
\underline{\mathbf{b}} &:= \sigma_1\sigma_2 \cdot \sigma_1\sigma_2 \cdot \sigma_2 \cdot \sigma_1\sigma_2 \cdot \sigma_1\sigma_2 \cdot \sigma_1^{-1}\sigma_2^{-1} \cdot \sigma_1\sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2,
\end{aligned}$$

whose letters are sliding braids, that both belong to $\mathbf{RNF}(\mathbf{B}_3)$, and that represent respectively the braids $\mathbf{a} = \Delta\sigma_2\sigma_1^3\sigma_2^5$ and $\mathbf{b} = \Delta\sigma_2\sigma_1^3\sigma_2^6 = \mathbf{a}\sigma_2$. The morphism ψ maps the words $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ to words

$$\begin{aligned}
\underline{\mathbf{a}}' &:= \sigma_1 \cdot \sigma_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \sigma_1 \cdot \sigma_1 \cdot \sigma_1 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \text{ and} \\
\underline{\mathbf{b}}' &:= \sigma_1 \cdot \sigma_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \sigma_1^{-1} \cdot \sigma_2^{-1} \cdot \sigma_1 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2,
\end{aligned}$$

whose letters are generators $\sigma_i^{\pm 1}$, and that both belong to $\overline{\mathbf{RNF}}(\mathbf{B}_3)$. However, the 12^{th} leftmost letter of $\underline{\mathbf{a}}'$ comes from the 9^{th} leftmost letter of $\underline{\mathbf{a}}$, while the 12^{th} leftmost letter of $\underline{\mathbf{b}}'$ comes only from the 7^{th} leftmost letter of $\underline{\mathbf{b}}$, as illustrated in Fig. 3.67. Hence, reading synchronously the letters of $\underline{\mathbf{a}}'$ and $\underline{\mathbf{b}}'$ is not the same as reading synchronously the letters of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

Hence, for all braids $\alpha, \beta \in \mathbf{B}_3$, consider the words $\underline{\mathbf{a}} := \mathbf{RNF}(\alpha)$, $\underline{\mathbf{b}} := \mathbf{RNF}(\beta)$, $\underline{\mathbf{a}}' := \overline{\mathbf{RNF}}(\alpha)$ and $\underline{\mathbf{b}}' := \overline{\mathbf{RNF}}(\beta)$. For each integer $k \geqslant 0$, let us define the integer $\varphi_{\underline{\mathbf{a}}}(k) := \min\{u \geqslant 0 : \sum_{i=1}^{u} |\lambda(\underline{\mathbf{a}}_i)| \geqslant \min\{k, |\underline{\mathbf{a}}'|\}\}$. We define similarly the integer $\varphi_{\underline{\mathbf{b}}}(k)$. Finally, we call *synchronisation difference* between α and β the integer

$$\Delta_{\mathbf{sync}}(\alpha, \beta) := \|\varphi_{\underline{\mathbf{a}}} - \varphi_{\underline{\mathbf{b}}}\|_\infty = \max_{k \geqslant 0} |\varphi_{\underline{\mathbf{a}}}(k) - \varphi_{\underline{\mathbf{b}}}(k)|\,.$$

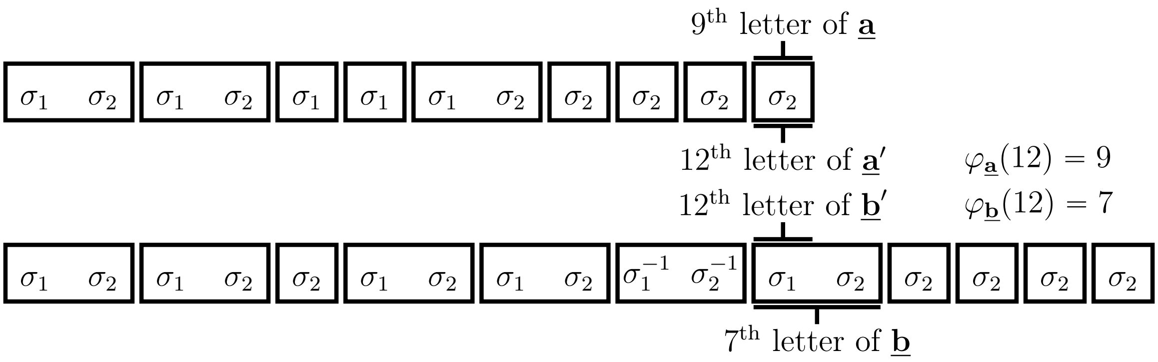


Figure 3.67 – Synchronisation on $\overline{\mathbf{RNF}}$ and partial desynchronisation on **RNF**

For instance, in the above example (illustrated in Fig. 3.67), we have $\varphi_{\underline{\mathbf{a}}}(12) = 9$, $\varphi_{\underline{\mathbf{b}}}(12) = 7$ and $\Delta_{\mathbf{sync}}(\alpha, \beta) = 2$.

Let us consider some integer $k \geqslant 0$ as well as the integers $A_k := \sum_{i=1}^{\min\{k, |\underline{\mathbf{a}}|\}} |\lambda(\underline{\mathbf{a}}_i)|$ and $B_k := \sum_{i=1}^{\min\{k, |\underline{\mathbf{b}}|\}} |\lambda(\underline{\mathbf{b}}_i)|$. Without loss of generality, we have $A_k \geqslant B_k$, and therefore

$$|A_k - B_k| \leqslant 2(k + 1 - \varphi_{\underline{\mathbf{a}}}(B_k)) \leqslant 2 + 2\Delta_{\mathbf{sync}}(\alpha, \beta).$$

We prove now that the normal form **RNF** is synchronously automatic as soon as each of the sets $\{\Delta_{\mathbf{sync}}(\gamma, \gamma\sigma_i) : \gamma \in \mathbf{B}_3\}$ and $\{\Delta_{\mathbf{sync}}(\gamma, \sigma_i\gamma) : \gamma \in \mathbf{B}_3\}$, for $i \in \{1, 2\}$, is finite. In order to do so, let us reuse the notations of Definition 2.56.

Consider some braid $\gamma \in \mathbf{B}_3$, as well as the words $\underline{\mathbf{a}} := \mathbf{RNF}(\gamma)$, $\underline{\mathbf{b}} := \mathbf{RNF}(\sigma_i\gamma)$, $\underline{\mathbf{a}}' := \overline{\mathbf{RNF}}(\gamma)$ and $\underline{\mathbf{b}}' := \overline{\mathbf{RNF}}(\sigma_i\gamma)$, and consider some integer $k \geqslant 0$, as well as integers A_k and B_k such as defined above. Since $\overline{\mathbf{RNF}}$ is synchronously automatic, the set $\overline{\Delta}_i^{\mathbf{left}} := \bigcup_{\gamma \in G} \Delta_{\overline{\mathbf{RNF}}}^{\mathbf{left}}(\sigma_i\gamma, \gamma)$ is finite. Moreover, using the notations introduced in Section 2.1, we have:

$$\begin{aligned}
\langle \underline{\mathbf{b}}_{k+1\ldots|\underline{\mathbf{b}}|} \rangle \langle \underline{\mathbf{a}}_{k+1\ldots|\underline{\mathbf{a}}|} \rangle^{-1} &= \langle \underline{\mathbf{b}}'_{B_k+1\ldots|\underline{\mathbf{b}}'|} \rangle \langle \underline{\mathbf{a}}'_{A_k+1\ldots|\underline{\mathbf{a}}'|} \rangle^{-1} \\
&= \langle \underline{\mathbf{b}}'_{B_k+1\ldots|\underline{\mathbf{b}}'|} \rangle \langle \underline{\mathbf{a}}'_{B_k+1\ldots|\underline{\mathbf{a}}'|} \rangle^{-1} \langle \underline{\mathbf{a}}'_{A_k+1\ldots B_k} \rangle^{-1} \text{ if } A_k \leqslant B_k, \text{ or} \\
&= \langle \underline{\mathbf{b}}'_{B_k+1\ldots|\underline{\mathbf{b}}'|} \rangle \langle \underline{\mathbf{a}}'_{B_k+1\ldots|\underline{\mathbf{a}}'|} \rangle^{-1} \langle \underline{\mathbf{a}}'_{B_k+1\ldots A_k} \rangle \text{ if } A_k \geqslant B_k + 1.
\end{aligned}$$

If the set $\{\Delta_{\mathbf{sync}}(\gamma, \sigma_i\gamma) : \gamma \in \mathbf{B}_3\}$, let M be an upper bound for this set. Since each braid $\langle \underline{\mathbf{b}}'_{B_k+1\ldots|\underline{\mathbf{b}}'|} \rangle \langle \underline{\mathbf{a}}'_{B_k+1\ldots|\underline{\mathbf{a}}'|} \rangle^{-1}$ belongs to the finite set $\overline{\Delta}_i^{\mathbf{left}}$ and each braid $\langle \underline{\mathbf{a}}'_{A_k+1\ldots B_k} \rangle^{-1}$ or $\langle \underline{\mathbf{a}}'_{B_k+1\ldots A_k} \rangle$ belongs to the finite Minkowski product $\{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \mathbf{1}\}^M$, it follows that the braid $\langle \underline{\mathbf{b}}_{k+1\ldots|\underline{\mathbf{b}}|} \rangle \langle \underline{\mathbf{a}}_{k+1\ldots|\underline{\mathbf{a}}|} \rangle^{-1}$ belongs to the finite set $\overline{\Delta}_i^{\mathbf{left}} \{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \mathbf{1}\}^M$.

This membership relation holds for all integers $k \geqslant 0$ and for all braids $\gamma \in \mathbf{B}_3$, which proves that the set $\Delta_i^{\mathbf{left}} := \bigcup_{\gamma \in G} \Delta_{\mathbf{RNF}}^{\mathbf{left}}(\sigma_i\gamma, \gamma)$ is finite. Repeating this argument proves that, if each the sets $\{\Delta_{\mathbf{sync}}(\gamma, \gamma\sigma_i) : \gamma \in \mathbf{B}_3\}$ and $\{\Delta_{\mathbf{sync}}(\gamma, \sigma_i\gamma) : \gamma \in \mathbf{B}_3\}$ are finite for $i \in \{1, 2\}$, then **RNF** is synchronously automatic.

Eventually, let F and G be the respective sets of final states of $\mathcal{A}$ and $\overline{\mathcal{A}}_1^{\mathbf{right}}$, and let $\mathbf{x}_s$ and $\mathbf{z}_s$ be the respective initial states of $\mathcal{A}$ and of $\overline{\mathcal{A}}_1^{\mathbf{right}}$. In addition, let $\Gamma_1^{\mathbf{right}}$ be the weighted graph defined as follows. The vertices of $\Gamma_1^{\mathbf{right}}$ are tuples $(\mathbf{x}, \alpha, i, \mathbf{y}, \beta, j, \mathbf{z})$, where $\mathbf{x}$ and $\mathbf{y}$ are states of $\mathcal{A}$, α and β are elements of the alphabet $\mathbf{RS}$, i and j are integers such that $1 \leqslant i \leqslant |\psi(\alpha)|$ and $1 \leqslant j \leqslant \psi(\beta)$, and $\mathbf{z}$ is a state of $\overline{\mathcal{A}}_1^{\mathbf{right}}$.

For every two vertices $\mathbf{v}_1 = (\mathbf{x}_1, \alpha_1, i_1, \mathbf{y}_1, \beta_1, j_1, \mathbf{z}_1)$ and $\mathbf{v}' = (\mathbf{x}_2, \alpha_2, i_2, \mathbf{y}_2, \beta_2, j_2, \mathbf{z}_2)$ of $\Gamma_1^{\mathbf{right}}$, if there exists two letters $u, v \in \{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}\}$ such that

- if $\psi(\alpha_1) = a_1 \cdot \ldots \cdot a_k$, then $1 \leqslant i_1 \leqslant k$ and $u = a_{i_1}$;
- if $1 \leqslant i_1 < k$, then $i_2 = i_1 + 1$, $\mathbf{x}_2 = \mathbf{x}_1$ and $\alpha_2 = \alpha_1$;
- if $i_1 = k$, then $i_2 = 1$, and the edge $\mathbf{x}_1 \xrightarrow{\alpha} \mathbf{x}_2$ exists in $\mathcal{A}$;
- if $\psi(\beta_1) = b_1 \cdot \ldots \cdot b_\ell$, then $1 \leqslant j_1 \leqslant \ell$ and $v = b_{j_1}$;
- if $1 \leqslant j_1 < \ell$, then $j_2 = j_1 + 1$, $\mathbf{y}_2 = \mathbf{y}_1$ and $\beta_2 = \beta_1$;
- if $j_1 = \ell$, then $j_2 = 1$, and the edge $\mathbf{y}_1 \xrightarrow{\beta} \mathbf{y}_2$ exists in $\mathcal{A}$;
- the edge $\mathbf{z}_1 \xrightarrow{(u,v)} \mathbf{z}_2$ exists in $\overline{\mathcal{A}}_1^{\mathbf{right}}$,

then we draw an edge $\mathbf{v} \to \mathbf{v}'$ in $\Gamma_1^{\mathbf{right}}$, with weight $\mathbf{1}_{i_1=1} - \mathbf{1}_{j_1=1}$.

Finally, we prune $\Gamma_1^{\mathbf{right}}$ and restrict it to those vertices that belong to some path going from the set $\{(\mathbf{x}_s, \alpha, 1, \mathbf{x}_s, \beta, 1, \mathbf{z}_s) \mid \alpha, \beta \in \mathbf{RS}\}$ to the set $\{(\mathbf{x}, \alpha, 1, \mathbf{y}, \beta, 1, \mathbf{z}) \mid \alpha, \beta \in \mathbf{RS}, \mathbf{x}, \mathbf{y} \in F, \mathbf{z} \in G\}$.

One shows easily that the set $\{\Delta_{\mathbf{sync}}(\beta, \beta\sigma_1) : \beta \in \mathbf{B}_3\}$ is finite if and only if $\Gamma_1^{\mathbf{right}}$ does not contain any cycle of non-zero weight. Although the whole graph $\Gamma_1^{\mathbf{right}}$ might be huge, its pruned version turns out to be very small and is very fast to compute. By computing analogous pruned weighted graphs $\Gamma_2^{\mathbf{right}}$, $\Gamma_1^{\mathbf{left}}$ and $\Gamma_2^{\mathbf{left}}$, and checking that all cycles in these graphs have weight zero (using the Bellman-Ford algorithm), we check in no time that **RNF** is indeed automatic. □

In addition, further experimental tests, which were not conclusive due to space issues, suggest the following generalisation of Proposition 3.66.

Conjecture 3.68.
Consider some integer $n \geqslant 2$. Let $\mathbf{RS}$ denote the set of right-oriented sliding braids in $\mathbf{B}_n$, i.e. $\mathbf{RS} := \{[k \rightharpoondown \ell] : 1 \leqslant k < \ell \leqslant n\} \cup \{[k \hookrightarrow \ell] : 1 \leqslant k < \ell \leqslant n\}$. The relaxation normal form $\mathbf{RNF} : \mathbf{B}_n \mapsto \mathbf{RS}$ is synchronously automatic.

Chapter 4

Counting Braids According to Their Geometric Norm

Résumé

Les tresses peuvent être représentées de manière géométrique, en tant que diagrammes de courbes. La complexité géométrique d'une tresse est la plus petite complexité d'un diagramme de courbes représentant cette tresse. Nous introduisons et étudions la notion de fonction génératrice géométrique associée. Nous calculons explicitement la fonction génératrice géométrique pour le groupe de tresses à trois brins et démontrons qu'elle n'est ni rationnelle ni algébrique, ni même holonome. Ce résultat peut sembler contre-intuitif. En effet, la complexité usuelle (liée à la présentation d'Artin des groupes de tresses) est algorithmiquement plus difficile à calculer que la complexité géométrique, alors que la fonction génératrice associée pour le groupe de tresses à trois brins est rationnelle.

La majeure partie du contenu de ce chapitre a été publiée dans [64].

Abstract

Braids can be represented geometrically as curve diagrams. The geometric complexity of a braid is the minimal complexity of a curve diagram representing it. We introduce and study the corresponding notion of geometric generating function. We compute explicitly the geometric generating function for the group of braids on three strands and prove that it is neither rational nor algebraic, nor even holonomic. This result may appear as counterintuitive. Indeed, the standard complexity (due to the Artin presentation of braid groups) is algorithmically harder to compute than the geometric complexity, yet the associated generating function for the group of braids on three strands is rational.

Most of the content of this chapter appeared in [64].

Chapter 4 is devoted to the problem of counting braids with a given geometric complexity. Standard notions of complexity on the elements of a monoid are algebraic: given a set S of generators of the group, the complexity of an element γ is the smallest integer $u \geqslant 0$ such that γ belongs to the Minkowski product S^u. In the context of n-strand braid groups, such algebraic complexities include the Artin complexity, where $S := \{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$, and the Garside complexity, where $S := \{\beta^{\pm 1} : \beta \in \mathbf{B}_n^+, \mathbf{1} \leqslant_\ell \beta \leqslant_\ell \Delta\}$.

The Garside normal form provides a complete answer to both the problem of computing the Garside complexity of a braid and the problem of counting braids with a given complexity [28]. The problem of counting braids with a given Artin complexity was settled when $n = 3$, and the associated generating function is rational [74, 84]. However, this problem remains open for $n \geqslant 4$, and even the problem of computing the Artin complexity of a braid seems intractable when n is large [76, 80].

Unlike the Artin complexity, the geometric complexity of a braid can be computed in polynomial time, using variants of Algorithm 3.9, of the Dynnikov coordinates or of the coordinates introduced below in Definition 4.5. Hence, we might expect that the associated generating function be at least as simple as the generating function associated with the Artin complexity. Yet, surprisingly, we prove below that, even in the case of the group of braids with three strands, the *geometric generating function* is not rational, nor even holonomic.

All the notions and results mentioned in Chapter 4 are essentially original, and most of them were published in [64], although results about Lambert series (Definition 4.27 and Proposition 4.28) and composition of curve diagrams (from Definition 4.34 to Theorem 4.39) are novel and were therefore unpublished prior to this thesis.

4.1 Counting Braids With a Given Norm

In Chapter 4, we will consider only curve diagrams, and cast both open and closed laminations aside, with the exception of the trivial open lamination. Henceforth, we denote by $\mathbf{L}$ the trivial open lamination, and denote by $\mathbf{L}_i$ the (vertical) curves of $\mathbf{L}$.

4.1.1 Generalising Curve Diagrams

We begin with introducing the original notion of *generalised curve diagram*, which generalises (standard) curve diagrams and are specific instances of families of curves such as treated in Section 2.4.

Definition 4.1 (Generalised curve diagram).
Let k be a positive integer, and let $p_1 < \ldots < p_n$ be mobile punctures inside the open interval $(-1, 1)$. We call k-generalised curve diagram*, and denote by $\mathcal{D}$, a union of k non-intersecting curves that consists of*

- *one open curve with endpoints -1 and $+1$;*
- *$k-1$ closed curves, none of which encircles any of the points ± 1*

and such that each puncture of the disk belongs to one of these k curves.

Curve diagrams, such as introduced in Definition 2.109, are exactly 1-generalised curve diagrams. If $\mathcal{D}$ is a (1-generalised) curve diagram, the endpoints of the curve $\mathcal{D}$ itself are -1 and $+1$. Therefore, two distinct $(\mathcal{D}, \mathbf{L})$-arcs can share at most one endpoint, where $\mathbf{L}$ is the trivial open lamination. Hence, if P and Q are $(\mathcal{D}, \mathbf{L})$-adjacent endpoints, there exists one unique $(\mathcal{D}, \mathbf{L})$-arc with endpoints P and Q, and we denote this arc by $[P, Q]_{\mathcal{D}}$.

Remember that Theorem 2.129 identifies the tightness of a curve diagram $\mathcal{D}$ (in the sense of Definition 2.119) with the tightness of $\mathcal{D}$ with respect to the trivial open lamination $\mathbf{L}$ and the to set of mobile punctures $\{p_1, \ldots, p_n\}$ (in the sense of Definition 2.128). Hence, we extend Definition 2.119 to generalised curve diagrams.

Definition 4.2 (Tight generalised curve diagram)**.**
Let $\mathcal{D}$ be a generalised curve diagram. We say that $\mathcal{D}$ is tight *if $\mathcal{D}$ is tight with respect to the trivial open lamination $\mathbf{L}$ and to the set of mobile punctures $\{p_1, \ldots, p_n\}$.*

Using the fact that each puncture must belong to a curve of $\mathcal{D}$, we deduce the following result.

Proposition 4.3.
A generalised curve diagram $\mathcal{D}$ is tight if and only if $\mathcal{D}$ is transverse to the trivial open lamination $\mathbf{L}$ and if, for every $(\mathcal{D}, \mathbf{L})$-arc A whose endpoints are $(\mathbf{L}, \mathcal{D})$-adjacent endpoints, the arc A contains one point among $p_1, \ldots, p_n$.

4.1.2 From Diagrams to Coordinates

We focus now on an original notion of *coordinates* of a (tight) curve diagram and of a braid, which is similar to the Dynnikov coordinates in its essence, although it is more suitable for tackling the problem of counting braids. This notion will be central in all Chapter 4.

Definition 4.4 (Endpoints ordering)**.**
Let $\mathcal{D}$ be a generalised curve diagram that is transverse to the trivial open lamination $\mathbf{L}$, and let $\mathbf{L}_i$ be a curve of $\mathbf{L}$, with $1 \leqslant i \leqslant n-1$.

The punctures p_1 and p_n belong to distinct connected components of $\mathbb{C} \backslash \mathbf{L}_i$. Hence, the curve $\mathbf{L}_i$ intersects the curve diagram $\mathcal{D}$ at least once. We orient each curve $\mathbf{L}_i$ from bottom to top *and thereby induce a linear ordering on $\mathbf{L}_i \cap \mathcal{D}$: we denote by $\mathbf{L}_i^j$ the j-th smallest element of $\mathbf{L}_i \cap \mathcal{D}$.*

For the sake of coherence, we also define the sets $\mathbf{L}_0 := \{-1\}$ and $\mathbf{L}_n := \{+1\}$. Then, we denote by $\mathbf{L}_0^1$ the left point -1, and by $\mathbf{L}_n^1$ the right point $+1$.

An immediate induction on i and on k shows that, if $\mathcal{D}$ is a k-generalised curve diagram transverse to $\mathbf{L}$, then the cardinality of the set $\mathbf{L}_i \cap \mathcal{D}$ is odd.

In addition, if $\mathcal{D}$ is transverse to $\mathbf{L}$, any two points P and Q lying on distinct lines $\mathbf{L}_{i-1}$ and $\mathbf{L}_i$ can be linked by at most one $(\mathcal{D}, \mathbf{L})$-arc. Therefore, we can unambiguously denote this arc by $[P, Q]_{\mathcal{D}}$, which gives rise to the following definition of *coordinates* of a tight generalised curve diagram.

Definition 4.5 (Curve diagram coordinates and braid coordinates).
Let $\mathcal{D}$ be a tight generalised curve diagram. The coordinates *of $\mathcal{D}$ are defined as the tuple* $\mathbf{sa} := (s_0, a_1, s_1, a_2, \ldots, a_n, s_n)$ *such that*

- $s_i = \frac{1}{2}(|\mathbf{L}_i \cap \mathcal{D}| - 1)$, *for all* $i \in \{0, \ldots, n\}$;
- $a_i = \max\{j \geqslant 0 : \forall k \in \{1, \ldots, j\}, \mathbf{L}_{i-1}^k$ *and* $\mathbf{L}_i^k$ *are* $(\mathcal{D}, \mathbf{L})$*-adjacent endpoints*$\}$, *for all* $i \in \{1, \ldots, n\}$ *such that* $s_{i-1} \neq s_i$;
- a_i *is the integer such that the puncture* p_i *lies on the arc* $[\mathbf{L}_{i-1}^{a_i+1}, \mathbf{L}_i^{a_i+1}]_{\mathcal{D}}$, *for all* $i \in \{1, \ldots, n\}$ *such that* $s_{i-1} = s_i$.

If, in addition, $\mathcal{D}$ is a (1-generalised) curve diagram, representing some braid β, then we also say that $\mathbf{sa}$ *are the coordinates of β.*

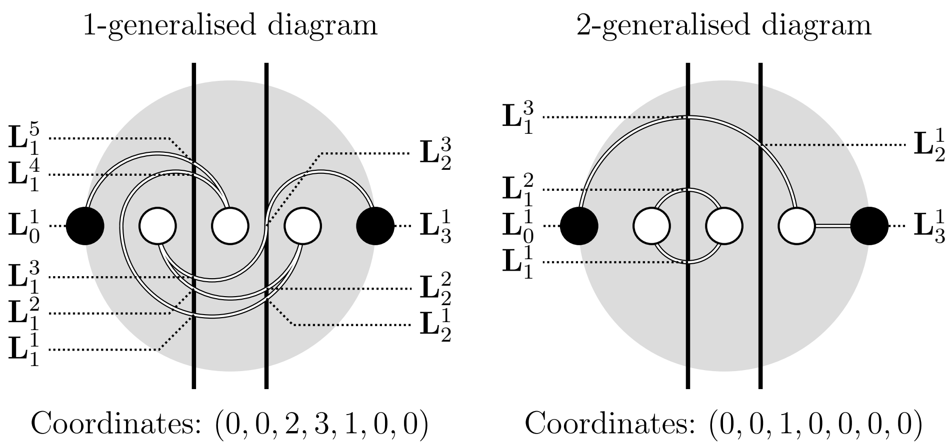


Figure 4.6 – Tight generalised curve diagram and associated coordinates

We show below that coordinates indeed characterise tight generalised curve diagrams.

Definition 4.7 (Zones and adjacent points).
*Let $\mathcal{D}$ be some tight generalised curve diagram, and let $i \in \{1, \ldots, n\}$ be some integer. We denote by $\mathcal{Z}_i$ be the area lying to the left of $\mathbf{L}_i$ (if $i \leqslant n-1$) and to the right of $\mathbf{L}_{i-1}$ (if $i \geqslant 2$): we call $\mathcal{Z}_i$ the i-*th zone *of the diagram. In addition, let A be some $(\mathcal{D}, \mathbf{L})$-arc lying inside the area $\mathcal{Z}_i$, and let P and Q be the endpoints of the arc A. We say that P and Q are i-*th zone adjacent points*, which we denote by $P \overset{i}{\sim} Q$, and denote the arc A by $[P, Q]_{\mathcal{D}}^i$.*

Observe that, although there may exist two $(\mathcal{D}, \mathbf{L})$-arcs with endpoints P and Q (when $\mathcal{D}$ is a tight k-generalised curve diagram with $k \geqslant 2$, e.g. the points $\mathbf{L}_1^1$ and $\mathbf{L}_1^2$ in the 2-generalised curve diagram of Fig. 4.6), the arc $[P, Q]_{\mathcal{D}}^i$ itself is uniquely defined. Indeed, remember that $\mathcal{D}$ and $\mathbf{L}$ are transverse to each other, which contradicts the fact that P and Q may be linked by two or more arcs lying in the same area $\mathcal{Z}_i$.

The notion of zones and of i-th zone adjacent points leads to the following results.

Lemma 4.8.
Let $\mathcal{D}$ be a tight generalised curve diagram with coordinates $\mathbf{sa} := (s_0, a_1, s_1, \ldots, s_n)$, and let $i \in \{1, \ldots, n\}$ be some integer. The $(\mathcal{D}, \mathbf{L})$-arcs contained in the area $\mathcal{Z}_i$ link respectively:

- *the points $\mathbf{L}_{i-1}^j$ and $\mathbf{L}_i^j$ such that $j \leqslant a_i$;*
- *the points $\mathbf{L}_{i-1}^j$ and $\mathbf{L}_{i-1}^k$ such that $j + k = 2(a_i + s_{i-1} - s_i) + 1$ and $\min\{j, k\} > a_i$, if $s_{i-1} > s_i$;*
- *the points $\mathbf{L}_i^j$ and $\mathbf{L}_i^k$ such that $j + k = 2(a_i + s_i - s_{i-1}) + 1$ and $a_i < j < k$, if $s_i > s_{i-1}$;*
- *the points $\mathbf{L}_{i-1}^j$ and $\mathbf{L}_i^k$ such that $j - k = 2(s_{i-1} - s_i)$ and $\min\{j, k\} > a_i$.*

Proof. We call *left-box* each interval $\{u, \ldots, v\}$ such that the points $\mathbf{L}_{i-1}^u \overset{i}{\sim} \mathbf{L}_{i-1}^v$, and *right-box* each interval $\{u, \ldots, v\}$ such that $\mathbf{L}_i^u \overset{i}{\sim} \mathbf{L}_i^v$.

If $\{u, \ldots, v\}$ is a left-box (with $u < v$), then for all $w \in \{u+1, \ldots, v-1\}$ there exists some $x \in \{u+1, \ldots, v-1\}$ such that $\{w, \ldots, x\}$ (or $\{x, \ldots, w\}$ if $x < w$) is also a left-box. An immediate induction then shows that the interval $\{u, \ldots, v\}$ necessarily contains some minimal left-box, which must be of the form $\{x, x+1\}$. Hence, Proposition 4.3 shows that the arc $[\mathbf{L}_{i-1}^x, \mathbf{L}_{i-1}^{x+1}]_{\mathcal{D}}^i$ must be *the* $(\mathcal{D}, \mathbf{L})$-arc containing the puncture p_i.

This proves that, among any two left-boxes, one must contain the other, and again an immediate induction proves that the family of left-boxes must be a set of intervals of the form $\{\{y - k, \ldots, y + k - 1\} : 1 \leqslant k \leqslant \ell\}$ for some integer $\ell \geqslant 0$. Similarly, the family of right-boxes must be a set of intervals of the form $\{\{z - k, \ldots, z + k - 1\} : 1 \leqslant k \leqslant m\}$ for some integer $m \geqslant 0$. In addition, since p_i belongs to only one $(\mathcal{D}, \mathbf{L})$-arc, we must either have $\ell = 0$ or $m = 0$.

Finally, consider the two sets $\mathbf{S}_1 = \{1, \ldots, y - \ell - 1, y + \ell, \ldots, 2s_{i-1} + 1\}$ and $\mathbf{S}_2 = \{1, \ldots, z - m - 1, z + m, \ldots, 2s_i + 1\}$. Each point $\mathbf{L}_{i-1}^u$ with $u \in \mathbf{S}_1$ must be i-th zone adjacent with some point $\mathbf{L}_i^v$ with $v \in \mathbf{S}_2$, and vice-versa. Since the arcs of $\mathcal{D}$ cannot cross each other, it comes immediately that the set $\mathbf{S}_1$ and $\mathbf{S}_2$ have the same cardinality and that, if u is the j-th smallest element of $\mathbf{S}_1$ and v is the j-th smallest element of $\mathbf{S}_2$, for some $j \in \{1, \ldots, |\mathbf{S}_1|\}$, then $\mathbf{L}_{i-1}^u \overset{i}{\sim} \mathbf{L}_i^v$. Considering the definition of the coordinates $\mathbf{sa}$, Lemma 4.8 follows. □

Proposition 4.9.
Let $\mathcal{D}$ and $\mathcal{D}'$ be two tight generalised curve diagrams with respective coordinates $\mathbf{sa}$ and

$\mathbf{sa}'$. *If* $\mathbf{sa} = \mathbf{sa}'$*, then there exists some isotopy of* $\mathbb{C}$*, preserving* $\mathbf{L}$ *setwise and* $\{p_i : 1 \leqslant n\}$ *pointwise, and that maps* $\mathcal{D}$ *to* $\mathcal{D}'$*.*

Proof. Let $(s_0, a_1, \ldots, s_n)$ be the common coordinates of $\mathcal{D}$ and $\mathcal{D}'$. Since $|\mathcal{D} \cap \mathbf{L}_i| = 2s_i + 1 = |\mathcal{D}' \cap \mathbf{L}_i|$ for all $i \in \{0, \ldots, n\}$, we assume without loss of generality that $\mathcal{D} \cap \mathbf{L} = \mathcal{D}' \cap \mathbf{L}$. Lemma 4.8 proves then that some isotopy of $\mathbb{C}$ preserving pointwise the lines $\mathbf{L}_i$ (for $1 \leqslant i \leqslant n-1$) maps the diagram $\mathcal{D}$ to the diagram $\mathcal{D}'$.

In addition, denote by b_i the integer $a_i + |s_{i-1} - s_i|$. Observe that the puncture p_i lies on the arcs

- $[\mathbf{L}_{i-1}^{b_i}, \mathbf{L}_{i-1}^{b_i+1}]_{\mathcal{D}}^i$ and $[\mathbf{L}_{i-1}^{b_i}, \mathbf{L}_{i-1}^{b_i+1}]_{\mathcal{D}'}^i$ if $s_{i-1} > s_i$;
- $[\mathbf{L}_{i}^{b_i}, \mathbf{L}_{i}^{b_i+1}]_{\mathcal{D}}^i$ and $[\mathbf{L}_{i}^{b_i}, \mathbf{L}_{i}^{b_i+1}]_{\mathcal{D}'}^i$ if $s_i > s_{i-1}$;
- $[\mathbf{L}_{i-1}^{a_i+1}, \mathbf{L}_{i}^{a_i+1}]_{\mathcal{D}}^i$ and $[\mathbf{L}_{i-1}^{a_i+1}, \mathbf{L}_{i}^{a_i+1}]_{\mathcal{D}'}^i$ if $s_{i-1} = s_i$.

Therefore, we can even assume that the above-mentioned isotopy preserves each puncture p_i, which is the statement of Proposition 4.9. □

Corollary 4.10.
Let β *and* β' *be two braids with coordinates* $\mathbf{sa}$ *and* $\mathbf{sa}'$*. If* $\mathbf{sa} = \mathbf{sa}'$*, then* $\beta = \beta'$*.*

These coordinates are therefore analogous to the Dynnikov coordinates (see [36, 39] for details) in several respects. First, both arise from counting intersection points between different collections of lines. Second, both provide an injective mapping from the braid group $\mathbf{B}_n$ into the set $\mathbb{Z}^{2n}$ or $\mathbb{Z}^{2n+1}$. Finally, both systems of coordinates come with very efficient algorithms, whose complexities are of the same order of magnitude. However, the coordinates used here are very closely linked with the notion of (diagrammatic or laminated) norm, whereas the process of computing the norm of a braid from its Dynnikov coordinates is less immediate.

4.1.3 From Coordinates to Diagrams

Proposition 4.9 and Corollary 4.10 allow us to identify each tight generalised curve diagram and each braid with a tuple of coordinates. Aiming to count (1-generalised) tight curve diagrams, we aim now at describing which coordinates correspond to generalised tight curve diagrams.

Lemma 4.11.
Let $\mathcal{D}$ *be a tight generalised curve diagram, with coordinates* $(s_0, a_1, \ldots, s_n)$*. We have* $s_0 = s_n = 0$*, and* $0 \leqslant a_i \leqslant 2\min\{s_{i-1}, s_i\} + \mathbf{1}_{s_{i-1} \neq s_i}$ *for all integers* $i \in \{1, \ldots, n\}$*.*

Proof. By definition, we have $2s_i + 1 = |\mathcal{D} \cap \mathbf{L}_i|$ whenever $0 \leqslant i \leqslant n$. Hence, whenever $s_{i-1} = s_i$, Definition 4.5 directly implies that $1 \leqslant a_i + 1 \leqslant |\mathcal{D} \cap \mathbf{L}_i| = 2s_i + 1$. However, if $s_{i-1} \neq s_i$, then the points $\mathbf{L}_{i-1}^j$ and $\mathbf{L}_i^j$ exist (and are $(\mathcal{D}, \mathbf{L})$-adjacent points) whenever $j \leqslant a_i$, which proves that

$$0 \leqslant a_i \leqslant \min\{|\mathcal{D} \cap \mathbf{L}_{i-1}|, |\mathcal{D} \cap \mathbf{L}_i|\} = 2\min\{s_{i-1}, s_i\} + 1.$$

□

In the following, we call *virtual coordinates* the tuples $(s_0, a_1, \dots, s_n)$ that satisfy the equalities and inequalities mentioned in Lemma 4.11. A natural question is, provided some virtual coordinates $\mathbf{sa}$, whether there exists some tight generalised curve diagram whose coordinates are $\mathbf{sa}$. We prove now that this is the case.

Proposition 4.12.
Let $\mathbf{sa} = (s_0, a_1, \dots, s_n)$ *be virtual coordinates. There exists some tight generalised curve diagram* $\mathcal{D}$ *whose coordinates are* $\mathbf{sa}$.

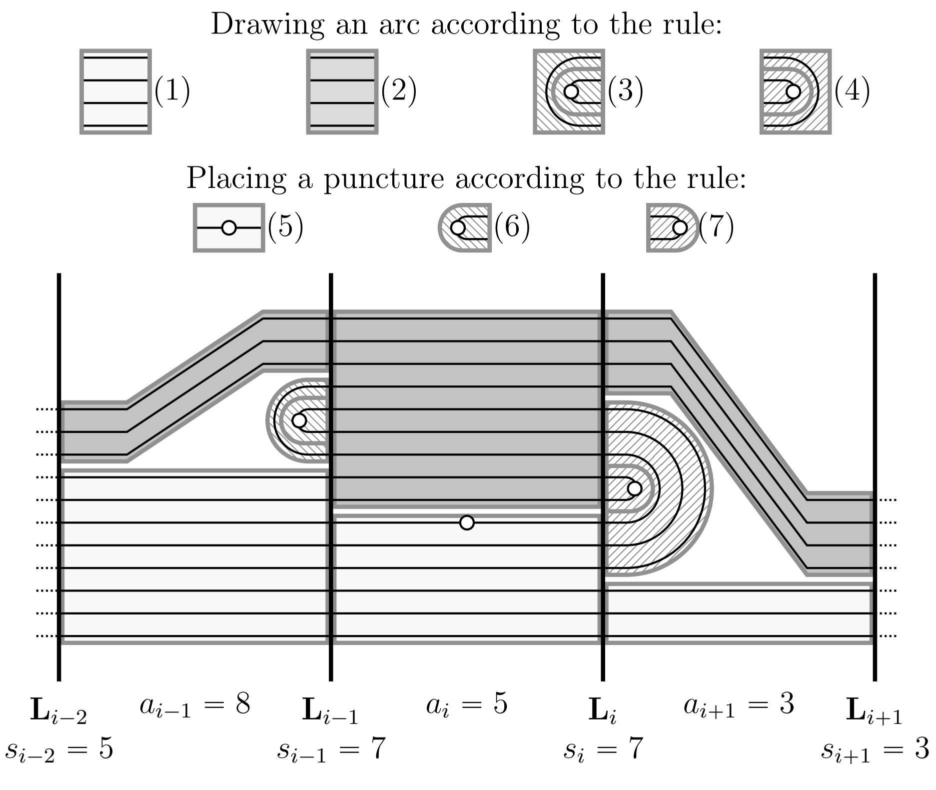


Figure 4.13 – Drawing lines and placing punctures of a diagram based on its coordinates

Proof. We just need to follow the recipe provided by Lemma 4.8 and draw the diagram $\mathcal{D}$. First, we call $\mathbf{L}_0^1$ and $\mathbf{L}_n^1$ the points -1 and $+1$. Then, on each line $\mathbf{L}_i$, for $i \in \{1, \dots, n-1\}$, let us place $2s_i + 1$ points $\mathbf{L}_i^1, \dots, \mathbf{L}_i^{2s_i+1}$, from bottom to top. Now, for each integer $i \in \{1, \dots, n\}$, define the integer $b_i := a_i + |s_{i-1} - s_i|$. We draw lines that lie inside the zone $\mathcal{Z}_i$ and that link

1. points $\mathbf{L}_{i-1}^j$ and $\mathbf{L}_i^j$ such that $j \leqslant a_i$;

2. points $\mathbf{L}_{i-1}^{j}$ and $\mathbf{L}_{i-1}^{k}$ such that $j+k=2b_i+1$ and $a_i<j<k$, if $s_{i-1}>s_i$;
3. points $\mathbf{L}_{i}^{j}$ and $\mathbf{L}_{i}^{k}$ such that $j+k=2b_i+1$ and $a_i<j<k$, if $s_i>s_{i-1}$;
4. points $\mathbf{L}_{i-1}^{j}$ and $\mathbf{L}_{i}^{k}$ such that $j-k=2(s_{i-1}-s_i)$ and $\min\{j,k\}>a_i$,

so that no two such lines intersect each other. Note that it is indeed possible to do so (e.g., if the points $(\mathbf{L}_i^j)_{1\leqslant j\leqslant 2s_i+1}$ are close enough to each other, one can draw straight segments $[\mathbf{L}_{i-1}^j,\mathbf{L}_i^k]$ and half-circles with diameter $[\mathbf{L}_{i-1}^j,\mathbf{L}_{i-1}^k]$ or $[\mathbf{L}_i^j,\mathbf{L}_i^k]$).

Then, place a point $\overline{p}_i$ on the arc

5. $[\mathbf{L}_{i-1}^{b_i},\mathbf{L}_{i-1}^{b_i+1}]_{\mathcal{D}}^i$ if $s_{i-1}>s_i$;
6. $[\mathbf{L}_{i}^{b_i},\mathbf{L}_{i}^{b_i+1}]_{\mathcal{D}}^i$ if $s_i>s_{i-1}$;
7. $[\mathbf{L}_{i-1}^{a_i+1},\mathbf{L}_{i}^{a_i+1}]_{\mathcal{D}}^i$ if $s_{i-1}=s_i$.

Up to an isotopy of $\mathbb{C}$ preserving the lamination $\mathbf{L}$ pointwise and mapping each point $\overline{p}_i$ to the actual position of the puncture p_i, we have just drawn the diagram $\mathcal{D}$. □

4.2 Actually Counting Braids

We introduce now the original notion of *geometric generating functions* or the braid group $\mathbf{B}_n$. We will compute exact values for this function and its coefficients when $n\leqslant 3$, then look for approximations of its coefficients when $n\geqslant 4$.

Definition 4.14 (Geometric generating functions)**.**
Let n be a positive integer. We respectively define the closed laminated generating function*, the* open laminated generating function *and the* diagrammatic generating function *of $\mathbf{B}_n$ as the functions $\mathcal{B}_n^{c,\ell}: z\mapsto\sum_{\beta\in\mathbf{B}_n} z^{\|\beta\|_\ell^c}$, $\mathcal{B}_n^{o,\ell}: z\mapsto\sum_{\beta\in\mathbf{B}_n} z^{\|\beta\|_\ell^o}$ and $\mathcal{B}_n^{d}: z\mapsto\sum_{\beta\in\mathbf{B}_n} z^{\|\beta\|_\ell^o}$.*

Recall that Corollary 2.116 and Proposition 2.120 state that $\|\beta\|_\ell^c=\|\beta\|_\ell^o+n+3=\|\beta^{-1}\|_d+n+3$, for all braids $\beta\in\mathbf{B}_n$. Hence, consider the integers $N_{n,k}^{c,\ell}=|\{\beta\in\mathbf{B}_n:\|\beta\|_\ell^c=k\}|$, $N_{n,k}^{o,\ell}=|\{\beta\in\mathbf{B}_n:\|\beta\|_\ell^o=k\}|$ and $N_{n,k}^{d}=|\{\beta\in\mathbf{B}_n:\|\beta\|_d=k\}|$. It follows that $N_{n,k}^d=N_{n,k}^{o,\ell}=N_{n,k+n-3}^{c,\ell}$ for all integers $n\geqslant 1$ and $k\geqslant 0$, and therefore that

$$\mathcal{B}_n^{c,\ell}(z)=z^{n+3}\mathcal{B}_n^{o,\ell}(z)=z^{n+3}\mathcal{B}_n^{d}(z).$$

Hence, we focus hereafter on the diagrammatic generating function and on its coefficients. The above equalities will then allow us to translate immediately our results in terms of the two other geometric generating functions.

By definition of the coordinates of braids, if a braid $\beta\in\mathbf{B}_n$ has coordinates $\mathbf{sa}:=(s_0,a_1,\ldots,s_n)$, then $\|\beta\|_d=n-1+2\sum_{i=1}^{n-1}s_i$. Therefore, instead of considering directly the function $\mathcal{B}_n^d(z)$, we focus on the integers $g_{n,k}:=|\{\beta\in\mathbf{B}_n:\|\beta\|_d=2k+n-1\}|$ and on the generating function $\mathcal{G}_n(z) := \sum_{k \geqslant 0} g_{n,k} z^k$. In particular, observe that $\mathcal{B}_n^d(z) = z^{n-1}\mathcal{G}_n(z^2)$, so that properties about the integers $g_{n,k}$ and on the function $\mathcal{G}_n(z)$ reflect on the function $\mathcal{B}_n^d(z)$.

Then, following Proposition 4.9 and 4.12, we want to count the tuples **sa** of virtual coordinates, with a given sum $\sum_{i=1}^{n-1} s_i$, and whose associated generalised curve diagram is a 1-generalised curve diagram.

Let $\mathcal{D}$ be the generalised diagram associated with **sa**. We denote by $\sim$ the relation of $(\mathcal{D}, \mathbf{L})$-adjacency (i.e., $P \sim Q$ if and only if P and Q are $(\mathcal{D}, \mathbf{L})$-adjacent endpoints), and we denote by $\equiv$ the reflexive transitive closure of $\sim$. Observe that, if $\mathcal{D}$ is a k-generalised curve diagram, then the relation $\equiv$ has exactly k equivalence classes. Therefore, we aim below at counting coordinates **sa** where the relation $\equiv$ has exactly one equivalence class; we will say that **sa** are *actual coordinates*.

Aiming to reduce the number of cases to look at, we will use the symmetries mentioned in Section 2.4.3. If a braid β has coordinates $(s_0, a_1, s_1, \ldots, a_n, s_n)$, then its horizontally symmetric braid $\mathbf{S}_h(\beta)$ has coordinates $(s_n, a_n, s_{n-1}, \ldots, a_1, s_0)$ and its vertically symmetric braid $\mathbf{S}_v(\beta)$ has coordinates $(s_0, a'_1, s_1, \ldots, a'_n, s_n)$, where $a'_i = 2\min\{s_{i-1}, s_i\} + \mathbf{1}_{s_{i-1} \neq s_i} - a_i$.

4.2.1 An Introductory Example: The Braid Group $\mathbf{B}_2$

In the braid group $\mathbf{B}_2$, everything is obvious. Indeed, the group $\mathbf{B}_2$ is isomorphic to $\mathbb{Z}$, and generated by the Artin generator σ_1. Since $\|\sigma_1^k\|_d = 1 + 2|k|$ for all integers $k \in \mathbb{Z}$, it follows that

$$g_{2,k} = \mathbf{1}_{k=0} + 2 \cdot \mathbf{1}_{k \geqslant 1},\ \mathcal{G}_2(z) = \frac{1+z}{1-z},\ \mathcal{B}_2^d(z) = \frac{z(1+z^2)}{1-z^2}.$$

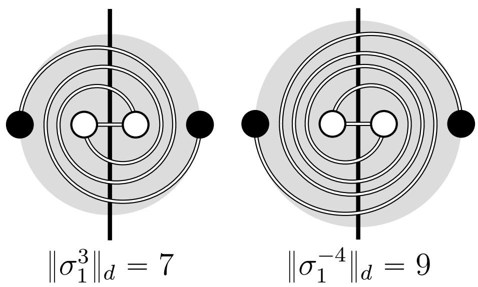


Figure 4.15 – $\|\sigma_1^k\|_d = 2k+1$

Let us recover this result with the tools introduced above, in particular the reflexive transitive closure (denoted by $\equiv$) of the $(\mathcal{D}, \mathbf{L})$-adjacency relation (denoted by $\sim$): we detail computations as a warm-up. A braid β with norm $2k+1$ has coordinates of the form $(0, a_1, k, a_2, 0)$, with $k \geqslant 0$ and $a_1, a_2 \in \{0, 1\}$.

If $k = 0$, then $\mathbf{sa} = (0,0,0,0,0)$, hence $\{\mathbf{L}_0^1, \mathbf{L}_1^1, \mathbf{L}_2^1\}$ is the only equivalence class of $\equiv$. It follows that $g_{2,0} = 0$.

We consider now the case $k \geqslant 1$. Using the vertical symmetry, we focus on the case $a_1 \leqslant a_2$. If $a_1 = 1$, then $a_2 = 1$, hence $\{\mathbf{L}_0^1, \mathbf{L}_1^1, \mathbf{L}_2^1\}$ is an equivalence class of $\equiv$ that does not contain $\mathbf{L}_1^{2k+1}$. Similarly, if $a_2 = 0$, then $a_1 = 0$, hence $\{\mathbf{L}_0^1, \mathbf{L}_1^{2k+1}, \mathbf{L}_2^1\}$ is an equivalence class of $\equiv$ that does not contain $\mathbf{L}_1^1$.

Therefore, we must have $\mathbf{sa} = (0,0,k,1,0)$ (or $\mathbf{sa} = (0,1,k,0,0)$, but we decided to let this case aside for now). Hence, Lemma 4.8 proves that

$$\mathbf{L}_0^1 \sim \mathbf{L}_1^{2k+1} \sim \mathbf{L}_1^2 \sim \mathbf{L}_1^{2k-1} \sim \ldots \sim \mathbf{L}_1^3 \sim \mathbf{L}_1^{2k-2} \sim \mathbf{L}_1^1 \sim \mathbf{L}_2^1.$$

This proves that $(0,0,k,1,0)$ and $(0,1,k,0,0)$ are actual coordinates, and therefore that $g_{2,k} = 2$. We deduce from these values of $g_{2,k}$ the above expression of the functions $\mathcal{G}_2(z) = \sum_{k \geqslant 0} z^k$ and $\mathcal{B}_2^d(z) = z\mathcal{G}_2(z^2)$.

This second proof is longer and more convoluted than the direct proof obtained by enumerating the braids in the group $\mathbf{B}_2$. However, enumerating the braids in $\mathbf{B}_3$ seems out of reach, whereas considering virtual coordinates and identifying which are actual coordinates will be possible, as shown in Section 4.2.2.

4.2.2 A Challenging Example: The Braid Group $\mathbf{B}_3$

Holonomic functions are univariate power series f that satisfy some linear differential equation

$$\sum_{i=0}^{k} c_i(z) \frac{\partial^i}{\partial z^i} f(z) = 0,$$

where $c_0(z), \ldots, c_k(z)$ are complex polynomials. This class of function generalises rational and algebraic functions, and is closed under various operations, such as addition, multiplication, term-wise multiplication and algebraic substitution (i.e. replacing the function $z \mapsto f(z)$ by some function $z \mapsto f(g(z))$ where g is an algebraic function, i.e. a solution of some equation $P(z, g(z)) = 0$, P being some non-degenerate polynomial).

A short introduction on holonomic series and their use in analytic combinatorics, including the associated tools for manipulating holonomic series, can be found in [49, Annex B.4].

The central result of this chapter is the following one.

Theorem 4.16.

The integers $g_{3,k}$ and the generating functions $\mathcal{G}_3(z)$ and $\mathcal{B}_3^d(z)$ are given by:

$$\begin{aligned}
\mathcal{G}_3(z) &= 2\frac{1+2z-z^2}{z^2(1-z^2)}\left(\sum_{n\geqslant 3}\varphi(n)z^n\right)+\frac{1-3z^2}{1-z^2},\\
\mathcal{B}_3^d(z) &= 2\frac{1+2z^2-z^4}{z^2(1-z^4)}\left(\sum_{n\geqslant 3}\varphi(n)z^{2n}\right)+\frac{z^2(1-3z^4)}{1-z^4},\\
g_{3,k} &= \mathbf{1}_{k=0}+2\left(\varphi(k+2)-\mathbf{1}_{k\in 2\mathbb{Z}}+2\sum_{i=1}^{\lfloor k/2\rfloor}\varphi(k+3-2i)\right)\mathbf{1}_{k\geqslant 1},
\end{aligned}$$

where φ denotes the Euler totient. The functions $\mathcal{G}_3(z)$ and $\mathcal{B}_3^d(z)$ are not holonomic.

In particular, the functions $\mathcal{G}_3(z)$ and $\mathcal{B}_3^d(z)$ are neither rational nor algebraic. Furthermore, and since the ordinary generating function $\mathcal{G}_3(z) = \sum_{k\geqslant 0} g_{3,k}z^k$ is not holonomic, neither are the exponential generating function $\sum_{k\geqslant 0}\frac{g_{3,k}}{k!}z^k$ nor the Poisson generating function $\exp(-z)\sum_{k\geqslant 0}\frac{g_{3,k}}{k!}z^k$. Likewise, the ordinary, exponential and Poisson generating functions associated with the sequences $(N_{3,k}^{c,\ell})_{k\geqslant 0}$, $(N_{3,k}^{o,\ell})_{k\geqslant 0}$ and $(N_{3,k}^{d})_{k\geqslant 0}$ are not holonomic.

We proceed now to the proof of Theorem 4.16, which we split into five steps.

Proof of Theorem 4.16 – Step 1: Simple Cases

Following the preliminary remarks we made at the beginning of this section, we focus on virtual coordinates the form $\mathbf{sa} = (0, a_1, k, a_2, \ell, a_3, 0)$ associated with some generalised curve diagram $\mathcal{D}$ whose induced relation $\equiv$ has a unique equivalence class. Henceforth, we consider ℓ and k as parameters, and compute the integer

$$C_{k,\ell} = |\{(a_1, a_2, a_3) : (0, a_1, k, a_2, \ell, a_3, 0) \text{ are actual coordinates}\}|.$$

Using the vertical symmetry, we know that $C_{k,\ell} = C_{\ell,k}$. We focus below on the case where $\ell \leqslant k$, and proceed to a disjunction of cases.

First, if $k = \ell = 0$, then $\mathbf{sa} = (0,0,0,0,0,0,0)$ are the coordinates of the trivial braid $\mathbf{1} \in \mathbf{B}_3$. Therefore, $C_{0,0} = 1$.

If $k = 0 < \ell$, then $a_1 = 0$, and $\mathbf{sa}$ are actual coordinates if and only if $(0, a_2, \ell, a_3, 0)$ are also actual coordinates. Indeed, as illustrated by Fig. 4.17, the virtual coordinates $(0, a_2, \ell, a_3, 0)$ can be obtained from $(0, a_1, 0, a_2, \ell, a_3, 0)$ by "shrinking" the edge $[\mathbf{L}_0^1, \mathbf{L}_1^1]_{\mathcal{D}}^1$. Hence, $\mathbf{sa}$ are actual coordinates if and only if $\{a_2, a_3\} = \{0, 1\}$. It follows that $C_{0,\ell} = C_{\ell,0} = 2$.

Similarly, if $1 \leqslant k = \ell$, then $\mathbf{sa}$ are actual coordinates if and only if $0 \leqslant a_2 \leqslant 2k$ and if $(0, a_1, k, a_3, 0)$ are also actual coordinates: as illustrated by Fig. 4.17, the virtual coordinates $(0, a_1, k, a_3, 0)$ can be obtained from $(0, a_1, k, a_2, k, a_3, 0)$ by "shrinking" each edge $[\mathbf{L}_1^j, \mathbf{L}_2^j]_{\mathcal{D}}^2$, when $1 \leqslant j \leqslant 2k+1$. We therefore have $2k+1$ ways of choosing a_2 and 2 ways of choosing (a_1, a_3), which proves that $C_{k,k} = 2(2k+1)$.

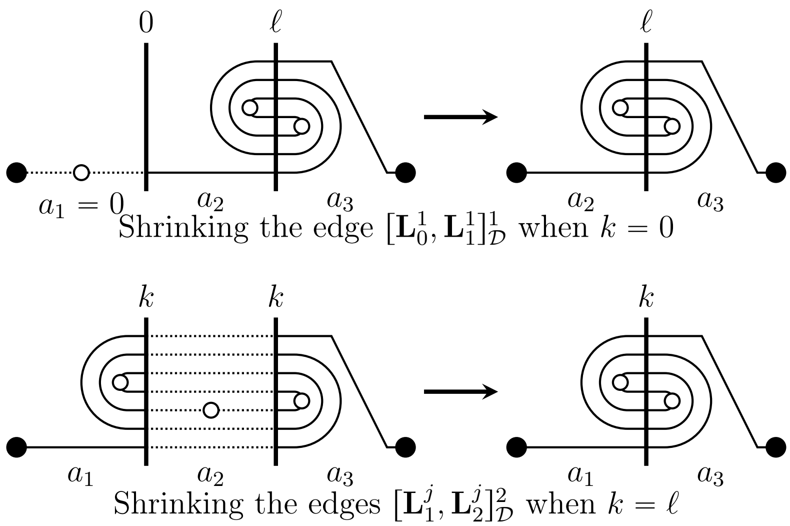


Figure 4.17 – Shrinking edges of tight generalised curve diagrams when $k = 0$ and $k = \ell$

Proof of Theorem 4.16 – Step 2: Towards Cyclic Permutations

We consider now the case where $1 \leqslant k < \ell$. Using the *horizontal* symmetry, we may focus on the case where $a_1 = 1$: doing so, we will find exactly half of the actual coordinates $(0, a_1, k, a_2, \ell, a_3, 0)$.

In order to ease subsequent computations, we decide here to modify slightly the generalised curve diagram $\mathcal{D}$ we drew from the coordinates $\mathbf{sa}$, as illustrated in Fig. 4.18. We proceed as follows:

- we add points $\mathbf{L}_1^{2k+2}$ and $\mathbf{L}_2^{2\ell+2}$ on the lines $\mathbf{L}_1$ and $\mathbf{L}_2$, *above* the points $\mathbf{L}_1^{2k+1}$ and $\mathbf{L}_2^{2\ell+1}$;
- we draw a curve (drawn in gray in Fig. 4.18) from $\mathbf{L}_0^1$ to $\mathbf{L}_3^1$, that does not cross the other curves of $\mathcal{D}$, and that crosses the lines $\mathbf{L}_1$ and $\mathbf{L}_2$ at $\mathbf{L}_1^{2k+2}$ and $\mathbf{L}_2^{2\ell+2}$.

Informally, we decided to "close by above" the unique open curve contained in $\mathcal{D}$. Recall that $\sim$ denotes the $(\mathcal{D}, \mathbf{L})$-adjacency relation, and that $\equiv$ denotes the reflexive transitive closure of $\sim$. What we just did was to add the relations $\mathbf{L}_0^1 \sim \mathbf{L}_1^{2k+2} \sim \mathbf{L}_2^{2\ell+2} \sim \mathbf{L}_3^1$. Since $\mathcal{D}$ already contained an open curve with endpoints $\mathbf{L}_0^1$ and $\mathbf{L}_3^1$, adding these points, curves and relations did not change the number of equivalence classes of the relation $\equiv$.

From now on, and in the rest of the proof of Theorem 4.16, we will only use such "closed by above" generalised diagrams, and we will identify $\mathcal{D}$ with this "closed by above" version.

Then, let us define the integer $m := \ell - k$. Since $0 \leqslant k \leqslant \ell$, observe that:

- $\mathbf{L}_0^1 \sim \mathbf{L}_1^{2k+2} \sim \mathbf{L}_2^{2\ell+2} \sim \mathbf{L}_3^1$;
- $\mathbf{L}_1^j \sim \mathbf{L}_2^j$ for all $j \in \{1, \ldots, a_2\}$;

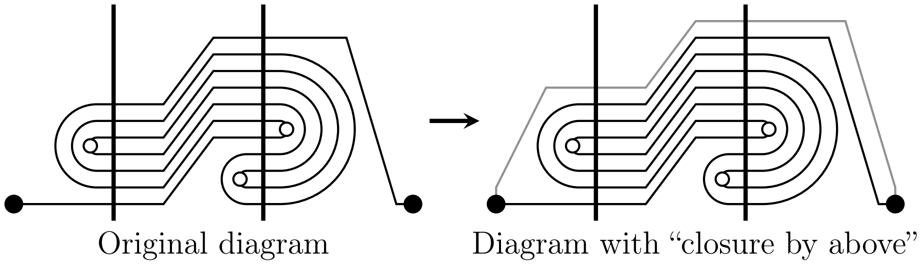


Figure 4.18 – Closing the open curve of $\mathcal{D}$ by above

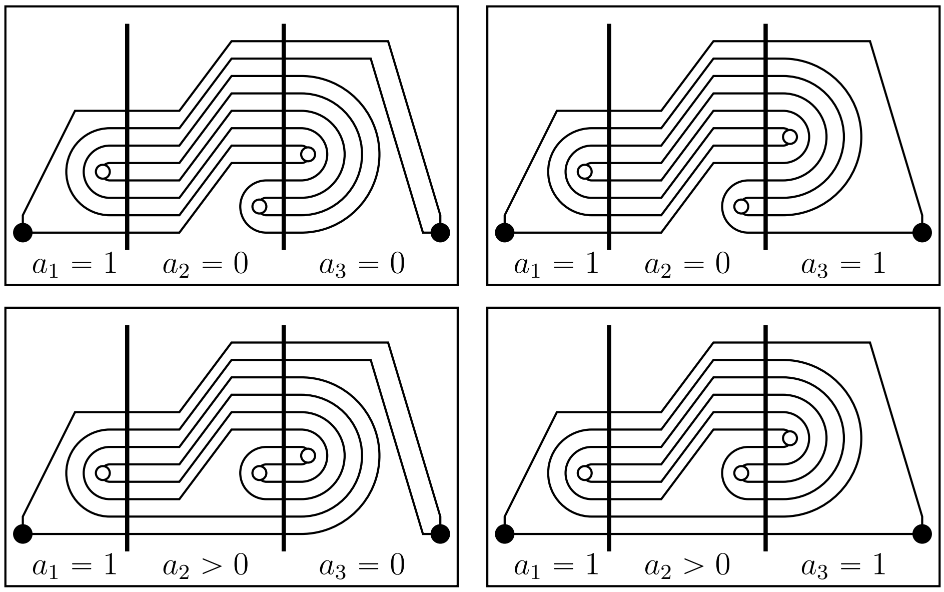


Figure 4.19 – Four different cases: $a_1 = 1$, $a_2 \overset{?}{=} 0$ and $a_3 \overset{?}{=} 0$

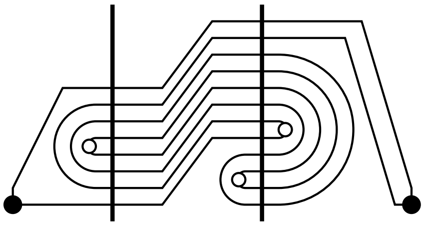
- $\mathbf{L}_1^j \sim \mathbf{L}_2^{j+2m}$ for all $j \in \{a_2 + 1, \ldots, 2k + 1\}$.

Hence, each equivalence class of the relation $\equiv$ contains points of the type $\mathbf{L}_2^m$, as illustrated by Fig. 4.19.

We then define additional relations on the set $\{1, \ldots, 2\ell + 2\}$. Let u and v be elements of $\{1, \ldots, 2\ell + 2\}$. We write $u \overset{\sqsubset}{\sim} v$ if some connected component of $\mathcal{D} \backslash \mathbf{L}_2$ has endpoints $\mathbf{L}_2^u$ and $\mathbf{L}_2^v$ and lies to the left of $\mathbf{L}_2$ (i.e. in the area $\mathcal{Z}_1 \cup \mathcal{Z}_2$). Similarly, we write $u \overset{\sqsupset}{\sim} v$ if some connected component of $\mathcal{D} \backslash \mathbf{L}_2$ has endpoints $\mathbf{L}_2^u$ and $\mathbf{L}_2^v$ and lies to the right of $\mathbf{L}_2$ (i.e. in the area $\mathcal{Z}_3$).

Alternatively, one might define the relations $\overset{\sqsubset}{\sim}$ and $\overset{\sqsupset}{\sim}$ by saying that $u \overset{\sqsubset}{\sim} v$ whenever $\mathbf{L}_2^u \overset{2}{\sim} \mathbf{L}_2^v$, $\mathbf{L}_2^u \overset{2}{\sim} \mathbf{L}_1^w \overset{1}{\sim} \mathbf{L}_1^x \overset{2}{\sim} \mathbf{L}_2^v$ or $\mathbf{L}_2^u \overset{2}{\sim} \mathbf{L}_1^w \overset{1}{\sim} \mathbf{L}_0^1 \overset{1}{\sim} \mathbf{L}_1^x \overset{2}{\sim} \mathbf{L}_2^v$ for some w and x, and that $u \overset{\sqsupset}{\sim} v$ whenever $\mathbf{L}_2^u \overset{3}{\sim} \mathbf{L}_2^v$ or $\mathbf{L}_2^u \overset{3}{\sim} \mathbf{L}_3^1 \overset{3}{\sim} \mathbf{L}_2^v$.

One checks easily that, whenever $u \overset{\sqsubset}{\sim} v$ or $u \overset{\sqsupset}{\sim} v$, the integers u and v have different parities. Hence, consider the permutation θ of $\{0, \ldots, \ell\}$ such that $\theta(u) = v$ if and only if there exists some (even) integer $w \in \{1, \ldots, 2\ell + 2\}$ such that $2u + 1 \overset{\sqsubset}{\sim} w \overset{\sqsupset}{\sim} 2v + 1$. By construction, there is a bijection between the equivalence classes of the relation $\equiv$ and the orbits of θ, as follows: we identify the equivalence class $\mathcal{C}$ of $\equiv$ with the orbit $\{u : \mathbf{L}_2^{2u+1} \in \mathcal{C}\}$ of θ.



$1 \overset{\sqsubset}{\sim} 4$, $2 \overset{\sqsubset}{\sim} 3$, $5 \overset{\sqsubset}{\sim} 12$, $6 \overset{\sqsubset}{\sim} 11$, $7 \overset{\sqsubset}{\sim} 10$, $8 \overset{\sqsubset}{\sim} 9$

$1 \overset{\sqsupset}{\sim} 10$, $2 \overset{\sqsupset}{\sim} 9$, $3 \overset{\sqsupset}{\sim} 8$, $4 \overset{\sqsupset}{\sim} 7$, $5 \overset{\sqsupset}{\sim} 6$, $11 \overset{\sqsupset}{\sim} 12$

$0 \xrightarrow{\theta} 3 \xrightarrow{\theta} 0$, $1 \xrightarrow{\theta} 4 \xrightarrow{\theta} 1$, $2 \xrightarrow{\theta} 5 \xrightarrow{\theta} 2$

Figure 4.20 – Relations $\overset{\sqsubset}{\sim}$ and $\overset{\sqsupset}{\sim}$, and permutation θ on a 3-generalised diagram

Hence, **sa** are actual coordinates if and only if θ is a cyclic permutation of $\{0, \ldots, \ell\}$. For the ease of the computation, we identify below the set $\{0, \ldots, \ell\}$ with the set $\mathbb{Z}_{\ell+1} := \mathbb{Z}/(\ell+1)\mathbb{Z}$.

Proof of Theorem 4.16 – Step 3: Which Permutations are Cyclic?

Let us define the real number $\alpha := \frac{a_2}{2}$. Note that α is not necessarily an integer, and that $a_2 = \lfloor \alpha \rfloor + \lceil \alpha \rceil$. In addition, recall that we defined above the integer $m := \ell - k$, such that $m > 0$. We consider separately various cases.

$\triangleright$ If $a_2 > 0$ and $a_3 = 1$, then $0 \xrightarrow{\theta} 0$, as shown in Fig. 4.19 (bottom-right case). It follows that θ is not a cyclic permutation of the set $\mathbb{Z}_{\ell+1}$.

$\triangleright$ If $a_2 = 0$, then one checks easily, as shown in Fig. 4.19 (top cases), that

a. if $0 \leqslant u < m$, then $2u + 1 \overset{\sqsubset}{\sim} 2(m - u) \overset{\sqsupset}{\sim} 2u + 1 + 2(k + a_3)$;

b. if $u = m$, then $2u + 1 \overset{\sqsubset}{\sim} 2\ell + 2 \overset{\sqsupset}{\sim} \mathbf{1}_{a_3=0} \cdot 2\ell + 1$;

c. if $m < u \leqslant \ell$, then $2u + 1 \overset{\sqsubset}{\sim} 2(\ell + 1 + m - u) \overset{\sqsupset}{\sim} 2u + 1 - 2(m + 1 - a_3)$.

It follows that $u \xrightarrow{\theta} u + (k + a_3)$ for all $u \in \mathbb{Z}_{\ell+1}$.

$\triangleright$ If $k + 1 \geqslant a_2 > a_3 = 0$, then one checks, as shown in Fig. 4.21 (top case), that

a. if $u = 0$, then $2u + 1 \overset{\sqsubset}{\sim} 2\ell + 2 \overset{\sqsupset}{\sim} 2\ell + 1$;

b. if $1 \leqslant u < \alpha$, then $2u + 1 \overset{\sqsubset}{\sim} 2(\ell + 1 - u) \overset{\sqsupset}{\sim} 2u - 1$;

c. if $\alpha \leqslant u < m + \alpha$, then $2u + 1 \overset{\sqsubset}{\sim} 2(m + a_2 - u) \overset{\sqsupset}{\sim} 2u + 1 + 2(k - a_2)$;

d. if $m + \alpha \leqslant u < \ell + 1 - \alpha$, then $2u + 1 \overset{\sqsubset}{\sim} 2(\ell + 1 + m - u) \overset{\sqsupset}{\sim} 2u + 1 - 2(m + 1)$;

e. if $\ell + 1 - \alpha \leqslant u \leqslant \ell$, then $2u + 1 \overset{\sqsubset}{\sim} 2(\ell + 1 - u) \overset{\sqsupset}{\sim} 2u - 1$.

It follows that

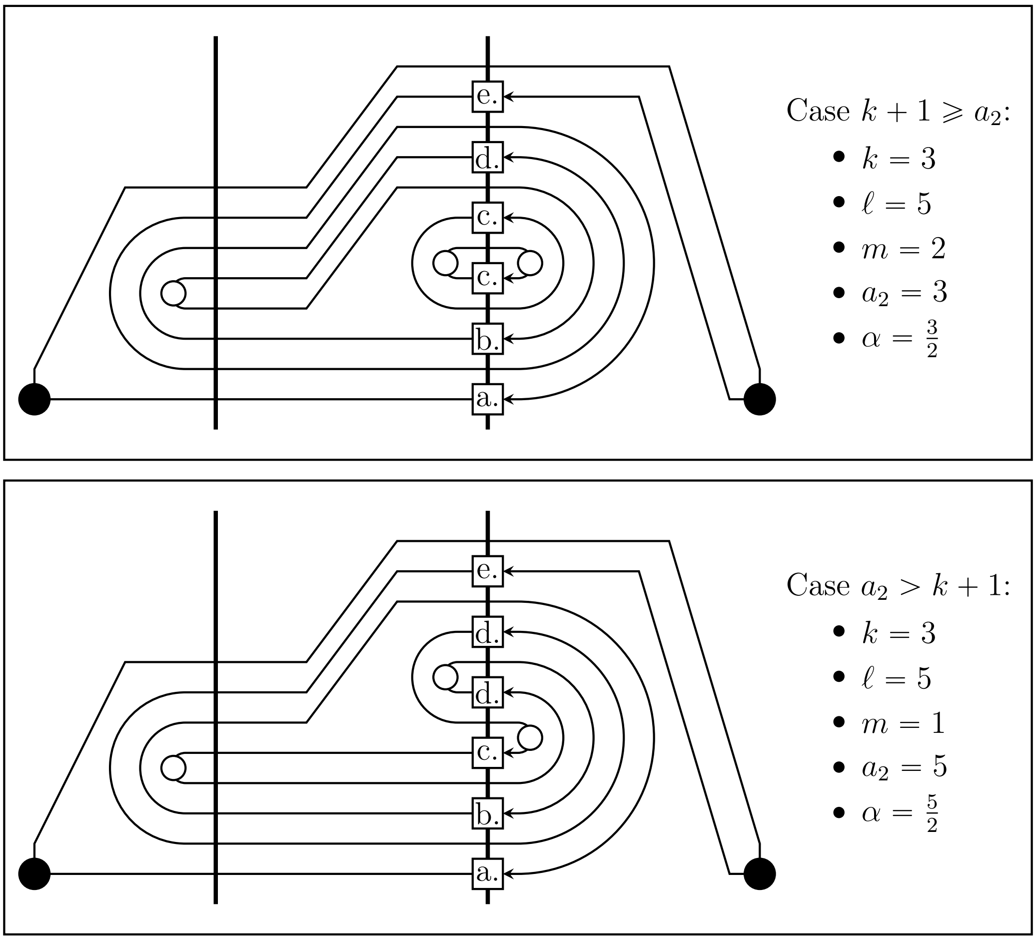


Figure 4.21 – Case $a_2 > 0$ and $a_3 = 0$: $k + 1 \geqslant a_2$ and $a_2 > k + 1$

- $u \to u - 1$ if $0 \leqslant u < \alpha$ or if $\ell + 1 - \alpha \leqslant u \leqslant \ell$;
- $u \to u + (k - a_2)$ if $\alpha \leqslant u < m + \alpha$;
- $u \to u - (m + 1)$ if $m + \alpha \leqslant u < \ell + 1 - \alpha$.

$\triangleright$ If $a_2 > k + 1 > a_3 = 0$, then one checks, as shown in Fig. 4.21 (bottom case), that

a. if $u = 0$, then $2u + 1 \overset{\sqsubset}{\sim} 0 \overset{\sqsupset}{\sim} 2\ell + 1$;
b. if $1 \leqslant u < k + 1 - \alpha$, then $2u + 1 \overset{\sqsubset}{\sim} 2(\ell + 1 - u) \overset{\sqsupset}{\sim} 2u - 1$;
c. if $k + 1 - \alpha \leqslant u < \alpha$, then $2u + 1 \overset{\sqsubset}{\sim} 2(k + 1 - u) \overset{\sqsupset}{\sim} 2u + 1 + 2(m - 1)$;
d. if $\alpha \leqslant u < m + \alpha$, then $2u + 1 \overset{\sqsubset}{\sim} 2(a_2 + m - u) \overset{\sqsupset}{\sim} 2u + 1 - 2(a_2 - k)$;
e. if $m + \alpha \leqslant u \leqslant \ell$, then $2u + 1 \overset{\sqsubset}{\sim} 2(\ell + 1 - u) \overset{\sqsupset}{\sim} 2u - 1$.

It follows that

- $u \to u - 1$ if $0 \leqslant u < k + 1 - \alpha$ or if $m + \alpha \leqslant u \leqslant \ell$;
- $u \to u + (m - 1)$ if $k + 1 - \alpha \leqslant u < \alpha$;
- $u \to u - (a_2 - k)$ if $\alpha \leqslant u < m + \alpha$.

In both pictures of Fig. 4.21, on each point $\mathbf{L}_2^{2u+1}$, is written the subcase (from a. to e.) to which we should refer. For instance, in the top picture, the label "b." is written on the point $\mathbf{L}_2^3$, which is associated with the case $i = 1$, i.e. $1 \leqslant u < \alpha$. Indeed, one checks that $\mathbf{L}_2^3 \overset{2}{\sim} \mathbf{L}_1^3 \overset{1}{\sim} \mathbf{L}_1^6 \overset{2}{\sim} \mathbf{L}_2^{10} \overset{2}{\sim} \mathbf{L}_2^0$, which shows that $3 \overset{\sqsubseteq}{\sim} 10 \overset{\sqsupseteq}{\sim} 0$, as mentioned in the above enumeration of cases.

Overall, in each case, we observe that each permutation ℓ has a specific structure, which we call *translation* and *translated cut.*

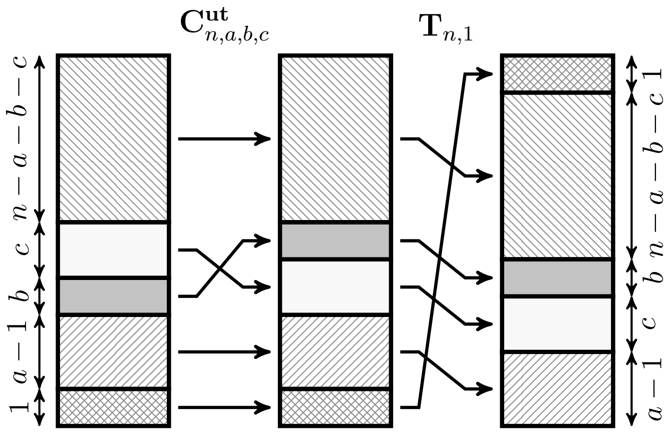


Figure 4.22 – Translated cut $\mathbf{TC}^{\mathbf{cut}}_{n,a,b,c}$

Definition 4.23 (Translation and translated cut)**.**
Let a, b, c and n be non-negative integers such that $a+b+c \leqslant n$, and let $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. We call translation*, and denote by $\mathbf{T}_{n,a}$, the permutation of $\mathbb{Z}_n$ such that $\mathbf{T}_{n,a} : k \mapsto k - a$. We call* translated cut*, and denote by $\mathbf{TC}^{\mathbf{cut}}_{n,a,b,c}$, the permutation $\mathbf{T}_{n,1} \circ \mathbf{C}^{\mathbf{cut}}_{n,a,b,c}$ of $\mathbb{Z}_n$, where $\mathbf{C}^{\mathbf{cut}}_{n,a,b,c}$ is the permutation such that*

$$\begin{aligned}\mathbf{C}^{\mathbf{cut}}_{n,a,b,c} : k \quad \mapsto \quad & k \text{ if } k \in \{0, \ldots, a-1, a+b+c, \ldots, n-1\} \\ & k + c \text{ if } k \in \{a, \ldots, a+b-1\} \\ & k - b \text{ if } k \in \{a+b, \ldots, a+b+c-1\}.\end{aligned}$$

We proved above that

- if $a_2 = 0$, then θ is the translation $\mathbf{T}_{\ell+1,m+1-a_3}$;
- if $k + 1 \geqslant a_2 > a_3 = 0$, then θ is the translated cut $\mathbf{TC}^{\mathbf{cut}}_{\ell+1,\lceil\alpha\rceil,m,k+1-a_2}$;
- if $a_2 > k + 1 > a_3 = 0$, then θ is the translated cut $\mathbf{TC}^{\mathbf{cut}}_{\ell+1,k+1-\lfloor\alpha\rfloor,a_2-k-1,m}$.

Hence, it remains to check which translations and translated cuts are cyclic permutations. The first case is immediate, whereas the second one is not. Both cases are expressed in terms of coprimality: for all relative integers a and b, we denote by $a \wedge b$ the largest common divisor of a and b, i.e. the (unique) non-negative integer d such that $\{ax + by : x, y \in \mathbb{Z}\} = d\mathbb{Z}$. In particular, note that $a \wedge b = |a| \wedge |b|$ and $0 \wedge b = |b|$ for all integers $a, b \in \mathbb{Z}$.

Lemma 4.24.
Let a and n be integers such that $0 \leqslant a \leqslant n$. The translation $\mathbf{T}_{n,a} : \mathbb{Z}_n \mapsto \mathbb{Z}_n$ is cyclic if and only if $a \wedge n = 1$.

Lemma 4.25.
Let a, b, c and n be non-negative integers such that $a + b + c \leqslant n$. The translated cut $\mathbf{TC}^{\mathbf{ut}}_{n,a,b,c} : \mathbb{Z}_n \mapsto \mathbb{Z}_n$ is cyclic if and only if $(c-1) \wedge (b+1) = 1$.

Proof. First, observe that $\mathbf{T}_{n,a} \circ \mathbf{TC}^{\mathbf{ut}}_{n,a,b,c} = \mathbf{TC}^{\mathbf{ut}}_{n,0,b,c} \circ \mathbf{T}_{n,a}$. This means that the translated cuts $\mathbf{TC}^{\mathbf{ut}}_{n,a,b,c}$ and $\mathbf{TC}^{\mathbf{ut}}_{n,0,b,c}$ are conjugate to each other. Therefore, the permutation $\mathbf{TC}^{\mathbf{ut}}_{n,a,b,c}$ is cyclic if and only if $\mathbf{TC}^{\mathbf{ut}}_{n,0,b,c}$ is cyclic too, and we henceforth assume that $a = 0$.

Second, observe that $b \xmapsto{\mathbf{TC}^{\mathbf{ut}}_{n,0,b,c}} n-1 \xmapsto{\mathbf{TC}^{\mathbf{ut}}_{n,0,b,c}} n-2 \xmapsto{\mathbf{TC}^{\mathbf{ut}}_{n,0,b,c}} \ldots \xmapsto{\mathbf{TC}^{\mathbf{ut}}_{n,0,b,c}} b+c-1$. Hence, the permutation $\mathbf{TC}^{\mathbf{ut}}_{n,0,b,c}$ is cyclic if and only if $\mathbf{TC}^{\mathbf{ut}}_{b+c,0,b,c}$ is cyclic too, and we henceforth assume that $n = b + c$.

Third, observe that $\mathbf{TC}^{\mathbf{ut}}_{b+c,0,b,c}$ is simply the translation $\mathbf{T}_{b+c,b+1}$. Consequently, the permutation $\mathbf{TC}^{\mathbf{ut}}_{b+c,0,b,c}$ is cyclic if and only if $(b+c) \wedge (b+1) = 1$, i.e. if and only if $(c-1) \wedge (b+1) = 1$. This completes the proof. $\square$

Remember that **sa** are actual coordinates if and only if θ is a cyclic permutation of $\mathbb{Z}_{\ell+1}$. Hence, Lemmas 4.24 and 4.25 prove that $\mathbf{sa} = (0, a_1, k, a_2, k+m, a_3, 0)$ are actual coordinates if and only if we are in one of the following cases:

(i) $a_2 = 0$, $a_3 = 0$ and $k \wedge (m+1) = 1$;
(ii) $a_2 = 0$, $a_3 = 1$ and $(k+1) \wedge m = 1$;
(iii) $k + 1 \geqslant a_2 \geqslant 1$, $a_3 = 0$ and $(k - a_2) \wedge (m+1) = 1$;
(iv) $2k + 1 \geqslant a_2 \geqslant k + 2$, $a_3 = 0$ and $(a_2 - k) \wedge (m-1) = 1$.

In particular, taking into account the virtual coordinates **sa** such that $a_1 = 0$ (and whose case was tackled in the first few lines of Section 4.2.2), it follows that, whenever $k \geqslant 1$ and $m \geqslant 1$, we obtain the following formula for the integers

$$C_{k,k+m} = C_{k+m,k} = |\{(a_1, a_2, a_3) : (0, a_1, k, a_2, k+m, a_3, 0) \text{ are actual coordinates}\}| :$$

$$\begin{aligned}
\frac{C_{k,k+m}}{2} &= \mathbf{1}_{k \wedge (m+1)=1} + \mathbf{1}_{(k+1) \wedge m=1} + \\
&\quad \sum_{a_2=1}^{k+1} \mathbf{1}_{(k-a_2) \wedge (m+1)=1} + \sum_{a_2=k+2}^{2k+1} \mathbf{1}_{(m-1) \wedge (a_2-k)=1} \\
&= \sum_{a=1}^{k} \mathbf{1}_{a \wedge (m+1)=1} + \mathbf{1}_{(k+1) \wedge m=1} + \sum_{a=1}^{k+1} \mathbf{1}_{a \wedge (m-1)=1}.
\end{aligned}$$

Proof of Theorem 4.16 – Step 4: Generating Functions

Focus now on the generating function $\mathcal{G}_3(z) = \sum_{k \geqslant 0} g_{3,k} z^k$. For the sake of clarity and conciseness, we only indicate the main steps of our computations, which are mainly based on rearranging terms.

We proved in Section 4.2.2 that $C_{0,0} = 1$, that $C_{0,\ell} = C_{\ell,0} = 2$ for $\ell \geqslant 1$, and that $C_{k,k} = 2(2k+1)$ for $k \geqslant 1$. It follows that

$$\mathcal{G}_3(z) = \sum_{k,\ell \geqslant 0} C_{k,\ell} z^{k+\ell} = 1 + \sum_{\ell \geqslant 1} 4 z^\ell + \sum_{k \geqslant 1} 2(2k+1) z^{2k} + 2 \sum_{k \geqslant 1, m \geqslant 1} C_{k,k+m} z^{2k+m}.$$

Using the above formula for $C_{k,k+m}$ (when $k, m \geqslant 1$), we can rewrite this as

$$\mathcal{G}_3(z) = 1 + \frac{4z}{1-z} - \frac{2z^2(z^2-3)}{(1-z^2)^2} + 4(H_1(z) + H_2(z) + H_3(z) - H_4(z)),$$

where

$$H_1(z) = \sum_{k \geqslant 1} \sum_{m \geqslant 1} \sum_{a=1}^{k} \mathbf{1}_{a \wedge (m+1) = 1} z^{2k+m}, \qquad H_2(z) = \sum_{k \geqslant 1} \sum_{m \geqslant 1} \mathbf{1}_{(k+1) \wedge m = 1} z^{2k+m},$$

$$H_3(z) = \sum_{k \geqslant 0} \sum_{m \geqslant 1} \sum_{a=1}^{k+1} \mathbf{1}_{a \wedge (m-1) = 1} z^{2k+m}, \qquad H_4(z) = \sum_{m \geqslant 1} z^m.$$

Then, let us define the function $F(z) := \sum_{\alpha \geqslant 1} \sum_{\beta \geqslant 1} \mathbf{1}_{\alpha \wedge \beta = 1} z^{2\alpha+\beta}$. Using simple substitutions ($t := k - a$, $u := k + 1$, $v := m + 1$ and $w := m - 1$), we get

$$\begin{aligned}
H_1(z) &= \sum_{a \geqslant 1} \sum_{m \geqslant 1} \sum_{t \geqslant 0} \mathbf{1}_{a \wedge (m+1) = 1} z^{2(a+t)+m} = \frac{1}{1-z^2} \sum_{a \geqslant 1} \sum_{m \geqslant 1} \mathbf{1}_{a \wedge (m+1) = 1} z^{2a+m} \\
&= \frac{1}{z(1-z^2)} \left(\sum_{a \geqslant 1} \sum_{v \geqslant 1} \mathbf{1}_{a \wedge v = 1} z^{2a+v} - \sum_{a \geqslant 1} z^{2a+1} \right) = \frac{F(z)}{z(1-z^2)} - \frac{z^2}{(1-z^2)^2}, \\
H_2(z) &= \frac{1}{z^2} \left(\sum_{u \geqslant 1} \sum_{m \geqslant 1} \mathbf{1}_{u \wedge m = 1} z^{2u+m} - \sum_{m \geqslant 1} z^{2+m} \right) = \frac{F(z)}{z^2} - \frac{z}{1-z}, \\
H_3(z) &= \sum_{a \geqslant 1} \sum_{m \geqslant 1} \sum_{t \geqslant -1} \mathbf{1}_{a \wedge (m-1) = 1} z^{2(a+t)+m} = \frac{1}{z^2(1-z^2)} \sum_{a \geqslant 1} \sum_{m \geqslant 1} \mathbf{1}_{a \wedge (m-1) = 1} z^{2a+m} \\
&= \frac{1}{z(1-z^2)} \left(\sum_{a \geqslant 1} \sum_{w \geqslant 1} \mathbf{1}_{a \wedge w = 1} z^{2a+w} + z^2 \right) = \frac{F(z)}{z(1-z^2)} + \frac{z}{1-z^2}, \text{ and} \\
H_4(z) &= \frac{z}{1-z}.
\end{aligned}$$

Moreover, consider the coefficients f_n of the series $F(z)$. Since $F(z) = \sum_{n \geqslant 3} f_n z^n$, we have

$$\begin{aligned}
f_n &= \left| \left\{ a < \frac{n}{2} : a \wedge (n - 2a) = 1 \right\} \right| = \left| \left\{ a < \frac{n}{2} : a \wedge n = 1 \right\} \right| \\
&= \frac{1}{2} \left| \left\{ a \leqslant n : a \neq \frac{n}{2}, a \wedge n = 1 \right\} \right|.
\end{aligned}$$

Observe that, if $n \geqslant 3$ is even, then $\frac{n}{2} \wedge n = \frac{n}{2} \neq 1$. It follows that $f_n = \frac{\varphi(n)}{2}$, i.e. that $F(z) = \frac{1}{2}\sum_{n \geqslant 3} \varphi(n) z^n$, where φ denotes the Euler totient. Collecting the above terms, we have

$$\begin{aligned}\mathcal{G}_3(z) &= 1 + \frac{4z}{1-z} - \frac{2z^2(z^2-3)}{(1-z^2)^2} + 4\left(\frac{2}{z(1-z^2)} + \frac{1}{z^2}\right)F(z) + \\ &\quad 4\left(\frac{z}{1-z^2} - \frac{z^2}{(1-z^2)^2} - \frac{2z}{1-z}\right) \\ &= 4\frac{1+2z-z^2}{z^2(1-z^2)}F(z) + \frac{1-3z^2}{1-z^2},\end{aligned}$$

and since $\mathcal{B}_3(z) = z^2\mathcal{G}_3(z^2)$, the two first parts of Theorem 4.16 are proved.

In addition, developing term-wise the series $\mathcal{G}_3(z) = \sum_{k \geqslant 0} g_{3,k} z^k$ gives

$$\mathcal{G}_3(z) = \left(\sum_{k \geqslant 0} z^{2k}\right)\left(2(1+2z-z^2)\sum_{k \geqslant 1}\varphi(k+2)z^k + (1-3z^2)\right),$$

which proves that $g_{3,k} = \sum_{i=0}^{\lfloor k/2 \rfloor} \gamma_{k-2i}$, with

$$\gamma_i = \mathbf{1}_{i=0} - 3 \cdot \mathbf{1}_{i=2} + 2\varphi(i+2)\mathbf{1}_{i \geqslant 1} + 4\varphi(i+1)\mathbf{1}_{i \geqslant 2} - 2\varphi(i)\mathbf{1}_{i \geqslant 3}.$$

It follows that

$$g_{3,k} = \mathbf{1}_{k=0} + 2\left(\varphi(k+2) - \mathbf{1}_{k \in 2\mathbb{Z}} + 2\sum_{i=1}^{\lfloor k/2 \rfloor}\varphi(k+3-2i)\right)\mathbf{1}_{k \geqslant 1},$$

which proves the third part of Theorem 4.16.

Proof of Theorem 4.16 – Step 5: Holonomy

Finally, we prove the last part of Theorem 4.16, i.e. that the generating functions

$$\mathcal{G}_3(z) = 2\frac{1+2z-z^2}{z^2(1-z^2)}\left(\sum_{n \geqslant 3}\varphi(n)z^n\right) + \frac{-1+3z^2}{1-z^2},\ \mathcal{B}_3(z) = z^2\mathcal{G}_3(z^2)$$

are not holonomic.

We do so by using standard tools and results of complex analysis (see [49, Annex B.4]) and of analytic number theory (see [60]).

For the sake of contradiction, let us assume henceforth that $\mathcal{G}_3(z)$ is holonomic. Then, so is the generating function $\sum_{n \geqslant 1}\varphi(n)z^n$, i.e. the sequence $(\varphi(n))_{n \geqslant 1}$ is P-recursive: this means that there exists some complex polynomials $A_0(X), \ldots, A_k(X)$ with such that $A_k \neq 0$ and $\sum_{i=0}^{k} A_i(n)\varphi(n+i) = 0$ for all integers $n \geqslant 1$. In addition, since each term

$\varphi(n)$ is a rational number, we may even assume that $A_0(X), \ldots, A_k(X)$ have integer coefficients.

Let $d := \max\{\deg A_i : 0 \leqslant i \leqslant k\}$, and let us write $A_i(X) = \sum_{j=0}^{d} a_{i,j} X^j$ for all $i \leqslant k$. Then, consider some integer $\ell \in \{0, \ldots, k\}$ such that $\deg A_\ell = d$, i.e. $a_{\ell,d} \neq 0$, and let us define the integer $a_\infty := \sum_{i=0}^{k} |a_{i,d}|$.

The Euler identity

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-1}} = \prod_{p \text{ prime}} \left(\sum_{j \geqslant 0} p^{-j} \right) = \sum_{n \geqslant 1} n^{-1} = +\infty$$

shows that $\prod_{p \text{ prime}} (1 - p^{-1}) = 0$. Hence, there exists pairwise disjoint sets $P_0, \ldots, P_k$ of primes numbers greater than k and such that $\prod_{p \in P_i} (1 - p^{-1}) \leqslant \frac{1}{2a_\infty}$ for all $i \leqslant k$. Consequently, the integers $b_i = \prod_{p \in P_i}$ are pairwise coprime integers such that $\varphi(b_i) = \prod_{p \in P_i} (p-1) \leqslant \frac{b_i}{2a_\infty}$.

The Chinese remainder theorem [60, Theorem 59] shows that there exists an integer $N \geqslant 0$ such that $N + i \equiv 0 \pmod{b_i}$ for all $i \neq \ell$. Since the prime factors of b_i are greater than k, it follows that $N + \ell$ is coprime with b_i, for all $i \neq \ell$. Hence, the Dirichlet theorem [60, Theorem 15] states that there exists arbitrarily large integers n (in the set $\{N + z \prod_{i \neq \ell} b_i : z \in \mathbb{N}\}$) such that $n + \ell$ is prime. For such integers n, remember that $n + i \equiv 0 \pmod{b_i}$ when $i \neq \ell$. It follows that $\varphi(n + \ell) = n + \ell - 1$ and $\varphi(n+i) \leqslant \frac{\varphi(b_i)}{b_i}(n+i) \leqslant \frac{n+i}{2a_\infty}$. Since $0 = \sum_{i=0}^{k} A_i(n) \varphi(n+i)$, we deduce that

$$\begin{aligned} (n-1)|A_\ell(n)| &\leqslant |A_\ell(n)\varphi(n+\ell)| \leqslant \sum_{i \neq \ell} |A_i(n)\varphi(i+\ell)| \\ &\leqslant \frac{n+k}{2a_\infty} \sum_{i \neq \ell} |A_i(n)| \leqslant \frac{n+k}{2a_\infty} \sum_{i=0}^{k} |A_i(n)|. \end{aligned}$$

When $n \to +\infty$, we have $|A_\ell(n)| \sim |a_{\ell,d}| n^d$ and $\sum_{i=0}^{k} |A_i(n)| \sim a_\infty n^d$, from which we deduce that

$$|a_{\ell,d}| n^{d+1} \sim (n-1)|A_\ell(n)| \leqslant \frac{n+k}{2a_\infty} \sum_{i=0}^{k} |A_i(n)| \sim \frac{1}{2} n^{d+1},$$

which is impossible since $|a_{\ell,d}| \geqslant 1$. This contradiction shows that the generating function $\mathcal{G}_3(z)$ could not be holonomic.

Finally, since $\mathcal{G}_3(z) = z^{-1} \mathcal{B}_3(z^{1/2})$ and since $z \mapsto z^{1/2}$ is algebraic, the generating function $\mathcal{B}_3(z)$ cannot be holonomic either. This was the last step of the proof of Theorem 4.16, which is now completed.

4.3 Estimated and Asymptotic Values

4.3.1 Asymptotic Values in $\mathbf{B}_3$

We use Theorem 4.16 to estimate precisely the terms $g_{3,k}$ of the series $\mathcal{G}_3(z)$.

Proposition 4.26.
When $k \to +\infty$, we have:

$$g_{3,k} \sim 4\left(1 + \mathbf{1}_{k \in 2\mathbb{Z}}\right) \frac{k^2}{\pi^2}.$$

Proof. For the sake of simplicity, let us introduce some notation. We define $\alpha = \frac{4}{\pi^2}$ and $\phi_k = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \varphi(k-2i)$, as well as real numbers ε_k, θ_k and η_k such that $\phi_{2k} = (\alpha + \varepsilon_k)k^2$, $\phi_{2k-1} = (2\alpha + \theta_k)k^2$ and $\eta_k = \varepsilon_k + \theta_k$. We first want to prove that $\varepsilon_k \to 0$ and that $\theta_k \to 0$ when $k \to +\infty$.

It is a standard result that

$$\sum_{k=1}^{n} \varphi(k) \sim \frac{3}{\pi^2} n^2$$

when $n \to +\infty$ (see [60, Theorem 330]). It follows that

$$(3\alpha + \eta_k)k^2 = \phi_{2k} + \phi_{2k-1} = \sum_{i=1}^{2k} \varphi(i) \sim \frac{12}{\pi^2} k^2 = 3\alpha k^2,$$

which means that $\eta_k \to 0$. Hence, it remains to prove that $\varepsilon_k \to 0$.

Then, let A be some positive constant, and let K be some positive integer such that $\frac{\alpha}{(2k+1)^2} \leqslant A$ and $|\eta_k| \leqslant A$ whenever $k \geqslant K$. In addition, for each integer $\ell \geqslant \log_2(K)$, we define $M_\ell = \max\{|\varepsilon_k| : 2^\ell \leqslant k \leqslant 2^{\ell+1}\}$. If $2^\ell \leqslant k \leqslant 2^{\ell+1}$, then

$$\phi_{4k} = \sum_{i=0}^{k} \varphi(4i) + \sum_{i=0}^{k-1} \varphi(4i+2) = 2\sum_{i=0}^{k} \varphi(2i) + \sum_{i=0}^{k-1} \varphi(2i+1) = 2\phi_{2k} + \phi_{2k-1},$$

i.e. $\varepsilon_{2k} = \frac{\varepsilon_k + \eta_k}{4}$. It follows that

$$|\varepsilon_{2k}| \leqslant \frac{M_\ell + A}{4} \leqslant 2A + \frac{3M_\ell}{4}.$$

Similarly, if $2^\ell \leqslant k < 2^{\ell+1}$, then $\phi_{4k+2} = \phi_{4k} + \varphi(4k+2) = 2\phi_{2k} + \phi_{2k+1}$, i.e.

$$\varepsilon_{2k+1} = \frac{\alpha}{(2k+1)^2} + \frac{2k^2}{(2k+1)^2}\varepsilon_k - \frac{(k+1)^2}{(2k+1)^2}\varepsilon_{k+1} + \frac{(k+1)^2}{(2k+1)^2}\eta_{k+1}.$$

Since $k \geqslant 2^\ell \geqslant K$, we know that $\frac{\alpha}{(2k+1)^2} \leqslant A$. Moreover, note that

$$\frac{2k^2 + (k+1)^2}{(2k+1)^2} = \frac{3}{4} - \frac{4k-1}{4(2k+1)^2} \leqslant \frac{3}{4}.$$

It follows that

$$|\varepsilon_{2k+1}| \leqslant A + \frac{2k^2 + (k+1)^2}{(2k+1)^2} M_\ell + \frac{(k+1)^2}{(2k+1)^2} A \leqslant 2A + \frac{3M_\ell}{4}.$$

Overall, $|\varepsilon_m| \leqslant 2A + \frac{3M_\ell}{4}$ whenever $2^{\ell+1} \leqslant m \leqslant 2^{\ell+2}$, which shows that $M_{\ell+1} \leqslant 2A + \frac{3M_\ell}{4}$. It follows that $\limsup M_\ell \leqslant 8A$ and, since A is an arbitrary positive constant, that $M_\ell \to 0$ when $\ell \to +\infty$. Recall that $|\varepsilon_k| \leqslant M_\ell$ whenever $2^\ell \leqslant k \leqslant 2^{\ell+1}$: this proves that $\varepsilon_k \to 0$, and therefore that $\theta_k = \eta_k - \varepsilon_k \to 0$. It follows that

$$\phi_k \sim \left(1 + \mathbf{1}_{k \in 2\mathbb{Z}+1}\right) \frac{k^2}{\pi^2}$$

when $k \to +\infty$.

With the above notations, and according to Theorem 4.16, we have

$$g_{3,k} = \mathbf{1}_{k=0} + 2\left(\varphi(k+2) - \mathbf{1}_{k \in 2\mathbb{Z}} + 2\sum_{i=1}^{\lfloor k/2 \rfloor} \varphi(k+3-2i)\right) \mathbf{1}_{k \geqslant 1} = 4\phi_{k+1} + \mathcal{O}(k).$$

Moreover, we just showed that $k^2 \leqslant \left(1 + \mathbf{1}_{k \in 2\mathbb{Z}+1}\right) k^2 \sim \pi^2 \phi_k$, and therefore that $k^2 = \mathcal{O}(\phi_k)$. It follows that $g_{3,k} = 4\phi_{k+1} + \mathcal{O}(k+1) = 4\phi_{k+1} + o(\phi_{k+1})$, i.e. that $g_{3,k} \sim 4\phi_{k+1}$, which proves Proposition 4.26. □

Definition 4.27 (Approximately polynomial sequence)**.**
Let $(u_n)_{n \geqslant 0}$ be a real-valued sequence. We say that (u_n) is approximately polynomial *if there exists a real number $\delta > 1$ and a periodic, non-zero sequence $(\omega_n)_{n \geqslant 0}$ whose terms are non-negative real numbers and such that $u_n \sim \omega_n n^\delta$.*

Proposition 4.26 proves that the sequence $(g_{3,k})_{k \geqslant 0}$ is approximately polynomial. Indeed, it suffices to consider $\delta = 2$ and $\omega_n = 4(1 + \mathbf{1}_{n \in 2\mathbb{Z}})\pi^{-2}$. It follows immediately that sequences $(N^{c,\ell}_{3,k})_{k \geqslant 0}$, $(N^{o,\ell}_{3,k})_{k \geqslant 0}$ and $(N^{d}_{3,k})_{k \geqslant 0}$ are also approximately polynomial.

Approximately polynomial sequences have *complicated* Lambert series, as shown by the following result.

Proposition 4.28.
Let $(u_n)_{n \geqslant 0}$ be an approximately polynomial sequence. The Lambert series $L : z \mapsto \sum_{n \geqslant 1} u_n \frac{z^n}{1-z^n}$ is not holonomic.

Proof. This proof relies on the tools that were used for proving that $\mathcal{G}_3(z)$ is not holonomic or for proving Proposition 4.26. However, it is substantially more technical, and we therefore begin by drawing a sketch of this proof.

We proceed by contradiction, and assume henceforth that $L(z)$ is holonomic. First, we build a sequence (V_n) that must be "approximately" recurrent linear. Second, we define some objects that we will not use before the end of the proof, but that give us lower bounds for integer constants that we want to use immediately. Third, we define a class of integers N by using congruence relations, and study the factors of integers of the form $N+i$. Finally, we consider the linear equation that (V_n) is supposed to "approximately" satisfy, split this equation in several parts whose values we estimate using the objects defined in the second part of the proof, and derive a contradiction.

Proof of Proposition 4.28 – First Part

Consider the real number $\delta > 1$ and the sequence $(\omega_n)_{n\geqslant 0}$ used in Definition 4.27. Let ε be some positive real constant. There exists an integer n_0 such that, for all $n \geqslant n_0$, $|u_n - \omega_n n^\delta| \leqslant \varepsilon \omega_n n^\delta$. Then consider the real constants $S := \sum_{n=0}^{n_0} |u_n - \omega_n n^\delta|$ and $\Omega := \max\{\omega_n : n \geqslant 0\}$.

For all integers $n \geqslant 1$, we define $U_n := \sum_{k|n} u_k$ and $V_n := \sum_{k|n} \omega_{n/k} k^{-\delta}$. Note that $|V_n| \leqslant \sum_{k\geqslant 1} \Omega k^{-\delta} = \Omega\zeta(\delta)$. Therefore,

$$|U_n - n^\delta V_n| \leqslant \sum_{k|n} \left|u_k - \omega_k k^\delta\right| \leqslant \varepsilon n^\delta |V_n| + S \leqslant \varepsilon \Omega n^\delta \zeta(d) + S.$$

Since ε is arbitrarily small, it follows that $U_n = n^\delta V_n + o(n^\delta)$.

We decided to assume that the Lambert series $L(z) = \sum_{n\geqslant 1} U_n z^n$ is holonomic. This means that the sequence $(U_n)_{n\geqslant 1}$ is P-recursive, i.e. that there exists real polynomials $A_0(X), \ldots, A_m(X)$ such that $A_m(X) \neq 0$ and $\sum_{i=0}^m A_i(n+i)U_{n+i} = 0$ for all integers $n \geqslant 1$. Let $d := \max\{\deg A_i : 0 \leqslant i \leqslant m\}$, and let us write $A_i(X) = \sum_{j=0}^d a_{i,j} X^j$. It follows that

$$0 = \sum_{i=0}^m A_i(n) U_{n+i} = \sum_{i=0}^m \sum_{j=0}^d \left(a_{i,j} V_{n+i} n^{j+\delta} + o(n^{j+\delta})\right) = n^{d+\delta} \sum_{i=0}^m a_{i,d} V_{n+i} + o(n^{d+\delta}),$$

and therefore that $\sum_{i=0}^m a_{i,d} V_{n+i} \to 0$ when $n \to +\infty$.

Proof of Proposition 4.28 – Second Part

Now, let ρ be the period of the sequence (ω_n), and let $z \in \{0, \ldots, \rho-1\}$ be an integer such that $\omega_z \neq 0$. We also define $\gamma := \lceil \log_2(2\rho + m)\rceil$ and $a_\infty := \sum_{i=0}^m |a_{i,d}|$, and choose some integer ℓ such that $a_{\ell,d} \neq 0$. Without loss of generality, we assume henceforth that $a_{\ell,d} = 1$.

Then, for all integers $i \in \{0, \ldots, m\}$ and all integers $k \geqslant 1$, we define the finite set $\mathcal{S}_{i,k} := \{0\} \cup \{a_{i,d}\omega_n k^{-\delta} : 0 \leqslant n < \rho\}$. We also define the Minkowski sum $\mathcal{S}_\infty := \sum_{i=0}^m \sum_{k=1}^{\rho^\gamma(m+1)} \mathcal{S}_{i,k}$, and choose some positive real constant Z_0 such that $Z_0 \leqslant |z|$ for all $z \in \mathcal{S}_\infty \backslash \{0\}$. The mysterious sets $\mathcal{S}_{i,k}$ and $\mathcal{S}_\infty$, as well as the constants $\rho^\gamma(m+1)$ and Z_0, will play a role at the end of the proof.

Finally, let q be some prime number such that $q \equiv 1 \pmod{\rho}$, $q \geqslant 2m+1$ and $q^\delta \geqslant \frac{4}{\Omega \rho^\gamma (m+1) Z_0}$. Note that such a prime number q exists, according to Dirichlet theorem [60, Theorem 15]. Then, we also consider some integer M such that $M^{\delta-1} \geqslant \frac{2 a_\infty \Omega q^\delta}{(\delta-1)\omega_z}$, $M > \rho + q + (m-1)$ and $M \geqslant (m+1)^2$.

Proof of Proposition 4.28 – Third Part

Let N be some positive integer such that

- $N \equiv \rho + z - \ell \pmod{\rho^{\gamma+1}}$;
- $N \equiv q - \ell \pmod{q^2}$;
- $N \equiv 1 \pmod{k}$ for all integers $k \leqslant M$ such that k is coprime with ρ and q.

The Chinese remainder theorem [60, Theorem 59] proves that arbitrarily large such integers N exist.

Let i be some element of $\{0, \ldots, m\}$, and let R_i be the largest divisor of $N+i$ whose prime factors divide ρ. Let r be some prime factor of R_i, and let $\alpha \geqslant 0$ be the maximal integer such that r^α divides $(N+i) \wedge \rho^\gamma$. Since r^α divides the positive integer $\rho + z + i - \ell$, it follows that $r^\alpha \leqslant \rho + z + i \leqslant 2\rho + m$, hence that $\alpha \leqslant \log_r(2\rho + m) \leqslant \log_2(2\rho + m) \leqslant \gamma$. This shows that $r^{\alpha+1}$ divides $\rho^{\gamma+1}$, hence that $r^{\alpha+1}$ does not divide $N+i$. Consequently, α is the maximal integer such that r^α divides $N+i$ or, equivalently, R_i It follows that R_i divides ρ^γ, hence that $R_i \leqslant \rho^\gamma$.

Then, let S_i be the largest divisor of $N+i$ that is coprime with ρ and q and whose prime factors are not greater than M. First, if k is a divisor of S_i such that $k \leqslant M$, then k divides $N+i$ and $N+i \equiv i+1 \pmod{k}$, so that $k \leqslant 1+i \leqslant m+1$. Then, since S_i is a product of integers not greater than M, hence not greater than $m+1$, and since $M \geqslant (m+1)^2$, an immediate induction shows that S_i has no divisor greater than M, i.e no divisor greater than $m+1$. It follows that $S_i \leqslant m+1$.

Finally, note that $(N+\ell) \wedge q^2 = q$, i.e. that $N+\ell$ is divisible by q but not by q^2, and that $(N+i) \wedge q = (\ell - i) \wedge q = 1$ if $i \neq \ell$. Hence, we factor each number $N+i$ as a product $N+i = q^{\mathbf{1}_{i=\ell}} R_i S_i T_i$ such that $R_i \leqslant \rho^\gamma$, $S_i \leqslant m+1$ and each prime factor of T_i is greater than M and coprime with ρ and q.

Proof of Proposition 4.28 – Fourth Part

Let us define the real numbers $W_n := \sum_{k|n} \mathbf{1}_{k>M} \omega_{n/k} k^{-\delta}$ for all integers $n \geqslant 1$, as well as

$$X := \sum_{i=0}^{m} a_{i,d} W_{N+i},\ Y := a_{\ell,d} q^{-\delta} \sum_{k|R_\ell S_\ell} \omega_{(N+\ell)/qk} k^{-\delta} \text{ and } Z := \sum_{i=0}^{m} \sum_{k|R_i S_i} a_{i,d} \omega_{(N+i)/k} k^{-\delta}.$$

Since $q \equiv 1 \pmod{\rho}$, and hence q is coprime with ρ, and since $M > \rho + q + (m-1)$, each term of the sum $\sum_{i=0}^{m} a_{i,d} V_{n+i} = \sum_{i=0}^{m} \sum_{k|N+i} a_{i,d} \omega_{(N+i)/k} k^{-\delta}$ appears exactly once in either the sums X, Y or Z. This shows that $\sum_{i=0}^{m} a_{i,d} V_{n+i} = X + Y + Z$.

We evaluate now the sums $|X|$, $|Y|$ and $|Z|$. First, note that $|W_n| \leqslant \Omega \sum_{k>M} k^{-\delta} \leqslant \frac{\Omega}{(\delta-1)M^{\delta-1}}$, and therefore that $|X| \leqslant \frac{a_\infty \Omega}{(\delta-1)M^{\delta-1}}$. Second, observe that

$$\Omega \rho^\gamma (m+1) \geqslant \Omega \sum_{k=1}^{\rho^\gamma(m+1)} k^{-\delta} \geqslant \sum_{k|R_\ell S_\ell} \omega_{(N+\ell)/qk} k^{-\delta} = q^\delta Y \geqslant \omega_{(N+\ell)/q} = \omega_z,$$

and therefore that $\omega_z q^{-\delta} \leqslant |Y| \leqslant \Omega \rho^\gamma (m+1) q^{-\delta}$. Finally, each term of the sum $Z = \sum_{i=0}^{m} \sum_{k=1}^{\rho^\gamma(m+1)} a_{i,d} \mathbf{1}_{k|R_i R_i} \omega_{(N+i)/k} k^{-\delta}$ belongs to the a $\mathcal{S}_{i,k}$ defined during the second part of the proof. It follows that $Z \in \mathcal{S}_\infty$, and that either $Z = 0$ or that $|Z| \geqslant Z_0$.

We deduce from this that

- $|X+Y+Z| \geqslant |Y| - |X| \geqslant \omega_z q^{-\delta} - \frac{(m+1)\Omega}{(\delta-1)M^{\delta-1}} \geqslant \frac{1}{2}\omega_z q^{-\delta}$ if $Z = 0$;
- $|X+Y+Z| \geqslant |Z| - |X| - |Y| \geqslant Z_0 - 2|Y| \geqslant Z_0 - 2\Omega\rho^\gamma(m+1)q^{-\delta} \geqslant \frac{1}{2} Z_0$ if $Z \neq 0$.

Consequently, and since N can be arbitrarily large, we cannot have $\sum_{i=0}^{k} a_{i,d} V_{n+i} \to 0$. This contradiction shows that $L(z)$ is not holonomic. □

Corollary 4.29.
Let $k \geqslant -1$ be some integer. The Lambert series $L_{3,k} : z \mapsto \sum_{n\geqslant 1} g_{3,n+k} \frac{z^n}{1-z^n}$ is not holonomic.

This result completes Theorem 4.16 by showing that neither the "standard" generating function nor the Lambert series $L_{3,k}(z)$ associated with the sequences $(g_{3,n+k})_{n\geqslant 1}$ are holonomic. In particular, provided that the sequence $(\varphi(n))_{n\geqslant 1}$ has a very simple Lambert series $\sum_{n\geqslant 1} \varphi(n) \frac{z^n}{1-z^n} = \frac{z}{(1-z)^2}$ and that the function $\mathcal{G}_3(z) = \sum_{n\geqslant 0} g_{3,n} z^n$ is the composition of the series $\sum_{n\geqslant 1} \varphi(n) z^n$ by a rational fraction, one might have hoped that the Lambert series $L_{3,k}(z)$ would be simple too. Corollary 4.29 proves that this is not the case, and shows that even variants of the series $L_{3,k}(z)$ such as the Lambert series of the sequences $(N_{3,k}^{c,\ell})_{k\geqslant 0}$, $(N_{3,k}^{o,\ell})_{k\geqslant 0}$ and $(N_{3,k}^{d})_{k\geqslant 0}$ are also non-holonomic.

4.3.2 Estimates in $\mathbf{B}_n$ ($n \geqslant 4$)

We did not manage to compute explicitly the generating functions $\mathcal{G}_n(z)$ nor the integers $g_{n,k}$ for $n \geqslant 4$. Hence, we settle for upper and lower bounds. We begin with simple, yet non-trivial estimates, then introduce the original notion of *composition of diagrams* to derive tighter lower bounds on the integers $g_{n,k}$.

We first identify an upper bound on the number of *virtual* coordinates $(s_0, a_1, s_1, \ldots, a_n, s_n)$ such that $\sum_{i=0}^{n} s_i = k$. Of course, this will also provide us with an upper bound on the integers $g_{n,k}$.

Proposition 4.30.
Let $n \geqslant 1$ and $k \geqslant 0$ be integers. Then, $g_{n,k} \leqslant 2^n \binom{k+n-1}{n-1}^{n-2} \binom{k+n-2}{n-2}$.

Proof. Let us bound above the number of ways in which a tuple $(s_0, a_1, \ldots, s_n)$ of actual coordinates can be chosen. First, there are exactly $\binom{k+n-2}{n-2}$ ways of choosing non-negative integers $s_1, \ldots, s_{n-1}$ whose sum is k.

Second, we know that $0 \leqslant a_i \leqslant 2\min\{s_{i-1}, s_i\}+1$ for all integers $i \in \{1, 2, \ldots, n\}$. Then, let u be an integer such that $s_u = \max\{s_1, \ldots, s_{n-1}\}$: we know that $0 \leqslant a_j \leqslant 2s_{j-1} + 1$ when $1 \leqslant j \leqslant u$ and that $0 \leqslant a_j \leqslant 2s_j + 1$ when $u + 1 \leqslant j \leqslant n$. Therefore, let S_u denote the set $\{1, \ldots, u-1, u+1, \ldots, n-1\}$: the tuple $(a_1, \ldots, a_n)$ must belong to the Cartesian product $\{0, 1\} \times \prod_{j \in S_u} \{0, \ldots, 2s_j + 1\} \times \{0, 1\}$, whose cardinality is $P = 2^n \prod_{j \in S_u} (s_j + 1)$.

By arithmetic-geometric inequality, it follows that

$$P \leqslant 2^n \left(\frac{\sum_{j \in S_u} (s_j + 1)}{n-2} \right)^{n-2} \leqslant 2^n \left(\frac{\sum_{j=1}^{n-1} (s_j + 1)}{n-1} \right)^{n-2} = 2^n \left(\frac{k+n-1}{n-1} \right)^{n-2},$$

which completes the proof. □

In order to compute a lower bound, we also prove a combinatorial result which is interesting in itself.

Proposition 4.31.
Let $(s_0, s_1, \ldots, s_n)$ be non-negative integers, with $s_0 = s_n = 0$. There exists integers $a_1, \ldots, a_n$ such that $(s_0, a_1, s_1, \ldots, a_n, s_n)$ are actual coordinates.

Proof. Let us choose $a_i = s_{i-1}$ if $s_{i-1} \leqslant s_i$, and $a_i = s_i + 1$ if $s_{i-1} > s_i$, and let $\mathcal{D}$ be the generalised curve diagram associated with the coordinates $\mathbf{sa} := (s_0, a_1, s_1, \ldots, a_n, s_n)$. We show below that $\mathcal{D}$ is a 1-generalised curve diagram. We do so by proving, using an induction on $i \in \{0, \ldots, n\}$, the following properties $\mathcal{P}_i$ and $\mathcal{Q}_i$:

$$\begin{aligned} \mathcal{P}_i &= \forall j \in \{1, \ldots, s_i\}, \mathbf{L}_i^j \equiv \mathbf{L}_i^{2s_i+1-j}; \\ \mathcal{Q}_i &= \forall \ell \in \{1, \ldots, 2s_{i-1}+1\}, \exists m \in \{1, \ldots, 2s_i+1\} \text{ such that } \mathbf{L}_{i-1}^\ell \equiv \mathbf{L}_i^m. \end{aligned}$$

First, $\mathcal{P}_0$ and $\mathcal{Q}_0$ are vacuously true. Now, let $i \in \{1, \ldots, n\}$ be some integer such that $\mathcal{P}_{i-1}$ and $\mathcal{Q}_{i-1}$ are true, and let us prove $\mathcal{P}_i$ and $\mathcal{Q}_i$.

If $s_{i-1} \leqslant s_i$, then it follows from $\mathcal{P}_i$ and Lemma 4.8 that

- $\mathbf{L}_i^j \sim \mathbf{L}_{i-1}^j \equiv \mathbf{L}_{i-1}^{2s_{i-1}+1-j} \sim \mathbf{L}_i^{2s_i+1-j}$ whenever $1 \leqslant j \leqslant s_{i-1}$;
- $\mathbf{L}_i^{s_i+1-j} \sim \mathbf{L}_i^{s_i+j}$ whenever $1 \leqslant j \leqslant s_i - s_{i-1}$;
- $\mathbf{L}_{i-1}^{2s_{i-1}+1} \sim \mathbf{L}_i^{2s_i+1}$,

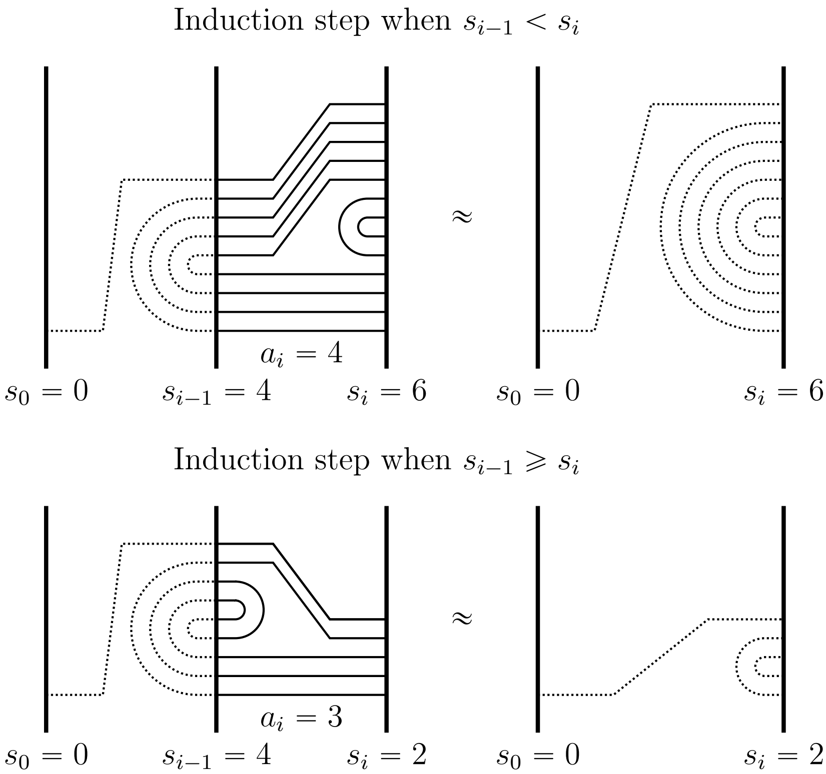


Figure 4.32 – Constructing actual coordinates

which proves $\mathcal{P}_i$ and $\mathcal{Q}_i$.

If $s_{i-1} > s_i$, then it follows from $\mathcal{P}_i$ and Lemma 4.8 that

- $\mathbf{L}_i^j \sim \mathbf{L}_{i-1}^j \equiv \mathbf{L}_{i-1}^{2s_{i-1}+1-j} \sim \mathbf{L}_i^{2s_i+1-j}$ whenever $1 \leqslant j \leqslant s_i - 1$;
- $\mathbf{L}_{i-1}^{2s_{i-1}+1} \sim \mathbf{L}_i^{2s_i+1}$,

which already proves $\mathcal{P}_i$ in the case $j \neq s_i$ and $\mathcal{Q}_i$ in the case $m \notin \{s_i, \ldots, 2s_{i-1} - s_i\}$.

Moreover, observe that $\mathbf{L}_{i-1}^j \equiv \mathbf{L}_{i-1}^{2s_{i-1}+1-j} \sim \mathbf{L}_{i-1}^{j+2}$ whenever $s_i \leqslant j \leqslant 2s_{i-1} - 1 - s_i$. An immediate induction on j then shows that $\mathbf{L}_{i-1}^{s_i} \equiv \mathbf{L}_{i-1}^j$ for all $j \in \{s_i, s_i + 2, \ldots, 2s_{i-1} - s_i\}$ and that $\mathbf{L}_{i-1}^{s_i+1} \equiv \mathbf{L}_{i-1}^j$ for all $j \in \{s_i + 1, s_i + 3, \ldots, 2s_{i-1} + 1 - s_i\}$. Therefore, it follows that $\mathbf{L}_i^{s_i} \sim \mathbf{L}_{i-1}^{s_i} \equiv \mathbf{L}_{i-1}^{2s_{i-1}+1-s_i} \equiv \mathbf{L}_{i-1}^{s_i+1} \sim \mathbf{L}_i^{s_i+1}$ and, incidentally, that $\mathbf{L}_{i-1}^{s_i} \equiv (\mathbf{L}_{i-1}^{s_i}$ and $\mathbf{L}_{i-1}^{s_i+1}) \equiv \mathbf{L}_{i-1}^j$ whenever $s_i \leqslant j \leqslant 2s_{i-1} - s_i$. These two remarks respectively complete the case $j = s_i$ of $\mathcal{P}_i$ and the case $s_i \leqslant m \leqslant 2s_{i-1} - s_i$ of $\mathcal{Q}_i$, which proves that both properties $\mathcal{P}_i$ and $\mathcal{Q}_i$ must hold.

Overall, we have proved that $\mathcal{Q}_i$ holds for all $i \in \{0, \ldots, n\}$, which proves, using an immediate induction on i, that $\mathbf{L}_i^j \equiv \mathbf{L}_n^1$ for all $i \in \{0, \ldots, n\}$ and for all $j \in \{1, \ldots, 2s_j + 1\}$. This means that $\equiv$ has one unique equivalence class, i.e. that $\mathcal{D}$ is a 1-generalised curve diagram, that is, that $\mathbf{sa}$ are actual coordinates. $\square$

Corollary 4.33.
Consider integers $n \geqslant 1$ and $k \geqslant 0$. Then, $g_{n,k} \geqslant \binom{k+n-2}{n-2}$.

This simple lower bound is not tight, as shown below.

Definition 4.34 (Composition of virtual coordinates and of generalised diagrams)**.**
Let $\mathbf{sa} = (s_0, a_1, s_1, \ldots, a_n, s_n)$ *and* $\mathbf{sa}' = (s'_0, a'_1, s'_1, \ldots, a'_k, s'_k)$ *be virtual coordinates. The* composition *of* $\mathbf{sa}$ *by* $\mathbf{sa}'$ *is defined as the tuple* $\mathbf{sa} \circ \mathbf{sa}' := (\sigma_0, \alpha_1, \sigma_1, \ldots, \alpha_{n+k-1}, \sigma_{n+k-1})$ *such that*

$$\begin{aligned}\sigma_i &= 0 && \text{if } i = 0 \\ &= s_1 + s'_i && \text{if } 1 \leqslant i \leqslant k-1 \\ &= s_{i+1-k} && \text{if } k \leqslant i\end{aligned} \qquad \text{and} \qquad \begin{aligned}\alpha_i &= a_1 \cdot \mathbf{1}_{s_1>0} + a'_1 \cdot \mathbf{1}_{s_1=0} && \text{if } i = 1 \\ &= s_1 + a_1 - a'_1 + a'_i && \text{if } 2 \leqslant i \leqslant k \\ &= a_{i+1-k} && \text{if } k+1 \leqslant i\end{aligned}$$

In particular, the tuple $\mathbf{sa} \circ \mathbf{sa}'$ also consists of virtual coordinates.

By extension, let $\mathcal{D}$ and $\mathcal{D}'$ be the generalised diagrams respectively associated with $\mathbf{sa}$ *and* $\mathbf{sa}'$*. The* composition *of $\mathcal{D}$ by $\mathcal{D}'$ is the generalised diagram $\mathcal{D} \circ \mathcal{D}'$ associated with the coordinates* $\mathbf{sa} \circ \mathbf{sa}'$*.*

Intuitively, composing $\mathcal{D}$ with $\mathcal{D}'$ consists in plugging the diagram $\mathcal{D}'$ inside the diagram $\mathcal{D}$ as follows. We remove the half-arc of $\mathcal{D}$ that goes to the puncture p_1 to the point $\mathbf{L}_1^{s_1+a_1-a'_1+1}$, and replace it by a copy of $\mathcal{D}'$, where the left endpoint of $\mathcal{D}'$ is glued on p_1 and the right endpoint of $\mathcal{D}'$. This process is illustrated by Fig. 4.35.

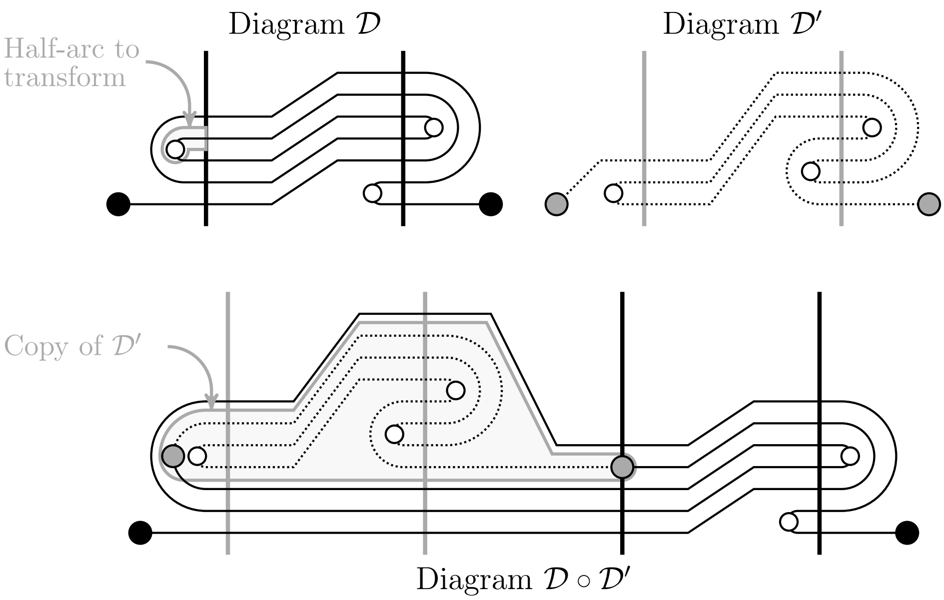


Figure 4.35 – Composing two generalised diagrams

Lemma 4.36 and Corollary 4.37 follow immediately from Definition 4.34 and from this intuition.

Lemma 4.36.
Let k and ℓ be positive integers, and let $\mathcal{D}$ and $\mathcal{D}'$ be respectively a k- and an ℓ-generalised diagrams. The composition $\mathcal{D} \circ \mathcal{D}'$ is a $(k+\ell-1)$-generalised diagram.

Corollary 4.37.
The composition of two (1-generalised) diagrams is a (1-generalised) diagram.

In addition, the composition is injective when restricted to a small class of diagrams.

Lemma 4.38.
Consider two positive integers m and n. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two (1-generalised) diagrams with n punctures, and let $\mathcal{D}'_1$ and $\mathcal{D}'_2$ be two generalised diagrams with m punctures. If $\mathcal{D}_1 \circ \mathcal{D}'_1 = \mathcal{D}_2 \circ \mathcal{D}'_2$, then $\mathcal{D}_1 = \mathcal{D}_2$ and $\mathcal{D}'_1 = \mathcal{D}'_2$.

Proof. Let $\mathbf{sa} = (s_0, a_1, \ldots, s_n)$, $\mathbf{sa}' = (s'_0, a'_1, \ldots, s'_m)$ and $\mathbf{sa} \circ \mathbf{sa}' = (\sigma_0, \alpha_1, \ldots, \sigma_{n+m-1})$ be the respective coordinates of $\mathcal{D}_1$, $\mathcal{D}'_1$ and $\mathcal{D}_1 \circ \mathcal{D}'_1$. We just need to show that $\mathbf{sa}$ and $\mathbf{sa}'$ depend only on m, n and $\mathbf{sa} \circ \mathbf{sa}'$.

First, note that $s_0 = s'_0 = s'_m = 0$, that $s_i = \sigma_{i+m-1}$ when $1 \leqslant i \leqslant n$, and that $s'_i = \sigma_i - \sigma_m$ when $1 \leqslant i \leqslant m-1$. Hence, one knows $(s_i)_{0 \leqslant i \leqslant n}$ and $(s'_i)_{0 \leqslant i \leqslant m}$. Then, observe that $a_1 = \alpha_1 \cdot \mathbf{1}_{s_1 > 0}$ and that $a_i = \alpha_{i+m-1}$ when $2 \leqslant i \leqslant n$.

Computing a'_1 is slightly more difficult. Let us denote by $\sim$ the relation of $(\mathcal{D}_1 \circ \mathcal{D}'_1, \mathbf{L})$-neighbourhood. In addition, let us consider the two statements

$$\begin{aligned}
\mathcal{S}_0 &= \forall i \in \{1, \ldots, m-1\}, \mathbf{L}_i^{s_1+a_1} \sim \mathbf{L}_{i+1}^{s_1+a_1} \\
\mathcal{S}_1 &= \forall i \in \{1, \ldots, m-1\}, \mathbf{L}_i^{s_1+2s'_i+a_1+1} \sim \mathbf{L}_{i+1}^{s_1+2s'_{i+1}+a_1+1}
\end{aligned}$$

If $s'_1 = 0$, then $a'_1 = 0$. However, if $s'_1 > 0$, then we know that $\mathcal{D}'_1$ is not trivial. If $s'_1 > 0$, and $a'_1 = 0$, then $\mathcal{S}_0$ holds but, since $\mathcal{D}'_1$ is not trivial and was plugged between the points $\mathbf{L}_1^{s_1+2s'_1+a_1+1}$ and $\mathbf{L}_m^{s_1+a_1+1}$, $\mathcal{S}_1$ does not hold. Conversely, if $s'_1 > 0$, and $a'_1 = 1$, then $\mathcal{S}_1$ holds but, since $\mathcal{D}'_1$ is not trivial and was plugged between the points $\mathbf{L}_1^{s_1+a_1}$ and $\mathbf{L}_m^{s_1+a_1}$, $\mathcal{S}_0$ does not hold. Hence, one can compute a'_1. Finally, since $a'_i = \alpha_i + a'_1 - s_1 - a_1$ when $2 \leqslant i \leqslant m$, Lemma 4.38 follows. □

Theorem 4.39.
Let k and n be positive integers. There exists two (computable) constants $\kappa_n \geqslant 0$ and $\gamma_n > 0$ such that, if $k \geqslant \kappa_n$, then $g_{k,n} \geqslant \gamma_n k^{\lceil 3n/2 \rceil}$.

Proof. This proof relies on tools of analytic number theory, similar to those used for proving Proposition 4.26. However, like the proof of Proposition 4.28, this proof is very technical, and we therefore begin by drawing a sketch of this proof.

First, we introduce univariate and bivariate generating functions as well as some term-wise comparison ordering of such functions. Second, we focus on some bivariate generating function $G_3(y, z)$ and approximate it by below with the help of a simple generating function $\overline{F}(y,z)$. Third, we approximate by below the terms of the functions $\overline{F}(y,y^k)$, for $k \geqslant 0$, in terms of the Euler totient φ. Fourth, we build a family of integers $\gamma_{a,b,c}$, thanks to which we find additional lower approximations of the functions $\overline{F}(y,y^k)$ and $G_3(y,y^k)$, and we find lower bounds for the integers $\gamma_{a,b,c}$. Finally, we derive Theorem 4.39 itself.

Proof of Theorem 4.39 – First Part

In what follows, we will use the "term-wise" comparison ordering $\succcurlyeq$ defined as follows. If $C(x) = \sum_{u \geqslant 0} c_u x^u$ and $D(x) = \sum_{u \geqslant 0} d_u x^u$ are two univariate generating series such that $c_u \geqslant d_u$ for all $u \geqslant 0$, then $C(x) \succcurlyeq D(x)$. Similarly, if $C(y,z) = \sum_{u,v \geqslant 0} c_{u,v} y^u z^v$ and $D(y,z) = \sum_{u,v \geqslant 0} d_{u,v} y^u z^v$ are two bivariate generating series such that $c_{u,v} \geqslant d_{u,v}$ for all $u, v \geqslant 0$, we write $C(y,z) \succcurlyeq D(y,z)$.

For all integers $s \geqslant 0$, $k \geqslant 0$ and $n \geqslant 1$, let $g_{n,k,s}$ be the number of (1-generalised) coordinates $(s_0, a_1, s_1, \ldots, a_n, s_n)$ such that $\sum_{i=0}^n s_i = k$ and $s_1 = s$. In addition, consider the generating functions $G_n(y,z) = \sum_{k,s \geqslant 0} g_{n,k,s} y^k z^s$. Corollary 4.37 proves that

$$g_{n+m-1,k,s} \geqslant \sum_{a,t \geqslant 0} g_{n,k-a-(m-1)t,t} g_{m,a,s-t},$$

i.e. that $G_{n+m-1}(y,z) \succcurlyeq G_n(y, y^{m-1}z) G_m(y,z)$. Then, let n be some integer such that $n \geqslant 3$, and let $n^* := 1 + \mathbf{1}_{n \in 2\mathbb{Z}}$. By construction, we have $\mathcal{G}_n(y) = G_n(y,1)$, and an immediate induction shows that

$$\mathcal{G}_n(y) \succcurlyeq G_3(y, y^{n-3}) G_3(y, y^{n-5}) \ldots G_3(y, y^{n^*-1}) \mathcal{G}_{n^*}(y).$$

Proof of Theorem 4.39 – Second Part

In Section 4.2, we computed the integers

$$g_{3,k,k+\ell} = C_{k,\ell} = |\{(a_1,a_2,a_3) : (0, a_1, k, a_2, \ell, a_3, 0) \text{ are actual coordinates}\}|.$$

We have $g_{k,2k} = 2(2k+1)$ if $k \geqslant 1$, $g_{3,k,2k+m} \geqslant \sum_{a=1}^k \mathbf{1}_{a \wedge (m+1) = 1}$ if $k, m \geqslant 1$, and $g_{3,k,\ell} \geqslant 0$ if $k = 0$ or $\ell < 2k$. It follows that $G_3(y,z) = \sum_{k,s \geqslant 0} g_{3,k,s} y^k z^s \succcurlyeq 2A(yz^2) + B(yz^2, z)$, where

$$A(y) = \frac{y(3-y)}{(1-y)^2} \text{ and } B(y,z) = \sum_{k \geqslant 1} \sum_{m \geqslant 1} \sum_{a=1}^{k} \mathbf{1}_{a \wedge (m+1) = 1} y^k z^m.$$

Using the auxiliary function $\overline{F}(y,z) := \sum_{\alpha \geqslant 1} \sum_{\beta \geqslant 1} \mathbf{1}_{\alpha \wedge \beta = 1} y^\alpha z^\beta$ and the substitution $t := k - a$, we obtain

$$\begin{aligned} B(y,z) &= \sum_{a \geqslant 1} \sum_{m \geqslant 1} \sum_{t \geqslant 0} \mathbf{1}_{a \wedge (m+1) = 1} y^{a+t} z^m = \frac{1}{1-y} \sum_{a \geqslant 1} \sum_{m \geqslant 1} \mathbf{1}_{a \wedge (m+1) = 1} y^a z^m \\ &= \frac{1}{z(1-y)} \left(\sum_{a \geqslant 1} \sum_{v \geqslant 1} \mathbf{1}_{a \wedge v = 1} y^a z^v - \sum_{a \geqslant 1} y^a z \right) = \frac{\overline{F}(y,z)}{z(1-y)} - \frac{y}{(1-y)^2}. \end{aligned}$$

Collecting the above terms, we have

$$2A(y)+B(y,z) \geqslant \frac{2y}{1-y}+\frac{3y}{(1-y)^2}+\frac{\overline{F}(y,z)}{z(1-y)} \geqslant \frac{\overline{F}(y,z)}{z(1-y)} \text{ and } G_3(y,y^n) \geqslant \frac{\overline{F}(y^{2n+1},y^n)}{y^n(1-y^{2n+1})}.$$

Proof of Theorem 4.39 – Third Part

For all $n \geqslant 0$, we have $\overline{F}(y^{2n+1},y^n) = \sum_{u\geqslant 1}\sum_{v\geqslant 1}\mathbf{1}_{u\wedge v=1}y^{u(2n+1)+vn} = \sum_{k\geqslant 0}\overline{f}_{n,k}y^k$, with

$$\overline{f}_{n,k} = \left|\left\{u : 0 < (2n+1)u < n, (2n+1)u \equiv k\ (n), u \wedge \frac{k-(2n+1)u}{n} = 1\right\}\right|.$$

Let us use the change of variable $w := \frac{k-u}{n}$. We obtain easily that $0 < (2n+1)u < k \Leftrightarrow \frac{2k}{2n+1} < w < \frac{k}{n}$, that $(2n+1)u \equiv k \pmod n \Leftrightarrow u \equiv k \pmod n \Leftrightarrow w \in \mathbb{Z}$ and that

$$\begin{aligned} u \wedge \frac{k-(2n+1)u}{k} = 1 &\Leftrightarrow nu \wedge (k-(2n+1)u) = n \Leftrightarrow nu \wedge (k-u) = n \\ &\Leftrightarrow n(k-nw) \wedge nw = n \Leftrightarrow (k-nw) \wedge w = 1 \Leftrightarrow k \wedge w = 1. \end{aligned}$$

This shows that $\overline{f}_{n,k} = \left|\left\{w : \frac{2k}{2n+1} < w < \frac{k}{n}, k \wedge w = 1\right\}\right|$.

We introduce now some arithmetic notions. We say that n is *square-free* if n is the prime factors of n are pairwise distinct, i.e. if n is not divisible by any integer of the form k^2. We denote by $\mathcal{D}_p(n)$ the set of prime factors of n, and by $\mathcal{D}_\mu(n)$ the set of square-free factors of n. We also denote by $p(n)$ the cardinality of the set $\Pi(n)$, and by $\mu(n)$ be the Möbius function of n, i.e. $\mu(n) = (-1)^{p(n)}$ if n is square-free and $\mu(n) = 0$ otherwise.

For all integers $m, n \geqslant 1$, let us define $\psi(m,k) := |\{w : 1 \leqslant w \leqslant m, k \wedge w = 1\}|$. For all integers $w \geqslant 1$, it comes easily that $\mathbf{1}_{w\wedge k=1} = \sum_{i\in\mathcal{D}_\mu(k)}\mu(i)\mathbf{1}_{w\in i\mathbb{Z}}$. This proves that $\psi(m,n) = \sum_{i\in\mathcal{D}_\mu(k)}\mu(i)\lfloor\frac{m}{i}\rfloor$. Moreover, recall that $\sum_{i\in\mathcal{D}_\mu(k)}\mu(i)\frac{1}{i} = \prod_{q\in\Pi(k)}(1-q^{-1}) = \frac{\varphi(k)}{k}$. Consequently, we have

$$\left|\psi(m,k) - \frac{\varphi(k)}{k}m\right| = \left|\psi(m,k) - \sum_{i\in\mathcal{D}_\mu(k)}\mu(i)\frac{m}{i}\right| < |\mathcal{D}_\mu(k)| \leqslant 2^{p(k)}.$$

Therefore, the equality $\overline{f}_{n,k} = \psi\left(\frac{k}{n},k\right) - \psi\left(\frac{2k}{2n+1},k\right) - \mathbf{1}_{k=n}$ shows that $\left|\overline{f}_{n,k} - \frac{\varphi(k)}{n(2n+1)}\right| \leqslant 2^{p(k)+1}$.

Now, consider the integer $N := (4^4n(2n+1)+1)!$, and let us assume that $k \geqslant N$. Let $q_1 < \ldots < q_{p(k)}$ be the prime factors of k. If $p(k) \leqslant 2$, then $k \geqslant 4^3n(2n+1) = n(2n+1)4^{p(k)+1}$. If $p(k) \geqslant 3$, consider the integers $k_1 := \prod_{i=1}^{p(k)} q_i$ and $k_2 = \frac{k}{k_1}$. Since $4 \leqslant q_3 < \ldots < q_{p(k)}$, it follows that

$$k = k_1k_2 \geqslant k_2q_{p(k)}\prod_{i=3}^{p(k)-1} q_i \geqslant k_2q_{p(k)}4^{p(k)-3} \geqslant 4^{p(k)-3}\max\{q_{p(k)},k_2\}.$$

Since $q_{p(k)}!k_2 \geqslant k_1 k_2 = k \geqslant (4^4 n(2n+1)+1)!$, it follows that $\max\{q_{p(k)}, k_2\} \geqslant 4^4 n(2n+1)$, and therefore that $k \geqslant n(2n+1)4^{p(k)+1}$.

Consequently, regardless of the value of $p(k)$, we have $k \geqslant n(2n+1)4^{p(k)+1}$. This proves that $\varphi(k) = k \prod_{q \in \Pi(k)} \frac{q-1}{q} \geqslant 2^{-p(k)} k \geqslant n(2n+1)2^{p(k)+2}$, and therefore that $\overline{f}_{n,k} \geqslant \frac{\varphi(k)}{2n(2n+1)}$.

Proof of Theorem 4.39 – Fourth Part

Consider the terms $\overline{g}_{n,k}$ of the generating function $G_3(y, y^n)$, i.e. $G_3(y, y^n) = \sum_{k \geqslant 0} \overline{g}_{n,k} y^k$. In addition, for all integers a, b, c, consider the integer

$$\gamma_{a,b,c} := |\{(u,v) : 1 \leqslant u, v \leqslant a, u \wedge c(a+bv) = 1\}|.$$

Finally, let ℓ be the largest integer such that $\ell \equiv k \pmod{2n+1}$ and $(2n+2)\ell \leqslant k$. If $\ell \geqslant N$, then the inequality $G_3(y, y^n) \geqslant \frac{\overline{F}(y^{2n+1}, y^n)}{y^n(1-y^{2n+1})}$ shows that

$$\begin{aligned}
\overline{g}_{n,n+k} &\geqslant \sum_{i=0}^{\lfloor k/(2n+1) \rfloor} \overline{f}_{n,k-(2n+1)i} \geqslant \sum_{i=1}^{\ell} \overline{f}_{n,\ell+(2n+1)i} \\
&\geqslant \frac{1}{2n(2n+1)} \sum_{i=1}^{\ell} \varphi(\ell + (2n+1)i), \text{ i.e.} \\
2n(2n+1)\overline{g}_{n,n+k} &\geqslant |\{(u,v) : 1 \leqslant u \leqslant \ell + (2n+1)v, 1 \leqslant v \leqslant \ell, \\
&\qquad u \wedge (\ell + (2n+1)v) = 1\}| \geqslant \gamma_{\ell,2n+1,1}.
\end{aligned}$$

We evaluate now the integers $\gamma_{a,b,c}$, and first focus on the case $a \wedge b = 1$. Let $Q := \lceil 3c\zeta(2) \rceil$ and let $P := \prod_{q \leqslant Q} \mathbf{1}_{q \text{ prime}} \mathbf{1}_{bc \notin q\mathbb{Z}}$. In addition, let us assume that $a \geqslant 9c\zeta(2)2^{p(P)+p(c)}$.

For all integers $u, v \geqslant 1$, and since $P \wedge c = 1$, it comes easily that

$$\mathbf{1}_{u \wedge c = 1} \mathbf{1}_{u \wedge (a+bv) \wedge P = 1} = \sum_{i \in \mathcal{D}_\mu(P)} \sum_{j \in \mathcal{D}_\mu(c)} (-1)^{\mu(i)+\mu(j)} \mathbf{1}_{u \in ij\mathbb{Z}} \mathbf{1}_{a+bv \in i\mathbb{Z}}.$$

Recall that $\sum_{u=1}^{a} \mathbf{1}_{u \in ij\mathbb{Z}} = \lfloor \frac{a}{ij} \rfloor$ and that $\lfloor \frac{a}{i} \rfloor \leqslant \sum_{v=1}^{a} \mathbf{1}_{a+bv \in i\mathbb{Z}} \leqslant \lfloor \frac{a}{i} \rfloor + 1$ whenever $i \wedge b = 1$.

Moreover, since $a \wedge b = 1$, all prime divisors of $u \wedge (a+bv)$ are either divisors of Pc

or are greater than Q. It follows that

$$\begin{aligned}
\gamma_{a,b,c} &\geqslant \sum_{u,v=1}^{a} \mathbf{1}_{u\wedge c=1}\mathbf{1}_{u\wedge(a+bv)\wedge P=1} - \sum_{u,v=1}^{a}\sum_{q>Q}\mathbf{1}_{q \text{ prime}}\mathbf{1}_{u\in q\mathbb{Z}}\mathbf{1}_{a+bv\in q\mathbb{Z}} \\
&\geqslant \sum_{i\in\mathcal{D}_\mu(P)}\sum_{j\in\mathcal{D}_\mu(c)} \left((-1)^{\mu(i)+\mu(j)}\frac{a}{ij}\frac{a}{i} - \frac{a}{ij} - \frac{a}{i} - 1\right) - \sum_{q>Q}\frac{a^2}{q^2} \\
&\geqslant a^2\left(\prod_{q\in\Pi(P)} 1-q^{-2}\right)\left(\prod_{q\in\Pi(c)} 1-q^{-1}\right) - 3\cdot 2^{p(P)+p(c)}a - \frac{a^2}{Q} \\
&\geqslant \left(\frac{\varphi(c)}{c\zeta(2)} - \frac{1}{Q}\right)a^2 - 3\cdot 2^{p(P)+p(c)}a \\
&\geqslant \left(\frac{1}{c\zeta(2)} - \frac{1}{3c\zeta(2)}\right)a^2 - \frac{a^2}{3c\zeta(2)} = \frac{a^2}{3c\zeta(2)}.
\end{aligned}$$

Now, let $d := \ell \wedge (2n+1)$. It comes immediately that

$$2n(2n+1)\overline{g}_{n,n+k} \geqslant \gamma_{\ell,2n+1,1} \geqslant \gamma_{\ell/d,(2n+1)/d,d} \geqslant \frac{\ell^2}{3d\zeta(2)} \geqslant \frac{\ell^2}{3(2n+1)\zeta(2)}$$

if ℓ is big enough, for instance if $\ell \geqslant 18c^2 2^{5(2n+1)}$. Moreover, since $\ell \geqslant \frac{k}{2n+2} - (2n+1)$, we know that if $n+k \geqslant 9(n+1)^3$, then $\ell \geqslant \frac{n+k}{2n+3}$. This proves that there exists an integer $\Lambda(n)$ such that, if $k \geqslant \Lambda(n)$, then $\overline{g}_{n,k} \geqslant \lambda_n k^2$, where $\lambda_n := \frac{1}{6n(2n+1)^2(2n+3)^2\zeta(2)}$.

Proof of Theorem 4.39 – Fifth Part

Recall that $\mathcal{G}_1(y) = 1$ and that $\mathcal{G}_3(y) \geqslant \sum_{k\geqslant 0} y^k$, as proved in Section 4.2. Now, let u be a positive integer, and let $\epsilon \in \{1,2\}$. We assume henceforth that $n = 2u+\epsilon$. We have $n^* = \epsilon$ and $u = \lfloor (n-1)/2 \rfloor$. First, observe that

$$\mathcal{G}_n(y) \geqslant G_3(y,y^{n-3})G_3(y,y^{n-5})\ldots G_3(y,y^{n^\epsilon-1})\mathcal{G}_\epsilon(y) \geqslant \lambda_{n-1}^u \left(\sum_{k\geqslant\Lambda(n-1)} k^2y^k\right)^u \mathcal{G}_\epsilon(y).$$

Let us also assume that $k \geqslant 2u\Lambda(n-1)$, and let $m := \lfloor k/(2u) \rfloor$. If $\epsilon = 1$, then

$$\begin{aligned}
g_{n,k} &\geqslant \lambda_{n-1}^u \sum_{k_1,\ldots,k_{u-1}=m}^{2m-1}\sum_{k_u\geqslant m} k_1^2\ldots k_u^2\mathbf{1}_{k_1+\ldots+k_u=k} \\
&\geqslant \lambda_{n-1}^u \sum_{k_1,\ldots,k_{u-1}=m}^{2m-1}\sum_{k_u\geqslant m} m^{2u}\mathbf{1}_{k_1+\ldots+k_u=k} = \lambda_{n-1}^u m^{3u-1}.
\end{aligned}$$

If $\epsilon = 2$, then

$$\begin{aligned} g_{n,k} &\geqslant \lambda_{n-1}^{u} \sum_{k_1,\ldots,k_u=m}^{2m-1} k_1^2 \ldots k_u^2 \mathbf{1}_{k_1+\ldots+k_u \leqslant k} \\ &\geqslant \lambda_{n-1}^{u} \sum_{k_1,\ldots,k_u=m}^{2m-1} m^{2u} \mathbf{1}_{k_1+\ldots+k_u \leqslant k} = \lambda_{n-1}^{u} m^{3u}. \end{aligned}$$

Adding these two cases, we obtain Theorem 4.39. □

4.4 Experimental Data, Conjectures and Open Questions

Proposition 4.30 and Corollary 4.33 prove that

$$\binom{k+n-2}{n-2} \leqslant g_{n,k} \leqslant 2^n \left(\frac{k+n-1}{n-1}\right)^{n-2} \binom{k+n-2}{n-2}.$$

Unfortunately, these lower and upper bounds do not match, since their ratio is equal to $2^n \left(\frac{k+n-1}{n-1}\right)^{n-2}$, hence grows arbitrarily when n and k grow. Theorem 4.39 also provides us with the following, tighter inequality when k is large enough:

$$\gamma_n k^{\lfloor 3n/2 \rfloor} \leqslant g_{n,k} \leqslant 2^n \left(\frac{k+n-1}{n-1}\right)^{n-2} \binom{k+n-2}{n-2}.$$

Nevertheless, the ratio between these upper and lower bound also grows arbitrarily when k grows.

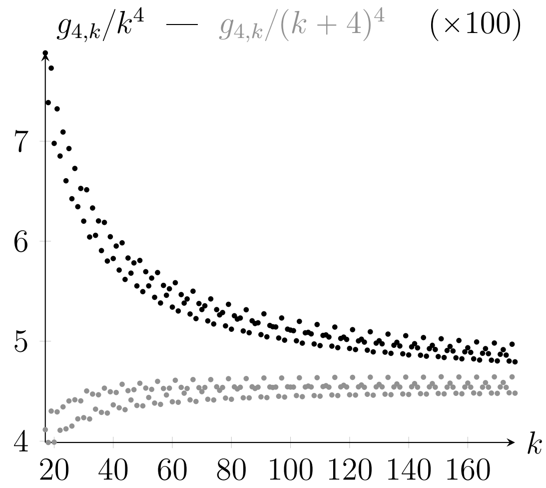
Therefore, aiming to identify simple asymptotic estimations of $g_{n,k}$ when n is fixed and $k \to +\infty$, we look for experimental data.

Figure 4.41 presents the ratios $g_{n,k}/k^{2(n-2)}$ (in black) and $g_{n,k}/(k+n)^{2(n-2)}$ (in gray). We computed $g_{n,k}$ by enumerating all the virtual coordinates, then checking individually which of them were actual coordinates (up to refinements such as using the above-mentioned symmetries to reduce the number of cases to look at).

The two series of points suggest the following conjecture, which was already proven to be true when $n = 2$ and $n = 3$.

Conjecture 4.40.
Let $n \geqslant 2$ be some integer. There exists two positive constants α_n and β_n such that $\alpha_n k^{2(n-2)} \leqslant g_{n,k} \leqslant \beta_n k^{2(n-2)}$ for all integers $k \geqslant 1$.

Figure 4.41 also suggests that the ratios $g_{n,k}/k^{2(n-2)}$ might be split into convergent clusters, according to the value of k mod 6 (when $n = 4$) or k mod 2 (when $n = 5$). Once again, this is coherent with the patterns noticed for $n = 2$ and $n = 3$, and therefore suggests a stronger conjecture.



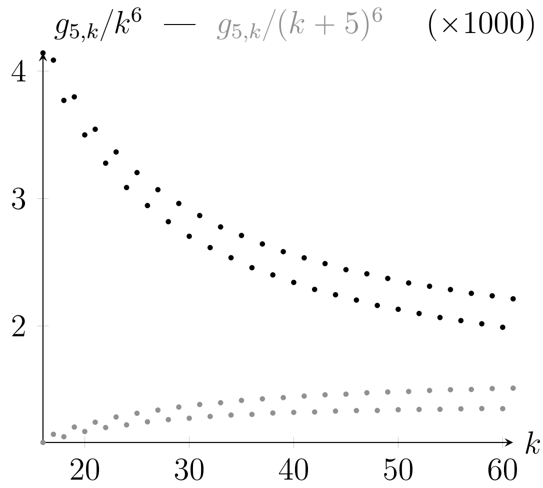


Figure 4.41 – Estimating $g_{n,k}$ — experimental data for $n = 4$ and $n = 5$

Conjecture 4.42.
Let $n \geqslant 2$ be some integer. There exists some positive integer ρ_n such that, for every integer $\ell \in \{0, 1, \ldots, \rho_n - 1\}$, the sequence of ratios $\frac{g_{n,k\rho_n+\ell}}{k^{2(n-2)}}$ has a positive limit $\lambda_{n,\ell}$ when $k \to +\infty$.

Assuming Conjecture 4.42, a natural further step would be to compute the limits $\lambda_{n,\ell}$ or to study more precisely the asymptotic behaviour of the ratios $g_{n,k}/k^{2(n-2)}$. In particular, we hope that computing arbitrarily precise approximations of the constants $\lambda_{n,\ell}$ for small values of n might help us guess analytic values of $\lambda_{n,\ell}$, thereby providing insight about the underlying combinatorial or number-theory-related structure of the integers $g_{n,k}$.

Chapter 5

Random Walks in Braid Groups Converge

Résumé

Considérons un groupe finiment engendré, associé à une notion de forme normale, et observons une marche aléatoire sur le groupe. La forme normale des éléments aléatoires obtenus lors de cette marche converge-t-elle ? Vershik et Malyutin apportent une réponse positive pour les groupes de tresses et la forme normale de Markov-Ivanovsky [90], et ils observent que : « Pour la forme normale de Garside (...) le problème de la convergence reste ouvert ». Nous répondons à cette question par l'affirmative pour les groupes d'Artin–Tits irréductibles de type sphérique et pour la forme normale de Garside. Nous étudions également la limite de la marche aléatoire, et montrons qu'elle est ergodique.

Le contenu de ce chapitre provient d'un travail en cours de rédaction, en collaboration avec Jean Mairesse.

Abstract

Consider a finitely generated group with an associated notion of normal form, and consider a random walk on the group. Does the normal form of the random elements converge? Vershik and Malyutin provide a positive answer for the braid groups with the Markov-Ivanovsky normal form [90], and they observe that: "For the Garside normal form (...) the stability problem is open". We answer the question by the affirmative for an Artin–Tits group of spherical type with the Garside normal form. We also study the limit of the random walks and show that it is ergodic.

The content of this chapter is the result of a paper in progress, written in collaboration with Jean Mairesse.

Chapter 5 is devoted to proving that the Garside normal forms of the random walk in irreducible Artin–Tits groups of spherical type is *stable*. In the early 2000s, Vershik asked the question of the stabilisation of normal forms in braid groups [89]. Since then, a first complete (positive) answer was given for the Markov-Ivanovsky normal form [90], while another partial (positive) answer was given for the Garside normal form in the Artin–Tits groups of dihedral type, including the group of braids with three strands [75]. We provide a complete (positive) answer for the Garside normal form in all irreducible Artin–Tits groups of spherical type.

In Section 5.1, we first revisit the analogous and easier case of random walks in heap monoids and heap groups. With the help of some tools that we will later reuse in the context of braids, we reprove results that are already mentioned in the literature [54, 77]. Then, in Sections 5.2, 5.3 and 5.5, we develop original tools that lead both to proving stabilisation results for the Garside normal form and to studying properties of the limit of the random walk.

5.1 Random Walk in Heap Monoids and Groups

In Chapter 5, we aim to study the convergence of a random walk in an Artin–Tits monoid of spherical type and in the associated group. However, we begin with studying the convergence of such random walks in heap monoids and groups. This study may be seen as a warm-up, and shows how to deal with a *natural* notion of convergence of random walks in a simple case. In particular, the results mentioned in Section 5.1.1 are already well-known (see [54, 77]) but we reprove them using tools that we will use later in the yet unexplored framework of braid monoids and groups.

5.1.1 Random Walk in Heap Monoids

In all Section 5.1.1, we will consider an irreducible heap monoid $\mathcal{M}^+$ with $n \geqslant 2$ generators $\sigma_1, \ldots, \sigma_n$ and dependency relation D. Let μ be a probability measure over $\mathcal{M}^+$ whose support contains the set $\{\sigma_1, \ldots, \sigma_n\}$, i.e. such that the real constant $\min \mu := \min_{1 \leqslant i \leqslant n} \mu(\sigma_i)$ is positive.

Definition 5.1 (Left random walk).
Let $(Y_k)_{k \geqslant 0}$ be i.i.d. random variables distributed with law μ. The left random walk *with* step-distribution *μ is the sequence $\mathbf{X} = (X_k)_{k \geqslant 0}$ defined by $X_0 := \mathbf{1}$ and $X_{k+1} := Y_k X_k$ for $k \geqslant 0$.*

When the probability measure μ is implicit from the context, the sequence $(X_k)_{k \geqslant 0}$ may also be called simply *left random walk*.

Does the random walk $(X_k)_{k \geqslant 0}$ converge? More precisely, let us identify each heap X_k with its left and right Garside normal forms $\mathbf{NF}_\ell(X_k)$ and $\mathbf{NF}_r(X_k)$, and let us consider some notion of convergence on words. Recall that the left Garside normal form $\mathbf{NF}_\ell$ is better known as the *Cartier-Foata* normal form **CF**. Notions of convergence for words include:

- the prefix-convergence, in which the distance between two words $\underline{\mathbf{a}} := a_1 \cdot \ldots \cdot a_k$ and $\underline{\mathbf{b}} := b_1 \cdot \ldots \cdot b_\ell$ such that $k \leqslant \ell$ is defined as $d_{\mathbf{pre}}(\underline{\mathbf{a}}, \underline{\mathbf{b}}) := 0$ if $\underline{\mathbf{a}} = \underline{\mathbf{b}}$, 2^{-k-1} if $\underline{\mathbf{a}}$ is a proper prefix of $\underline{\mathbf{b}}$, and $2^{-\min\{i : a_i \neq b_i\}}$ otherwise;
- the suffix-convergence, in which the distance between two words $\underline{\mathbf{a}} := a_{-k} \cdot \ldots \cdot a_{-1}$ and $\underline{\mathbf{b}} := b_{-\ell} \cdot \ldots \cdot b_{-1}$ such that $k \leqslant \ell$ is defined as $d_{\mathbf{suf}}(\underline{\mathbf{a}}, \underline{\mathbf{b}}) := 0$ if $\underline{\mathbf{a}} = \underline{\mathbf{b}}$, 2^{-k-1} if $\underline{\mathbf{a}}$ is a proper suffix of $\underline{\mathbf{b}}$, and $2^{-\min\{i : a_{-i} \neq b_{-i}\}}$ otherwise.

Some convergence and divergence results come quickly.

Proposition 5.2.
Let $\mathcal{M}^+$ be an irreducible heap monoid with at least 2 generators, and let $(X_k)_{k \geqslant 0}$ be the left random walk on $\mathcal{M}^+$. For all integers $i \geqslant 1$, the sequence $\mathbf{suf}_i(\mathbf{NF}_r(X_k))_{k \geqslant 0}$ is necessarily *convergent, and the sequences $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))_{k \geqslant 0}$ and $\mathbf{pre}_i(\mathbf{NF}_r(X_k))_{k \geqslant 0}$ are almost surely divergent.*

Proof. First, the sequence $\mathbf{suf}_i(\mathbf{NF}_r(X_k))_{k \geqslant 0}$ is non-decreasing (for the order $\geqslant_{r,\mathrm{rev-lex}}$) and belongs to a finite set, hence it *must* converge. Second, consider some even integer $k \geqslant 1$. The event $\mathcal{E}_k := \{Y_k \leqslant_\ell \mathbf{pre}_1(\mathbf{NF}_r(X_k))\} \cap \{(Y_k, Y_{k+1}) \in D\}$ occurs with probability at least $(\min \mu)^2$, regardless of the values of $(Y_i)_{i \notin \{k,k+1\}}$. Consequently, with probability one, the events $\mathcal{E}_k$ hold for infinitely many integers $k \in \{2, 4, 6, \ldots\}$.

When $\mathcal{E}_k$ occurs, then $\mathbf{NF}_r(X_{k+2}) = (Y_{k+1} Y_k) \cdot \mathbf{NF}_r(X_k)$. This proves that $Y_{k+1} \leqslant_\ell \mathbf{pre}_1(\mathbf{NF}_r(X_{k+2})) \leqslant_\ell \mathbf{pre}_1(\mathbf{NF}_\ell(X_{k+2}))$. However, since Y_k is a left-divisor of both simple braids $\mathbf{pre}_1(\mathbf{NF}_r(X_k))$ and $\mathbf{pre}_1(\mathbf{NF}_\ell(X_k))$ and since $(Y_k, Y_{k+1}) \in D$, it follows that Y_{k+1} cannot left-divide $\mathbf{pre}_1(\mathbf{NF}_r(X_k))$ nor $\mathbf{pre}_1(\mathbf{NF}_\ell(X_k))$, whence $\mathbf{pre}_1(\mathbf{NF}_r(X_k)) \neq \mathbf{pre}_1(\mathbf{NF}_r(X_{k+2}))$ and $\mathbf{pre}_1(\mathbf{NF}_\ell(X_k)) \neq \mathbf{pre}_1(\mathbf{NF}_\ell(X_{k+2}))$. This completes the proof. □

Furthermore, it follows from Lemma 2.37, Proposition 2.49 and Corollary 2.51 that the word $\mathbf{pre}_i(\mathbf{NF}_\ell(X_{k+1}))$ is a function of $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))$ and of Y_{k+1}. This shows that the sequence $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))$ is a Markov chain. However, the sequences $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k))_{k \geqslant 0}$ and $\mathbf{pre}_i(\mathbf{NF}_r(X_k))_{k \geqslant 0}$ are not necessarily monotonous (even for $i = 1$) and are not Markov chains, hence their behaviour is not so easy to capture.

Example 5.3.
In the dimer model $\mathcal{M}_3^+$, assume that the five first terms of $(Y_k)_{k \geqslant 0}$ are σ_1, σ_3, σ_3, σ_2 and σ_1, in this order. A direct computation shows that $\mathbf{NF}_\ell(X_4) = \sigma_2 \cdot \sigma_1\sigma_3 \cdot \sigma_3$, $\mathbf{NF}_r(X_4) = \sigma_2 \cdot \sigma_3 \cdot \sigma_1\sigma_3$, $\mathbf{NF}_\ell(X_5) = \sigma_1 \cdot \sigma_2 \cdot \sigma_1\sigma_3 \cdot \sigma_3$, and $\mathbf{NF}_r(X_5) = \sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_1\sigma_3$. Hence, neither $(\mathbf{suf}_1(\mathbf{NF}_\ell(X_k)))_{k \geqslant 0}$ nor $(\mathbf{pre}_1(\mathbf{NF}_r(X_k)))_{k \geqslant 0}$ is non-decreasing. In addition, neither are them finite-state Markov chains.

The situation is illustrated in Fig. 5.4. As usual, heaps, which we read from left to right, are represented by heap diagrams, which we read from bottom to top. The Garside words

$\mathbf{NF}_\ell(X_k)$ and $\mathbf{NF}_r(X_k)$ are represented by heap diagrams cut into several layers, each layer representing one heap letter of the word $\mathbf{NF}_\ell(X_k)$ or $\mathbf{NF}_r(X_k)$. Hence, the heap $\mathbf{suf}_1(\mathbf{NF}_\ell(X_k))$ is represented by the top layer of the left Garside normal form of X_k, and the braid $\mathbf{pre}_1(\mathbf{NF}_r(X_k))$ is represented by the bottom layer of the right Garside normal form of X_k. Both these layers are represented over a gray background.

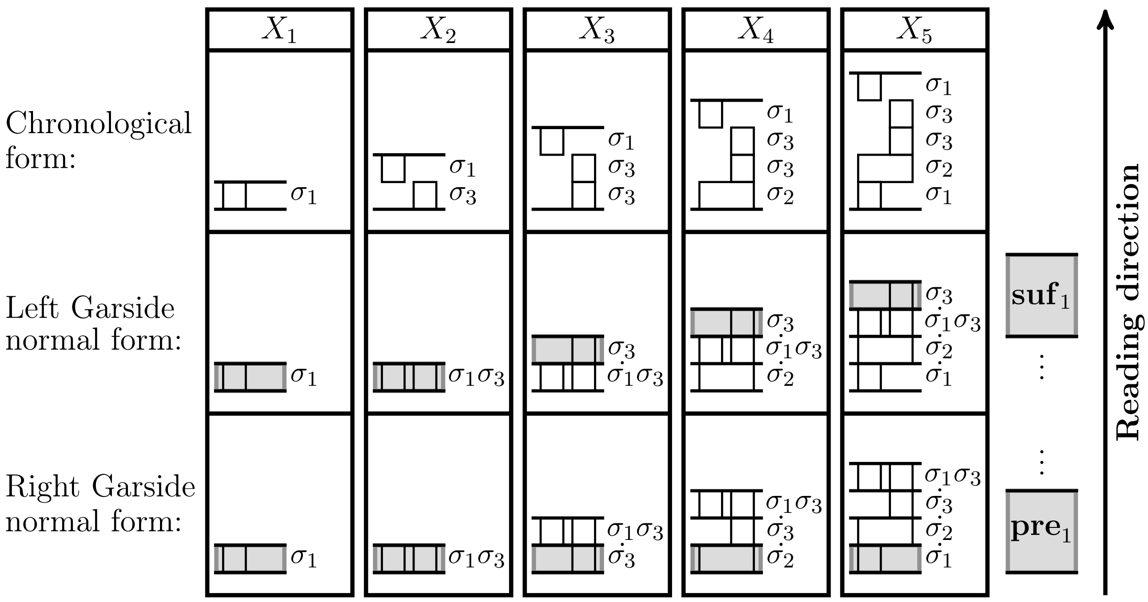


Figure 5.4 – Non-monotonic evolution of the Garside normal forms

We prove below that, for all integers $i \geqslant 0$, the sequence $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k))_{k \geqslant 0}$ is almost surely ultimately constant, i.e.

$$\mathbb{P}[\exists k \geqslant 0, \forall m \geqslant k, \mathbf{suf}_i(\mathbf{NF}_\ell(X_k)) = \mathbf{suf}_i(\mathbf{NF}_\ell(X_m))] = 1.$$

Definition 5.5 (Blocking permutation and blocking heap).
Let τ be a permutation of the set $\{\sigma_1, \ldots, \sigma_n\}$. We say that τ is a blocking *permutation if, for all $i \in \{1, \ldots, n-1\}$, there exists an integer $j \in \{i+1, \ldots, n\}$ such that $(\tau(i), \tau(j)) \in D$, i.e. such that $\tau(i) \cdot \tau(j)$ is a left (or right) Garside word. We say that the heap $\underline{\mathbf{t}} := \tau(1) \ldots \tau(n)$ is the* blocking heap *associated with the permutation τ.*

Observe that, since $\mathcal{M}^+$ is a connected heap monoid, there exists such blocking permutations. For instance, if, by performing some depth-first exploration of the Coxeter graph of $\mathcal{M}^+$, we denote by $\tau(i)$ the $(n+1-i)$-th generator of $\mathcal{M}^+$ that we uncover, then τ is a blocking permutation. In particular, for each generator σ_i of $\mathcal{M}^+$, there exists a blocking permutation τ such that $\tau(n) = \sigma_i$, and there exists a blocking permutation $\overline{\tau}$ such that $\overline{\tau}(1) = \sigma_i$.

Lemma 5.6.
Let τ be a blocking permutation with associated blocking heap $\mathbf{t}$, and let $\mathbf{a} \in \mathcal{M}^+$ be some heap. The heap $\tau(n)$ is the rightmost letter of both words $\mathbf{NF}_\ell(\mathbf{at})$ and $\mathbf{NF}_r(\mathbf{at})$.

Proof. Let i be some element i of $\{1,\ldots,n-1\}$, and let $\theta(i)$ be some element of $\{i+1,\ldots,n\}$ such that $\tau(i)\cdot\tau(\theta(i))$ is a left Garside word. We say that a factorisation $a_1\ldots a_{\lambda(\mathbf{a})+n}$ of $\mathbf{at}$ into generators of $\mathcal{M}^+$ is *good* if $\max\{j : a_j = \tau(\theta(i))\} > \max\{j : a_j = \tau(i)\}$. First, for each factorisation $a_1\ldots a_{\lambda(\mathbf{a})}$ of $\mathbf{a}$ into generators of $\mathcal{M}^+$, the factorisation $a_1\ldots a_{\lambda(\mathbf{a})}\tau(1)\ldots\tau(n)$ of the heap $\mathbf{at}$ is good.

Second, let us assume that there exist two factorisations $a_1\ldots a_{\lambda(\mathbf{a})+n}$ and $b_1\ldots b_{\lambda(\mathbf{a})+n}$ of the heap $\mathbf{at}$ into generators of $\mathcal{M}^+$ and an integer $I\in\{1,\ldots,\lambda(\mathbf{a})+n-1\}$ such that:

- the generators a_I and a_{I+1} commute, $b_I = a_{I+1}$ and $b_{I+1} = a_I$;
- we have $a_j = b_j$ for all $j\in\{1,\ldots,I-1,I+2,\ldots,\lambda(\mathbf{a})+n\}$.

It comes immediately that the former factorisation is good if and only if the latter factorisation is also good. Consequently, an immediate induction proves that each factorisation of $\mathbf{at}$ into generators of $\mathcal{M}^+$ is good, which proves that $\tau(i)\notin \mathbf{right}(\mathbf{at})$.

Since the set $\mathbf{right}(\mathbf{at})$ cannot be empty, it follows that $\mathbf{right}(\mathbf{at}) = \{\tau(n)\}$. Moreover, consider the words $w_1\cdot\ldots\cdot w_k := \mathbf{NF}_\ell(\mathbf{at})$ and $x_1\cdot\ldots\cdot x_k := \mathbf{NF}_r(\mathbf{at})$. Since $\tau(n) = \Delta_{\mathbf{right}(\mathbf{at})} = x_k \geqslant_r w_k >_r \mathbf{1}$, it follows that $w_k = x_k = \tau(n)$. □

Corollary 5.7.
Let τ be a blocking permutation with associated blocking heap $\mathbf{t}$, and let $\mathbf{t}^*$ be the mirror heap of $\mathbf{t}$, i.e. $\mathbf{t}^* := \tau(n)\tau(n-1)\ldots\tau(1)$. For all heaps $\mathbf{a},\mathbf{b}\in\mathcal{M}^+$, we have $\mathbf{NF}_\ell(\mathbf{att}^*\mathbf{b}) = \mathbf{NF}_\ell(\mathbf{at})\cdot\mathbf{NF}_\ell(\mathbf{t}^*\mathbf{b})$.

Proof. Let us recall the notations $\underline{\mathbf{a}}\lhd\underline{\mathbf{b}}$ and $\underline{\mathbf{a}}\rhd\underline{\mathbf{b}}$, which respectively mean that $\underline{\mathbf{a}}$ is a prefix of $\underline{\mathbf{b}}$, or that $\underline{\mathbf{b}}$ is a suffix of $\underline{\mathbf{a}}$. Lemma 5.6 proves that $\mathbf{NF}_\ell(\mathbf{at})\rhd\tau(n)$ and that $\mathbf{NF}_r(\mathbf{b}^*\mathbf{t})\rhd\tau(n)$, i.e. that $\tau(n)\lhd\mathbf{NF}_\ell(\mathbf{t}^*\mathbf{b})$. Since $\tau(n)\cdot\tau(n)$ is a left Garside word, it follows from Corollary 2.38 that $\mathbf{NF}_\ell(\mathbf{at})\cdot\mathbf{NF}_\ell(\mathbf{t}^*\mathbf{b})$ is a left Garside word, which completes the proof. □

Theorem 5.8.
Let $\mathcal{M}^+$ be a heap monoid, and let $(X_k)_{k\geqslant 0}$ be the left random walk on $\mathcal{M}^+$. For all integers $i\geqslant 0$, the sequence $(\mathbf{suf}_i(\mathbf{NF}_\ell(X_k)))_{k\geqslant 0}$ is almost surely ultimately constant.

Proof. Let τ be a blocking permutation of $\mathcal{M}^+$. For all integers $j\geqslant 0$, we consider the event $E_j := \{\forall i\in\{1,\ldots,n\}, Y_{2nj+i} = Y_{2n(j+1)+1-i} = \tau(i)\}$. The family of events $(E_j)_{j\geqslant 0}$ is independent, and each event E_j happens with probability at least $(\min\mu)^{2n}$. Hence, there almost surely exist infinitely many integers $j\geqslant 0$ such that E_j holds.

Now, consider some integer $j\geqslant i$ such that E_j holds: such an integer j almost surely exists. Corollary 5.7 proves that $\mathbf{NF}_\ell(X_{n(2j+1)})$ is a suffix of $\mathbf{NF}_\ell(X_k)$ for all $k\geqslant 2n(j+1)$. Note that $|\mathbf{NF}_\ell(X_{n(2j+1)})| = \|X_{n(2j+1)}\| \geqslant \frac{\lambda(X_{n(2j+1)})}{n} = 2j+1$. It follows that $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k)) = \mathbf{suf}_i(\mathbf{NF}_\ell(X_{n(2j+1)}))$ whenever $k\geqslant 2n(j+1)$, which completes the proof. □

Theorem 5.8 completes our picture of which prefixes and suffixes of the Garside normal forms of (X_k) have stabilisation properties (when $\mathcal{M}^+$ is an irreducible heap monoid with at least 2 generators). We sum up these results in Fig. 5.9.

	Convergence of the words	
	$\mathbf{NF}_\ell(X_k)_{k\geqslant 0}$	$\mathbf{NF}_r(X_k)_{k\geqslant 0}$
prefix-	✗	✗
suffix-	✓	✓

Figure 5.9 – Convergence of the normal forms of the random walk in the heap monoid

5.1.2 Random Walk in Heap Groups

Having studied the convergence of random walks in heap monoid, we change our point of view and study random walks in a heap group $\mathcal{M}$. In particular, we restrict our study to the case where the measure μ is a probability measure whose range is equal to $\{\sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1}\}$, and we still denote by $\min \mu$ the positive constant $\min_{1\leqslant i\leqslant n} \mu(\sigma_i^{\pm 1})$. In this context, we say that the left random on $\mathcal{M}$ with step-distribution μ is a *left random walk with Artin steps.*

The sequence $\mathbf{suf}_i(\mathbf{NF}_r(X_k))_{k\geqslant 0}$ is not non-decreasing (e.g. if $Y_1 = Y_0^{-1}$), hence we aim to proving convergence results for both families of sequences $\mathbf{suf}_i(\mathbf{NF}_r(X_k))_{k\geqslant 0}$ and $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k))_{k\geqslant 0}$. We begin with an immediate result.

Lemma 5.10.
Let $\mathcal{M}$ be a heap group, let σ_i^ϵ be some generator of $\mathcal{M}$, with $\epsilon = \pm 1$, and let Ω_i the set of generators of $\mathcal{M}$ that are dependent of σ_i, i.e. $\Omega_i = \{\sigma_i^{\pm 1}\} \cup \{\sigma_j^{\pm 1} : i \neq j \text{ and } m_{i,j} = \infty\}$. Consider some heap $\mathbf{a} \in \mathcal{M}$, as well as the word $a_1 \cdot \ldots \cdot a_k := \mathbf{NF}_r(\mathbf{a})$.

In addition, let us define cliques $a_0 := \mathbf{1}$ and $a_{k+1} := \sigma_i^\epsilon$, and let u be the smallest integer such that the clique a_u contains one letter $\sigma_j^{\pm 1}$ that belongs to Ω_i. We have

$$\begin{aligned} \mathbf{NF}_r(\sigma_i^\epsilon \mathbf{a}) &= a_1 \cdot \ldots \cdot a_{u-1} \cdot (\sigma_i^\epsilon a_u) \cdot a_{u+1} \cdot \ldots \cdot a_k \text{ if } \sigma_i^{-\epsilon} \text{ is a letter of } a_u; \\ &= a_1 \cdot \ldots \cdot a_{u-2} \cdot (\sigma_i^\epsilon a_{u-1}) \cdot a_u \cdot \ldots \cdot a_k \text{ otherwise.} \end{aligned}$$

From Lemma 5.10 follows immediately a result analogous to the case of heap monoids, whose proof is exactly analogous to the second part of the proof of Proposition 5.2.

Proposition 5.11.
Let $\mathcal{M}$ be a heap group with at least 2 generators and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathcal{M}$, with Artin steps. For all integers $i \geqslant 1$, the sequences $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))_{k\geqslant 0}$ and $\mathbf{pre}_i(\mathbf{NF}_r(X_k))_{k\geqslant 0}$ are almost surely divergent.

In addition, the first part of Proposition 5.2 also has a variant in the case of heap groups.

Theorem 5.12.
Let $\mathcal{M}$ be a heap group and let $(X_k)_{k \geqslant 0}$ be a left random walk in $\mathcal{M}$, with Artin steps. For all integers $i \geqslant 0$, the sequence $\mathbf{suf}_i(\mathbf{NF}_r(X_k))_{k \geqslant 0}$ is almost surely ultimately constant.

Proof. Let i and j be two distinct elements of $\{1, \ldots, n\}$ such that $m_{i,j} = \infty$. The subgroup $\mathcal{M}_{i,j}$ generated by $\{\sigma_i, \sigma_j\}$ is a 2-generator free monoid. In particular, let $\mathbf{p}_{i,j} : \mathcal{M} \mapsto \mathcal{M}_{i,j}$ be the canonical projection, i.e. the group morphism such that $\mathbf{p}_{i,j} : \sigma_i \mapsto \sigma_i$, $\mathbf{p}_{i,j} : \sigma_j \mapsto \sigma_j$ and $\mathbf{p}_{i,j} : \sigma_k \mapsto \sigma_k$ if $k \notin \{i, j\}$.

The random walk $(X_k)_{k \geqslant 0}$ on $\mathcal{M}$ induces a random walk $(\mathbf{p}_{i,j}(X_k))_{k \geqslant 0}$ on $\mathcal{M}_{i,j}$. This latter random walk must be transient [67, 88, 93], i.e. $\chi(\mathbf{p}_{i,j}(X_k)) \to +\infty$ when $k \to +\infty$, where χ denotes the product length (i.e. the length of the shortest factorisation into generators of $\mathcal{M}$). Hence, consider the function $\chi_{\min} : \mathbf{x} \mapsto \min\{\chi(\mathbf{p}_{i,j}(\mathbf{x})) : m_{i,j} = \infty\}$. We have just proven that $\chi_{\min}(X_k) \to +\infty$ almost surely when $k \to +\infty$.

Moreover, for all heaps $\mathbf{a} \in \mathcal{M}$, all the occurrences of the letters $\sigma_i^{\pm 1}$ and $\sigma_j^{\pm 1}$ appearing in the shortest factorisations of $\mathbf{a}$ must belong to pairwise distinct cliques of the word $\mathbf{NF}_r(\mathbf{a})$. Hence, for all pairs (i, j) such that $m_{i,j} = \infty$, at least $\chi_{\min}(\mathbf{a})$ letters of the word $\mathbf{NF}_r(\mathbf{a})$ are cliques where some letter $\sigma_i^{\pm 1}$ or $\sigma_j^{\pm 1}$ appears. In particular, for all generators σ_i^{ϵ} of the group $\mathcal{M}$, it follows from Lemma 5.10 that $d_{\mathbf{suf}}(\mathbf{NF}_r(\mathbf{a}), \mathbf{NF}_r(\sigma_i^{\epsilon}\mathbf{a})) \leqslant 2^{1-\chi_{\min}(\mathbf{a})}$.

Consequently, consider some integer $I \geqslant 0$. Since $\chi_{\min}(X_k) \to +\infty$ almost surely, there almost surely exists some integer $K \geqslant 0$ such that $\chi_{\min}(X_k) \geqslant I + 1$ whenever $k \geqslant K$. It means that $d_{\mathbf{suf}}(\mathbf{NF}_r(X_k), \mathbf{NF}_r(X_{k+1})) \leqslant 2^{1-\chi_{\min}(X_k)} \leqslant 2^{-I}$, i.e. that $\mathbf{suf}_I(\mathbf{NF}_r(X_k)) = \mathbf{suf}_I(\mathbf{NF}_r(X_{k+1}))$, for all $k \geqslant K$. This completes the proof. □

We will now use both Theorem 5.12 and a construction analogous to the above-defined blocking heaps in order to prove that the sequence $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k))_{k \geqslant 0}$ almost surely converges.

Lemma 5.13.
Let τ be a blocking permutation, and let $(\epsilon_1, \ldots, \epsilon_n)$ and $(\eta_1, \ldots, \eta_n)$ be two elements of $\{-1, 1\}^n$ such that $\epsilon_n = \eta_n$. Consider the heaps

$$\mathbf{t} := \tau(1)^{\epsilon_1} \ldots \tau(n)^{\epsilon_n} \text{ and } \mathbf{t}^* := \tau(n)^{\eta_n} \tau(n-1)^{\eta_{n-1}} \ldots \tau(1)^{\eta_1}.$$

For all heaps $\mathbf{a}, \mathbf{b} \in \mathcal{M}^+$ such that $\mathbf{a} \leqslant_\ell \mathbf{at}$ and $\mathbf{t}^*\mathbf{b} \geqslant_r \mathbf{b}$, then $\tau(n)^{\epsilon_n}$ is the rightmost letter of the words $\mathbf{NF}_\ell(\mathbf{at})$ and $\mathbf{NF}_r(\mathbf{at})$, $\tau(n)^{\epsilon_n}$ is also the leftmost letter of the words $\mathbf{NF}_\ell(\mathbf{t}^*\mathbf{b})$ and $\mathbf{NF}_r(\mathbf{t}^*\mathbf{b})$, and $\mathbf{NF}_\ell(\mathbf{att}^*\mathbf{b}) = \mathbf{NF}_\ell(\mathbf{at}) \cdot \mathbf{NF}_\ell(\mathbf{t}^*\mathbf{b})$.

Proof. The proof is entirely analogous to the proofs of Lemma 5.6 and of Corollary 5.7. □

Lemma 5.14.
Let $\mathcal{M}$ be a heap group and let $(X_k)_{k \geqslant 0}$ be a left random walk in $\mathcal{M}$, with Artin steps. There exists an integer $K \geqslant 0$ such that the event $\mathcal{E} := \{\forall k \geqslant K, \mathbf{NF}_\ell(X_k) \rhd \sigma_1\}$ holds with probability $\mathbb{P}[\mathcal{E}] > 0$.

Proof. Let $\mathbf{S}^{\pm 1}\backslash\{\mathbf{1}\}$ be the set of non-empty cliques of $\mathcal{M}$. Theorem 5.8 states that

$$P\left[\bigcup_{L\geqslant 0}\bigcup_{\mathbf{c}\in\mathbf{S}^{\pm1}\backslash\{\mathbf{1}\}}\bigcap_{k\geqslant L}\{\mathbf{NF}_r(X_k)\rhd\mathbf{c}\}\right]=1.$$

This proves that there exists an integer $L \geqslant 0$ and a clique $\mathbf{c} \in \mathbf{S}^{\pm1}\backslash\{\mathbf{1}\}$ such that the event $\mathcal{F}_{L,\mathbf{c}} := \{\forall k \geqslant L, \mathbf{NF}_r(X_k) \rhd \mathbf{c}\}$ holds with probability $\mathbb{P}[\mathcal{F}_{L,\mathbf{c}}] > 0$.

Now, let τ be a blocking permutation such that $\tau(n) = 1$. Consider the heaps $\mathbf{t} := \tau(1)^{\epsilon_1}\ldots\tau(n)^{\epsilon_n}$, where $\epsilon_i := -1$ if $1 \leqslant i \leqslant n-1$ and $\tau(i)^{-1} \leqslant_\ell \mathbf{c}$, and $\epsilon_i := 1$ otherwise. In addition, consider the events

$$\mathcal{F}_1 := \{X_n = \mathbf{t}\} \text{ and } \mathcal{F}_2 := \{\forall k \geqslant n+L, \mathbf{NF}_r(X_kX_n^{-1}) \rhd \mathbf{c}\}.$$

The events $\mathcal{F}_1$, and $\mathcal{F}_2$ are independent, so that $\mathbb{P}[\mathcal{F}_1 \cap \mathcal{F}_2] = \mathbb{P}[\mathcal{F}_1]\mathbb{P}[\mathcal{F}_2] > 0$.

Let us now assume that the event $\mathcal{F}_1 \cap \mathcal{F}_2$ holds. For all $k \geqslant n+L$, we have $X_kX_n^{-1} \leqslant_\ell X_k$, i.e. $X_k \geqslant_r X_n$, and therefore Lemma 5.13 proves that $\mathbf{NF}_\ell(X_k) \rhd \sigma_1$. Hence, choosing $K := n+L$ completes the proof. □

Lemma 5.15.
Let $\mathcal{M}$ be a heap group and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathcal{M}$, with Artin steps. The set $\{k \geqslant 0 : \sigma_1 \lhd \mathbf{NF}_\ell(X_k)\}$ is almost surely infinite.

Proof. Let τ be a blocking permutation such that $\tau(n) = \sigma_1$, and let $\epsilon := (\epsilon_1, \ldots, \epsilon_n)$ be an element of $\{-1, 1\}^n$. We denote by $\mathbf{t}^*_\epsilon$ the heap $\tau(n)^{\epsilon_n}\tau(n-1)^{\epsilon_{n-1}}\ldots\tau(1)^{\epsilon_1}$. For all tuples $\epsilon \in \{1,-1\}^n$ and all integers $K \geqslant 0$, we denote by $\mathcal{E}_{K,\epsilon}$ the event $\{\forall j \in \{0, \ldots, n-1\}, Y_{Kn+j} = \tau(j+1)^{\epsilon_{j+1}}\}$. The event $\mathcal{E}_{K,\epsilon}$ holds with probability $\mathbb{P}[\mathcal{E}_{K,\epsilon}] \geqslant (\min \mu)^n$ and is independent from the random variables $(Y_k)_{k<Kn \text{ or } K(n+1)\leqslant k}$.

Then, for all $K \geqslant 0$, consider the tuple $\epsilon^K := (\epsilon^K_1, \ldots, \epsilon^K_n)$, where $\epsilon^K_i := -1$ if $1 \leqslant i \leqslant n-1$ and if $\tau(i)^{-1} \in \mathbf{left}(X_{nK})$. Moreover, if $\mathcal{E}_{K,\epsilon^K}$ holds, then Lemma 5.13 proves that $\sigma_1 \lhd \mathbf{NF}_\ell(X_{(K+1)n})$. It follows that $\mathbb{P}[\sigma_1 \lhd \mathbf{NF}_\ell(X_{(K+1)n}) \mid (X_k)_{0\leqslant k\leqslant Kn}] \geqslant \mathbb{P}[\mathcal{E}_{K,\epsilon^K} \mid (X_k)_{0\leqslant k\leqslant Kn}] \geqslant (\min \mu)^n$. This proves that the set $\{K \geqslant 0 : \sigma_1 \lhd \mathbf{NF}_\ell(X_{Kn})\}$ is almost surely infinite, which completes the proof. □

Theorem 5.16.
Let $\mathcal{M}$ be a heap group and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathcal{M}$, with Artin steps. For all integers $i \geqslant 0$, the sequence $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k))_{k\geqslant 0}$ is almost surely ultimately constant.

Proof. Let us build an increasing sequence of random variables $(\tau_k)_{k\geqslant 0}$ and $(\theta_k)_{k\geqslant 0}$ in $\mathbb{N} \cup \{+\infty\}$ as follows. Let K be an integer such as defined in Lemma 5.14. We set $\tau_0 := 0$ and, for all integers $k \geqslant 0$ such that $\tau_k < +\infty$, we define (inductively) $\theta_k := \min\{j > \tau_k + K : \sigma_1 \text{ is not the rightmost letter of } \mathbf{NF}_\ell(X_jX_{\tau_k}^{-1})\}$ and $\tau_{k+1} := \min\{j \geqslant \theta_k : \|X_j\| \geqslant i \text{ and } \sigma_1 \lhd \mathbf{NF}_\ell(X_j)\}$. By construction, the random variables θ_k and τ_k are stopping times, the family $(\theta_k - \tau_k)_{k\geqslant 0}$ is i.i.d, and Lemma 5.15 proves that $\mathbb{P}[\tau_{k+1} < +\infty \mid \theta_k < +\infty] = 1$.

Moreover, for all integers $k \geqslant 0$, the random walk $(X_j X_{\tau_k}^{-1})_{j \geqslant \tau_k}$ follows the same distribution law as the random walk $(X_j)_{j \geqslant 0}$. Consequently, Lemma 5.14 proves that $\mathbb{P}[\theta_k = +\infty \mid \tau_k < +\infty] = \mathbb{P}[\mathcal{E}] > 0$, where $\mathcal{E} = \{\forall k \geqslant K, \mathbf{NF}_\ell(X_k) \rhd \sigma_1\}$. In particular, since the family $(\theta_k - \tau_k)_{k \geqslant 0}$ is i.i.d, there almost surely exists some integer k such that $\tau_k < +\infty$ and $\theta_k = +\infty$. For such an integer k, and for all integers $j \geqslant \tau_k + K$, it follows immediately that $\mathbf{NF}_\ell(X_j) = \mathbf{NF}_\ell(X_j X_{\tau_k}^{-1}) \cdot \mathbf{NF}_\ell(X_{\tau_k})$, and therefore that $\mathbf{suf}_i(\mathbf{NF}_\ell(X_j)) = \mathbf{suf}_i(\mathbf{NF}_\ell(X_{\tau_k}))$, which completes the proof. □

Proposition 5.11 and Theorems 5.12 and 5.16 are stated above in the context of left random walks with Artin steps, i.e. whose steps are distributed according to a probability measure μ with range $\{\sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1}\}$. In fact, the proof of Proposition 5.11 holds as soon as range(μ) generates (positively) the heap group $\mathcal{M}$, and the variants of the proofs of Theorems 5.12 and 5.16 hold as soon as range(μ) generates (positively) the heap group $\mathcal{M}$ *and* μ has a finite first moment, i.e. $\mathbb{E}_\mu[|\mathbf{NF}_\ell(X)|] < +\infty$.

Theorem 5.16 completes our picture of which prefixes and suffixes of the Garside normal forms of (X_k) have stabilisation properties, which we sum up in Fig. 5.17. Note that, although some of the arguments that we used were different from the case of heap monoids, the results that we obtain are entirely similar.

	Convergence of the words	
	$\mathbf{NF}_\ell(X_k)_{k \geqslant 0}$	$\mathbf{NF}_r(X_k)_{k \geqslant 0}$
prefix-	✗	✗
suffix-	✓	✓

Figure 5.17 – Convergence of the normal forms of the random walk in the heap group

5.2 Combinatorics of Garside Normal Forms

Having studied the stabilisation of Garside normal forms in the context of irreducible heap monoids and groups, we aim now to prove similar results in the context of irreducible Artin–Tits monoids of spherical type and groups. Hence, from this point on, we focus on introducing and using original notions that will lead to the results of Chapter 5. Among these notions are two key concepts, the *bilateral Garside automaton* and the *blocking braids*.

Before going further, we need to recall some notation introduced in Section 2.3 and introduce some additional notation. In what follows, let $\mathbf{A}^+$ be an Artin–Tits monoid, generated by the elements $\sigma_1, \ldots, \sigma_n$ (with $n \geqslant 2$, so as to avoid the cases of the monoids $\{0\}$ and $\mathbb{Z}_{\geqslant 0}$), and whose Garside group $\mathcal{W}$ is finite and irreducible, i.e. follows the classification of Theorem 2.26. The monoid $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, and call its elements *braids*. We also denote by $\mathbf{A}$ the group associated with $\mathbf{A}^+$, by $\mathbf{1}$ the unit element of $\mathbf{A}^+$, and by Δ the Garside element $\mathbf{LCM}_{\leqslant_\ell}(\sigma_1, \ldots, \sigma_n) = \mathbf{LCM}_{\geqslant_r}(\sigma_1, \ldots, \sigma_n)$ of $\mathbf{A}^+$.

In addition, for each set $S \subseteq \{\sigma_1, \ldots, \sigma_n\}$, we denote by $\mathbf{A}_S^+$ the Artin–Tits monoid generated by S, and by $\mathcal{W}_S$ the associated Coxeter group. Since $\mathcal{W}_S$ is a subgroup of $\mathcal{W}$, and therefore is finite, the monoid $\mathbf{A}_S^+$ is an Artin–Tits monoid of spherical type. We denote by Δ_S the braid $\mathbf{LCM}_{\leqslant_\ell}(S) = \mathbf{LCM}_{\geqslant_r}(S)$.

Finally, we denote by $\mathcal{S}$ the set of simple elements of $\mathbf{A}^+$, i.e. the set of divisors of Δ, and we denote by $\mathcal{S}^\circ$ the set of *proper simple elements* of $\mathbf{A}^+$, i.e. the set $\mathcal{S}^\circ := \mathcal{S} \backslash \{\mathbf{1}, \Delta\}$.

5.2.1 Connectedness of the Bilateral Garside Automaton

Let $\mathbf{a}$ be a proper simple element of $\mathbf{A}^+$. Since $\mathbf{1} <_\ell \mathbf{a} <_\ell \Delta$ and $\Delta >_r \mathbf{a} >_r \mathbf{1}$, it follows that $\varnothing \subsetneq \mathbf{left}(\mathbf{a}), \mathbf{right}(\mathbf{a}) \subsetneq \{\sigma_1, \ldots, \sigma_n\}$, i.e. that both sets $\mathbf{left}(\mathbf{a})$ and $\mathbf{right}(\mathbf{a})$ are proper subsets of $\{\sigma_1, \ldots, \sigma_n\}$. This gives rise to the notion of *bilateral Garside automaton* of the monoid $\mathbf{A}^+$, which is a variant of the left Garside acceptor automaton introduced in Definition 2.42.

Definition 5.18 (Bilateral Garside automaton and paths).
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathcal{S}^\circ$ be the set of its proper simple elements. The bilateral Garside automaton *is defined as the non-deterministic, finite-state automaton $\mathcal{G}_{\text{gar}} := (A, V, \delta, V, V)$, with*

- *alphabet $A = \mathcal{S}^\circ$;*
- *set of states $V = \{\mathbf{right}(\mathbf{a}) : \mathbf{a} \in A\}$;*
- *transition function δ with domain $\{(P, \mathbf{a}) : \mathbf{left}(\mathbf{a}) = P\}$ and such that $\delta : (P, \mathbf{a}) \mapsto \mathbf{right}(\mathbf{a})$ if $\mathbf{left}(\mathbf{a}) = P$;*
- *set of initial states V;*
- *set of accepting states V.*

A bilateral Garside path *is a word $b_1 \cdot \ldots \cdot b_k$ with letters in $\mathcal{S}^\circ$ such that $\mathbf{right}(b_i) = \mathbf{left}(b_{i+1})$ for all $i < k$. We then say that $\underline{\mathbf{b}}$ is a bilateral Garside path from $\mathbf{left}(b_k)$ to $\mathbf{right}(b_1)$.*

Note that each arc of $\mathcal{G}_{\text{gar}}$ may have several labels. Proposition 2.59 proves that the bilateral Garside paths are precisely the Δ-free words that are both left Garside words and right Garside words (i.e. that are left Garside words, regardless of whether they are read from left to right or from right to left, whence the adjective “bilateral”).

In addition, for each braid $\mathbf{a} \in \mathcal{S}^\circ$, we have $\mathbf{a}^* \in \mathcal{S}^\circ$ as well as $\mathbf{left}(\mathbf{a}) = \mathbf{right}(\mathbf{a}^*)$ and $\mathbf{right}(\mathbf{a}) = \mathbf{left}(\mathbf{a}^*)$, where $\mathbf{a}^*$ denotes the reversal of the braid $\mathbf{a}$. Therefore, for every arc (P, Q) in of $\mathcal{G}_{\text{gar}}$, the pair (Q, P) is also an arc of $\mathcal{G}_{\text{gar}}$, and the graph $\mathcal{G}_{\text{gar}}$ can be seen as non-directed. Examples of bilateral Garside automata are shown in Fig. 5.19 and by Fig. 6.2, page 211.

The left Garside acceptor automaton recognises the left Garside normal form, and its mirror, in which the arcs are reversed, recognises the right Garside normal form. Considering automata as labelled oriented graphs with marked initial and final states, the bilateral Garside automaton is a subgraph of both these automata, from which the state $\{\sigma_1, \ldots, \sigma_n\}$ has been deleted.[1] Figure 5.19, which presents the bilateral Garside automaton and the left Garside acceptor automaton associated with the monoid $\mathbf{B}_4^+$, where loops and labels have been omitted.

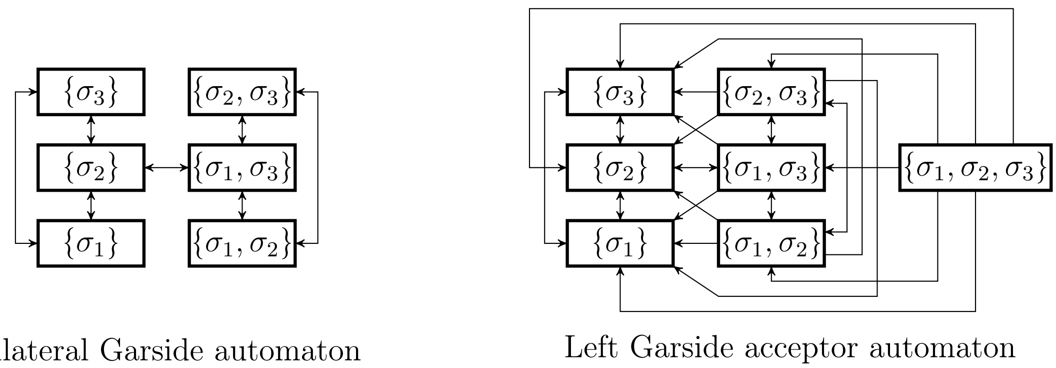


Figure 5.19 – Bilateral Garside automaton and left Garside acceptor automaton of the monoid $\mathbf{B}_4^+$

We prove below that the graph $\mathcal{G}_{\mathrm{gar}}$ is connected. This result was left implicit in [11, 56], where the following proof is used to show the connectedness of the left Garside acceptor automaton.

Lemma 5.20.
Let S be a proper subset of $\{\sigma_1, \ldots, \sigma_n\}$ and let τ be an element of $\{\sigma_1, \ldots, \sigma_n\} \backslash S$. In addition, let $S^\tau := \{\sigma \in S : \sigma\tau = \tau\sigma\}$ be the set of the elements of S that commute with τ.

If $S^\tau \neq S$, then the braid $\mathbf{b} := \tau\Delta_S$ is a simple braid such that $\mathbf{left}(\mathbf{b}) = \{\tau\} \cup S^\tau$ *and* $\mathbf{right}(\mathbf{b}) = S$.

Proof. Since $\tau \notin S = \mathbf{left}(\Delta_S)$, Lemma 2.16 proves that $\mathbf{b} \in \mathcal{S}$.

It is clear that $\{\tau\} \cup S^\tau \subseteq \mathbf{left}(\mathbf{b}) \subseteq \{\tau\} \cup S$. Moreover, if $\sigma \in S \cap \mathbf{left}(\mathbf{b})$, let m be the integer such that $\Delta_{\{\sigma,\tau\}} = [\sigma, \tau]^m$. We have $\tau[\sigma, \tau]^{m-1} = \Delta_{\{\sigma,\tau\}} \leqslant_\ell \mathbf{b} = \tau\Delta_S$, whence $[\sigma, \tau]^{m-1} \leqslant_\ell \Delta_S$. Since τ is not a factor of Δ_S, this proves that $m = 2$, i.e. that $\sigma \in S^\tau$. Hence, we have $\mathbf{left}(\mathbf{b}) = \{\tau\} \cup S^\tau$.

It is also clear that $S \subseteq \mathbf{right}(\mathbf{b}) \subseteq \{\tau\} \cup S$. Moreover, since $\mathbf{left}(\mathbf{b}) \subsetneq \{\tau\} \cup S$, we know that $\mathbf{b}$ is a proper divisor of $\Delta_{\{\tau\}\cup S}$, whence $\mathbf{right}(\mathbf{b}) \subsetneq \{\tau\} \cup S$. This proves that $\mathbf{right}(\mathbf{b}) = S$. $\square$

[1] The systematic study of the monoids with at most 7 generators suggests that the bilateral Garside automaton is *exactly* the intersection or the left and right Garside acceptor automata, from which the state $\{\sigma_1, \ldots, \sigma_n\}$ has been deleted.

Proposition 5.21.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathcal{G}_{\mathrm{gar}}$ be the bilateral Garside automaton of $\mathbf{A}^+$. The graph $\mathcal{G}_{\mathrm{gar}}$ is connected. Moreover, for all vertices P and Q of $\mathcal{G}_{\mathrm{gar}}$, there exists a bilateral Garside path of length $\mathfrak{D}$ from P to Q, where $\mathfrak{D}$ is the diameter of $\mathcal{G}_{\mathrm{gar}}$.

Proof. Consider some set $P \in V$, and consider the Coxeter diagram $\mathbf{G}$ of the Coxeter group $\mathcal{W}$ associated with the braid monoid $\mathbf{A}^+$, i.e. the graph with set of vertices $\{\sigma_1, \ldots, \sigma_n\}$ and set of edges $\{(\sigma, \tau) : \sigma\tau \neq \tau\sigma\}$. Since $\mathcal{W}$ is irreducible, the graph $\mathbf{G}$ must be connected. In addition, we denote by $\delta_{\mathbf{G}}(s,t)$ the distance between the nodes s and t in $\mathbf{G}$.

We identify hereafter P with a set of vertices of $\mathbf{G}$. Let $c(P)$ be the number of connected components of P, and let $d(P) := \sum_{s \in P} \delta_{\mathbf{G}}(\sigma_1, s)$. We prove now that P is connected to $\{\sigma_1\}$ in $\mathcal{G}_{\mathrm{gar}}$.

1. If $d(P) = 0$, then $P = \{\sigma_1\}$ is obviously connected to itself.
2. If $c(P) = |P|$ and $d(P) > 0$, the set P is independent and contains some vertex $v \neq \sigma_1$. Let $s_0, s_1, \ldots, s_k$ be a path in $\mathbf{G}$ such that $s_0 = \sigma_1$, $s_k = v$ and $k = \delta_{\mathbf{G}}(\sigma_1, v)$. Since P is independent, the vertex s_{k-1} does not belong to P. Hence, consider the set $Q := \{s_{k-1}\} \cup \{t \in P : \delta_{\mathbf{G}}(t, s_{k-1}) \geqslant 2\}$.
   Lemma 5.20 states that the simple braid $\mathbf{b} := s_{k-1}\Delta_P$ satisfies $\mathbf{left}(\mathbf{b}) = Q$ and $\mathbf{right}(\mathbf{b}) = P$. By construction, the set Q is independent, whence $c(Q) = |Q|$. Finally, observe that
   $$d(P) - d(Q) = \sum_{t \in P} \mathbf{1}_{\delta_{\mathbf{G}}(t, s_{k-1}) = 1} \delta_{\mathbf{G}}(\sigma_1, t) - \delta_{\mathbf{G}}(\sigma_1, s_{k-1}) \geqslant \delta_{\mathbf{G}}(\sigma_1, v) - \delta_{\mathbf{G}}(\sigma_1, s_{k-1}) = 1.$$
   Hence, an immediate induction on $d(P)$ shows that P is connected to $\{\sigma_1\}$ in $\mathcal{G}_{\mathrm{gar}}$.
3. If $c(P) < |P|$, then consider some connected component S of P with cardinality $|S| \geqslant 2$, and τ be some neighbour of S in $\mathbf{G}$. In addition, let k and ℓ be the respective cardinalities of the sets $\{s \in P : \delta_{\mathbf{G}}(\tau, s) = 1\}$ and $\{s \in P : \delta_{\mathbf{G}}(\tau, s) = 1 \text{ and } \forall t \in P, \delta_{\mathbf{G}}(s,t) \neq 1\}$. Note that, since $|S| \geqslant 2$, we have $k > \ell$. In addition, consider the set $Q := \{\tau\} \cup \{s \in P : \delta_{\mathbf{G}}(s, \tau) \geqslant 2\}$.
   Lemma 5.20 states that the simple braid $\mathbf{b} := \tau\Delta_P$ satisfies $\mathbf{left}(\mathbf{b}) = Q$ and $\mathbf{right}(\mathbf{b}) = P$. Moreover, note that $c(Q) = c(P) + 1 - \ell$ and that $|Q| = |P| + 1 - k$, whence $|Q| - c(Q) < |P| - c(P)$. Hence, an immediate induction on $|P| - c(P)$ shows that P is connected to $\{\sigma_1\}$ in $\mathcal{G}_{\mathrm{gar}}$.

This proves that the bilateral Garside automaton $\mathcal{G}_{\mathrm{gar}}$ is indeed connected. The last part of Proposition 5.21 comes from the fact that, for each proper subset P of $\{\sigma_1, \ldots, \sigma_n\}$, the braid Δ_P satisfies $\mathbf{left}(\Delta_P) = \mathbf{right}(\Delta_P) = P$, which shows that $\mathcal{G}_{\mathrm{gar}}$ contains a loop around the node P. □

The proof of Proposition 5.21 even provides us with the crude upper bound $\mathfrak{D} \leqslant 2(n^2 + n)$. In Chapter 6, we compute explicitly the value of the diameter $\mathfrak{D}$ for all Artin–Tits monoids of spherical type $\mathbf{A}^+$. In particular, we show that we have in fact $\mathfrak{D} \leqslant 4$ in all cases, which provides us an upper bound on $\mathbf{D}$ regardless of the value of n.

5.2.2 Blocking patterns

Let us now define a notion of *blocking pattern* analogous to the above notion of blocking heap, and that is a variant of the blocking braids introduced by Caruso and Wiest [26]: Proposition 5.21 will be helpful for proving that blocking patterns actually exist. Note that, in the literature of braids, the notion of "blocking" may have had several meanings, depending on the context. For instance, our blocking braids are not related to the *blocked-braid groups* of Maglia, Sabadini and Walters [73], to the *blocked punctures* used by González-Meneses and Wiest in [59], nor to the *separator braids* of Fromentin [50].

They are, however, somehow related to *barriers* of Fromentin [52]. A barrier is, in the context of *dual braid monoids* [15] and of the cycling normal form, a braid α meant to prevent certain pseudo-commutations of the type $\beta\alpha = \alpha\gamma$. Then, in [26], blocking braids are meant to prevent similar pseudo-commutations, in the context of standard braid monoids and of the (left) Garside normal form. Below, we modify this latter notion of blocking braid, by preventing pseudo-commutations in both left and right Garside normal forms.

Definition 5.22 (Blocking patterns)**.**
Let us place ourselves in an irreducible Artin–Tits monoid of spherical type $\mathbf{A}^+$ *with* n *generators. Let* $\underline{\mathbf{w}} := w_1 \cdot \ldots \cdot w_k$ *be a* Δ*-free, non empty left Garside word. If* $\mathbf{right}(w_k)$ *has cardinality* $n-1$*, then we say that* $\underline{\mathbf{w}}$ *is a* blockable word.

If $\underline{\mathbf{w}}$ *is a blockable word, we call* $\underline{\mathbf{w}}$-blocking patterns *the words* $\underline{\mathbf{w}} \cdot \underline{\mathbf{x}} \cdot \sigma$ *where* σ *is an Artin generator of the monoid* $\mathbf{A}^+$ *and where* $\underline{\mathbf{x}}$ *is a bilateral Garside path of length* $\mathfrak{D}$ *from* $\mathbf{right}(w_k)$ *to* $\{\sigma\}$*. We denote by* $\mathcal{B}_{\underline{\mathbf{w}}}$ *the set of all* $\underline{\mathbf{w}}$*- or* $\phi_\Delta(\underline{\mathbf{w}})$*-blocking patterns.*

Finally, we call blocking patterns *all the* $\underline{\mathbf{w}}$*-blocking patterns, for all blockable words* $\underline{\mathbf{w}}$*.*

Lemma 5.23.
Let $\underline{\mathbf{p}}$ *be a blocking pattern and let* $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$ *be positive braids such that* $\mathbf{ab}$ *is* Δ*-free and that* $\mathbf{NF}_\ell(\mathbf{b}) \rhd \underline{\mathbf{p}}$*. We have* $|\underline{\mathbf{p}}| \geqslant \mathfrak{D} + 2$ *and* $\mathbf{NF}_r(\mathbf{ab}) \rhd \mathbf{suf}_{\mathfrak{D}+1}(\underline{\mathbf{p}})$*.*

Proof. Let $\underline{\mathbf{w}}$ be a blockable word such that $\underline{\mathbf{p}}$ is a $\underline{\mathbf{w}}$-blocking pattern. It comes immediately that $k = |\underline{\mathbf{w}}| + \mathfrak{D} + 1 \geqslant \mathfrak{D} + 2$.

Hence, consider the word $\underline{\mathbf{q}} := \mathbf{suf}_{\mathfrak{D}+1}(\underline{\mathbf{p}})$ and the braid $\mathbf{u} := \mathbf{abq}^{-1}$. Since $\underline{\mathbf{p}}$ is a blocking pattern, the set $\mathbf{right}(p_{-\mathfrak{D}-2})$ has cardinality $n-1$. Moreover, Δ does not divide $\mathbf{ab}$, hence does not divide $\mathbf{u}$ either. It follows that $\mathbf{right}(p_{-\mathfrak{D}-2}) \subseteq \mathbf{right}(\mathbf{u}) \subsetneq \{\sigma_1, \ldots, \sigma_n\}$, whence $\mathbf{right}(\mathbf{u}) = \mathbf{right}(p_{-\mathfrak{D}-2}) = \mathbf{left}(p_{-\mathfrak{D}-1})$. Therefore, Corollary 2.41 states that $\mathbf{NF}_r(\mathbf{ab}) = \mathbf{NF}_r(\mathbf{u}) \cdot \mathbf{NF}_r(\mathbf{q}) = \mathbf{NF}_r(\mathbf{u}) \cdot \underline{\mathbf{q}}$, which proves Lemma 5.23. □

Lemma 5.24.
Let $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$ *be positive braids such that* $\mathbf{ab}$ *is* Δ*-free and such that* $\mathbf{NF}_\ell(\mathbf{b})$ *can be factored into a product* $\mathbf{NF}_\ell(\mathbf{b}) = \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{b}}_2$*, where* $\underline{\mathbf{p}}$ *is a blocking pattern. We have* $\mathbf{NF}_\ell(\mathbf{ab}) \rhd \underline{\mathbf{b}}_2$*. Similarly, if* $\mathbf{NF}_r(\mathbf{b})$ *can be factored into a product* $\mathbf{NF}_r(\mathbf{b}) = \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{b}}_2$*, where* $\underline{\mathbf{p}}$ *is a blocking pattern, then* $\mathbf{NF}_r(\mathbf{ab}) \rhd \underline{\mathbf{b}}_2$*.*

Proof. Let λ be the rightmost letter of $\mathbf{NF}_\ell(\mathbf{ab}_1\mathbf{p})$, and let σ be the rightmost letter of $\underline{\mathbf{p}}$. Since $\mathbf{ab}$ is Δ-free, so is $\mathbf{ab}_1\mathbf{p}$. Therefore, Lemma 5.23 proves that $\alpha_r(\mathbf{ab}_1\mathbf{p}) = \sigma$. Hence, $\sigma = \alpha_r(\mathbf{ab}_1\mathbf{p}) \geqslant_r \lambda >_r \mathbf{1}$. Since $\mathbf{p}$ is a blocking pattern, σ is an Artin generator, and thus $\sigma = \lambda$. This proves that $\mathbf{right}(\lambda) = \mathbf{right}(\sigma) \supseteq \mathbf{left}(\underline{\mathbf{b}}_2)$, and Corollary 2.38 then states that $\mathbf{NF}_\ell(\mathbf{ab}_1\mathbf{pb}_2) = \mathbf{NF}_\ell(\mathbf{ab}_1\mathbf{p}) \cdot \underline{\mathbf{b}}_2$, which shows the first part of Lemma 5.24.

Then, let us assume that $\mathbf{NF}_r(\mathbf{b})$ can be factored into a product $\mathbf{NF}_r(\mathbf{b}) = \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{b}}_2$. Lemma 5.23 proves that $\mathbf{NF}_r(\mathbf{ab}_1\mathbf{p}) \rhd \mathbf{suf}_{\mathfrak{D}+1}(\underline{\mathbf{p}})$, hence that $\mathbf{NF}_r(\mathbf{ab}) \rhd \mathbf{suf}_{\mathfrak{D}+1}(\underline{\mathbf{p}}) \cdot \underline{\mathbf{b}}_2$, which completes the proof. □

Corollary 5.25.
Let σ be an Artin generator of an irreducible Artin–Tits monoid of spherical type $\mathbf{A}^+$, and let $\mathbf{b} \in \mathbf{A}^+$ be a positive braid such that $\mathbf{NF}_\ell(\mathbf{b})$ can be factored into a product $\mathbf{NF}_\ell(\mathbf{b}) = \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{b}}_2$, where $\underline{\mathbf{p}}$ is a blocking pattern. We have $\mathbf{NF}_\ell(\sigma\mathbf{b}) \rhd \underline{\mathbf{b}}_2$.

Proof. Since ϕ_Δ leaves the set of Artin generators invariant and since $\underline{\mathbf{p}}$ is Δ-free (as is every blocking pattern), we may assume, without loss of generality, that $\mathbf{NF}_\ell(\mathbf{b})$ is Δ-free.

Let λ be the leftmost letter of $\underline{\mathbf{b}}_1 \cdot \underline{\mathbf{p}}$. If $\sigma\lambda = \Delta$, then $\mathbf{NF}_\ell(\sigma\mathbf{b})$ is obtained by dropping the first letter of $\mathbf{NF}_\ell(\mathbf{b})$, hence $\underline{\mathbf{b}}_2$ is a suffix of $\mathbf{NF}_\ell(\sigma\mathbf{b})$. If $\sigma\lambda \neq \Delta$, then $\sigma\mathbf{b}$ is Δ-free, hence Lemma 5.24 applies and proves the result as well. □

Definition 5.26 (Flags)**.**
*Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type. Let $\underline{\mathbf{w}}$ be a left Garside word, let $\mathbf{b} \in \mathbf{A}^+$ be a positive braid, and let $b_1 \cdot \ldots \cdot b_k = b_{-k} \cdot \ldots \cdot b_{-1}$ be the left Garside normal form of $\mathbf{b}$. We call $\underline{\mathbf{w}}$-*flags *of the braid $\mathbf{b}$ the integers $i \in \mathbb{Z}$ such that $\underline{\mathbf{w}} = b_i \cdot \ldots \cdot b_{i+|\underline{\mathbf{w}}|-1}$. We denote by $\mathbf{F}^+_{\underline{\mathbf{w}}}(\mathbf{b})$ the set of all positive $\underline{\mathbf{w}}$-flags of $\mathbf{b}$, and we denote by $\mathbf{F}^-_{\underline{\mathbf{w}}}(\mathbf{b})$ the set of all negative $\underline{\mathbf{w}}$-flags of $\mathbf{b}$. In addition, we denote by $\mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{b})$ the cardinality of both sets $\mathbf{F}^+_{\underline{\mathbf{w}}}(\mathbf{b})$ and $\mathbf{F}^-_{\underline{\mathbf{w}}}(\mathbf{b})$.*

We then extend the notion of flags to sets. If Ω is a set of Δ-free left Garside words, and Ω-flag is a $\underline{\mathbf{w}}$-flag for some word $\underline{\mathbf{w}} \in \Omega$. We define similarly the sets $\mathbf{F}^+_\Omega(\mathbf{b}) = \bigcup_{\underline{\mathbf{w}} \in \Omega} \mathbf{F}^+_{\underline{\mathbf{w}}}(\mathbf{b})$ and $\mathbf{F}^-_\Omega(\mathbf{b}) = \bigcup_{\underline{\mathbf{w}} \in \Omega} \mathbf{F}^-_{\underline{\mathbf{w}}}(\mathbf{b})$, and we denote by $\mathfrak{f}_\Omega(\mathbf{b})$ the cardinality of the sets $\mathbf{F}^+_\Omega(\mathbf{b})$ and $\mathbf{F}^-_\Omega(\mathbf{b})$.

Lemma 5.27.
Let $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$ be positive braids such that $\mathbf{ab}$ is Δ-free, and let $\underline{\mathbf{p}}$ be a blocking pattern. We have $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) + \mathfrak{D} + 1$ and $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) + |\underline{\mathbf{p}}|$.

Proof. We first prove that $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) + \mathfrak{D} + 1$. If $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}) = 0$, then the result holds obviously. However, if $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}) \geqslant 1$, let us factor $\mathbf{NF}_\ell(\mathbf{a})$ into a product $\mathbf{NF}_\ell(\mathbf{a}) = \underline{\mathbf{a}}_1 \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{a}}_2$ such that $\underline{\mathbf{a}}_1$ is as long as possible.

Let $\underline{\mathbf{q}} := \mathbf{suf}_{\mathfrak{D}+1}(\underline{\mathbf{p}})$ be the suffix of $\underline{\mathbf{p}}$ of length $\mathfrak{D}+1$. In addition, consider the braids $\mathbf{d} := \mathbf{a}_1\mathbf{pq}^{-1}$ and $\mathbf{e} := \mathbf{qa}_2\mathbf{b}$. Since $\underline{\mathbf{p}}$ is a blocking pattern, we know that $\mathbf{left}(p_{-\mathfrak{D}-1})$ has cardinality $n-2$. Moreover, since $\mathbf{ab}$ is Δ-free, so is $\mathbf{e}$, so that $\mathbf{left}(p_{-\mathfrak{D}-1}) \subseteq \mathbf{left}(\mathbf{e}) \subsetneq \{\sigma_1, \ldots, \sigma_n\}$. It follows that $\mathbf{left}(\mathbf{e}) = \mathbf{left}(p_{-\mathfrak{D}-1}) = \mathbf{right}(p_{-\mathfrak{D}-2})$.

Since $\mathbf{NF}_\ell(\mathbf{d})$ is obtained by removing the $\mathfrak{D}+1$ rightmost letters of $\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{p}}$, we know that $p_{-\mathfrak{D}-2}$ is the rightmost letter of $\mathbf{NF}_\ell(\mathbf{d})$. Therefore, Corollary 2.41 states that $\mathbf{NF}_\ell(\mathbf{ab}) = \mathbf{NF}_\ell(\mathbf{d}) \cdot \mathbf{NF}_\ell(\mathbf{e})$. It follows that $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) \geqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{d}) \geqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}_1\mathbf{p}) - (\mathfrak{D}+1) = \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}) - (\mathfrak{D}+1)$, by maximality of $\underline{\mathbf{a}}_1$.

We prove now that $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) + |\underline{\mathbf{p}}|$. If $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}) = 0$, then again the result holds. However, if $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}) \geqslant 1$, let us factor $\mathbf{NF}_\ell(\mathbf{b})$ as a product $\mathbf{NF}_\ell(\mathbf{b}) = \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{b}}_2$ such that $\underline{\mathbf{b}}_2$ is as long as possible. Lemma 5.24 states that $\underline{\mathbf{b}}_2$ is a suffix of $\mathbf{NF}_\ell(\mathbf{ab})$. Moreover, $\underline{\mathbf{b}}_2$ is obtained by removing the $|\underline{\mathbf{p}}|$ leftmost letters of $\underline{\mathbf{p}} \cdot \underline{\mathbf{b}}_2$. Hence, $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) \geqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}_2) \geqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{pb}_2) - |\underline{\mathbf{p}}| = \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}) - |\underline{\mathbf{p}}|$, by maximality of $\underline{\mathbf{b}}_2$. □

Lemma 5.28.
Let $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$ *be positive braids such that* $\mathbf{ab}$ *is* Δ*-free. We have* $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}) + \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}) + |\underline{\mathbf{p}}|$.

Proof. We proceed by induction on $|\mathbf{NF}_\ell(\mathbf{b})|$. First, if $\mathbf{b} = \mathbf{1}$, then the result is obvious. Henceforth, we assume that $|\mathbf{NF}_\ell(\mathbf{b})| \geqslant 1$.

Let λ_b be the leftmost letter of $\mathbf{NF}_\ell(\mathbf{b})$. We factor $\mathbf{NF}_\ell(\mathbf{a}\lambda_b)$ into a product $\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{a}}_2$ such that $\underline{\mathbf{a}}_1$ is as long as possible. Consider the braid $\mathbf{q} := \mathbf{a}_1\mathbf{p}$. By maximality of $\underline{\mathbf{a}}_1$, we have $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{q}) = \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}\lambda_b)$. Moreover, the rightmost letter of $\underline{\mathbf{p}}$ is an Artin generator, let us say σ_u. According to Lemma 5.23, we know that $\sigma_u = \alpha_r(\mathbf{q})$ and that $\mathbf{right}(\sigma_u) = \mathbf{right}(\mathbf{q}) = \{\sigma_u\}$.

Now, assume that $\mathbf{a}_2 = \mathbf{1}$. Then, $\sigma_u = \alpha_r(\mathbf{q}) = \alpha_r(\mathbf{a}\lambda_b) \geqslant_r \lambda_b >_r \mathbf{1}$, and therefore $\lambda_b = \sigma_u$ is the rightmost letter of $\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{p}} = \mathbf{NF}_\ell(\mathbf{a}\lambda_b)$. Since λ_b is also the leftmost letter of $\mathbf{NF}_\ell(\mathbf{b})$, Corollary 2.41 states that $\mathbf{NF}_\ell(\mathbf{ab}) = \mathbf{NF}_\ell(\mathbf{a}) \cdot \mathbf{NF}_\ell(\mathbf{b})$. It follows that $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}) + \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}) + |\underline{\mathbf{p}}|$.

Then, assume that, on the contrary, $\mathbf{a}_2 \neq \mathbf{1}$, and let ρ be the leftmost letter of $\mathbf{NF}_r(\mathbf{a}_2)$. It follows from the relations $\varnothing \subsetneq \mathbf{left}(\rho) \subseteq \mathbf{left}(\mathbf{a}_2) \subseteq \mathbf{right}(\sigma_u) = \{\sigma_u\}$ that $\mathbf{left}(\rho) = \{\sigma_u\} = \mathbf{right}(\mathbf{q})$. Hence, Corollary 2.41 proves that $\alpha_r(\mathbf{qa}_2) = \alpha_r(\mathbf{a}_2)$, and consequently $\mathbf{a}_2 \geqslant_r \alpha_r(\mathbf{a}_2) = \alpha_r(\mathbf{qa}_2) = \alpha_r(\mathbf{a}\lambda_b) \geqslant_r \lambda_b$. Therefore, $\mathbf{a}_2\lambda_b^{-1}$ belongs to $\mathbf{A}^+$, and $\mathbf{left}(\mathbf{a}_2\lambda_b^{-1}) \subseteq \mathbf{left}(\mathbf{a}_2) \subseteq \mathbf{right}(\sigma_u)$.

Again, Corollary 2.41 states that $\mathbf{NF}_\ell(\mathbf{a}) = \mathbf{NF}_\ell(\mathbf{qa}_2\lambda_b^{-1}) = \mathbf{NF}_\ell(\mathbf{q}) \cdot \mathbf{NF}_\ell(\mathbf{a}_2\lambda_b^{-1})$, and therefore $\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}\lambda_b) = \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{q}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a})$. Moreover, note that $\mathbf{NF}_\ell(\lambda_b^{-1}\mathbf{b})$ is a suffix of $\mathbf{NF}_\ell(\mathbf{b})$, and therefore that $\mathfrak{f}_{\underline{\mathbf{p}}}(\lambda_b^{-1}\mathbf{b}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b})$. By induction hypothesis, it follows that

$$\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{ab}) = \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}\lambda_b\lambda_b^{-1}\mathbf{b}) \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}\lambda_b) + \mathfrak{f}_{\underline{\mathbf{p}}}(\lambda_b^{-1}\mathbf{b}) + |\underline{\mathbf{p}}| \leqslant \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}) + \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}) + |\underline{\mathbf{p}}|.$$

□

Proposition 5.29.
Consider an irreducible Artin–Tits monoid of spherical type $\mathbf{A}^+$. *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$ *be two*

positive braids, and let $\underline{\mathbf{w}}$ *be a blockable word. We have*

$$\mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{ab}) \leqslant \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{a}) + \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{b}) + 2|\mathcal{S}|^{\mathfrak{D}+1}(2|\underline{\mathbf{w}}| + 3\mathfrak{D} + 3).$$

Proof. Let u be the largest integer such that $\mathbf{ab} \geqslant \Delta^u$. Lemma 2.52 states that there exists positive braids $\mathbf{a}', \mathbf{a}'', \mathbf{b}', \mathbf{b}'' \in \mathbf{A}^+$ such that $\mathbf{a} = \mathbf{a}'\mathbf{a}''$, $\mathbf{b} = \mathbf{b}'\mathbf{b}''$ and $\mathbf{a}''\mathbf{b}' = \Delta^u$. Therefore, $\mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{ab}) = \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\Delta^u \phi_\Delta^u(\mathbf{a}')\mathbf{b}'') = \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\phi_\Delta^u(\mathbf{a}')\mathbf{b}'')$, and $\phi_\Delta^u(\mathbf{a}')\mathbf{b}''$ is Δ-free.

In addition, observe that each $\underline{\mathbf{w}}$-blocking pattern is a product $\underline{\mathbf{w}} \cdot \underline{\mathbf{p}}$ or $\phi_\Delta(\underline{\mathbf{w}}) \cdot \underline{\mathbf{p}}$, where $|\underline{\mathbf{p}}| = \mathfrak{D} + 1$. Therefore, there exist at most $2|\mathcal{S}|^{\mathfrak{D}+1}$ $\underline{\mathbf{w}}$-blocking patterns. In addition, by construction, the set $\mathcal{B}_{\underline{\mathbf{w}}}$ is closed under ϕ_Δ. Hence, Lemmas 5.27 and 5.28 prove that

$$\begin{aligned}
\mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{ab}) &= \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\phi_\Delta^u(\mathbf{a}')\mathbf{b}'') = \sum_{\underline{\mathbf{p}} \in \mathcal{B}_{\underline{\mathbf{w}}}} \mathfrak{f}_{\underline{\mathbf{p}}}(\phi_\Delta^u(\mathbf{a}')\mathbf{b}'') \leqslant \sum_{\underline{\mathbf{p}} \in \mathcal{B}_{\underline{\mathbf{w}}}} \Big(\mathfrak{f}_{\underline{\mathbf{p}}}(\phi_\Delta^u(\mathbf{a}')) + \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}'') + |\underline{\mathbf{p}}|\Big) \\
&\leqslant \sum_{\underline{\mathbf{p}} \in \mathcal{B}_{\underline{\mathbf{w}}}} \Big(\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}') + \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}'') + |\underline{\mathbf{p}}|\Big) \leqslant \sum_{\underline{\mathbf{p}} \in \mathcal{B}_{\underline{\mathbf{w}}}} \Big(\mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{a}'\mathbf{a}'') + \mathfrak{f}_{\underline{\mathbf{p}}}(\mathbf{b}'\mathbf{b}'') + \mathfrak{D} + 2|\underline{\mathbf{p}}| + 1\Big) \\
&\leqslant \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{a}) + \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{b}) + |\mathcal{B}_{\underline{\mathbf{w}}}|(2|\underline{\mathbf{w}}| + 3\mathfrak{D} + 3) \\
&\leqslant \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{a}) + \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{w}}}}(\mathbf{b}) + 2|\mathcal{S}|^{\mathfrak{D}+1}(2|\underline{\mathbf{w}}| + 3\mathfrak{D} + 3).
\end{aligned}$$

□

5.3 Stabilisation of the Random Walk

5.3.1 First Results

Let μ be a probability measure over the set $\{\sigma_1, \ldots, \sigma_n\}$, finite support, such that the real constant $\min \mu := \mu(\sigma_i)$ is positive.

Definition 5.30 (Left random walk).
Let $(Y_k)_{k \geqslant 0}$ *be i.i.d. random variables distributed with law* μ. *The* left random walk with step-distribution μ *is the sequence* $\mathbf{X} = (X_k)_{k \geqslant 0}$ *defined by* $X_0 := \mathbf{1}$ *and* $X_{k+1} := Y_k X_k$ *for* $k \geqslant 0$.

When the probability measure μ is implicit from the context, we simply say that $\mathbf{X}$ is a *left random walk*. In addition, since μ is assumed to have its range equal to $\{\sigma_1, \ldots, \sigma_n\}$, we may also reuse the terminology of Section 5.1 and say that $\mathbf{X}$ is a *left random walk with Artin steps.*

Note that, here and in the sequel of Chapter 5, we always focus on the *left random walk*, and not its dual *right random walk*, which is the random walk usually considered, and which we mentioned in the introductory chapter of this thesis. The main reason for focusing here on the left random walk rather than the right random walk is that

Theorem 5.40 will provide us a convergence result for the left Garside normal form, which is the usual Garside normal form. Of course, all the results mentioned below can be instantaneously translated by exchanging the notions of left and right, hence imply dual results about the right random walk.

First, like in the case of heap monoids, some convergence and divergence results come quickly.

Proposition 5.31.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, and let $(X_k)_{k \geqslant 0}$ be a left random walk in $\mathbf{A}^+$ with Artin steps. For all integers $i \geqslant 1$, the sequences $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))_{k \geqslant 0}$ and $\mathbf{suf}_i(\mathbf{NF}_r(X_k))_{k \geqslant 0}$ is almost surely convergent, with limit $\Delta \cdot \ldots \cdot \Delta$, and the sequence and $\mathbf{pre}_i(\mathbf{NF}_r(X_k))_{k \geqslant 0}$ is almost surely divergent.

Proof. First, let $\lambda(\Delta)$ be the Artin length of the Garside element Δ. For all integers $k \geqslant 0$, if $Y_{k+i\lambda(\Delta)-1} \ldots Y_k = \Delta^i$, which happens with probability at least $(\min \mu)^{i\lambda(\Delta)}$, then it follows that $\mathbf{pre}_i(\mathbf{NF}_\ell(X_m)) = \mathbf{suf}_i(\mathbf{NF}_r(X_m)) = \Delta \cdot \ldots \cdot \Delta$ for all integers $m \geqslant k + i\lambda(\Delta)$.

Second, since the random walk $(X_m)_{m \geqslant 0}$ is transient, there almost surely exists some integer $M \geqslant 0$ such that $\mathbf{pre}_1(\mathbf{NF}_r(X_k)) \neq \mathbf{1}$ whenever $k \geqslant M$. Consider some even integer $k \geqslant M$. The event $\mathcal{E}_k := \{Y_k \leqslant_\ell \mathbf{pre}_1(\mathbf{NF}_r(X_k))\} \cap \{\exists i, \{Y_k, Y_{k+1}\} = \{\sigma_i, \sigma_{i+1}\}\}$ occurs with probability at least $(\min \mu)^2$, regardless of the values of $(Y_i)_{i \notin \{k,k+1\}}$. Consequently, with probability one, the events $\mathcal{E}_k$ hold for infinitely many integers $k \in \{2, 4, 6, \ldots\}$. When $\mathcal{E}_k$ occurs, then $\mathbf{NF}_r(X_{k+2}) = (Y_{k+1}Y_k) \cdot \mathbf{NF}_r(X_k)$, hence Y_k is a left-divisor of $\mathbf{pre}_1(\mathbf{NF}_r(X_k))$ but not of $\mathbf{pre}_1(\mathbf{NF}_r(X_{k+2}))$, which proves that $\mathbf{pre}_1(\mathbf{NF}_r(X_k)) \neq \mathbf{pre}_1(\mathbf{NF}_r(X_{k+2}))$. This completes the proof. $\square$

For all integers $i \geqslant 1$, it follows from Lemma 2.37, Proposition 2.49 and Corollary 2.51 that the word $\mathbf{pre}_i(\mathbf{NF}_\ell(X_{k+1}))$ is a function of $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))$ and of Y_{k+1}. This shows that the sequence $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))$ is a Markov chain. However, the sequences $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k))$ and $\mathbf{pre}_i(\mathbf{NF}_r(X_k))$ are not necessarily monotonous (even for $i = 1$) and are not Markov chains, hence their behaviour is not so easy to capture.

Example 5.32.
In the braid monoid $\mathbf{B}_4^+$, assume that the five first terms of $(Y_k)_{k \geqslant 0}$ are σ_2, σ_1, σ_2, σ_3 and σ_3, in this order. A direct computation shows that $\mathbf{NF}_\ell(X_4) = \mathbf{NF}_r(X_4) = \sigma_3\sigma_1\sigma_2\sigma_1$, $\mathbf{NF}_\ell(X_5) = \sigma_1\sigma_3 \cdot \sigma_3\sigma_2\sigma_1$, and $\mathbf{NF}_r(X_5) = \sigma_3 \cdot \sigma_3\sigma_1\sigma_2\sigma_1$. Hence, neither $(\mathbf{suf}_1(\mathbf{NF}_\ell(X_k)))_{k \geqslant 0}$ nor $(\mathbf{pre}_1(\mathbf{NF}_r(X_k)))_{k \geqslant 0}$ is non-decreasing. In addition, neither are they finite-state Markov chains.

The situation is illustrated in Fig. 5.33. As usual, braids, which we read from left to right, are represented by braid diagrams, which we read from top to bottom. The Garside words $\mathbf{NF}_\ell(X_k)$ and $\mathbf{NF}_r(X_k)$ are represented by braid diagrams cut into several layers, each layer representing one braid letter of the word $\mathbf{NF}_\ell(X_k)$ or $\mathbf{NF}_r(X_k)$ Hence, the braid $\mathbf{suf}_1(\mathbf{NF}_\ell(X_k))$ is represented by the bottom layer of the left Garside normal form of X_k, and the braid $\mathbf{pre}_1(\mathbf{NF}_r(X_k))$ is represented by the top layer of the right Garside normal form of X_k. Both these layers are represented over a gray background.

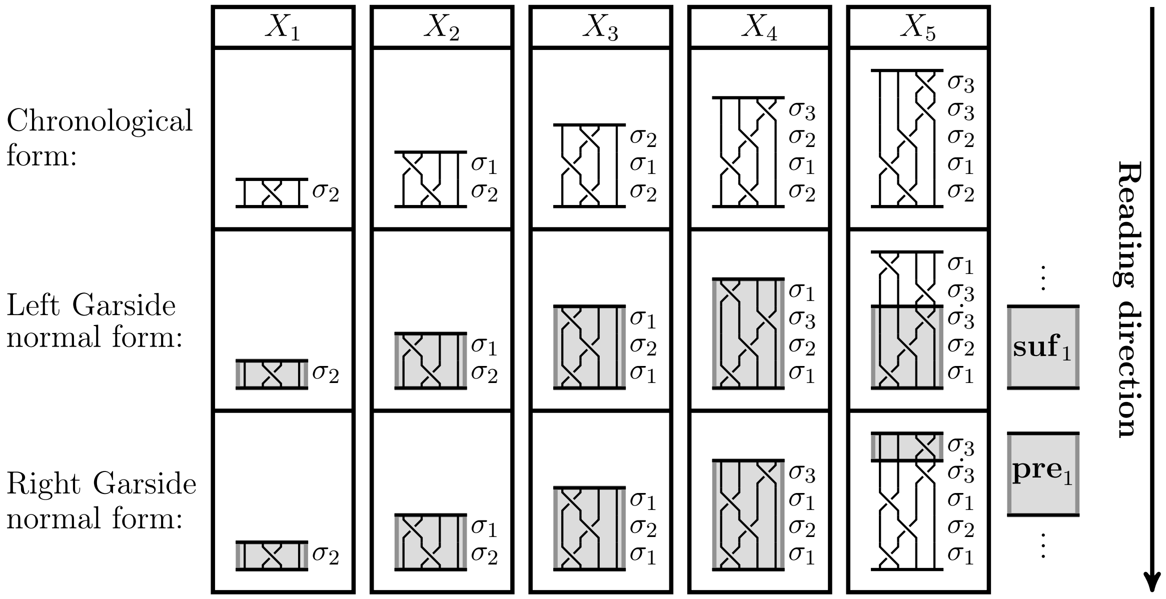


Figure 5.33 – Non-monotonic evolution of the Garside normal forms

5.3.2 Density of Garside Words

Below, we study the sequences $(\mathbf{suf}_i(\mathbf{NF}_\ell(X_k)))_{k\geqslant 0}$. We begin by introducing the additional integer $\Lambda := (2\mathfrak{D}+1)\lambda(\Delta)$. Furthermore, in what follows, we will often consider words of the form $\mathbf{NF}_\ell(\delta(\mathbf{a}))$, where $\mathbf{a}$ is some braid, and δ is the natural projection on the set $\{\mathbf{x} \in \mathbf{A}^+, \Delta^2 \text{ does not divide } \mathbf{x}\}$. Hence, we will only write $\mathbf{NF}^\delta_\ell(\mathbf{a})$: $\mathbf{NF}^\delta_\ell$ is a variant of the left Garside normal form in the context of the quotient monoid $\mathbf{A}^+/\Delta^2$.

Lemma 5.34.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, let $\mathbf{b} \in \mathbf{A}^+$ be a positive braid, and let $\underline{\mathbf{w}}$ be a Δ-free left Garside word. There exists a braid $\mathbf{a} \in \mathbf{A}^+$ of Artin length $\Lambda + \lambda(\mathbf{w})$ such that $\underline{\mathbf{w}} \lhd \mathbf{NF}^\delta_\ell(\mathbf{ab})$.

Proof. First, let $z \in \{0,1\}$ be such that $\delta(\Delta^z\mathbf{b})$ is Δ-free, and consider letters of the words $\underline{\mathbf{w}} =: w_1 \cdot \ldots \cdot w_u$ and $\mathbf{NF}^\delta_\ell(\Delta^z\mathbf{b}) =: b_1 \cdot \ldots \cdot b_v$. We define the sets $P = \mathbf{right}(w_u)$ (or $P = \{\sigma_1\}$ if $\underline{\mathbf{w}}$ is the empty word) and $Q = \mathbf{left}(b_1)$ (or $Q = \{\sigma_1\}$ if $\delta(\Delta^z\mathbf{b}) = \mathbf{1}$).

According to Proposition 5.21, there exists bilateral Garside paths $\underline{\mathbf{s}}$ and $\underline{\mathbf{t}}$ of length $\mathfrak{D}$, respectively from P to $\{\sigma_1\}$ and from $\{\sigma_1\}$ to Q. Since $\lambda(\mathbf{s}) \leqslant \mathfrak{D}\lambda(\Delta)$ and $\lambda(\mathbf{t}) \leqslant \mathfrak{D}\lambda(\Delta)$, the integer $y = \Lambda - \lambda(\mathbf{s}) - \lambda(\mathbf{t}) - z\lambda(\Delta)$ is non-negative.

Hence, consider the braid $\mathbf{a} = \mathbf{ws}\sigma_1^y\mathbf{t}\Delta^z$. We obviously have $\lambda(\mathbf{a}) = \Lambda + \lambda(\mathbf{w})$, and Corollary 2.41 proves that $\mathbf{NF}^\delta_\ell(\mathbf{ab}) = \mathbf{NF}^\delta_\ell(\mathbf{a}\delta(\mathbf{b})) = \underline{\mathbf{w}} \cdot \underline{\mathbf{s}} \cdot (\sigma_1)^y \cdot \underline{\mathbf{t}} \cdot \mathbf{NF}^\delta_\ell(\Delta^z\mathbf{b})$, which completes the proof. □

Lemma 5.34 provides us with lower bounds on the probability of finding a given Garside word in the left Garside normal form of a braid X_m when m is big enough.

Proposition 5.35.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathbf{A}^+$ with Artin steps. Let $\underline{\mathbf{w}}$ be a Δ-free left Garside word and let m be an integer such that $m \geqslant \Lambda + \lambda(\mathbf{w})$. We have $\mathbb{P}\left[\underline{\mathbf{w}} \lhd \mathbf{NF}^\delta_\ell(X_m)\right] \geqslant (\min\mu)^{\Lambda+\lambda(\mathbf{w})}$, where $\min\mu$ denotes the positive real constant $\min_i \mu(\sigma_i)$.

Proof. Consider the event $E := \{\underline{\mathbf{w}} \lhd \mathbf{NF}^\delta_\ell(X_m)\}$ and the integer $j := m - \Lambda - \lambda(\mathbf{w})$. For every braid $\mathbf{b} \in \mathbf{A}^+$, Lemma 5.34 states that there exists a braid $\mathbf{a}$ such that $\lambda(\mathbf{a}) = \Lambda + \lambda(\mathbf{w})$ and $\underline{\mathbf{w}} \lhd \mathbf{NF}^\delta_\ell(\mathbf{ab})$. Therefore, we have

$$\mathbb{P}[E \mid X_j = \mathbf{b}] \geqslant \mathbb{P}[Y_{m-1}\ldots Y_j = \mathbf{a} \mid X_j = \mathbf{b}] \geqslant (\min\mu)^{\Lambda+\lambda(\mathbf{w})},$$

whence $\mathbb{P}[E] \geqslant \inf_{\mathbf{b}\in\mathbf{A}^+} \mathbb{P}[E \mid X_j = \mathbf{b}] \geqslant (\min\mu)^{\Lambda+\lambda(\mathbf{w})}$. ☐

Theorem 5.36.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, Let $\underline{\mathbf{w}}$ be a Δ-free left Garside word, and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathbf{A}^+$ with Artin steps. We have

$$\liminf_{k\to\infty} \frac{\mathbb{E}[\mathfrak{f}_{\underline{\mathbf{w}}}(X_k)]}{k} > 0.$$

Proof. First, let λ be the leftmost letter of $\underline{\mathbf{w}}$, or $\lambda = \sigma_1$ if $\underline{\mathbf{w}}$ is empty. In addition, let $\underline{\mathbf{x}}$ be a bilateral Garside path from $\{\sigma_1\}$ to $\mathbf{left}(\lambda)$, and let $\underline{\mathbf{y}} := \underline{\mathbf{x}} \cdot \underline{\mathbf{w}}$.

Let j and k be integers such that $\Lambda + \lambda(\mathbf{y}) \leqslant j \leqslant k - \Lambda - 1$, and let $Z_{j,k} := Y_{k-1}\ldots Y_j$. Since X_j and $Z_{j,k}$ are independent, so are the events $E^j_1 := \{\underline{\mathbf{y}} \lhd \mathbf{NF}^\delta_\ell(X_j)\}$ and $E^j_2 := \{\mathbf{NF}_r(\delta(Z_{j,k})) \rhd (\sigma_1)\}$. By construction, the random braids $Z_{j,k}$ and X_{k-j} follow the same law. Therefore, Proposition 5.35 states that $\mathbb{P}[E^j_1] \geqslant (\min\mu)^{\Lambda+\lambda(\mathbf{y})}$ and that $\mathbb{P}[E^j_2] \geqslant (\min\mu)^{\Lambda+1}$, hence that $\mathbb{P}[E^j_1 \cap E^j_2] \geqslant (\min\mu)^{2\Lambda+\lambda(\mathbf{y})+1}$.

Let us assume in this paragraph that the event $E^j_1 \cap E^j_2$ holds. Let $z_1 \cdot \ldots \cdot z_v$ be the left Garside normal form of $Z_{j,k}$, and let $u \geqslant 0$ be the integer such that $Z_{j,k} = \Delta^{2u}\delta(Z_{j,k})$. It follows that $\mathbf{NF}^\delta_\ell(Z_{j,k}) = z_{2u+1} \cdot \ldots \cdot z_v$, hence that $\sigma_1 = \alpha_r(\delta(Z_{j,k})) \geqslant_r z_v >_r \mathbf{1}$, i.e. $z_v = \sigma_1$. Consequently, Corollary 2.41 applies to the braid $\delta(X_k) = \delta(Z_{j,k})\delta(X_j)$, which proves that $\mathbf{NF}^\delta_\ell(X_k) = \mathbf{NF}^\delta_\ell(Z_{j,k}) \cdot \mathbf{NF}^\delta_\ell(X_j)$, and therefore that $-|\mathbf{NF}^\delta_\ell(X_j)|$ is a (negative) $\underline{\mathbf{y}}$-flag (and thus a $\underline{\mathbf{w}}$-flag) of X_k.

We enter now the second step of the proof. Let σ_i and σ_j be two generators such that $m_{i,j} \neq 2$. The subgroup of $\mathbf{A}^+/\Delta^2$ generated by σ_i^2 and $\sigma_i\sigma_j^2\sigma_i$ is isomorphic to a free group $\mathbb{Z} * \mathbb{Z}$. The random walk in the group $\mathbf{A}^+/\Delta^2$ is therefore transient [67, 88, 93], i.e. there exists a real constant $\theta < 1$ such that $\mathbb{P}[\exists i \geqslant 1, \delta(X_i) = \mathbf{1}] = \theta$.

Let q be some integer such that $q > (2\Lambda + \lambda(\mathbf{y}) + 1)\frac{\ln(\min\mu)}{\ln(\theta)}$. We say that an integer i has q returns in the random walk (X_k) if the set $\{j > i : \delta(X_j) = \delta(X_i)\}$ has cardinality

q or more. Such an event happens with probability θ^q, which means that $\mathbb{P}[j \in R_q] = \theta^q$, where R_q is the set of integers $i \in \mathbb{N}$ with q returns.

For all integers $u \geqslant 0$, consider the sets $\mathcal{I}_{j,u} := E_1^j \cap E_2^j \cap \{|\mathbf{NF}_\ell^\delta(X_j)| = u\}$ and $\mathcal{J}_u := \{j \in \{0,\ldots,k\} : j \in \mathcal{I}_{j,u}\}$. For each $j \in \mathcal{J}_u$, note that $\mathbf{NF}_\ell^\delta(X_j) = \mathbf{suf}_u(\mathbf{NF}_\ell^\delta(X_k))$. Hence, if the elements of $\mathcal{J}_u$ are $j_1 < \ldots < j_r$, the integers $j_1, \ldots, j_{r-q}$ must belong to R_q. This proves that $|\mathcal{J}_u \backslash R_q| \leqslant q$, and it follows that

$$\begin{aligned}
\sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1} \mathbf{1}_{\mathcal{I}_{j,u}} &= \sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1} \left(\mathbf{1}_{j\in\mathcal{J}_u\backslash R_q} + \mathbf{1}_{j\in R_q, |\mathbf{NF}_\ell^\delta(X_j)|=u}\right)\mathbf{1}_{\mathcal{I}_{j,u}} \\
&\leqslant \left(\sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1} \mathbf{1}_{j\in\mathcal{J}_u\backslash R_q}\right)\mathbf{1}_{\bigcup_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1}\mathcal{I}_{j,u}} + \sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1} \mathbf{1}_{j\in R_q, |\mathbf{NF}_\ell^\delta(X_j)|=u} \\
&\leqslant q\mathbf{1}_{\bigcup_{j\leqslant k}\mathcal{I}_{j,u}} + \sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1} \mathbf{1}_{j\in R_q, |\mathbf{NF}_\ell^\delta(X_j)|=u}.
\end{aligned}$$

Furthermore, note that $E_1^j \cap E_2^j = \bigcup_{u\geqslant 0}\mathcal{I}_{j,u}$ and that $-u \in \mathbf{F}_{\underline{\mathbf{w}}}^-$ whenever $\bigcup_{j\leqslant k}\mathcal{I}_{j,u}$ is satisfied. It follows that

$$\begin{aligned}
\mathfrak{f}_{\underline{\mathbf{w}}}(X_k) &= \sum_{u\geqslant 0}\mathbf{1}_{-u\in\mathbf{F}_{\underline{\mathbf{w}}}^-(X_k)} \geqslant \sum_{u\geqslant 0}\mathbf{1}_{\bigcup_{j\leqslant k}\mathcal{I}_{j,u}} \geqslant \frac{1}{q}\sum_{u\geqslant 0}\left(\sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1}\mathbf{1}_{\mathcal{I}_{j,u}} - \mathbf{1}_{j\in R_q,|\mathbf{NF}_\ell^\delta(X_j)|=u}\right) \\
&\geqslant \frac{1}{q}\sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1}\left(\sum_{u\geqslant 0}\mathbf{1}_{\mathcal{I}_{j,u}} - \sum_{u\geqslant 0}\mathbf{1}_{j\in R_q,|\mathbf{NF}_\ell^\delta(X_j)|=u}\right) = \frac{1}{q}\sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1}(\mathbf{1}_{E_1^j\cap E_2^j} - \mathbf{1}_{j\in R_q}),
\end{aligned}$$

hence that

$$\mathbb{E}\left[\mathfrak{f}_{\underline{\mathbf{w}}}(X_k)\right] \geqslant \sum_{j=\Lambda+\lambda(\mathbf{y})}^{k-\Lambda-1}\frac{\mathbb{P}\left[E_1^j \cap E_2^j\right] - \mathbb{P}\left[j\in R_q\right]}{q} \geqslant (k-2\Lambda-\lambda(\mathbf{y}))\frac{(\min\mu)^{2\Lambda+\lambda(\mathbf{y})+1} - \theta^q}{q}.$$

Since $q > (2\Lambda + \lambda(\mathbf{y}) + 1)\frac{\ln(\min\mu)}{\ln(\theta)}$, it follows that

$$\liminf_{k\to\infty}\frac{\mathbb{E}[\mathfrak{f}_{\underline{\mathbf{w}}}(X_k)]}{k} \geqslant \frac{(\min\mu)^{2\Lambda+\lambda(\mathbf{y})+1} - \theta^q}{q} > 0.$$

□

Corollary 5.37.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathbf{A}^+$ with Artin steps. Let Ω be a set of Δ-free left Garside words. We have

$$\liminf_{k\to\infty}\frac{\mathbb{E}[\mathfrak{f}_\Omega(X_k)]}{k} > 0.$$

5.3.3 Stabilisation in the Artin–Tits Monoid

Lemma 5.38.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators. Let $\underline{\mathbf{w}}$ be a Δ-free left Garside word $\underline{\mathbf{w}}$. There exists a blockable word $\underline{\mathbf{z}}$ such that, for all braids $\mathbf{b} \in \mathbf{A}^+$, $\mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(\mathbf{b}) \leqslant 2\mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{b})$.

Proof. If $\underline{\mathbf{w}}$ is the empty word, then the blockable word $\underline{\mathbf{z}} := \sigma_n^{-1}\Delta$ is just fine. Henceforth, we assume that the word $\underline{\mathbf{w}} = w_1 \cdot \ldots \cdot w_k$ is not empty. Using Proposition 5.21, let $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ be bilateral Garside paths of length $\mathfrak{D}$, respectively from $\mathbf{right}(\phi_\Delta(w_k))$ to $\mathbf{left}(w_1)$ and from $\mathbf{right}(w_k)$ to $\{\sigma_1, \ldots, \sigma_{n-2}\}$. The word $\underline{\mathbf{z}} := \phi_\Delta(\underline{\mathbf{w}}) \cdot \underline{\mathbf{x}} \cdot \underline{\mathbf{w}} \cdot \underline{\mathbf{y}}$ is a blockable word.

Then, let i be some $\mathcal{B}_{\underline{\mathbf{z}}}$-flag of $\mathbf{b}$, and let $\underline{\mathbf{p}}$ be some $\underline{\mathbf{z}}$- or $\phi_\Delta(\underline{\mathbf{z}})$-blocking pattern such that i is a $\underline{\mathbf{p}}$-flag. If $\underline{\mathbf{p}}$ is a $\underline{\mathbf{z}}$-blocking pattern, then $i + |\underline{\mathbf{w}}| + \mathfrak{D}$ is a $\underline{\mathbf{w}}$-flag; if $\underline{\mathbf{p}}$ is a $\phi_\Delta(\underline{\mathbf{z}})$-blocking pattern, then i is a $\underline{\mathbf{w}}$-flag. Consequently,

$$\mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(\mathbf{b}) = \sum_{i \geqslant 0} \mathbf{1}_{i \in \mathbf{F}^+_{\mathcal{B}_{\underline{\mathbf{z}}}}(\mathbf{b})} \leqslant \sum_{i \geqslant 0} \mathbf{1}_{i \in \mathbf{F}^+_{\underline{\mathbf{w}}}(\mathbf{b})} + \mathbf{1}_{i + |\underline{\mathbf{w}}| + \mathfrak{D} \in \mathbf{F}^+_{\underline{\mathbf{w}}}(\mathbf{b})} \leqslant 2 \sum_{i \geqslant 0} \mathbf{1}_{i \in \mathbf{F}^+_{\underline{\mathbf{w}}}(\mathbf{b})} = 2\mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{b}).$$

□

Theorem 5.39.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, and let $(X_k)_{k \geqslant 0}$ be a left random walk in $\mathbf{A}^+$ with Artin steps. For each integer $k \geqslant 0$, let $\underline{\mathbf{s}}(k)$ be the largest common suffix of all words $\mathbf{NF}_\ell(X_j)$ such that $j \geqslant k$. In addition, let $\underline{\mathbf{w}}$ be a Δ-free left Garside word. Then, $\liminf \frac{1}{k}\mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{s}(k))$ is almost surely a positive constant.

Proof. According to Lemma 5.38, there exists a blockable word $\underline{\mathbf{z}}$ such that $\mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(\mathbf{b}) \leqslant 2\mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{b})$ for all positive braids $b \in \mathbf{A}^+$.

We consider now the random variables $\mathfrak{q}_k := \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(X_k) + 2|\mathcal{S}|^{\mathfrak{D}+1} + 2|\underline{\mathbf{z}}| + 3\mathfrak{D} + 3$, for all integers $k \geqslant 0$. In addition, let $\mathbf{S} : (Y_0, Y_1, \ldots) \mapsto (Y_1, Y_2, \ldots)$ be the *shift* operator. Proposition 5.29 states that $\mathfrak{q}_{k+\ell} \leqslant \mathfrak{q}_k + \mathfrak{q}_\ell \circ \mathbf{S}^k$. Since $\mathbf{S}$ is measure-preserving, Kingman ergodic sub-additive theorem states that $\frac{\mathfrak{q}_k}{k} \to \liminf \mathbb{E}\left[\frac{\mathfrak{q}_k}{k}\right]$ almost surely. Moreover, Corollary 5.37 states that $\liminf \mathbb{E}\left[\frac{\mathfrak{f}_{\mathcal{B}_{\mathbf{z}}}(X_k)}{k}\right] > 0$, hence that $\liminf \mathbb{E}\left[\frac{\mathfrak{q}_k}{k}\right] > 0$.

Now, consider the positive real constant $\theta = \liminf \mathbb{E}\left[\frac{\mathfrak{q}_k}{k}\right]$. In what follows, for all integers $k \geqslant 0$, we denote by θ_k the integer $\lceil \frac{k}{2}\theta \rceil$. We almost surely have $\frac{\mathfrak{q}_k}{k} \to \theta$, whence the existence of some integer $K \geqslant 0$ such that $\mathfrak{q}_k \geqslant \theta_k + 2|\mathcal{S}|^{\mathfrak{D}+1} + 3|\underline{\mathbf{z}}| + 4\mathfrak{D} + 4$, i.e. $\mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(X_k) \geqslant \theta_k + |\underline{\mathbf{z}}| + \mathfrak{D} + 1$, for all integers $k \geqslant K$.

Then, consider integers $k \geqslant m \geqslant K$, and let u and v be respectively the θ_m-th and $(\theta_m + |\underline{\mathbf{z}}| + \mathfrak{D} + 1)$-th largest elements of $\mathbf{F}^-_{\mathbf{B}_{\underline{\mathbf{z}}}}(X_k)$. Note that $u \geqslant v + |\underline{\mathbf{z}}| + \mathfrak{D} + 1$, and that $|\underline{\mathbf{z}}| + \mathfrak{D} + 1$ is the length of all $\underline{\mathbf{z}}$-blocking patterns. Therefore, we can factor $\mathbf{NF}_\ell(X_k)$ into a product $\mathbf{NF}_\ell(X_k) = \underline{\mathbf{a}} \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{a}}' \cdot \underline{\mathbf{q}} \cdot \underline{\mathbf{a}}''$, where $\underline{\mathbf{p}}$ and $\underline{\mathbf{q}}$ are $\underline{\mathbf{z}}$-blocking patterns, $|\underline{\mathbf{q}} \cdot \underline{\mathbf{a}}''| = -u$ and $|\underline{\mathbf{p}} \cdot \underline{\mathbf{a}}' \cdot \underline{\mathbf{q}} \cdot \underline{\mathbf{a}}''| = -v$.

Corollary 5.25 states that the word $\underline{\mathbf{a}}' \cdot \underline{\mathbf{q}} \cdot \underline{\mathbf{a}}''$ is a suffix of $\mathbf{NF}_\ell(X_{k+1})$, hence that u is the θ_m-th smallest element of $\mathbf{F}_{\mathcal{B}_{\underline{\mathbf{z}}}}(X_{k+1})$. An immediate induction proves then that $\underline{\mathbf{q}} \cdot \underline{\mathbf{a}}''$ is a suffix of all words $\mathbf{NF}_\ell(X_k)$ for $k \geqslant m$. Therefore, $\underline{\mathbf{q}} \cdot \underline{\mathbf{a}}''$ is a suffix of $\underline{\mathbf{s}}(k)$, and $2\mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{s}(k)) \geqslant \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(\mathbf{s}(k)) \geqslant \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(\mathbf{qa}'') \geqslant \theta_m$. Since this is true for all integers $k \geqslant m \geqslant K$, it follows that $2\mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{s}(k)) \geqslant \theta_k \geqslant \frac{k}{2}\theta$, hence that $\liminf \frac{1}{k}\mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{s}(k)) \geqslant \frac{\theta}{4} > 0$ almost surely. □

We derive the following result from Theorem 5.39.

Theorem 5.40.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathbf{A}^+$ with Artin steps. Let $i \geqslant 0$ be an integer. The sequence $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k))_{k\geqslant 0}$ is almost surely ultimately constant, i.e., with probability one, there exists an integer $K \geqslant 0$ such that $\mathbf{suf}_i(\mathbf{NF}_\ell(X_k)) = \mathbf{suf}_i(\mathbf{NF}_\ell(X_K))$ for all $k \geqslant K$.

Proof. For each integer $k \geqslant 0$, let $\underline{\mathbf{s}}(k)$ be the largest common suffix of all words $\mathbf{NF}_\ell(X_j)$ such that $j \geqslant k$. Let $\underline{\varepsilon}$ be the empty word. According to Theorem 5.39, there almost surely exists a real constant $\theta > 0$ such that $\liminf \frac{1}{k}\mathfrak{f}_{\underline{\varepsilon}}(\mathbf{s}(k)) \geqslant \theta$. Since $\mathfrak{f}_{\underline{\varepsilon}}(\mathbf{s}(k)) \leqslant |\underline{\mathbf{s}}(k)|$, this proves that $|\underline{\mathbf{s}}(k)| \to +\infty$. □

Theorem 5.40 completes our picture of which prefixes and suffixes of the Garside normal forms of (X_k) have stabilisation properties, which we sum up in Fig. 5.41. Note that, this time, the results that we obtain are not similar to those in the heap groups and monoids, since the sequence of words $\mathbf{NF}_\ell(X_k)_{k\geqslant 0}$ is almost surely prefix-convergent.

	Convergence of the words	
	$\mathbf{NF}_\ell(X_k)_{k\geqslant 0}$	$\mathbf{NF}_r(X_k)_{k\geqslant 0}$
prefix-	✓	✗
suffix-	✓	✓

Figure 5.41 – Convergence of the normal forms of the random walk in the braid monoid

5.3.4 From Artin–Tits Monoids to Groups

In Sections 5.3.1, 5.3.2 and 5.3.3, we considered random walks in the monoid $\mathbf{A}^+$, supported by a probability measure μ over the set of *generators*. We aim now at translating those results in the context of random walks in the monoid $\mathbf{A}^+$ (or the group $\mathbf{A}$), supported by any probability measure μ with finite first moment, and whose support generates the whole monoid (or the whole group).

We begin with variants of Corollary 5.25 and of Theorem 5.40.

Corollary 5.42.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators. Let $\underline{\mathbf{w}}$ be a blockable *word, let $\mathbf{b}, \mathbf{c} \in \mathbf{A}^+$ be two positive braids, and let k be the Artin length of the braid $\mathbf{b}$. Let us assume that we can write $\mathbf{NF}_\ell(\mathbf{c})$ as a product $\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{p}}_1 \cdot \underline{\mathbf{a}}_2 \cdot \ldots \cdot \underline{\mathbf{a}}_k \cdot \underline{\mathbf{p}}_k \cdot \underline{\mathbf{a}}_{k+1}$, where each sub-word $\underline{\mathbf{p}}_i$ is a blocking pattern. We have $\mathbf{NF}_\ell(\mathbf{bc}) \rhd \underline{\mathbf{a}}_{k+1}$.*

Proof. Corollary 5.25 implies Corollary 5.42 when $k = 1$. An immediate induction on k completes the proof. □

Proposition 5.43.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators. Let μ be a probability measure over $\mathbf{A}^+$ with finite first moment such that $\{\sigma_1, \ldots, \sigma_n\} \subseteq$ supp(μ)*, and let $(X_k)_{k \geqslant 0}$ be the left random walk with step-distribution μ. For all integers $i \geqslant 1$, the sequence $(\mathbf{suf}_i(\mathbf{NF}_\ell(X_k)))_{k \geqslant 0}$ is almost surely ultimately constant.*

Proof. The proofs in Sections 5.2, 5.3.2 and Lemma 5.38 still apply. Using Corollary 5.42 instead of Corollary 5.25, we adapt the proof of Theorem 5.39 as follows.

Like in the proof of Theorem 5.39, we have $\frac{\mathfrak{q}_k}{k} \to \theta$ for some real number $\theta > 0$, and have fixed some words $\underline{\mathbf{w}}$ and $\underline{\mathbf{z}}$. We define now the real number $\lambda := \frac{\theta}{3(|\underline{\mathbf{z}}| + \mathfrak{D} + 1)}$ and the events $\mathcal{L}_k^\lambda = \{\lambda(Y_k) \geqslant \lambda k\}$, for $k \geqslant 0$. Since $\sum_{k \geqslant 0} \mathbb{P}[\mathcal{L}_k^\lambda] \leqslant (1 + \lambda^{-1})\, \mathbb{E}_\mu[\lambda(X)] < +\infty$, Borel-Cantelli theorem states that there almost surely exists some integer $L \geqslant 0$ such that none of the events $(\mathcal{L}_k^\lambda)_{k \geqslant L}$ holds.

Since $\frac{\mathfrak{q}_k}{k} \to \theta$, there exists some integer $K \geqslant L$ such that $\mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(X_k) \geqslant 2\Theta_k$ for all integers $k \geqslant K$, where $\Theta_k := \left\lceil \frac{\theta k}{3} \right\rceil + z + \mathfrak{D} + 1$. Consider integers $k \geqslant m \geqslant K$, and let u and v be respectively the Θ_m-th and $2\Theta_k$-th largest elements of $\mathbf{F}^-_{\mathcal{B}_{\underline{\mathbf{z}}}}(X_k)$. Note that $u \geqslant v + \Theta_k \geqslant v + (z + \mathfrak{D} + 1)(\lambda k + 1) \geqslant v + (z + \mathfrak{D} + 1)(\lambda(Y_k) + 1)$ and that $|\underline{\mathbf{z}}| + \mathfrak{D} + 1$ is the length of all $\underline{\mathbf{z}}$-blocking patterns.

Since $\mathcal{L}_k^\lambda$ does not hold, we have $\lambda(Y_k) \leqslant \lambda k$, and we can therefore factor $\mathbf{NF}_\ell(X_k)$ into a product $\mathbf{NF}_\ell(X_k) = \underline{\mathbf{a}}_1 \cdot \underline{\mathbf{p}}_1 \cdot \underline{\mathbf{a}}_2 \cdot \ldots \cdot \underline{\mathbf{a}}_{\lambda(Y_k)+1} \cdot \underline{\mathbf{p}}_{\lambda(Y_k)+1} \cdot \underline{\mathbf{a}}_{\lambda(Y_k)+2}$, where $|\underline{\mathbf{p}}_{\lambda(Y_k)+1} \cdot \underline{\mathbf{a}}_{\lambda(Y_k)+2}| = -u$, $|\underline{\mathbf{p}}_1 \cdot \underline{\mathbf{a}}_2 \cdot \ldots \cdot \underline{\mathbf{a}}_{\lambda(Y_k)+2}| = -v$ and each $\underline{\mathbf{p}}_i$ is a $\underline{\mathbf{z}}$-blocking pattern. Hence, Corollary 5.42 states that $\underline{\mathbf{p}}_{\lambda(Y_k)+1} \cdot \underline{\mathbf{a}}_{\lambda(Y_k)+2}$ is a suffix of $\mathbf{NF}_\ell(X_{k+1})$, hence of all words $\mathbf{NF}_\ell(X_k)$ for $k \geqslant m$. From this, we conclude that $\liminf \frac{1}{k} \mathfrak{f}_{\mathcal{B}_{\underline{\mathbf{z}}}}(\mathbf{s}(k)) \geqslant \frac{\theta}{3}$ and that $\liminf \frac{1}{k} \mathfrak{f}_{\underline{\mathbf{w}}}(\mathbf{s}(k)) \geqslant \frac{\theta}{6} > 0$ almost surely.

In particular, the case where $\underline{\mathbf{w}}$ is the empty word implies Proposition 5.43. □

Theorem 5.44.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators. Let μ be a probability measure over $\mathbf{A}$ with finite first moment such that supp(μ) *generates $\mathbf{A}$ as a monoid, and let $(X_k)_{k \geqslant 0}$ be the left random walk with step-distribution μ. For all integers $i \geqslant 1$, the sequence $(\mathbf{suf}_i(\mathbf{NF}_\ell(X_k)))_{k \geqslant 0}$ is almost surely ultimately constant.*

Proof. First, we leave the context of random walks in the monoid $\mathbf{A}^+$ to focus on random walks in the quotient monoid $\mathbf{A}^+/\Delta^2$. This step is straightforward: we just need to identify the left Garside normal form of a positive braid $\mathbf{b} \in \mathbf{A}^+$ with the left Garside normal form of the braid $\delta(\mathbf{b})$.

Second, we consider probability measures μ on $\mathbf{A}^+/\Delta^2$ with finite moment and such that $\mathrm{supp}(\mu)$ generates the whole monoid $\mathbf{A}^+/\Delta^2$. Indeed, let K be a positive integer such that each braid σ_i is a product of at most K elements of $\mathrm{supp}(\mu)$. Then, for each integer $i \in \{1, \ldots, n\}$, consider the stopping time $\tau_i := \min\{k : Y_k = \sigma_i\}$. We also consider the stopping time $\tau := \min\{K, \tau_1, \tau_2, \ldots, \tau_{n-1}\}$ and the convolution power $\mu^{*\tau}$. The probability measure $\mu^{*\tau}$ has a finite first moment and $\{\sigma_1, \ldots, \sigma_n\} \subseteq \mathrm{supp}(\mu^{*\tau})$, hence Proposition 5.43 applies to the random walk supported by $\mu^{*\tau}$ in the monoid $\mathbf{A}^+/\Delta^2$. Then, since τ is a bounded stopping time and since μ has a finite first moment, the arguments used in the end of the proof Proposition 5.43 help us generalise Proposition 5.43 to the random walk supported by the distribution μ itself.

Finally, and since $\mathbf{A}^+/\Delta^2$ and $\mathbf{A}/\Delta^2$ are isomorphic groups, we step back from a random walk in $\mathbf{A}^+/\Delta^2$ to a random walk in $\mathbf{A}$ itself, and consider probability measures μ on $\mathbf{A}$ such that μ has a finite first moment and $\mathrm{supp}(\mu)$ is generates the monoid $\mathbf{A}$ itself. Indeed, we just proved that arbitrarily long suffixes of $\mathbf{NF}_\ell^\delta(X_k)$ have a stabilisation property. Since such suffixes are also suffixes of $\mathbf{NF}_\ell(X_k)$, we obtain precisely Theorem 5.44. □

Variants of Theorem 5.44 also apply for different normal forms, e.g. the symmetric Garside normal form. Indeed, for each integer $k \geqslant 0$, let u_k be the infimum of the braid X_k, i.e. the smallest integer such that $\Delta^{-u_k} X_k$ belongs to $\mathbf{A}^+$. According to [88], there exists real constants δ and γ (with $\gamma > 0$), that depend on the distribution μ, and such that the limits $\frac{u_k}{k} \to \delta$ and $\frac{|\mathbf{NF}_\ell^\delta(X_k)|}{k} \to \gamma$ almost surely hold.

If $\delta > 0$, then, almost surely, all but finitely many values of the random walk $(X_k)_{k \geqslant 0}$ will be positive braids, i.e. belong to $\mathbf{A}^+$. In such a case, we only have to consider the standard left Garside normal form on positive braids. However, if $\delta \leqslant 0$, then, almost surely, infinitely many values of the random walk $(X_k)_{k \geqslant 0}$ are not positive braids.

If $-\gamma < \delta < 0$, then, almost surely, all but finitely many of the braids X_k have a non-empty positive part, whose length almost surely grows at a rate $\delta - \gamma$. This proves that the suffixes of $\mathbf{NF}^+_{\mathrm{sym}}(X_k)$ have a stabilisation property. If $\delta < -\gamma$, then, almost surely, all but finitely many of the braids X_k are negative, i.e inverses of positive braids, and therefore have an empty positive part. Then, the words $\mathbf{NF}^-_{\mathrm{sym}}(X_k)$ begin with prefixes of the form $\Delta \cdot \ldots \cdot \Delta$, whose length L_k almost surely grows at a rate $-\gamma - \delta$. These prefixes are followed by subwords $\underline{\mathbf{w}}$, whose length grows at a positive rate, and such that $\phi_\Delta^{L_k}(\underline{\mathbf{w}})$ has a stabilisation property.

Finally, if $\delta = -\gamma$, then, almost surely, infinitely many of the braids X_k have a non-empty positive part, and infinitely many have an empty positive part. Therefore, the suffixes of $\mathbf{NF}^+_{\mathrm{sym}}(X_k)$ and the prefixes of $\mathbf{NF}^-_{\mathrm{sym}}(X_k)$ certainly have no stabilisation property, even up to applying the conjugation morphism ϕ_Δ.

5.3.5 Deleting Occurrences of Δ

Proposition 5.31 proves that, for all integers $i \geqslant 1$, both sequences $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))_{k\geqslant 0}$ and $\mathbf{suf}_i(\mathbf{NF}_r(X_k))_{k\geqslant 0}$ must almost surely converge towards the word $\Delta \cdot \ldots \cdot \Delta$. Hence, we might informally describe the situation by saying that, when studying the asymptotic behaviour of the sequences $\mathbf{pre}_i(\mathbf{NF}_\ell(X_k))_{k\geqslant 0}$ and $\mathbf{suf}_i(\mathbf{NF}_r(X_k))_{k\geqslant 0}$, we are "blinded" by the occurrences of Δ. Although counting these occurrences of Δ would provide some relevant information about the sequences $\mathbf{NF}_\ell(X_k)_{k\geqslant 0}$ and $\mathbf{NF}_r(X_k)_{k\geqslant 0}$, the presence of Δ letters obfuscates all informations that might be contained in the leftmost part of the words $\mathbf{NF}_\ell(X_k)$ or in the rightmost part of $\mathbf{NF}_r(X_k)$. Consequently, it is tempting to consider the words $\mathbf{NF}_\ell(X_k)$ from which occurrences of Δ have been deleted. This gives rise to the notion of Δ-free Garside normal form, which is analogous to the words $\mathbf{NF}_\ell^\delta$ defined above.

Definition 5.45 (Δ-free Garside normal forms in the braid group $\mathbf{A}$)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type. Let $\mathbf{a} \in \mathbf{A}$ be a braid. There exists an integer $\inf(\mathbf{a}) \in \mathbb{Z}$ *and a Δ-free, positive braid $\tilde{\mathbf{a}} \in \mathbf{A}^+$ such that $\mathbf{a} = \tilde{\mathbf{a}}\Delta^{\inf(\mathbf{a})}$. The* Δ-free left Garside normal form *of $\mathbf{a}$ is defined as the word $\mathbf{NF}_\ell^\Delta(\mathbf{a}) := \mathbf{NF}_\ell(\tilde{\mathbf{a}})$, and the* Δ-free right Garside normal form *of $\mathbf{a}$ is defined as the word $\mathbf{NF}_r^\Delta(\mathbf{a}) := \mathbf{NF}_r(\tilde{\mathbf{a}})$.*

Considering the Δ-free Garside normal forms instead on the usual Garside normal forms consists in moving from the group $\mathbf{A}$ to the quotient space $\langle\Delta\rangle/\mathbf{A}$. Note that this quotient space is itself a group if and only if Δ belongs to the centre of $\mathbf{A}$. In this new framework, the above study of the random walk in the braid group and monoid has direct implications. For the sake of simplicity, we step back here to the case where μ has range $\{\sigma_1, \ldots, \sigma_n\}$. Using the same process as in the above section, we might easily generalise these results to the case where μ has finite first moment.

Proposition 5.46.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathbf{A}^+$ with Artin steps. For all integers $i \geqslant 1$, the sequences $\mathbf{pre}_i(\mathbf{NF}_\ell^\Delta(X_k))_{k\geqslant 0}$ and $\mathbf{pre}_i(\mathbf{NF}_r^\Delta(X_k))_{k\geqslant 0}$ are almost surely divergent.

Proof. It follows immediately from Proposition 5.35 that both sets $\Omega_1 := \{k \geqslant 0 : \sigma_1 \lhd \mathbf{NF}_\ell^\Delta(X_k)\}$ and $\Omega_2 := \{k \geqslant 0 : \sigma_2 \lhd \mathbf{NF}_\ell^\Delta(X_k)\}$ are almost surely infinite, which proves that $\mathbf{pre}_i(\mathbf{NF}_\ell^\Delta(X_k))_{k\geqslant 0}$ is almost surely divergent.

Furthermore, since $\mathbf{pre}_1(\mathbf{NF}_r^\Delta(X_k))$ is a left-divisor of $\mathbf{pre}_i(\mathbf{NF}_\ell^\Delta(X_k))$ for all $k \geqslant 0$, it also follows that $\mathbf{pre}_1(\mathbf{NF}_r^\Delta(X_k)) = \sigma_1$ when $k \in \Omega_1$ and that $\mathbf{pre}_1(\mathbf{NF}_r^\Delta(X_k)) = \sigma_2$ when $k \in \Omega_2$, which proves that $\mathbf{pre}_i(\mathbf{NF}_r^\Delta(X_k))_{k\geqslant 0}$ is almost surely divergent too. □

Proposition 5.47.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators,

and let $(X_k)_{k\geqslant 0}$ be a left random walk in $\mathbf{A}^+$ with Artin steps. For all integers $i \geqslant 1$, the sequences $\mathbf{suf}_i(\mathbf{NF}_\ell^\Delta(X_k))_{k\geqslant 0}$ and $\mathbf{suf}_i(\mathbf{NF}_r^\Delta(X_k))_{k\geqslant 0}$ are almost surely convergent.

Proof. First, since the random walk $(X_k)_{k\geqslant 0}$ in the quotient group $\mathbf{A}/\Delta^2$ is transient, we know that $|\mathbf{NF}_\ell^\Delta(X_k)| \to +\infty$ almost surely. Moreover, if $|\mathbf{NF}_\ell^\Delta(X_k)| \geqslant i$, then $\mathbf{suf}_i(\mathbf{NF}_\ell^\Delta(X_k)) = \mathbf{suf}_i(\mathbf{NF}_\ell(X_k))$, hence the convergence of $\mathbf{suf}_i(\mathbf{NF}_\ell^\Delta(X_k))_{k\geqslant 0}$ follows from Theorem 5.40.

Second, Theorem 5.39 proves that $\mathfrak{f}_{\underline{\varepsilon}}(\mathbf{s}(k)) \to +\infty$ almost surely, where $\underline{\varepsilon}$ denotes the empty word. Consequently, it follows from the second part of Lemma 5.24 that the sequence $\mathbf{suf}_i(\mathbf{NF}_r^\Delta(X_k))_{k\geqslant 0}$ converges almost surely. □

Propositions 5.46 and 5.47 provide us with a different situation, which we sum up in Fig. 5.48. Observe that this situation is different from that of standard Garside normal forms, and analogous to that of heap monoids and groups.

	Convergence of the words	
	$\mathbf{NF}_\ell^\Delta(X_k)_{k\geqslant 0}$	$\mathbf{NF}_r^\Delta(X_k)_{k\geqslant 0}$
prefix-	✗	✗
suffix-	✓	✓

Figure 5.48 – Convergence of the Δ-free normal forms of the random walk in the braid monoid

5.4 The Limit of the Random Walk

5.4.1 The Limit as a Markov Process

In what follows, we consider a probability measure μ over the quotient group $\mathbf{A}^+/\Delta^2$, with finite first moment, and such that $\operatorname{supp}(\mu)$ generates (positively) the whole group $\mathbf{A}^+/\Delta^2$. We will say that a left random walk $\mathbf{X}$ with step-distribution μ is a *left random walk with finite first moment* on $\mathbf{A}+/\Delta^2$. Moreover, if $\operatorname{supp}(\mu)$ is finite, then we say that $\mathbf{X}$ is a *left random walk with bounded steps* on $\mathbf{A}^+/\Delta^2$. In addition, we also consider the stopping time τ and the convolution power $\mu^{*\tau}$ introduced in the proof of Theorem 5.44, such that $\{\sigma_1, \ldots, \sigma_n\} \subseteq \operatorname{supp}(\mu^{*\tau})$.

Definition 5.49 (Stable limit).
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. We call stable limit *of $\mathbf{X}$ the left-infinite word $\lim(\mathbf{X}) = \ldots \cdot w_{-2} \cdot w_{-1}$ such that*

$$\forall i \geqslant 1, \exists j \geqslant 0, \forall k \geqslant j, \lim(\mathbf{X}) \rhd \mathbf{suf}_i(\mathbf{NF}_\ell(X_k)),$$

if such a word exists (which almost surely happens).

Note that, following the proof of Theorem 5.44, the left-random walks associated with the probability measures μ and $\mu^{*\tau}$ almost surely have the same stable limit. Henceforth, we assume that $\mu = \mu^{*\tau}$, i.e. that $\{\sigma_1, \ldots, \sigma_n\} \subseteq \mathrm{supp}(\mu)$.

In what follows, we will also consider "shifted" left-random walks associated with μ, of the form $(X_k(X_\kappa)^{-1})_{k \geqslant \kappa}$, where κ is some non-negative integer. We denote such a random walk by $\overline{\mathbf{X}}^{(\kappa)}$, and denote by $\overline{X}_k^{(\kappa)}$ the braid $X_k(X_\kappa)^{-1}$. In particular, we have $\mathbf{X} = \overline{\mathbf{X}}^{(0)}$ and $X_k = \overline{X}_k^{(0)}$.

Lemma 5.50.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{s}$ and $\mathbf{t}$ be two proper simple braids, i.e. two elements of $\mathcal{S}^\circ = \{\mathbf{x} \in \mathbf{A}^+ : \mathbf{1} <_\ell \mathbf{x} <_\ell \Delta\}$. There exists a braid $\mathbf{a}_{\mathbf{s},\mathbf{t}}$ of Artin length Λ such that $\mathbf{right}(\mathbf{s}) = \mathbf{left}(\mathbf{a}_{\mathbf{s},\mathbf{t}})$ and $\mathbf{NF}_\ell(\mathbf{a}_{\mathbf{s},\mathbf{t}}) \rhd \mathbf{t}$.

Proof. According to Proposition 5.21, there exists bilateral Garside paths $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ of length $\mathfrak{D}$, respectively from $\mathbf{right}(\mathbf{s})$ to $\{\sigma_1\}$ and from $\{\sigma_1\}$ to $\mathbf{left}(\mathbf{t})$. Since $\Lambda = (2\mathfrak{D} + 1)\lambda(\Delta)$, the integer $y = \Lambda - \lambda(\mathbf{u}) - \lambda(\mathbf{v}) - \lambda(\mathbf{t})$ is non-negative. Hence, the braid $\mathbf{a}_{\mathbf{s},\mathbf{t}} := \mathbf{u}\sigma_1^y\mathbf{v}\mathbf{t}$ is a braid of Artin length $\lambda(\mathbf{a}_{\mathbf{s},\mathbf{t}}) = \Lambda$ such that $\mathbf{NF}_\ell(\mathbf{a}_{\mathbf{s},\mathbf{t}}) = \underline{\mathbf{u}} \cdot (\sigma_1)^y \cdot \underline{\mathbf{v}} \cdot \mathbf{t}$, which completes the proof. □

Proposition 5.51.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k \geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. Let μ be the step-distribution of $\mathbf{X}$, and let $\lim(\mathbf{X})$ be the stable limit of $\mathbf{X}$. For all braids $\mathbf{s} \in \mathcal{S}^\circ$, we have $\mathbb{P}[\lim(\mathbf{X}) \rhd \mathbf{s}] \geqslant (\min \mu)^\Lambda$.

Proof. Let $\mathbf{t} \in \mathcal{S}^\circ$ be the rightmost letter of $\lim(\overline{\mathbf{X}}^{(\Lambda)})$, and let $\mathbf{s} \in \mathcal{S}^\circ$ be an arbitrary proper simple braid. If $X_\Lambda = \mathbf{a}_{\mathbf{t},\mathbf{s}}$, then $\lim(\mathbf{X}) = \lim(\overline{\mathbf{X}}^{(\Lambda)}) \cdot \mathbf{NF}_\ell(\mathbf{a}_{\mathbf{t},\mathbf{s}}) \rhd \mathbf{s}$. Since X_Λ is independent of the random walk $\overline{\mathbf{X}}^{(\Lambda)}$, hence of $\lim(\overline{\mathbf{X}}^{(\Lambda)})$, it follows that

$$\begin{aligned} \mathbb{P}[\lim(\mathbf{X}) \rhd \mathbf{s}] &\geqslant \sum_{\mathbf{t} \in \mathcal{S}^\circ} \mathbb{P}[\lim(\overline{\mathbf{X}}^{(\Lambda)}) \rhd \mathbf{t}, X_\Lambda = \mathbf{a}_{\mathbf{t},\mathbf{s}}] = \sum_{\mathbf{t} \in \mathcal{S}^\circ} \mathbb{P}[\lim(\overline{\mathbf{X}}^{(\Lambda)}) \rhd \mathbf{t}]\mathbb{P}[X_\Lambda = \mathbf{a}_{\mathbf{t},\mathbf{s}}] \\ &\geqslant \sum_{\mathbf{t} \in \mathcal{S}^\circ} \mathbb{P}[\lim(\overline{\mathbf{X}}^{(\Lambda)}) \rhd \mathbf{t}](\min \mu)^\Lambda = (\min \mu)^\Lambda. \end{aligned}$$

□

Definition 5.52 (Stretched integer, suffix time, witness time and witness word)**.**
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k \geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$.

We say that an integer $k \geqslant 1$ is stretched *in $\mathbf{X}$ if, for all integers $j < k$, the word $\mathbf{NF}_\ell^\delta(X_k)$ is not a suffix of $\mathbf{NF}_\ell^\delta(X_j)$. If, in addition, the word $\mathbf{NF}_\ell^\delta(X_k)$ is a suffix of $\lim(\mathbf{X})$, then we say that k is a* suffix time *of $\mathbf{X}$.*

We call witness time *of* $\mathbf{X}$ *each integer* k *that is stretched in* $\mathbf{X}$ *and such that* $\lim(\mathbf{X}) \rhd \mathbf{s} \cdot \mathbf{NF}^{\delta}_{\ell}(X_k)$*, where* $\mathbf{s}$ *is the rightmost letter of* $\lim(\mathbf{X})$*. We also denote by* $\mathcal{W}_{\mathbf{X}}$ *the set of witness times for* $\mathbf{X}$*, and by* ω_k *the* k*-th smallest element of* $\mathcal{W}_{\mathbf{X}}$ *(if* $|\mathcal{W}_{\mathbf{X}}| \geqslant k$*).*

Finally, we call witness words of $\mathbf{X}$ *the word* $\lim(\mathbf{X})$ *itself and the words* $\mathbf{NF}^{\delta}_{\ell}(X_k)$ *such that* k *is a witness time for* $\mathbf{X}$*.*

Proposition 5.53.
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of spherical type, and let* $\mathbf{X} = (X_k)_{k \geqslant 0}$ *be a left random walk with finite first moment on* $\mathbf{A}^+/\Delta^2$*. Let* $\mathbf{s}$ *be an element of* $\mathcal{S}^{\circ}$*, and let* $\underline{\mathbf{x}}$ *be the smallest witness word of* $\mathbf{X}$*. We denote by* $E_{\mathbf{s}}$ *the event* $\{\lim(\mathbf{X}) \rhd \mathbf{s}\}$*.*

The random variables $(\underline{\mathbf{x}} \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$ *and* $(\lim(\mathbf{X})\underline{\mathbf{x}}^{-1} \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$ *are independent, and the random variables* $(\lim(\mathbf{X}) \mid E_{\mathbf{s}})$ *and* $(\lim(\mathbf{X})\underline{\mathbf{x}}^{-1} \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$ *are identically distributed.*

The random variables $(\{\omega_i - \omega_1\}_{i \geqslant 2} \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$ *and* $(\omega_1 \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$ *are independent, and the random variables* $(\{\omega_i\}_{i \geqslant 1} \mid E_{\mathbf{s}})$ *and* $(\{\omega_i - \omega_1\}_{i \geqslant 2} \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$ *are identically distributed.*

Proof. For all integers $k \geqslant 0$, the event $E_{\mathbf{s}} \cap \{\omega_1 = k\}$ holds if and only if:

- the braid $\mathbf{s}$ is the last letter of the word $\lim(\overline{\mathbf{X}}^{(k)})$;
- we have both $\mathbf{left}(\delta(X_k)) \subseteq \mathbf{right}(\mathbf{s})$ and $\mathbf{NF}^{\delta}_{\ell}(X_k) \rhd \mathbf{s}$;
- the integer k is stretched in $\mathbf{X}$ and, for all integers $j \in \{1, \ldots, k-1\}$ that are stretched in $\mathbf{X}$, the word $\mathbf{s} \cdot \mathbf{NF}^{\delta}_{\ell}(X_j)$ is not a suffix of $\mathbf{s} \cdot \mathbf{NF}^{\delta}_{\ell}(X_k)$.

Hence, the variable $(\overline{\mathbf{X}}^{(\omega_1)} \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$ is independent of $((X_j)_{0 \leqslant j \leqslant \omega_1} \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$. Moreover, it also follows that $(\overline{\mathbf{X}}^{(\omega_1)} \mid E_{\mathbf{s}} \cap \{\omega_1 < +\infty\})$ follows the same law as $(\mathbf{X} \mid E_{\mathbf{s}})$. This completes the proof. □

It follows directly from Proposition 5.53 that the random variable $\lim(\mathbf{X})$ can be described as a Markov process.

Definition 5.54 (Stable Markov process)**.**
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of spherical type, and let* $\mathbf{X} = (X_k)_{k \geqslant 0}$ *be a left random walk with finite first moment on* $\mathbf{A}^+/\Delta^2$*.*

Let us first choose a simple braid $\mathbf{s} \in \mathcal{S}^{\circ}$ *with probability* $\mathbf{s} \mapsto \mathbb{P}[\lim(\mathbf{X}) \rhd \mathbf{s}]$*. Then, for each integer* $k \geqslant 1$*, we choose a (finite or infinite) braid word* $\underline{\mathbf{x}}_k$ *with probability*

$$\mathbb{P}[\underline{\mathbf{x}}_k \textit{ is the smallest witness word of } \mathbf{X} \mid \lim(\mathbf{X}) \rhd \mathbf{s}].$$

Finally, we denote by $\underline{\mathbf{x}}_{\infty}$ *the product* $\ldots \cdot \underline{\mathbf{x}}_2 \cdot \underline{\mathbf{x}}_1$*, where the product is infinite if every word* $\underline{\mathbf{x}}_k$ *is finite, or ends with the first infinite word* $\underline{\mathbf{x}}_k$ *we choose.*

The random variable $\underline{\mathbf{x}}_{\infty}$ *is called the* stable Markov process *for the law* μ*.*

Corollary 5.55.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k \geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. The stable limit $\lim(\mathbf{X})$ *and the stable Markov process $\underline{\mathbf{x}}_\infty$ are identically distributed.*

5.4.2 The Stable Markov Process is Infinite

We aim now at proving that the factorisation $\underline{\mathbf{x}}_\infty = \ldots \cdot \underline{\mathbf{x}}_2 \cdot \underline{\mathbf{x}}_1$ of the stable Markov process contains infinitely many factors of finite expected length.

Lemma 5.56.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k \geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. Let k be a suffix time of $\mathbf{X}$, and let k' be some integer such that $k' > k$. The integer k' is a suffix time of $\mathbf{X}$ if and only if $k' - k$ is a suffix time of the shifted random walk $\overline{\mathbf{X}}^{(k)}$.

Proof. Since k is a suffix time of $\mathbf{X}$, it follows that $\lim(\mathbf{X}) = \lim(\overline{\mathbf{X}}^{(k)}) \cdot \mathbf{NF}^\delta_\ell(X_k)$.

Let us first assume that k' is a suffix time of $\mathbf{X}$. The word $\mathbf{NF}^\delta_\ell(X_k)$ must be a suffix of $\mathbf{NF}^\delta_\ell(X_{k'})$, hence we can factor $\mathbf{NF}^\delta_\ell(X_{k'})$ as the product $\mathbf{NF}^\delta_\ell(X_{k'}) = \mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'}) \cdot \mathbf{NF}^\delta_\ell(X_k)$, and therefore

$$\lim(\overline{\mathbf{X}}^{(k)}) \cdot \mathbf{NF}^\delta_\ell(X_k) = \lim(\mathbf{X}) \rhd \mathbf{NF}^\delta_\ell(X_{k'}) = \mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'}) \cdot \mathbf{NF}^\delta_\ell(X_k),$$

which proves that $\mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'})$ is a suffix of $\lim(\overline{\mathbf{X}}^{(k)})$.

In addition, consider some integer $j \in \{k, \ldots, k'\}$ such that $\mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_j) \rhd \mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'})$. Since

$$\mathbf{NF}^\delta_\ell(X_j) = \mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_j) \cdot \mathbf{NF}^\delta_\ell(X_k) \rhd \mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'}) \cdot \mathbf{NF}^\delta_\ell(X_k) = \mathbf{NF}^\delta_\ell(X_{k'}),$$

it follows that $j = k'$. This proves that $k' - k$ is stretched in the random walk $\overline{\mathbf{X}}^{(k)}$, and therefore is a suffix time of $\overline{\mathbf{X}}^{(k)}$.

Conversely, let us assume that $k' - k$ is a suffix time of the shifted random walk $\overline{\mathbf{X}}^{(k)}$. Since $\mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'})$ is a suffix of $\lim(\overline{\mathbf{X}}^{(k)})$, it follows that $\mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'}) \cdot \mathbf{NF}^\delta_\ell(X_k)$ is a left Garside word, and therefore that $\mathbf{NF}^\delta_\ell(X_{k'}) = \mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'}) \cdot \mathbf{NF}^\delta_\ell(X_k)$ is a suffix of $\lim(\mathbf{X}) = \lim(\overline{\mathbf{X}}^{(k)}) \cdot \mathbf{NF}^\delta_\ell(X_k)$.

Moreover, consider some integer $j \in \{0, \ldots, k'\}$. If $j < k$, then $\mathbf{NF}^\delta_\ell(X_k)$ is not a suffix of $\mathbf{NF}^\delta_\ell(X_j)$, hence neither is $\mathbf{NF}^\delta_\ell(X_{k'})$. If $k \leqslant j \leqslant k'$ and if $\mathbf{NF}^\delta_\ell(X_{k'})$ is a suffix of $\mathbf{NF}^\delta_\ell(X_j)$, then $\mathbf{NF}^\delta_\ell(X_k)$ is a suffix of $\mathbf{NF}^\delta_\ell(X_j)$ too. Hence, we can factor $\mathbf{NF}^\delta_\ell(X_j)$ as the product

$$\mathbf{NF}^\delta_\ell(X_j) = \mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_j) \cdot \mathbf{NF}^\delta_\ell(X_k) \rhd \mathbf{NF}^\delta_\ell(X_{k'}) = \mathbf{NF}^\delta_\ell(\overline{X}^{(k)}_{k'}) \cdot \mathbf{NF}^\delta_\ell(X_k),$$

and therefore $\mathbf{NF}_\ell^\delta(\overline{X}_{k'}^{(k)})$ is a suffix of $\mathbf{NF}_\ell^\delta(\overline{X}_j^{(k)})$. It follows that $j = k'$, which shows that k' is a suffix time of $\mathbf{X}$. □

Corollary 5.57.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. Let k be a witness time of $\mathbf{X}$, and let k' be some integer such that $k' > k$. The integer k' is a witness time of $\mathbf{X}$ if and only if $k' - k$ is a witness time of the shifted random walk $\overline{\mathbf{X}}^{(k)}$.

Proof. If k' is a witness time of $\mathbf{X}$, then

$$\lim(\overline{\mathbf{X}}^{(k)}) \cdot \mathbf{NF}_\ell^\delta(X_k) = \lim(\mathbf{X}) \rhd \mathbf{s} \cdot \mathbf{NF}_\ell^\delta(X_{k'}) = \mathbf{s} \cdot \mathbf{NF}_\ell^\delta(\overline{X}_{k'}^{(k)}) \cdot \mathbf{NF}_\ell^\delta(X_k),$$

hence $\lim(\overline{\mathbf{X}}^{(k)}) \rhd \mathbf{s} \cdot \mathbf{NF}_\ell^\delta(\overline{X}_{k'}^{(k)})$, and $k' - k$ is a witness time of $\overline{\mathbf{X}}^{(k)}$.

Conversely, if $k' - k$ is a witness time of $\overline{\mathbf{X}}^{(k)}$, then $\mathbf{NF}_\ell^\delta(\overline{X}_{k'}^{(k)}) \rhd \mathbf{s}$, hence

$$\lim(\mathbf{X}) = \lim(\overline{\mathbf{X}}^{(k)}) \cdot \mathbf{NF}_\ell^\delta(X_k) \rhd \mathbf{s} \cdot \mathbf{NF}_\ell^\delta(\overline{X}_{k'}^{(k)}) \cdot \mathbf{NF}_\ell^\delta(X_k) = \mathbf{s} \cdot \mathbf{NF}_\ell^\delta(X_{k'}),$$

and k' is a witness time of $\mathbf{X}$. □

Corollary 5.58.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$.

Let us define ω_0, and let $\omega_1 < \omega_2 < \ldots$ be the witness times of $\mathbf{X}$. In addition, for all integers $k \geqslant 1$, consider the word $\underline{\mathbf{x}}_k := \mathbf{NF}_\ell^\delta(\overline{X}_{\omega_k}^{(\omega_{k-1})})$. The (finite or infinite product) $\ldots \cdot \underline{\mathbf{x}}_2 \cdot \underline{\mathbf{x}}_1$ is equal to $\lim(\mathbf{X})$, and for each integer $k \geqslant 1$, the word $\underline{\mathbf{x}}_k$ is the smallest witness word of the shifted random walk $\overline{\mathbf{X}}^{()}\omega_{k-1}$.

Lemma 5.59.
For all braids $\mathbf{a} \in \mathbf{A}^+$ and $\sigma \in \{\sigma_1, \ldots, \sigma_n\}$, we have $\lambda(\delta(\mathbf{a})) - 2\lambda(\Delta) \leqslant \lambda(\delta(\mathbf{a}\sigma)) \leqslant \lambda(\delta(\mathbf{a})) + 1$ and $\lambda(\delta(\mathbf{a})) - 2\lambda(\Delta) \leqslant \lambda(\delta(\sigma\mathbf{a})) \leqslant \lambda(\delta(\mathbf{a})) + 1$.

Proof. Consider the Δ-free braid $\mathbf{b} \in \mathbf{A}^+$ and the integer $z \in \{0, 1\}$ such that $\delta(\mathbf{a}) = \mathbf{b}\Delta^z$. We define $\tau := \phi_\Delta^z(\sigma)$, which yields the equality $\delta(\mathbf{a}\sigma) = \delta(\mathbf{b}\tau\Delta^z)$.

If $\mathbf{b}\tau \geqslant_r \Delta$, then $\mathbf{b} \geqslant_r \Delta\tau^{-1}$. Hence, defining $\mathbf{c} := \mathbf{b}\tau\Delta^{-1}$, we know that $\mathbf{c}$ is Δ-free and that $\delta(\mathbf{a}\sigma) = \delta(\mathbf{c}\Delta^{z+1})$, which proves that $\lambda(\delta(\mathbf{a}\sigma)) \geqslant \lambda(\mathbf{c}) \geqslant \lambda(\mathbf{b}) - \lambda(\Delta) \geqslant \lambda(\delta(\mathbf{a})) - 2\lambda(\Delta)$. However, if $\mathbf{b}\tau$ is Δ-free, then $\delta(\mathbf{a}\sigma) = \mathbf{b}\tau\Delta^z$, and therefore we also have $\lambda(\delta(\mathbf{a}\sigma)) = \lambda(\mathbf{b}\tau\Delta^z) \geqslant \lambda(\mathbf{b}) \geqslant \lambda(\delta(\mathbf{a})) - 2\lambda(\Delta)$. The converse inequality $\lambda(\delta(\mathbf{a}\sigma)) \leqslant \lambda(\delta(\mathbf{a})) + 1$ is immediate.

An analogous argument shows that $\lambda(\delta(\mathbf{a})) - 2\lambda(\Delta) \leqslant \lambda(\delta(\sigma\mathbf{a})) \leqslant \lambda(\delta(\mathbf{a})) + 1$. □

Proposition 5.60.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k \geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$.

For all braids $\mathbf{s} \in \mathcal{S}^\circ$, we have $\mathbb{E}[|\mathcal{W}_{\mathbf{X}}| \mid \lim(\mathbf{X}) \rhd \mathbf{s}] = +\infty$. Moreover, if $\mathbf{X}$ has bounded steps, then $\liminf_{k \to +\infty} \frac{1}{k}\mathbb{E}[|\mathcal{W}_{\mathbf{X}} \cap \{0,\ldots,k\}| \mid \lim(\mathbf{X}) \rhd \mathbf{s}] > 0$.

Proof. Let $k \geqslant 0$ be some integer, and let E_k be the event $\{\forall j \in \{0,\ldots,k\}, \lambda(\delta(\overline{X}_k^{(\Lambda)})) \geqslant \lambda(\delta(\overline{X}_j^{(\Lambda)}))\}$. Following Lemma 5.34, there exists a braid $\mathbf{a}$ of Artin length $\Lambda+1$ such that $\sigma_1 \lhd \mathbf{NF}_\ell^\delta(\mathbf{a}\overline{X}_k^{(\Lambda)})$. Let $a_{-\Lambda-1}\ldots a_{-2}a_{-1}$ be a factorisation of $\mathbf{a}$ into the Artin generators σ_i.

Then, let $\mathbf{t} \in \mathcal{S}^\circ$ be the last letter of the word $\mathbf{NF}_\ell^\delta(\mathbf{a}\overline{X}_k^{(\Lambda)})$. Lemma 5.50 proves that there exists some braids $\mathbf{b}$ and $\mathbf{c}$ of Artin length Λ such that $\mathbf{left}(\mathbf{b}) \subseteq \mathbf{right}(\mathbf{t})$, $\mathbf{NF}_\ell(\mathbf{b}) \rhd \mathbf{s}$, $\mathbf{left}(\mathbf{c}) \subseteq \mathbf{right}(\mathbf{s})$ and $\mathbf{NF}_\ell(\mathbf{c}) \rhd \sigma_1$. Let $b_{-\Lambda}\ldots b_{-2}b_{-1}$ and $c_{-\Lambda}\ldots c_{-2}c_{-1}$ be respective factorisations of $\mathbf{b}$ and $\mathbf{c}$ into Artin generators.

Finally, consider the integer $\Pi := \Lambda + 2 + 2(2\Lambda+1)\lambda(\Delta)$, as well as the events

$$\begin{array}{ll} F := \bigcap_{i=0}^{\Lambda-1}\{Y_i = b_{-i-1}\}; & G := \bigcap_{i=0}^{\Lambda}\{Y_{k+\Lambda+i} = a_{-i-1}\}; \\ H := \bigcap_{i=1}^{\Pi}\{Y_{k+2\Lambda+i} = \sigma_1\}; & I := \bigcap_{i=1}^{\Lambda}\{Y_{k+\Pi+2\Lambda+i} = c_{-i}\} \text{ and} \\ J := \{\lim(\overline{\mathbf{X}}^{(k+\Pi+3\Lambda+1)}) \rhd \mathbf{s}\}. & \end{array}$$

By construction, the events E_k, F, G, H, I and J are independent. It follows, using Proposition 5.51, that $\mathbb{P}[E_k \cap F \cap G \cap H \cap I \cap J] \geqslant (\min \mu)^{\Pi+3\Lambda+1}\mathbb{P}[E_k]$.

We prove now that, if the event $E_k \cap F \cap G \cap H \cap I \cap J$ is satisfied, then $k + \Pi + 3\Lambda + 1 \in \mathcal{W}_{\mathbf{X}}$. Indeed, it follows from $F \cap G \cap H \cap I$ that $\mathbf{NF}_\ell^\delta(X_{k+\Pi+3\Lambda+1}) = \underline{\mathbf{c}} \cdot (\sigma_1)^\Pi \cdot \mathbf{NF}_\ell^\delta(\mathbf{a}\overline{X}_k^{(\Lambda)}) \cdot \underline{\mathbf{b}} \rhd \mathbf{s}$, where $(\sigma_1)^\Pi$ denotes the word with Π letters σ_1 in a row. Since $\mathbf{left}(\mathbf{c}) \subseteq \mathbf{right}(\mathbf{s})$ and since $\lim(\overline{\mathbf{X}}^{(k+\Pi+3\Lambda+1)}) \rhd \mathbf{s}$ under the event J, it also follows that $\lim(\mathbf{X}) = \lim(\overline{\mathbf{X}}^{(k+\Pi+3\Lambda+1)}) \cdot \mathbf{NF}_\ell^\delta(X_{k+\Pi+3\Lambda+1}) \rhd \mathbf{s} \cdot \mathbf{NF}_\ell^\delta(X_{k+\Pi+3\Lambda+1})$.

Hence, it remains to prove that $k + \Pi + 3\Lambda + 1$ is stretched in $\mathbf{X}$. We do so by showing that $\lambda(\delta(X_{k+\Pi+3\Lambda+1})) > \lambda(\delta(X_j))$ whenever $0 \leqslant j \leqslant k + \Pi + 3\Lambda$. We proceed by distinguishing cases:

- if $0 \leqslant j \leqslant \Lambda$, then $\lambda(\delta(X_j)) \leqslant j \leqslant \Lambda$;
- if $\Lambda \leqslant j \leqslant \Lambda + k$, then Lemma 5.59 shows that $\lambda(\delta(X_j)) \leqslant \lambda(\delta(\overline{X}_{j-\Lambda}^{(\Lambda)})) + \Lambda \leqslant \lambda(\delta(\overline{X}_k^{(\Lambda)})) + \Lambda$;
- if $\Lambda + k \leqslant j \leqslant 2\Lambda + k + 1$, then $\lambda(\delta(X_j)) \leqslant \lambda(\delta(\overline{X}_k^{(\Lambda)})) + 2\Lambda + 1$;
- if $2\Lambda + k + 1 \leqslant j \leqslant \Pi + 3\Lambda + k + 1$, then $\lambda(\delta(X_j)) = (j - 2\Lambda - k - 1) + \lambda(\delta(X_{2\Lambda+k+1}))$.

In particular, note that $\lambda(\delta(X_{\Pi+3\Lambda+k+1})) > \delta(X_j)$ whenever $2\Lambda + k + 1 \leqslant j \leqslant \Pi + 3\Lambda + k$.

In addition, Lemma 5.59 proves that

$$\begin{aligned}\lambda(\delta(X_{\Pi+3\Lambda+k+1})) &= \lambda(\delta(X_{2\Lambda+k+1})) + \Pi + \Lambda \\ &\geqslant \lambda(\delta(\overline{X}_k^{(\Lambda)})) + \Pi + \Lambda - 2(2\Lambda+1)\lambda(\Delta) > \lambda(\delta(\overline{X}_k^{(\Lambda)})) + 2\Lambda + 1,\end{aligned}$$

and therefore that $\lambda(\delta(X_{\Pi+3\Lambda+k+1})) > \delta(X_j)$ if $0 \leqslant j \leqslant 2\Lambda + k + 1$ too. This proves that k is a witness time in $\mathbf{X}$.

Moreover, remember that the group $\mathbf{A}^+/\Delta^2$ contains a free subgroup and that the distribution μ has a finite first moment. Hence, the random walk $(\delta(\overline{X}_k^{(\Lambda)}))_{k\geqslant 0}$ has a positive escape rate. Therefore, there exists positive real constants u and v such that the event $L := \{\forall k \geqslant 0, \frac{k}{u} \leqslant \lambda(\delta(\overline{X}_k^{(\Lambda)})) \geqslant uk\}$ holds with probability v.

If L is true, then $\sum_{i=0}^k \mathbf{1}_{E_i} \geqslant \frac{\ln(k)}{2\ln(u+1)}$. If, in addition, the set $\operatorname{supp}(\mu)$ is finite, then $\sum_{i=0}^k \mathbf{1}_{E_i} \geqslant \frac{k}{uM}$, where $M := \max\{\lambda(\delta(\beta)) : \beta \in \operatorname{supp}(\mu)\}$. This shows that $\sum_{i=0}^k \mathbb{P}[E_i] \geqslant \frac{\ln(k)v}{2\ln(u+1)}$, and that $\sum_{i=0}^k \mathbb{P}[E_i] \geqslant \frac{kv}{uM}$ if $\operatorname{supp}(\mu)$ is finite. Since

$$\begin{aligned}\mathbb{E}[|\mathcal{W}_{\mathbf{X}} \cap \{0,\ldots,\pi+3\Lambda+1+k\}| \cdot \mathbf{1}_{\lim(\mathbf{X})\rhd\mathbf{s}}] &\geqslant \sum_{j\leqslant k} \mathbb{P}[E_j \cap F \cap G \cap H \cap I \cap J] \\ &\geqslant (\min\mu)^{\Pi+3\Lambda+1} \sum_{j=0}^{k} \mathbb{P}[E_j],\end{aligned}$$

Proposition 5.60 follows. □

Theorem 5.61.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. The random walk $\mathbf{X}$ almost surely has infinitely many witness times. Moreover, if $\mathbf{X}$ has bounded steps, then $\mathbb{E}[\omega_1] < +\infty$ and $\mathbb{E}[|\underline{\mathbf{x}}_1|] < +\infty$.

Proof. Consider some braid $\mathbf{s} \in \mathcal{S}^\circ$, and let $p_{\mathbf{s}}$ denote the probability $\mathbb{P}[\omega_1 = +\infty \mid \lim(\mathbf{X})\rhd\mathbf{s}]$. If $p_{\mathbf{s}} > 0$, then it comes immediately that $\mathbb{E}[|\mathcal{W}_{\mathbf{X}}| \mid \lim(\mathbf{X})\rhd\mathbf{s}] = \frac{1-p_{\mathbf{s}}}{p_{\mathbf{s}}} < +\infty$, in contradiction with Proposition 5.60. This proves that $\mathcal{W}_{\mathbf{s}}$ is almost surely infinite.

We assume now that $\operatorname{supp}(\mu)$ is finite. Consider the generating function $f : z \mapsto \mathbb{E}[z^{\omega_1} \mid \lim(\mathbf{X}) \rhd \mathbf{s}]$. According to Proposition 5.60, there exists an integer $L \geqslant 0$ and a real number $q > 0$ such that $\mathbb{E}[|\mathcal{W}_{\mathbf{X}} \cap \{0,\ldots,k\}| \mid \lim(\mathbf{X}) \rhd \mathbf{s}] > qk$ for all $k \geqslant L$. Hence, for all $z \in (0,1)$, we have

$$\begin{aligned}\frac{1}{1-f(z)} &= \sum_{i\geqslant 0} f(z)^i = \mathbb{E}\left[1 + \sum_{i\geqslant 1} z^{\omega_i} \mid \lim(\mathbf{X}) \rhd \mathbf{s}\right] = 1 + \mathbb{E}\left[\sum_{j\in\mathcal{W}_{\mathbf{X}}} z^j \mid \lim(\mathbf{X}) \rhd \mathbf{s}\right] \\ &\geqslant 1 + \sum_{k\geqslant L} qz^k = 1 + q\frac{z^L}{1-z}, \text{ i.e.} \\ f(z) &\geqslant \frac{qz^L}{(1-z)+qz^L}.\end{aligned}$$

Since f is convex, it follows that f is continuous and differentiable at $z = 1$, with $f(1) = 1$ and $f'(1) \leqslant g'(1)$, where $g : z \mapsto \frac{qz^L}{2(1-z)+qz^L}$. It follows that $\mathbb{E}[\omega_1 \mid \lim(\mathbf{X}) \rhd \mathbf{s}] = f'(1) \leqslant g'(1) = \frac{2}{q} < +\infty$, and therefore that $\mathbb{E}[\omega_1] < +\infty$. Finally, observe that $|\underline{\mathbf{x}}_1| \leqslant \lambda(X_{\omega_1}) \leqslant M\omega_1$, where $M := \max\{\lambda(\beta) : \beta \in \mathrm{supp}(\mu)\}$. It follows that $\mathbb{E}[|\underline{\mathbf{x}}_1|] < +\infty$. □

5.4.3 Ergodicity

Definition 5.62 ($\mathbf{t}$-witness time and $\mathbf{t}$-witness word)**.**
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. Consider some braid $\mathbf{t} \in \mathcal{S}^\circ$. We call $\mathbf{t}$-witness time of $\mathbf{X}$ *each integer k that is a suffix time in $\mathbf{X}$ and such that $\mathbf{t} \lhd \mathbf{NF}_\ell^\delta(X_k)$. In addition, we denote by $\mathcal{W}_{\mathbf{X}}^{\mathbf{t}}$ the set of $\mathbf{t}$-witness times of $\mathbf{X}$.*

We also call $\mathbf{t}$-witness words of $\mathbf{X}$ *the word* $\lim(\mathbf{X})$ *itself and each word $\mathbf{NF}_\ell^\delta(X_k)$ where k is a $\mathbf{t}$-witness time for $\mathbf{X}$.*

Lemma 5.63.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. Let k be a suffix time in $\mathbf{X}$, and let k' be some integer such that $k' > k$. The integer k' is a $\mathbf{t}$-witness time of $\mathbf{X}$ if and only if $k' - k$ is a $\mathbf{t}$-witness time of $\overline{\mathbf{X}}^{(k)}$.

Proof. Lemma 5.56 indicates that k' is a suffix time in $\mathbf{X}$ if and only if $k' - k$ is a suffix time in $\overline{\mathbf{X}}^{(k)}$. Hence, if either $k' \in \mathcal{W}_{\mathbf{X}}$ or $k' - k \in \mathcal{W}_{\overline{\mathbf{X}}^{(k)}}$, we know that k' is a suffix time in $\mathbf{X}$, that $k' - k$ is a suffix time in $\overline{\mathbf{X}}^{(k)}$ and that $\mathbf{t} \lhd \mathbf{NF}_\ell^\delta(\overline{X}_{k'}^{(k)}) \cdot \mathbf{NF}_\ell^\delta(X_k) = \mathbf{NF}_\ell^\delta(X_{k'})$, which proves that k' is a $\mathbf{t}$-witness time of $\mathbf{X}$ and that $k' - k$ is a $\mathbf{t}$-witness time of $\overline{\mathbf{X}}^{(k)}$.
□

Proposition 5.64.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with n generators, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. Moreover, consider some braid $\mathbf{t} \in \mathcal{S}^\circ$ and some set $T \subseteq \{1, \ldots, n\}$. Let $\underline{\mathbf{y}}$ be the smallest $\mathbf{t}$-witness word of $\mathbf{X}$, let $\mathbf{s}$ be the rightmost letter of $\lim(\mathbf{X})$*, and let E_T be the event $\{T \subseteq \mathbf{right}(\mathbf{s})\}$.*

The word $\underline{\mathbf{y}}$ is almost surely finite. In addition, the random variables $(\underline{\mathbf{y}} \mid E_T \cap \{|\underline{\mathbf{y}}| < +\infty\})$ and $(\lim(\mathbf{X})\underline{\mathbf{y}}^{-1} \mid E_T \cap \{|\underline{\mathbf{y}}| < +\infty\})$ are independent, and the random variables $(\lim(\mathbf{X}) \mid E_{\mathbf{left}(\mathbf{t})})$ and $(\lim(\mathbf{X})\underline{\mathbf{y}}^{-1} \mid E_T \cap \{|\underline{\mathbf{y}}| < +\infty\})$ are identically distributed.

Furthermore, if $\mathbf{X}$ has bounded steps, then $\mathbb{E}[|\underline{\mathbf{y}}|] < +\infty$.

Proof. Let $\mathbf{s} \in \mathcal{S}^\circ$ be some proper simple braid, and let $\ldots \cdot \underline{\mathbf{x}}_2 \cdot \underline{\mathbf{x}}_1$ be the factorisation of $\lim(\mathbf{X})$ into witness words of $\mathbf{X}$, i.e. into words $\underline{\mathbf{x}}_i := \mathbf{NF}_\ell^\delta(\overline{X}_{\omega_i}^{(\omega_{i-1})})$, where $\omega_0 = 0$ and $\omega_1 < \omega_2 < \ldots$ are the witness times of $\mathbf{X}$.

We first prove that $\mathbb{E}[|\underline{\mathbf{y}}|] < +\infty$. If $\mathbf{s} \neq \mathbf{t}$, let $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ be shortest bilateral Garside paths from $\mathbf{right}(\mathbf{s})$ to $\mathbf{left}(\mathbf{t})$ and from $\mathbf{right}(\mathbf{t})$ to $\mathbf{left}(\mathbf{s})$. They cannot contain the letter $\mathbf{s}$. With probability at least $\mathbb{P}[\lim(\mathbf{X}) \rhd \mathbf{s}](\min \mu)^{\lambda(\mathbf{atbs})}$, we have simultaneously $\mathbf{NF}_\ell(X_{\lambda(\mathbf{tbs})}) = \mathbf{t} \cdot \underline{\mathbf{b}} \cdot \mathbf{s}$, $\mathbf{NF}_\ell(X_{\lambda(\mathbf{atbs})}) = \underline{\mathbf{a}} \cdot \mathbf{t} \cdot \underline{\mathbf{b}} \cdot \mathbf{s}$, and $\lim(\mathbf{X}) \rhd \mathbf{s} \cdot \underline{\mathbf{a}} \cdot \mathbf{t} \cdot \underline{\mathbf{b}} \cdot \mathbf{s}$. In that case, it follows that $\underline{\mathbf{x}}_1 = \underline{\mathbf{a}} \cdot \mathbf{t} \cdot \underline{\mathbf{b}} \cdot \mathbf{s} \rhd \mathbf{t} \cdot \underline{\mathbf{b}} \cdot \mathbf{s} = \underline{\mathbf{y}}$.

However, if $\mathbf{s} = \mathbf{t}$, then, with probability at least $\mathbb{P}[\lim(\mathbf{X}) \rhd \mathbf{s}](\min \mu)^{\lambda(\mathbf{s})}$, we have $X_{\lambda(\mathbf{s})} = \mathbf{s}$ and $\lim(\mathbf{X}) \rhd \mathbf{s} \cdot \mathbf{s}$, whence $\underline{\mathbf{x}}_1 = \mathbf{s} = \underline{\mathbf{y}}$.

Hence, there exists a positive real number $q > 0$ such that $\mathbb{P}[\underline{\mathbf{x}}_1 \rhd \underline{\mathbf{y}} \mid \lim(\mathbf{X}) \rhd \mathbf{s}] > q$. Moreover, consider the random variable $a := \min\{u : \underline{\mathbf{x}}_u \cdot \ldots \cdot \underline{\mathbf{x}}_1 \rhd \underline{\mathbf{y}}\}$. For all $i \geqslant 1$, the event $\{a = i \mid a \geqslant i\}$ depends only on the shifted random walk $\overline{\mathbf{X}}^{(\omega_{i-1})}$, hence is independent from the random variable $(X_j)_{0 \leqslant j \leqslant \omega_{i-1}}$. Therefore, the variable a is dominated by an exponential law of parameter $1 - q$, and therefore a is almost surely finite. Since each word $\underline{\mathbf{x}}_i$ is almost surely finite, the word $\underline{\mathbf{y}}$ itself is almost surely finite.

Moreover, if μ has finite support, then Theorem 5.61 states that $\mathbb{E}[|\underline{\mathbf{x}}_1|] < +\infty$, and since $\mathbb{E}[a] \leqslant \frac{q}{1-q}$, it follows that $\mathbb{E}[|\underline{\mathbf{y}}|] \leqslant \mathbb{E}[|\underline{\mathbf{x}}_1| \mid a = 1] + (\mathbb{E}[a] - 1)\mathbb{E}[|\underline{\mathbf{x}}_1| \mid a > 1] < +\infty$.

Then, for all integers $k \geqslant 1$, let F_k denote the event $\{k = \min \mathcal{W}_{\mathbf{X}}^{\mathbf{t}}\}$. We just proved that there almost surely exists some integer $k \geqslant 1$ such that F_k holds. Moreover, the event $E_T \cap F_k$ holds if and only if:

- there exists some braid $\mathbf{s} \in \mathcal{S}^\circ$ such that $\lim(\overline{\mathbf{X}}^{(k)}) \rhd \mathbf{s}$ and $\mathbf{right}(\mathbf{s}) \supseteq \mathbf{left}(\mathbf{t})$;
- there exists some braid $\mathbf{u} \in \mathcal{S}^\circ$ such that $\mathbf{NF}_\ell^\delta(X_k) \rhd \mathbf{u}$ and $\mathbf{right}(\mathbf{u}) \supseteq T$;
- we have $\mathbf{t} \lhd \mathbf{NF}_\ell^\delta(X_k)$;
- the integer k is stretched in $\mathbf{X}$;
- for all integers $j \in \{1, \ldots, k-1\}$ that are stretched in $\mathbf{X}$, the word $\mathbf{NF}_\ell^\delta(X_j)$ is not a suffix of $\mathbf{NF}_\ell^\delta(X_k)$.

Hence, the random variables $(\underline{\mathbf{x}} \mid E_T \cap F_k)$ and $(\lim(\mathbf{X})\underline{\mathbf{x}}^{-1} \mid E_T \cap F_k)$ are independent, and the random variables $(\lim(\mathbf{X}) \mid E_{\mathbf{left}(\mathbf{t})})$ and $(\lim(\mathbf{X})\underline{\mathbf{x}}^{-1} \mid E_T \cap F_k)$ are identically distributed. □

It follows directly from Proposition 5.64 that the random variable $\lim(\mathbf{X})$ can be described as an *alternative* Markov process.

Definition 5.65 (Alternative stable Markov process).
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k \geqslant 0}$ be a left random walk with finite first moment on $\mathbf{A}^+/\Delta^2$. Consider the braid $\mathbf{t}_0 = \mathbf{1}$, as well as a sequence of braids $(\mathbf{t}_k)_{k \geqslant 1}$ chosen independently and uniformly at random in the set $\mathcal{S}^\circ$. Then, for each integer $k \geqslant 1$, we choose a finite braid word $\underline{\mathbf{y}}_k$ with probability $\mathbb{P}[\underline{\mathbf{y}}_k$ is the smallest $\mathbf{t}_k$-witness word of $\lim(\mathbf{X}) \mid E_{\mathbf{left}(\mathbf{t}_{k-1})}]$. Finally, we denote by $\underline{\mathbf{y}}_\infty$ the product $\ldots \cdot \underline{\mathbf{y}}_2 \cdot \underline{\mathbf{y}}_1$, where the product is infinite if every word $\underline{\mathbf{y}}_k$ is finite, or ends with the first infinite word $\underline{\mathbf{y}}_k$ we choose.

The random variable $\underline{\mathbf{y}}_{\infty}$ *is called* alternative stable Markov process *for the law* μ.

Corollary 5.66.
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of spherical type, and let* $\mathbf{X} = (X_k)_{k \geqslant 0}$ *be a left random walk with finite first moment on* $\mathbf{A}^+/\Delta^2$. *The stable limit* $\lim(\mathbf{X})$ *and the alternative stable Markov process* $\underline{\mathbf{y}}_{\infty}$ *are identically distributed.*

Let $\mathcal{G}_n^\infty$ be the set of left-infinite Δ-free Garside words. The set $\mathcal{G}_n^\infty$, equipped with the product topology, is a compact set. Let E be a Borel subset of $\mathcal{G}_n^\infty$ that is shift-invariant, i.e. such that $E = \{\ldots \cdot \mathbf{s}_3 \cdot \mathbf{s}_2 : \ldots \cdot \mathbf{s}_3 \cdot \mathbf{s}_2 \cdot \mathbf{s}_1 \in E\}$.

Theorem 5.67.
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of spherical type, and let* $\mathbf{X} = (X_k)_{k \geqslant 0}$ *be a left random walk with bounded steps on* $\mathbf{A}^+/\Delta^2$. *The stable limit* $\lim(\mathbf{X})$ *is ergodic, i.e.* $\mathbb{P}[\lim(\mathbf{X}) \in E] \in \{0,1\}$ *for all shift-invariant Borel subsets* E *of* $\mathcal{G}_n^\infty$.

Proof. For all braids $\mathbf{s} \in \mathcal{S}^\circ$, let us denote by $\mathbb{W}_\mathbf{s}$ the set $\{\underline{\mathbf{x}} : \mathbb{P}[\underline{\mathbf{x}}$ is the smallest $\mathbf{s}$-witness word of $\lim(\mathbf{X})] > 0\}$. We also denote by $\mathbb{W}$ the set $\bigcup_{\mathbf{s} \in \mathcal{S}^\circ} \mathbb{W}_\mathbf{s}$, by p the positive constant $\frac{1}{|\mathcal{S}^\circ|}$, and by $\underline{\varepsilon}$ the empty word. Then, consider the probabilistic graph Γ whose

- state space is $S_\Gamma := \{(\underline{\varepsilon}, 0, \mathbf{1})\} \cup \{(\underline{\mathbf{x}}, k, \mathbf{s}) : \mathbf{s} \in \mathcal{S}^\circ, \underline{\mathbf{x}} \in \mathbb{W}_\mathbf{s}, 1 \leqslant k \leqslant |\underline{\mathbf{x}}|\}$;
- initial state is $(\underline{\varepsilon}, 0, \mathbf{1})$;
- probabilistic transition function $\tau : S_\Gamma \times S_\Gamma \mapsto \mathbb{R}$ is such that

$$\begin{aligned} \tau : ((\underline{\mathbf{x}}, k, \mathbf{s}), (\underline{\mathbf{y}}, \ell, \mathbf{t})) &\mapsto 1 \text{ if } \underline{\mathbf{x}} = \underline{\mathbf{y}}, \mathbf{s} = \mathbf{t} \text{ and } \ell = k+1 \\ &\mapsto p\, \mathbb{P}[\underline{\mathbf{y}} \text{ is the smallest } \mathbf{t}\text{-witness word} \\ &\qquad \text{of } \lim(\mathbf{X}) \mid E_{\mathbf{left(s)}}] \text{ if } k = |\underline{\mathbf{x}}| \text{ and } \ell = 1 \\ &\mapsto 0 \text{ otherwise.} \end{aligned}$$

Let $\mathbf{S} := (\underline{\mathbf{x}}^{(i)}, k^{(i)}, \mathbf{s}^{(i)})_{i \geqslant 0}$ be the sequence of states followed by an infinite run in Γ. By construction, and due to Proposition 5.64, each state besides $(\underline{\varepsilon}, 0, \mathbf{1})$ is positive recurrent. In addition, since $\sigma_1 \in \mathbb{W}_{\sigma_1}$, the graph Γ is contains a loop around the state $(\sigma_1, 1, \sigma_1)$ with positive weight. Hence, Γ is aperiodic, and the sequence $\mathbf{S}$ follows an ergodic law.

Moreover, let $\varphi : \mathbb{W}^\mathbb{N} \mapsto \mathcal{G}_n^\infty$ be the function such that $\varphi : (\underline{\mathbf{x}}^{(i)}, k^{(i)}, \mathbf{s}^{(i)})_{i \geqslant 0} \mapsto \ldots \cdot x^{(i)}_{-k^{(i)}} \cdot \ldots \cdot x^{(2)}_{-k^{(2)}} \cdot x^{(1)}_{-k^{(1)}}$, where $x^{(i)}_{-k^{(i)}}$ denotes the $k^{(i)}$-th rightmost letter of the word $\underline{\mathbf{x}}^{(i)}$. The random variable $\varphi(\mathbf{S})$ also follows an ergodic law, and Corollary 5.66 proves that $\lim(\mathbf{X})$ and $\varphi(\mathbf{S})$ follow the same law, which completes the proof. □

5.4.4 Consequences of Ergodicity

From this point on, we assume that the probability distribution μ has a finite support, i.e. we focus on left random walks with bounded steps. In particular, Theorem 5.67 applies. We prove now that all Δ-free left Garside words appear in the limit $\lim(\mathbf{X})$ with a positive density, and that their rightmost occurrence cannot be very far from the right end of $\lim(\mathbf{X})$.

Proposition 5.68.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with bounded steps on $\mathbf{A}^+/\Delta^2$. Let $\underline{\mathbf{a}}$ be a Δ-free left Garside word, and let $\mathbf{d}_{\underline{\mathbf{a}}}(k)$ be the density *of $\underline{\mathbf{a}}$ factors among the k last letters of* $\lim(\mathbf{X})$*, i.e.* $\mathbf{d}_{\underline{\mathbf{a}}}(k) := \frac{1}{k}|\{\underline{\mathbf{b}} : \lim(\mathbf{X}) \rhd \underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$ *and* $|\underline{\mathbf{b}}| \leqslant k\}|$*. There exists a positive real constant* $\mathbf{D}_{\underline{\mathbf{a}}} > 0$ *such that* $\mathbf{d}_{\underline{\mathbf{a}}}(k) \to \mathbf{D}_{\underline{\mathbf{a}}}$ *almost surely.*

Proof. Let Γ be the graph used in the proof of Theorem 5.67, and let π be the (unique) invariant probability on Γ. For each sequence of states $\mathbf{S} := (\underline{\mathbf{x}}^i, k^i, \mathbf{s}^i)_{0\leqslant i\leqslant|\underline{\mathbf{a}}|} \in S_\Gamma^{|\underline{\mathbf{a}}|+1}$, we denote by $\Pi(\mathbf{S})$ the product $\pi(\underline{\mathbf{x}}^0, k^0, \mathbf{s}^0)\prod_{i=0}^{|\underline{\mathbf{a}}|-1}\tau((\underline{\mathbf{x}}^i, k^i, \mathbf{s}^i), (\underline{\mathbf{x}}^{i+1}, k^{i+1}, \mathbf{s}^{i+1}))$ and by $\varphi(\mathbf{S})$ the word $x^{|\underline{\mathbf{a}}|}_{-k^{|\underline{\mathbf{a}}|}} \cdot \ldots \cdot x^2_{-k^2} \cdot x^1_{-k^1}$. Since Γ is an ergodic graph, it follows that $\mathbf{d}_{\underline{\mathbf{a}}}(k) \to \mathbf{D}_{\underline{\mathbf{a}}}$, where

$$\mathbf{D}_{\underline{\mathbf{a}}} := \sum_{\mathbf{S}:\varphi(\mathbf{S})=\underline{\mathbf{a}}} \Pi(\mathbf{S}) > 0.$$

□

Proposition 5.69.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with bounded steps on $\mathbf{A}^+/\Delta^2$. Let $\underline{\mathbf{a}}$ be a Δ-free left Garside word, and let $\mathfrak{d}_{\underline{\mathbf{a}}}$ be the depth *of the word $\underline{\mathbf{a}}$ in the limit* $\lim(\mathbf{X})$*, i.e.* $\mathfrak{d}_{\underline{\mathbf{a}}} := \min\{|\underline{\mathbf{w}}| : \underline{\mathbf{a}} \lhd \underline{\mathbf{w}}$ *and* $\lim(\mathbf{X}) \rhd \underline{\mathbf{w}}\}$*. We have* $\mathbb{E}[\mathfrak{d}_{\underline{\mathbf{a}}}] < +\infty$*.*

Proof. Let $a_{-\ell} \cdot \ldots \cdot a_{-2} \cdot a_{-1}$ be the letters of the word $\underline{\mathbf{a}}$. Then, let us focus on the braids $(\mathbf{t}_i)_{i\geqslant 0}$ and words $(\underline{\mathbf{y}}_i)_{i\geqslant 1}$ chosen when building the alternative Markov process in Definition 5.65. Finally, consider the set $L_{\underline{\mathbf{a}}} := \{k \geqslant 0 : \mathbf{t}_{k\ell+1} = a_{-1}$ and $\mathbf{t}_{k\ell+i} = \underline{\mathbf{y}}_{k\ell+i} = a_{-i}$ for all $i \in \{2, \ldots, \ell\}\}$.

The random variables $\mathbf{1}_{k\in L_{\underline{\mathbf{a}}}}$ are i.i.d. Bernoulli variables whose success probability is a positive real constant $p > 0$. It follows that $\mathbb{E}[\min L_{\underline{\mathbf{a}}}] = \frac{1}{p}$, hence that

$$\mathbb{E}[\mathfrak{d}_{\underline{\mathbf{a}}}] \leqslant \mathbb{E}[|y_\ell \cdot \ldots \cdot y_1| \mid 0 \in L_{\underline{\mathbf{a}}}] + (\mathbb{E}[\min L_{\underline{\mathbf{a}}}] - 1)\mathbb{E}[|y_\ell \cdot \ldots \cdot y_1| \mid 0 \notin L_{\underline{\mathbf{a}}}] < +\infty.$$

□

In addition, let us define here the notion of penetration distance, mentioned and studied in [55, 56].

Definition 5.70 (Penetration distance)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type, let $\underline{\mathbf{w}}$ be a Δ-free (finite or left-infinite) left Garside word, and let σ be an Artin generator of $\mathbf{A}^+$. Consider the factorisations of $\underline{\mathbf{w}}$ into a product $\underline{\mathbf{w}} = \underline{\mathbf{x}} \cdot \underline{\mathbf{y}}$ such that either:

- $\underline{\mathbf{x}} = \varepsilon$ *and* $\underline{\mathbf{w}} = \underline{\mathbf{y}}$*;*
- $\underline{\mathbf{x}}$ *is non-empty,* $\underline{\mathbf{y}}$ *is finite and* $\mathbf{left}(\mathbf{y}\sigma) \subseteq \mathbf{right}(x_{-1})$*;*
- $\underline{\mathbf{x}}$ *is non-empty,* $\underline{\mathbf{y}}$ *is finite and* Δ *divides* $\mathbf{y}\sigma$*.*

The penetration distance *of the braid σ in the word $\underline{\mathbf{w}}$, denoted by $\mathbf{pd}(\underline{\mathbf{w}}, \sigma)$, is defined as the smallest possible length of such a suffix $\underline{\mathbf{y}}$.*

Then, we prove variants of [55, Conjecture 3.3], when considering the distribution of the stable limit.

Proposition 5.71.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k \geqslant 0}$ be a left random walk with bounded steps on $\mathbf{A}^+/\Delta^2$. The expected penetration distance into the stable limit of $\mathbf{X}$ is uniformly bounded, i.e.

$$\forall \mathbf{s} \in \mathcal{S}^\circ, \mathbb{E}[\mathbf{pd}(\lim(\mathbf{X}), \mathbf{s})] < +\infty.$$

Proof. Let us denote by $\mathbf{b}$ the simple braid $\sigma_1^{-1}\Delta$. We prove that $\mathbf{pd}(\lim(\mathbf{X}), \mathbf{s}) \leqslant \mathfrak{d}_{\mathbf{b}}$. Indeed, consider a factorisation of $\lim(\mathbf{X})$ into a product $\underline{\mathbf{a}} \cdot \mathbf{b} \cdot \underline{\mathbf{c}}$.

If Δ divides $\mathbf{bcs}$, it comes immediately that $\mathbf{pd}(\lim(\mathbf{X}), \mathbf{s}) \leqslant |\mathbf{b} \cdot \underline{\mathbf{c}}|$. However, if $\mathbf{bcs}$ is Δ-free, then $\{\sigma_2, \ldots, \sigma_n\} = \mathbf{left}(\mathbf{b}) \subseteq \mathbf{left}(\mathbf{bcs}) \subsetneq \{\sigma_1, \ldots, \sigma_n\}$. It follows that $\mathbf{left}(\mathbf{bcs}) = \mathbf{left}(\mathbf{b}) \subseteq \mathbf{right}(a_{-1})$, and that $\mathbf{pd}(\lim(\mathbf{X}), \mathbf{s}) \leqslant |\mathbf{b} \cdot \underline{\mathbf{c}}|$. By choosing the suffix $\underline{\mathbf{c}}$ to be as small as possible, we complete the proof. □

Corollary 5.72.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let μ be a probability measure on $\mathbf{A}^+$ whose range is finite and generates $\mathbf{A}^+$ as a monoid. In addition, let k be a positive integer, and let μ^{*k} be the k-th convolution power of μ. The expected penetration distance into a braid β chosen according to μ^{*k} is asymptotically dominated by k, i.e.

$$\forall \mathbf{s} \in \mathcal{S}^\circ, \frac{1}{k}\mathbb{E}_{\mu^{*k}}[\mathbf{pd}(\mathbf{NF}_\ell(\beta), \mathbf{s})] \to 0 \textit{ when } k \to +\infty.$$

Proof. Let $\mathbf{X} = (X_u)_{u \geqslant 0}$ be the left random walk with step-distribution μ: the braid X_k is chosen according to the distribution μ^{*k}. Moreover, for all $k \geqslant 1$ and $\mathbf{s} \in \mathcal{S}^\circ$, consider the function $\phi_k^{\mathbf{s}} : \mathbf{X} \mapsto \frac{1}{k}\mathbf{pd}(\mathbf{NF}_\ell(X_k), \mathbf{s})$. In addition, let $\mathbf{s}_k$ be the longest common suffix of $\mathbf{NF}_\ell(X_k)$ and of $\lim(\mathbf{X})$, and let $M := \max\{|\mathbf{NF}_\ell(\mathbf{x})| : \mu(\mathbf{x}) > 0\}$.

By construction of $\lim(\mathbf{X})$, we know that $\mathbf{s}_k \to +\infty$ almost surely when $k \to +\infty$. Moreover, if $\mathbf{s}_k \geqslant \mathbf{pd}(\lim(\mathbf{X}), \mathbf{s}) + 1$, then $\mathbf{pd}(\lim(\mathbf{X}), \mathbf{s}) = \mathbf{pd}(\mathbf{NF}_\ell(X_k), \mathbf{s})$. It follows that $\phi_k^{\mathbf{s}}(\mathbf{X})$ is almost surely asymptotically equivalent to $\frac{1}{k}\mathbf{pd}(\lim(\mathbf{X}), \mathbf{s})$, and therefore that $\phi_k^{\mathbf{s}}(\mathbf{X}) \to 0$ almost surely when $k \to +\infty$. Since $0 \leqslant \phi_k^{\mathbf{s}} \leqslant M$, the dominated convergence theorem proves that $\frac{1}{k}\mathbb{E}[\mathbf{pd}(\mathbf{NF}_\ell(X_k), \mathbf{s})] = \mathbb{E}[\phi_k^{\mathbf{s}}(\mathbf{X})] \to 0$ when $k \to +\infty$, which completes the proof. □

Note that Corollary 5.72 applies in particular when μ is the uniform probability distribution on the set of generators $\{\sigma_1, \ldots, \sigma_n\}$. Consequently, the expected cost $\mathbf{C}_k$ of computing the Garside normal form of a braid of length k, whose k factors have been chosen uniformly at random one after the other, is asymptotically dominated by k^2. However, instead of only showing that $\mathbf{C}_k = o(k^2)$, we might have wished to prove that $\mathbf{C}_k = \mathcal{O}(k)$. A first possible step towards this result is Corollary 5.73.

Corollary 5.73.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k \geqslant 0}$ be a left random walk with bounded steps on $\mathbf{A}^+/\Delta^2$. Let ω_i be the i-th smallest witness time of $\mathbf{X}$. If ω_1 has a finite second moment, i.e. if $\mathbb{E}[\omega_1^2] < +\infty$, then

$$\sup_{\mathbf{s} \in \mathcal{S}^\circ} \sup_{k \geqslant 1} \mathbb{E}[\mathbf{pd}(\mathbf{NF}_\ell(X_k), \mathbf{s})] < +\infty.$$

Proof. Let $\mathbf{b}$ be the braid $\sigma^{-1}\Delta$, and let $\mathbf{b} \cdot p_{-\Lambda} \cdot \ldots \cdot p_{-1} = \underline{\mathbf{p}}$ be a $\mathbf{b}$-blocking pattern. In addition, let $\mathbf{t}$ be the rightmost letter of $\lim(\mathbf{X})$, let $\underline{\mathbf{u}}$ be a bilateral Garside path of length Λ that goes from $\mathbf{right}(p_{-1})$ to $\mathbf{left}(\mathbf{t})$, and let Θ denote the integer $2(\Lambda+1) = |\underline{\mathbf{p}} \cdot \underline{\mathbf{u}} \cdot \mathbf{t}|$. Then, consider the left Garside words $(\underline{\mathbf{z}}_i)_{i \geqslant 1}$ such that $\mathbf{NF}_\ell^\delta(X_{\omega_{i\Theta}}) = \underline{\mathbf{z}}_i \cdot \ldots \cdot \underline{\mathbf{z}}_1$, and the integers $\zeta_i := \omega_{i\Theta} - \overline{\omega}_{(i-1)\Theta}$, as well as the set $\mathcal{P} := \{i \geqslant 1 : \underline{\mathbf{z}}_i \rhd \underline{\mathbf{p}} \cdot \underline{\mathbf{u}} \cdot \mathbf{t}\}$ and the random variable $\theta_i := \mathbf{1}_{i \in \mathcal{P}}$.

By construction, the random variables $(\underline{\mathbf{z}}_i, \zeta_i, \theta_i)_{i \geqslant 1}$ are i.i.d variables, and θ_i is a non-zero Bernoulli variable. In addition, since ω_1 has a finite second moment, the random variable $\zeta_1 = \omega_\Theta$ has a finite second moment too.

Now, let us choose some integer $k \geqslant 0$, and let us define the integer $M := \max\{\lambda(\beta) : \beta \in \mathrm{supp}(\mu)\}$. In addition, consider the smallest integer $u \geqslant 0$ such that $\omega_{(u+1)\Theta} \geqslant k$, as well as the integer $\lambda_u := |\mathcal{P} \cap \{1, \ldots, u\}| = \sum_{i=1}^u \mathbf{1}_{i \in \mathcal{P}}$. For each integer $i \in \mathcal{P} \cap \{1, \ldots, u\}$, the blocking pattern $\underline{\mathbf{p}}$ is a subword of $\underline{\mathbf{z}}_i$. Hence, we factor the word $\mathbf{NF}_\ell^\delta(X_{\omega_{\Theta u}}) = \underline{\mathbf{z}}_u \cdot \ldots \cdot \underline{\mathbf{z}}_1$ into a product $\underline{\mathbf{a}}_0 \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{a}}_1 \cdot \underline{\mathbf{p}} \cdot \ldots \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{a}}_{\lambda_u}$, where $\underline{\mathbf{a}}_{\lambda_u}$ is as short as possible.

An immediate induction shows that, for all $j \leqslant \lambda_u/M$, the word $\underline{\mathbf{a}}_{Mj} \cdot \underline{\mathbf{p}} \cdot \ldots \cdot \underline{\mathbf{p}} \cdot \underline{\mathbf{a}}_{\lambda_u}$ is a suffix of $\mathbf{NF}_\ell^\delta(X_{\omega_{u\Theta}+j})$. Consequently, if $M\zeta_u < \lambda_u$, and since $0 \leqslant k - \omega_{u\Theta} < \zeta_{u+1}$, it follows that $\mathbf{NF}_\ell^\delta(X_k) \rhd \underline{\mathbf{p}} \cdot \underline{\mathbf{a}}_{\lambda_u}$, and therefore that $\mathbf{pd}(X_k, \mathbf{s}) \leqslant \mathfrak{d}_{\underline{\mathbf{p}}}$.

From this point on, we assume that $M\zeta_u \geqslant \lambda_u$. Let us define the real numbers $U := \left\lfloor \frac{k}{2\mathbb{E}[\zeta_1]} \right\rfloor$ and $V := \frac{U}{2}\mathbb{E}[\theta_1]$. Moreover, since $M\zeta_u \geqslant \lambda_u$, it follows that either $u < U$, or $M\zeta_u > V$, or $\lambda_U \leqslant V$. If $k > 2\mathbb{E}[\zeta_1]$, then $4\mathbb{E}[\zeta_1]U \geqslant k$, hence direct computations show that

$$\begin{aligned}
\mathbb{P}[u < U] &= \mathbb{P}[\omega_{U\Theta} \geqslant k] \leqslant \mathbb{P}[\zeta_1 + \ldots + \zeta_U \geqslant 2U\mathbb{E}[\zeta_1]] \leqslant \frac{U\mathrm{Var}(\zeta_1)}{U^2\mathbb{E}[\zeta_1]^2} \leqslant \frac{4\mathrm{Var}(\zeta_1)}{\mathbb{E}[\zeta_1]^2 k}; \\
\mathbb{P}[\lambda_U \leqslant V] &\leqslant \frac{U\mathrm{Var}(\theta_1)}{V^2} \leqslant \frac{16\mathrm{Var}(\theta_1)\mathbb{E}[\zeta_1]}{\mathbb{E}[\theta_1]k}; \\
\mathbb{P}[M\zeta_u > V] &\leqslant \sum_{v=1}^k \mathbb{P}[M\zeta_v > V] = k\mathbb{P}[M\zeta_1 > V] \leqslant \frac{kM^2\mathbb{E}[\zeta_1^2]}{V^2} \leqslant \frac{64M^2\mathbb{E}[\zeta_1^2]\mathbb{E}[\zeta_1]^2}{\mathbb{E}[\theta_1]k}.
\end{aligned}$$

Moreover, we still know that $\mathbf{pd}(X_k, \mathbf{s}) \leqslant \lambda(X_k) \leqslant Mk$. It follows that

$$\begin{aligned}
\mathbb{E}[\mathbf{pd}(X_k,\mathbf{s})] \leqslant\ & \mathbb{E}[\mathfrak{d}_{\underline{\mathbf{p}}} \cdot \mathbf{1}_{M\zeta_u<\lambda_u}] + 2M\mathbb{E}[\zeta_1 \cdot \mathbf{1}_{k \leqslant 2\mathbb{E}[\zeta_1]}] + \\
& Mk\mathbb{E}\left[(\mathbf{1}_{u<U} + \mathbf{1}_{\lambda_U \leqslant V} + \mathbf{1}_{M\zeta_u > V})\mathbf{1}_{M\zeta_u \geqslant \lambda_u} \cdot \mathbf{1}_{k>2\mathbb{E}[\zeta_1]}\right] \\
\leqslant\ & \mathbb{E}[\mathfrak{d}_{\underline{\mathbf{p}}}] + 2M\mathbb{E}[\zeta_1] + Mk\mathbb{E}[\mathbf{1}_{u<U} \cdot \mathbf{1}_{k>2\mathbb{E}[\zeta_1]}] + \\
& Mk\mathbb{E}[\mathbf{1}_{\lambda_U \leqslant V} \cdot \mathbf{1}_{k>2\mathbb{E}[\zeta_1]}] + Mk\mathbb{E}[\mathbf{1}_{M\zeta_u>V} \cdot \mathbf{1}_{k>2\mathbb{E}[\zeta_1]}] \\
\leqslant\ & \mathbb{E}[\mathfrak{d}_{\underline{\mathbf{p}}}] + 2M\mathbb{E}[\zeta_1] + \frac{4M\mathrm{Var}(\zeta_1)}{\mathbb{E}[\zeta_1]^2} + \\
& \frac{16M\mathrm{Var}(\theta_1)\mathbb{E}[\zeta_1]}{\mathbb{E}[\theta_1]} + \frac{64M^3\mathbb{E}[\zeta_1^2]\mathbb{E}[\zeta_1]^2}{\mathbb{E}[\theta_1]},
\end{aligned}$$

which completes the proof. □

5.4.5 The Stable Suffix Grows Quickly

Provided that both the sequences of words $(\mathbf{NF}_\ell^\delta(X_k))_{k\geqslant 0}$ and $(\underline{\mathbf{s}}(k))_{k\geqslant 0}$ tend to grow at linear speed, and that each word $\underline{\mathbf{s}}(k)$ is necessarily a suffix of the associated word $\mathbf{NF}_\ell^\delta(X_k)$, a natural question is that of comparing the growth speeds of both these families of words. In particular, whereas it is known that $\mathbf{NF}_\ell^\delta(X_k) \sim \gamma k$ (almost surely and in L^1) for some positive constant γ, we have only proved so far that there exists some positive constants α and β such that $\liminf \frac{1}{k}|\underline{\mathbf{s}}(k)| \geqslant \alpha$ and $\limsup \frac{1}{k}|\underline{\mathbf{s}}(k)| \leqslant \beta$ (almost surely and in L^1).

However, when μ has finite support, the above results of Section 5.4 allow us to derive much more precise results, which we state now.

Theorem 5.74.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type, and let $\mathbf{X} = (X_k)_{k\geqslant 0}$ be a left random walk with bounded steps on $\mathbf{A}^+/\Delta^2$.

For all integers $k \geqslant 0$, let $\underline{\mathbf{s}}(k)$ be the largest common suffix of the words $(X_j)_{j\geqslant k}$, i.e. the "invariant suffix after k steps". When $k \to +\infty$, the sequence of ratios $\frac{|\underline{\mathbf{s}}(k)|}{k}$ converges towards $\gamma := \lim \frac{1}{k}|\mathbf{NF}_\ell^\delta(X_k)|$ almost surely and in L^1.

Proof. First, recall that, as mentioned in Section 5.4.3, the constant γ is well-defined and positive. Then, consider some blocking pattern $\underline{\mathbf{p}}$, and let $\mathbf{s} \in \mathcal{S}^\circ$ be some proper simple braid. From now on, we assume that $\lim(\mathbf{X}) \rhd \mathbf{s}$, and implicitly condition all our events by the fact that $\lim(\mathbf{X}) \rhd \mathbf{s}$.

We proved in Section 5.4.3 that there exists witness times $\omega_1 < \omega_2 < \ldots$ such that $\mathbb{E}[\omega_i - \omega_{i-1}] < +\infty$, where $\omega_0 := 0$, and such that each word $\mathbf{NF}_\ell^\delta(\overline{X}_{\omega_i}^{(\omega_{i-1})})$ contains an occurrence of $\underline{\mathbf{p}}$. In addition, we even proved that the random variables $(\omega_i - \omega_{i-1})$ are i.i.d.

Now, consider some real constant $\varepsilon \in (0,1)$ and some positive integer I, and let $\mathcal{E}_I$ denote the event $\{\forall i \geqslant I, \omega_i - \omega_{i-1} < \varepsilon i\}$. Since the variables $(\omega_i - \omega_{i-1})$ are i.i.d. L^1 random variables, it follows that $\lim \mathbb{P}[\mathcal{E}_I] \to 1$ when $I \to +\infty$, and therefore there almost surely exists some integer $I \geqslant 1$ such that $\mathcal{E}_I$ holds.

In addition, consider some integer $i \geqslant I$, some integer $k \in \{\omega_i, \ldots, \omega_{i+1}\}$, and $\eta := \lfloor (1-\varepsilon)i \rfloor$. If $\mathcal{E}_I$ holds, then

$$\mathbf{NF}_\ell^\delta(X_k) \rhd \mathbf{NF}_\ell^\delta(X_{\omega i-k}) \rhd \mathbf{NF}_\ell^\delta(X_{\omega_\eta}),$$

which proves that $\underline{\mathbf{s}}(\omega_i) \rhd \mathbf{NF}_\ell^\delta(X_{\omega_\eta})$. Since some event $\mathcal{E}_I$ almost surely holds, it indeed follows that

$$\liminf \frac{|\underline{\mathbf{s}}(\omega_i)|}{|\mathbf{NF}_\ell^\delta(X_{\omega_\eta})|} \geqslant 1$$

almost surely when $i \to +\infty$. Moreover, when $i \to +\infty$, we almost surely have $\omega_i \sim i\mathbb{E}[\omega_1]$, and therefore $|\mathbf{NF}_\ell^\delta(X_{\omega_\eta})| \sim \gamma\eta\mathbb{E}[\omega_1]$ and $\liminf \frac{|\underline{\mathbf{s}}(\omega_i)|}{\omega_{i+1}} \geqslant \gamma$.

Then, if k is an integer, and if $k \to +\infty$, let $i := \max\{j : k \geqslant \omega_j\}$. On the one hand, we have $\frac{|\underline{\mathbf{s}}(k)|}{k} \geqslant \frac{|\underline{\mathbf{s}}(\omega_i)|}{\omega_{i+1}}$. On the other hand, we have $\frac{|\underline{\mathbf{s}}(k)|}{k} \leqslant \frac{|\mathbf{NF}_\ell^\delta(X_k)|}{k} \to \gamma$. It follows that $|\underline{\mathbf{s}}(k)| \sim \gamma k$ almost surely.

In addition, since the random variables $\underline{\mathbf{s}}(k)$ take values inside the closed interval $[0,1]$, the dominated convergence theorem even proves that $\mathbb{E}[\underline{\mathbf{s}}(k)] \to \gamma$ when $k \to +\infty$, which completes the proof. □

In particular, since $|\mathbf{NF}_\ell^\delta(X_k)| \geqslant |\underline{\mathbf{s}}(k)|$, it comes immediately that $\limsup \frac{1}{k}|\underline{\mathbf{s}}(k)| \leqslant \gamma$. Consequently, Theorem 5.74 can be rephrased by saying that, asymptotically, the stable suffixes $\underline{\mathbf{s}}(k)$ grow *as fast as possible.*

5.5 Experimental Data in the Braid Monoid $\mathbf{B}_n^+$

We provide now some experimental data obtained for the random walk with distribution μ uniform over the set of generators $\Sigma_n := \{\sigma_1, \ldots, \sigma_{n-1}\}$, when $4 \leqslant n \leqslant 7$.

For each value of n, we perform the following experiment. We simulate the start of 100000 independent random walks, by drawing the first 12000 values of $(Y_k)_{k\geqslant 0}$ at random. For each of them, and each integer $k \in \{1, \ldots, 10000\}$, we approximate the "invariant suffix" $\underline{\mathbf{s}}(k)$ to be the common suffix of the words $\mathbf{NF}_\ell(X_j)_{k\leqslant j\leqslant 12000}$. Figure 5.75 presents mean values of the ratio $\mathbf{r}_n(k) := \frac{|\underline{\mathbf{s}}(k)|}{k}$, where k ranges from 1 to 10000.

Observe that, if Theorems 5.36 and 5.40 provide a lower bound $\tilde{\mathbf{r}}_n$ on the value of $\liminf_{k\to+\infty} \mathbf{r}_n(k)$. For $n \geqslant 4$, this lower bound must be smaller than $(\min \mu)^{2\Lambda} \leqslant (n-1)^{-9n(n-1)} \leqslant 3 \cdot 10^{-52}$. Hence, there is still a huge margin between the above experimental approximations of $\mathbf{r}_n$ and the only theoretical lower bound $\tilde{\mathbf{r}}_n$ that we computed so far.

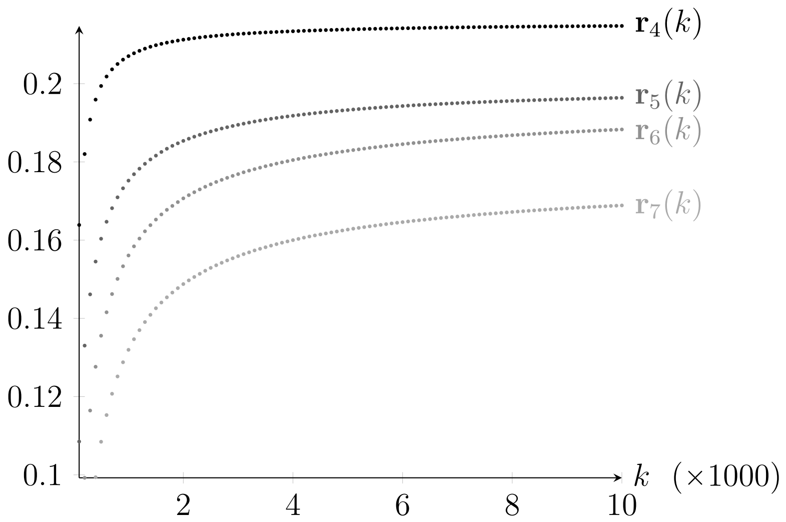


Figure 5.75 – Estimating $\mathbf{r}_n(k)$ — experimental data for $n \in \{4, 5, 6, 7\}$

Theorem 5.74 means exactly that the random variables $\mathbf{d}(k) := \frac{|\mathbf{NF}^{\delta}_{\ell}(X_k)|-|\underline{\mathbf{s}}(k)|}{k}$ must converge towards 0. Nevertheless, the above-mentioned experiments, which provided us with Fig. 5.75, indicate that $\mathbf{d}(5000) \approx 0.119$ in B_4, whereas $\mathbf{r}_4(5000) \approx 0.214$. Hence, the convergence of $\mathbf{d}(k)$ towards 0 is blatantly slow, and this suggests that computing the value of γ with great precision is to be a difficult task.

While performing the above experiment, we also collected some statistical data about the invariant suffix. Let $\underline{\mathbf{s}}$ be the pointwise limit of the suffixes $\underline{\mathbf{s}}(k)$, and let $\underline{\mathbf{s}}_{-j}$ denote the j-th rightmost letter of $\underline{\mathbf{s}}$. Figure 5.76 indicates approximate values of:

- the "typical cardinality" of right sets of the letters of $\underline{\mathbf{s}}$, i.e.

$$\mathbf{c}^r_n := \lim_{j \to +\infty} \mathbb{E}[|\mathbf{right}(\underline{\mathbf{s}}_{-j})|];$$

- the "typical cardinality" of left sets of the letters of $\underline{\mathbf{s}}$, i.e.

$$\mathbf{c}^{\ell}_n := \lim_{j \to +\infty} \mathbb{E}[|\mathbf{left}(\underline{\mathbf{s}}_{-j})|].$$

These values were obtained by approximating $\underline{\mathbf{s}}$ to be $\underline{\mathbf{s}}(10000)$, then $\mathbf{c}^r_n$ to be the mean value of $|\mathbf{right}(\underline{\mathbf{s}}_{-j})|$ when $50 \leqslant j \leqslant 100$, and $\mathbf{c}^{\ell}_n$ to be the mean value of $|\mathbf{left}(\underline{\mathbf{s}}_{-j})|$ when $50 \leqslant j \leqslant 100$. Since the letters of $\underline{\mathbf{s}}$ must all satisfy the relation $\mathbf{left}(\underline{\mathbf{s}}_{-j}) \subseteq \mathbf{right}(\underline{\mathbf{s}}_{-j-1})$, it comes with no surprise that $\mathbf{c}^{\ell}_n \leqslant \mathbf{c}^r_n$.

Furthermore, since the stable limit $\underline{\mathbf{s}}$ of the random walk was shown to be a Markov process on an infinite state space, we might hope to show that the sequence $(\underline{\mathbf{s}}_{-i})_{i \geqslant 1}$ is also a Markov process or a reversed Markov process, where $\underline{\mathbf{s}}_{-i}$ denotes the i-th rightmost letter of the word $\underline{\mathbf{s}}$. However, experimental data tend to disprove this hypothesis. We checked the values of $\underline{\mathbf{s}}_{-1}, \ldots, \underline{\mathbf{s}}_{-5}$ for 1 500 000 000 random walks, approximated by taking

n	$\mathbf{c}_n^r$	$\mathbf{c}_n^\ell$
4	1.04	1.00
5	1.62	1.28
6	2.10	1.60
7	2.51	1.89

Figure 5.76 – Some characteristics about the invariant suffix

the 200 first values of $(Y_k)_{k\geqslant 0}$. According to Fig. 5.75, such an approximation is sufficient to ensure with a reasonably small margin of error that we indeed obtained the values of $\underline{\mathbf{s}}_{-1}, \ldots, \underline{\mathbf{s}}_{-5}$). We found that:

- $\mathbb{P}[\underline{\mathbf{s}}_{-2} = \sigma_2\sigma_1 \mid \underline{\mathbf{s}}_{-1} = \sigma_2] = 0.0973 \pm 0.0001$ and $\mathbb{P}[\underline{\mathbf{s}}_{-5} = \sigma_2\sigma_1 \mid \underline{\mathbf{s}}_{-4} = \sigma_1] = 0.0954 \pm 0.0001$;
- $\mathbb{P}[\underline{\mathbf{s}}_{-1} = \sigma_1 \mid \underline{\mathbf{s}}_{-2} = \sigma_2\sigma_1] = 0.4984 \pm 0.0001$ and $\mathbb{P}[\underline{\mathbf{s}}_{-4} = \sigma_1 \mid \underline{\mathbf{s}}_{-5} = \sigma_2\sigma_1] = 0.5633 \pm 0.0001$.

The values 0.0973 and 0.0954 are clearly separated by our error margin of ± 0.0001, which indicates with absolute certainty that $\mathbb{P}[\underline{\mathbf{s}}_{-2} = \sigma_2\sigma_1 \mid \underline{\mathbf{s}}_{-1} = \sigma_2] \neq \mathbb{P}[\underline{\mathbf{s}}_{-5} = \sigma_2\sigma_1 \mid \underline{\mathbf{s}}_{-4} = \sigma_1]$. Likewise, our experiments makes certain that $\mathbb{P}[\underline{\mathbf{s}}_{-1} = \sigma_1 \mid \underline{\mathbf{s}}_{-2} = \sigma_2\sigma_1] \neq \mathbb{P}[\underline{\mathbf{s}}_{-4} = \sigma_1 \mid \underline{\mathbf{s}}_{-5} = \sigma_2\sigma_1]$. However, proving that $(\underline{\mathbf{s}}_{-i})_{i\geqslant 1}$ is neither a Markov process nor a reversed Markov process seems out of reach for now, although our experiments definitely show that this is the direction towards which we should aim.

Chapter 6

The Diameter of the Bilateral Garside Automaton

Résumé

Nous procédons ici à l'étude détaillée et systématique du diamètre du graphe de Garside bilatère des groupes d'Artin–Tits de type sphérique irréductibles. Le diamètre du graphe de Garside bilatère intervient explicitement dans le calcul d'une borne inférieure sur la vitesse de convergence des marches aléatoires étudiées dans le chapitre 5.

Nous considérons séparément les différents types de groupes de Coxeter associés aux groupes d'Artin–Tits de type sphérique irréductibles, et calculons à chaque fois le diamètre du graphe de Garside bilatère associé, dont nous montrons en particulier qu'il est nécessairement compris entre 1 et 4. Cette étude est directe dans le cas des familles infinies de groupes de Coxeter de type A_n, B_n, D_n, et à l'aide de l'ordinateur dans le cas des familles finies de groupes de Coxeter exceptionnels.

Abstract

We perform here a detailed and systematic study of the diameter of the bilateral Garside automaton of irreducible Artin–Tits groups of spherical type. This diameter is used explicitly for deriving lower bounds on the speed of convergence of random walks studied in Chapter 5.

We treat separately the different types of Coxeter groups associated with irreducible Artin–Tits groups of spherical type; and compute for each of them the diameter of the bilateral Garside automaton which we show to belong to the interval $\{1, 2, 3, 4\}$. This study was performed directly for infinite families of Coxeter groups of type A_n, B_n, D_n, and with the help of computers for finite families of exceptional Coxeter groups.

In Chapter 5, Proposition 5.21 consists in proving that the bilateral Garside automaton $\mathcal{G}_{\text{gar}}$ of an irreducible Artin–Tits monoid of spherical type $\mathbf{A}^+$ with n generators is connected, and shows that this diameter cannot be greater than $2(n^2+n)$. Since this diameter is later used for deriving lower bounds on the speed of convergence of random walks, it is meaningful to look for better upper bounds, or even for exact values of the diameter of $\mathcal{G}_{\text{gar}}$.

Consequently, and using the classification of finite irreducible Coxeter groups provided in Theorem 2.26, we proceed to a disjunction of cases, according to the type of the Coxeter group $\mathbf{W}$ associated with the Artin–Tits monoid $\mathbf{A}^+$. Below, we mimic the study of [56], in which upper bounds on the diameter of the *essential part* of the left Garside acceptor automaton are provided. However, here, we provide exact evaluations of the diameter of $\mathcal{G}_{\text{gar}}$, and do not provide only upper bounds on this diameter.

Before proceeding to a detailed study for each family of Coxeter groups, we first prove a combinatorial property satisfied by the bilateral Garside automaton, and that we will use several times later on.

Lemma 6.1.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with generators $\sigma_1, \ldots, \sigma_n$, and let $\mathcal{G}_{\text{gar}}$ be the bilateral Garside automaton of $\mathbf{A}^+$. Let P and Q be two proper subsets of $\{\sigma_1, \ldots, \sigma_n\}$, and let $\overline{P} := \{\sigma_1, \ldots, \sigma_n\} \backslash P$ and $\overline{Q} := \{\sigma_1, \ldots, \sigma_n\} \backslash Q$ be their complements. In addition, let $M : \mathbf{A}^+ \mapsto \mathbf{A}^+$ be an isomorphism of monoids. The pair (P, Q) is an arc of $\mathcal{G}_{\text{gar}}$ if and only if $(\phi_\Delta(\overline{P}), \overline{Q})$ is an arc of $\mathcal{G}_{\text{gar}}$, if and only if $(M(P), M(Q))$ is an arc of $\mathcal{G}_{\text{gar}}$.

Proof. Let us first assume that (P, Q) is an arc of $\mathcal{G}_{\text{gar}}$. Consider some braid $\mathbf{a} \in \mathcal{S}^\circ$ such that $P = \mathbf{left}(\mathbf{a})$ and $Q = \mathbf{right}(\mathbf{a})$. Since M is an isomorphism of monoids, M induces a permutation of the sets $\{\sigma_1, \ldots, \sigma_n\}$ and $\mathcal{S}^\circ$. It follows that $M(\mathbf{a}) \in \mathcal{S}^\circ$, that $\mathbf{left}(M(\mathbf{a})) = M(P)$ and that $\mathbf{right}(M(\mathbf{a})) = M(Q)$. Hence, $(M(P), M(Q))$ is an arc of $\mathcal{G}_{\text{gar}}$.

In addition, following Lemma 2.16, the braid $\mathbf{a}\Delta_{\overline{Q}}$ is a simple braid, where $\Delta_{\overline{Q}} := \mathbf{LCM}_{\leqslant_\ell}(\overline{Q})$. Hence, consider the braid $\partial_\Delta(\mathbf{a})$ such that $\Delta = \mathbf{a}\partial_\Delta(\mathbf{a})$. Since $\mathbf{a}\partial_\Delta(\mathbf{a})$ is simple, it follows that $Q \cap \mathbf{left}(\partial_\Delta(\mathbf{a})) = \emptyset$. Moreover, since $\mathbf{a}\Delta_{\overline{Q}} \leqslant_\ell \Delta = \mathbf{a}\partial_\Delta(\mathbf{a})$, we have $\overline{Q} \subseteq \mathbf{left}(\partial_\Delta(\mathbf{a}))$, and therefore $\mathbf{left}(\partial_\Delta(\mathbf{a})) = \overline{Q}$.

Similarly, we have $\mathbf{right}(\partial_\Delta(\mathbf{a})) = \overline{\mathbf{left}(\partial^2_\Delta(\mathbf{a}))}$. Since $\partial^2_\Delta = \phi_\Delta$, this means that $\mathbf{right}(\partial_\Delta(\mathbf{a})) = \phi_\Delta(\overline{P})$, hence that $(\overline{Q}, \phi_\Delta(\overline{P}))$ and $(\phi_\Delta(\overline{P}), \overline{Q})$ are arcs of $\mathcal{G}_{\text{gar}}$.

Finally, since M induces a permutation of $\{\sigma_1, \ldots, \sigma_n\}$ and since $(P, Q) \mapsto (\phi_\Delta(\overline{P}), \overline{Q})$ is an involution, Lemma 6.1 follows. $\square$

6.1 Case $\mathbf{W} = A_n$

Let us begin with the case where the Coxeter group of the monoid $\mathbf{A}^+$ is of type A_n, i.e. $\mathbf{A}^+ = \mathbf{B}_{n+1}$. Figure 6.2 presents the bilateral Garside automata of the monoids $\mathbf{B}_4^+$ and $\mathbf{B}_5^+$ (the first graph was also represented in Fig. 5.19). For the sake of readability, we omit labelling the edges and representing loops or multiple edges.

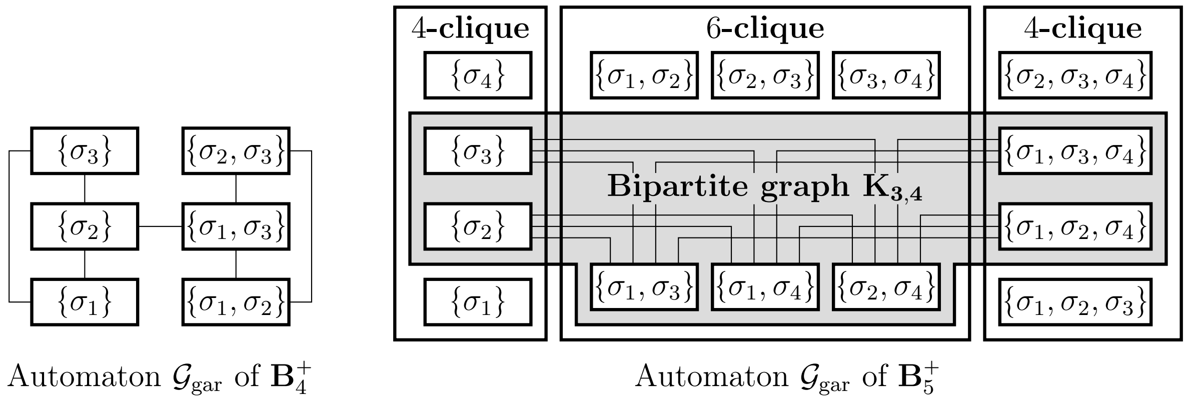


Figure 6.2 – Bilateral Garside automata of the monoids $\mathbf{B}_4^+$ and $\mathbf{B}_5^+$

Observe that the bilateral Garside automata of the braid monoids $\mathbf{B}_4^+$ and $\mathbf{B}_5^+$ have respective diameters 3 and 4. Henceforth, let $\mathcal{G}_{\text{gar}}$ be the bilateral Garside automaton of the braid monoid $\mathbf{B}_n^+$. We prove below that the diameter of $\mathcal{G}_{\text{gar}}$ is bounded (uniformly in n), by using repeatedly Proposition 2.29.

Lemma 6.3.
Let $\mathcal{G}_{\text{gar}}$ be the bilateral Garside automaton of the braid monoid $\mathbf{B}_n^+$. Let P and Q be two proper subsets of $\{1,\ldots,n-1\}$. If $|P| = |Q|$, then the sets $\{\sigma_i : i \in P\}$ and $\{\sigma_i : i \in Q\}$ are neighbours in $\mathcal{G}_{\text{gar}}$.

Proof. We prove by induction over n that some permutation $\varphi \in \mathfrak{S}_n$ satisfies $\mathbf{d}_{>0}(\varphi) = P$ and $\mathbf{d}_{>0}(\varphi^{-1}) = Q$ for all subsets P and Q of $\{1,\ldots,n-1\}$ (including $\emptyset$ and $\{1,\ldots,n-1\}$) such that $|P| = |Q|$. Note that P and Q play symmetric roles.

If $n = 2$, the result is vacuously true, and if $n = 3$, it is still immediate. If $n = 4$ or $n = 5$, then Fig. 6.2 proves the result. Henceforth, we assume that $n \geqslant 6$.

- If $n-1 \notin P \cup Q$, then $P, Q \subseteq \{1,\ldots,n-2\}$ and, by induction hypothesis, some permutation $\theta \in \mathfrak{S}_{n-1}$ satisfies $\mathbf{d}_{>0}(\theta) = P$ and $\mathbf{d}_{>0}(\theta^{-1}) = Q$. Hence, the permutation $\varphi \in \mathfrak{S}_n$ such that $\varphi : i \mapsto \theta(i)$ if $1 \leqslant i \leqslant n-1$ and $\varphi : n \mapsto n$ satisfies $\mathbf{d}_{>0}(\varphi) = P$ and $\mathbf{d}_{>0}(\varphi^{-1}) = Q$.
- Since P and Q play symmetric roles, it remains to treat the case where $n-1 \in Q$. Then, $P \neq \emptyset$, so we can consider the integer $p := \min P$. Note that $p - 1 \leqslant n-1-|P| = n-1-|Q|$, hence the set $R := \{1,\ldots,n-1\}\backslash Q$ has at least $p-1$

elements. Let r_i denote the i-th smallest element of R, for $i \leqslant p-1$, and set $R^- := \{r_i : 1 \leqslant i \leqslant p-1\}$.
In addition, set $S := \{1, \ldots, n-1\} \backslash R^-$. Observe that $R^- \subseteq R \subseteq \{1, \ldots, n-1\}$, whereby $|S| = n-p$, and let s_i denote the i-th smallest element of S, for $i \leqslant n-1-p$. Finally, set $Q^* := \{i \in \{1, \ldots, n-p-1\} : s_i \in Q\}$ and $P* := \{i \in \{1, \ldots, n-p-1\} : p+i \in P\}$. By induction hypothesis, some permutation $\theta \in \mathfrak{S}_{n-p}$ satisfies $\mathbf{d}_{>0}(\theta) = P^*$ and $\mathbf{d}_{>0}(\theta^{-1}) = Q^*$. Hence, one checks easily that the permutation $\varphi \in \mathfrak{S}_n$ such that

$$\begin{aligned} \varphi : i &\longmapsto r_i \text{ if } 1 \leqslant i \leqslant p-1 \\ p &\longmapsto n \\ i &\longmapsto s_{\theta(i-p)} \text{ if } p+1 \leqslant i \leqslant n \end{aligned}$$

satisfies $\mathbf{d}_{>0}(\varphi) = P$ and $\mathbf{d}_{>0}(\varphi^{-1}) = Q$.

□

Lemma 6.4.
Let $\mathcal{G}_{\mathrm{gar}}$ be the bilateral Garside automaton of the braid monoid $\mathbf{B}_n^+$. Let $P := (2\mathbb{Z}+1) \cap \{1, \ldots, n-1\}$ and $P' := \{\sigma_i : i \in P\}$. For each integer $k \in \{1, \ldots, n-1\}$, there exists a neighbour Q of P' in $\mathcal{G}_{\mathrm{gar}}$ such that $|Q| = k$.

Proof. Let $m := \lfloor n/2 \rfloor$. We treat two cases separately, depending on whether $k \geqslant m$ or $k \leqslant m$.

- If $m \leqslant k \leqslant n-1$, let $\ell := n-1-k$ and consider the permutation $\theta \in \mathfrak{S}_n$ such that
$$\begin{aligned} \theta : i \longmapsto\ & 2(m+1-i) \text{ if } 1 \leqslant i \leqslant m \\ & 2(i-m)-1 \text{ if } m+1 \leqslant i \leqslant m+\ell \\ & 2(n+\ell-i)+1 \text{ if } m+\ell+1 \leqslant i \leqslant n \end{aligned}$$
One checks easily that $\mathbf{d}_{>0}(\theta) = \{1, \ldots, m, m+\ell+1, \ldots, n-1\}$, whence $|\mathbf{d}_{>0}(\theta)| = k$, and that $\mathbf{d}_{>0}(\theta^{-1}) = P$.
- If $1 \leqslant k \leqslant m$, consider the permutation $\theta \in \mathfrak{S}_n$ such that
$$\begin{aligned} \theta : i \longmapsto\ & 2i \text{ if } 1 \leqslant i \leqslant m \\ & 2(m+k-i)+1 \text{ if } m+1 \leqslant i \leqslant m+k \\ & 2(i-m)-1 \text{ if } m+k+1 \leqslant i \leqslant n \end{aligned}$$
One checks easily that $\mathbf{d}_{>0}(\theta) = \{m, \ldots, m+k-1\}$, whence $|\mathbf{d}_{>0}(\theta)| = k$, and that $\mathbf{d}_{>0}(\theta^{-1}) = P$.

□

Lemma 6.5.
Let $\mathcal{G}_{\mathrm{gar}}$ be the bilateral Garside automaton of the braid monoid $\mathbf{B}_n^+$. If $n \geqslant 5$, then the sets $\{\sigma_1\}$ and $\{\sigma_1, \ldots, \sigma_{n-2}\}$ are at distance at least 4 from each other in $\mathcal{G}_{\mathrm{gar}}$.

Proof. Let a be some element of $\{1, \ldots, n-1\}$ and let $\psi \in \mathfrak{S}_n$ be some permutation such that $\mathbf{d}_{>0}(\psi) = \{a\}$. We have $\psi(1) < \psi(2) < \ldots < \psi(a)$ and $\psi(a+1) < \psi(a+2) < \ldots < \psi(n)$. Hence, let j be an element of $\mathbf{d}_{>0}(\psi^{-1})$.

If $j = \psi(i)$ for some $i \leqslant a$, then $1 \leqslant \psi^{-1}(j+1) < i \leqslant a$, and therefore $j+1 = \psi(\psi^{-1}(j+1)) < \psi(i) = j$, which is impossible. Similarly, if $j = \psi(i)-1$ for some $i \geqslant a+1$, then $n \geqslant \psi^{-1}(j) > i \geqslant a+1$, and therefore $j = \psi(\psi^{-1}(j)) > \psi(i) = j+1$, which is impossible. This shows that neither $\{\psi(i) : 1 \leqslant i \leqslant a\}$ nor $\{\psi(i)-1 : a+1 \leqslant i \leqslant n\}$ intersects $\mathbf{d}_{>0}(\psi^{-1})$. It follows that $|\mathbf{d}_{>0}(\theta^{-1})| \leqslant \min\{n-a, a\} \leqslant \lfloor n/2 \rfloor$.

Moreover, let $\varphi \in \mathfrak{S}_n$ be some permutation such that $\mathbf{d}_{>0}(\varphi) = \{1, \ldots, n-2\}$. We have $\psi(1) > \psi(2) > \ldots > \psi(n-1)$, so that, like in the above case, $\{\psi(i)-1 : 1 \leqslant i \leqslant n-2\} \subseteq \mathbf{d}_{>0}(\theta^{-1})$, and therefore that $|\mathbf{d}_{>0}(\varphi^{-1})| \geqslant n-2$.

In particular, in $\mathcal{G}_{\mathrm{gar}}$, each neighbour of $\{\sigma_1\}$ has cardinality 1, and each node at distance 2 from $\{\sigma_1\}$ has cardinality at most $\lfloor n/2 \rfloor$. However, each neighbour of $\{\sigma_1, \ldots, \sigma_{n-2}\}$ has cardinality $n-2 > \lfloor n/2 \rfloor$. Hence, $\{1\}$ and $\{\sigma_1, \ldots, \sigma_{n-2}\}$ are at distance at least 4 from each other. □

Proposition 6.6.
If $n \geqslant 5$, then the bilateral Garside automaton $\mathcal{G}_{\mathrm{gar}}$ of the braid monoid $\mathbf{B}_n^+$ has diameter 4.

Proof. Consider the set $P := (2\mathbb{Z}+1) \cap \{1, \ldots, n-1\}$. Combining Lemmas 6.3 and 6.4 proves that the eccentricity of $\{\sigma_i : i \in P\}$ in $\mathcal{G}_{\mathrm{gar}}$ is at most 2 (i.e. no node of $\mathcal{G}_{\mathrm{gar}}$ is at distance more than 2 from $\{\sigma_i : i \in P\}$), from which it follows that the diameter of $\mathcal{G}_{\mathrm{gar}}$ is at most 4. Then, Lemma 6.5 proves that this diameter is at least 4, which completes the proof. □

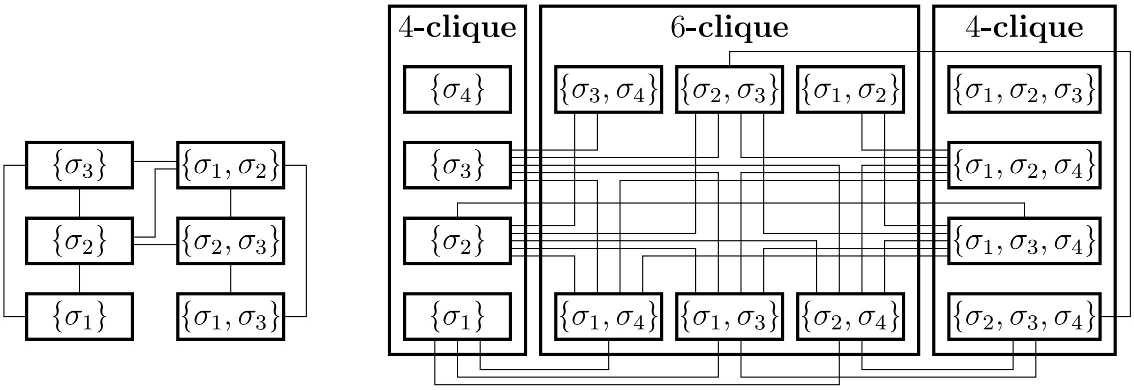
6.2 Case $\mathbf{W} = B_n$

We focus now on the bilateral Garside automaton $\mathcal{G}_{\mathrm{gar}}$ of the monoid $\mathbf{A}^+$ whose Coxeter group $\mathbf{W}$ is of type B_n. Figure 6.8 represents $\mathcal{G}_{\mathrm{gar}}$ when $n = 3$ and $n = 4$, where the labeling of the edges has been omitted and loops or multiple edges are not represented.

The bilateral Garside automaton of the monoid $\mathbf{A}^+$ has diameter 3 when the Coxeter group associated with $\mathbf{A}$ is either $\mathbf{W} = B_3$ or $\mathbf{W} = B_4$. Henceforth, let $\mathcal{G}_{\mathrm{gar}}$ be the bilateral Garside automaton of the monoid $\mathbf{A}^+$, whose Coxeter group is B_n. We prove below that the diameter of $\mathcal{G}_{\mathrm{gar}}$ is bounded (uniformly in n), by using repeatedly Proposition 2.31.

Lemma 6.7.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type with Coxeter group B_n, and let $\mathcal{G}_{\mathrm{gar}}$ be the bilateral automaton of $\mathbf{A}^+$. Let P and Q be two proper subsets of $\{0, \ldots, n-1\}$. If $|P| = |Q|$, then the sets $\{\sigma_i : i-1 \in P\}$ and $\{\sigma_i : i-1 \in Q\}$ are neighbours in $\mathcal{G}_{\mathrm{gar}}$.

Proof. We prove that some permutation $\varphi \in \mathfrak{S}_n^{\pm}$ satisfies $\mathbf{d}_{\geqslant 0}(\varphi) = P$ and $\mathbf{d}_{\geqslant 0}(\varphi^{-1}) = Q$ for all subsets P and Q of $\{0, \ldots, n-1\}$ (including $\varnothing$ and $\{0, \ldots, n-1\}$) such that $|P| = |Q|$.



Automaton $\mathcal{G}_{\text{gar}}$ of $\mathbf{A}^+$ $(n = 3)$ Automaton $\mathcal{G}_{\text{gar}}$ of $\mathbf{A}^+$ $(n = 4)$

Figure 6.8 – Bilateral Garside automata of $\mathbf{A}^+$ (when $\mathbf{W} = B_3$ and $\mathbf{W} = B_4$)

Indeed, let $p_1 < \ldots < p_k$ be the elements of the set $\{i : 1 \leqslant i \leqslant n, i-1 \in P\}$, where $k = |P|$, and let $\overline{p}_1 < \ldots < \overline{p}_{n-k}$ be the elements of the set $\{i : 1 \leqslant i \leqslant n, i-1 \notin P\}$. Similarly, let $q_1 < \ldots < q_k$ be the elements of the set $\{i : 1 \leqslant i \leqslant n, i-1 \in Q\}$, and let $\overline{q}_1 < \ldots < \overline{q}_{n-k}$ be the elements of the set $\{i : 1 \leqslant i \leqslant n, i-1 \notin Q\}$. One checks easily that the permutation $\varphi \in \mathfrak{S}_n^{\pm}$ such that

$$\begin{aligned} \varphi : p_i &\longmapsto -q_i \text{ if } 1 \leqslant i \leqslant k \\ \overline{p}_i &\longmapsto \overline{q}_i \text{ if } 1 \leqslant i \leqslant n-k \end{aligned}$$

satisfies $\mathbf{d}_{\geqslant 0}(\varphi) = P$ and $\mathbf{d}_{\geqslant 0}(\varphi^{-1}) = Q$. □

Lemma 6.9.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type with Coxeter group B_n, and let $\mathcal{G}_{\text{gar}}$ be the bilateral automaton of $\mathbf{A}^+$. Consider the set $P := (2\mathbb{Z}+1) \cap \{1,\ldots,n-1\}$. For each integer $k \in \{1,\ldots,n-1\}$, there exists a neighbour Q of the set $\{\sigma_i : i-1 \in P\}$ in $\mathcal{G}_{\text{gar}}$ such that $|Q| = k$.

Proof. Let $m := \lceil n/2 \rceil$. We treat two cases separately, depending on whether $k \geqslant n-2$ or $k = n-1$.

- If $k \leqslant n-2$, then Lemma 6.4 already proves that there exists a permutation $\varphi \in \mathfrak{S}_n$ such that $|\mathbf{d}_{>0}(\varphi)| = k$ and $\mathbf{d}_{>0}(\varphi^{-1}) = P$. Hence, the permutation $\theta \in \mathfrak{S}_n^{\pm}$ such that $\theta : i \mapsto \varphi(i)$ if $i \geqslant 1$ satisfies $|\mathbf{d}_{\geqslant 0}(\theta)| = k$ and $\mathbf{d}_{\geqslant 0}(\theta^{-1}) = P$.
- If $k = n-1$, consider the permutation $\theta \in \mathfrak{S}_n^{\pm}$ such that
$$\begin{aligned} \theta : i &\longmapsto 1-2i \text{ if } 1 \leqslant i \leqslant m \\ & 2(m-i) \text{ if } m+1 \leqslant i \leqslant n. \end{aligned}$$
One checks easily that $\mathbf{d}_{\geqslant 0}(\theta) = \{0,\ldots,m-1,m+1,\ldots,n-1\}$, whence $|\mathbf{d}_{\geqslant 0}(\theta)| = k$, and that $\mathbf{d}_{\geqslant 0}(\theta^{-1}) = P$

□

Lemma 6.10.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type with Coxeter group B_n, and let $\mathcal{G}_{\text{gar}}$

be the bilateral automaton of $\mathbf{A}^+$*. If* $n \geqslant 5$*, then the sets* $\{\sigma_n\}$ *and* $\{\sigma_1, \ldots, \sigma_{n-1}\}$ *are at distance at least* 4 *from each other in* $\mathcal{G}_{\mathrm{gar}}$*.*

Proof. Let a be some element of $\{0, \ldots, n-1\}$ and let $\psi \in \mathfrak{S}_n^{\pm}$ be some permutation such that $\mathbf{d}_{\geqslant 0}(\psi) = \{a\}$. In addition, let κ be the cardinality of the set $\{j \in \{0, \ldots, n\} : \psi^{-1}(j) < 0\}$. We have $0 = \psi(0) < \ldots < \psi(a)$ and $\psi(a+1) < \ldots < \psi(a+\kappa) < 0 < \psi(a+\kappa+1) < \ldots < \psi(n)$.

Hence, let j be an element of $\mathbf{d}_{\geqslant 0}(\psi^{-1})$. Let k and ℓ be non-negative integers such that $j = \pm\psi(k)$ and $j+1 = \pm\psi(\ell)$.

- If $\psi^{-1}(j) < 0$, then $0 > \psi(k) = -j > -(j+1) = \psi(\ell)$, hence $a+1 \leqslant \ell < k \leqslant a+\kappa$. It follows that $\psi^{-1}(j) = -k < -\ell = \psi^{-1}(j+1)$, which is impossible. Therefore, we have $\psi^{-1}(k) \geqslant 0$, i.e. $j = \psi(k)$.
- If $k \leqslant a$ and $\psi(\ell) = j+1$, then $a \geqslant k = \psi^{-1}(j) > \psi^{-1}(j+1) = \ell \geqslant 0$, hence $j = \psi(k) > \psi(\ell) = j+1$, which is impossible. Therefore, if $k \leqslant a$, we have $\psi^{-1}(j+1) < 0$.
- If $\ell \geqslant a+\kappa+1$, then $\psi(\ell) > 0$, hence $n \geqslant k > \ell \geqslant a+1$ and $\psi(k) = j$. This implies that $j = \psi(k) > \psi(\ell) = j+1$, which is impossible. Therefore, we have $\ell \leqslant a+\kappa$.

This shows that neither the set $\{j \in \{0, \ldots, n-1\} : \psi^{-1}(j) < 0$ or $(0 \leqslant \psi^{-1}(j) \leqslant a$ and $0 \leqslant \psi^{-1}(j+1))\}$ nor the set $\{j \in \{0, \ldots, n-1\} : a+\kappa+1 \leqslant |\psi^{-1}(j+1)|\}$ intersects $\mathbf{d}_{\geqslant 0}(\psi^{-1})$.

Moreover, the set $\{j \in \{0, \ldots, n\} : \psi^{-1}(j) \leqslant a\}$ is of cardinality $a+1+\kappa$, hence $\{j \in \{0, \ldots, n-1\} : \psi^{-1}(j) < 0$ or $(0 \leqslant \psi^{-1}(j) \leqslant a$ and $0 \leqslant \psi^{-1}(j+1))\}$ is of cardinality at least $\max\{\kappa-1, a\}$. In addition, the set $\{j \in \{0, \ldots, n-1\} : a+\kappa+1 \leqslant |\psi^{-1}(j+1)|\}$ is of course of cardinality $n-a-\kappa$. This shows that

$$\begin{aligned} |\mathbf{d}_{\geqslant 0}(\psi^{-1})| &\leqslant \min\{n+1-\kappa, n-a, a+\kappa\} \leqslant n-a, \text{ whence} \\ |\mathbf{d}_{\geqslant 0}(\psi^{-1})| &\leqslant \frac{(n+1-\kappa)+(n-a)+(a+\kappa)}{3} = \frac{2n+1}{3}. \end{aligned}$$

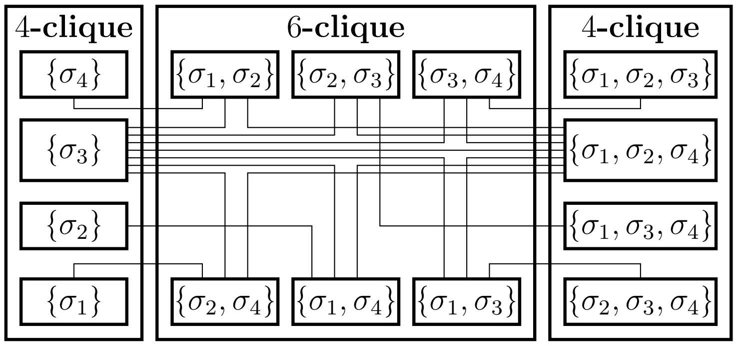
In particular, it follows that the neighbours of $\{\sigma_n\}$ in $\mathcal{G}_{\mathrm{gar}}$ are all of cardinality 1 and, using Lemma 6.1, that the neighbours of $\{\sigma_1, \ldots, \sigma_{n-1}\}$ are all of cardinality $n-1$. Moreover, each node at distance 2 from $\{\sigma_n\}$ has cardinality at most $\lfloor (2n+1)/3 \rfloor \leqslant n-2$. Hence, $\{\sigma_n\}$ and $\{\sigma_1, \ldots, \sigma_{n-1}\}$ are at distance at least 4 from each other. □

Proposition 6.11.
For each integer $n \geqslant 5$*, the bilateral Garside automaton* $\mathcal{G}_{\mathrm{gar}}$ *of the monoid* $\mathbf{A}^+$ *with Coxeter group* B_n *has diameter* 4*.*

Proof. Consider the set $P := (2\mathbb{Z}+1) \cap \{1, \ldots, n-1\}$. Combining Lemmas 6.7 and 6.9 proves that the eccentricity of $\{\sigma_i : i-1 \in P\}$ in $\mathcal{G}_{\mathrm{gar}}$ is at most 2, from which it follows that the diameter of $\mathcal{G}_{\mathrm{gar}}$ is at most 4. Then, Lemma 6.10 proves that this diameter is at least 4, which completes the proof. □

6.3 Case $\mathbf{W} = D_n$

We focus here on the bilateral Garside automaton $\mathcal{G}_{\text{gar}}$ of the monoid $\mathbf{A}^+$ whose Coxeter group $\mathbf{W}$ is of type D_n. Figure 6.12 presents $\mathcal{G}_{\text{gar}}$ when $n = 4$, where the labelling of the edges has been omitted and loops or multiple edges are not represented.



Automaton $\mathcal{G}_{\text{gar}}$ of $\mathbf{A}^+$ $(n = 4)$

Figure 6.12 – Bilateral Garside automaton of $\mathbf{A}^+$ (when $\mathbf{W} = D_4$)

Let D_n be the Coxeter group associated with $\mathbf{A}$. Figure 6.12 shows that the diameter of $\mathcal{G}_{\text{gar}}$ is 3 when $n = 4$. Brute-force computations show that the graphs $\mathcal{G}_{\text{gar}}$ (not shown in Fig. 6.12) have respective diameters 3, 4 and 4 when $n = 5$, 6 and 7. Henceforth, let $\mathcal{G}_{\text{gar}}$ be the bilateral Garside automaton of the monoid $\mathbf{A}^+$, whose Coxeter group is D_n. We prove below that the diameter of $\mathcal{G}_{\text{gar}}$ is bounded (uniformly in n), by using repeatedly Proposition 2.33.

Lemma 6.13.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type with Coxeter group D_n, and let $\mathcal{G}_{\text{gar}}$ be the bilateral automaton of $\mathbf{A}^+$. Let P and Q be two proper subsets of $\{0, \ldots, n-1\}$. If $|P| = |Q|$, then the sets $\{\sigma_i : i - 1 \in P\}$ and $\{\sigma_i : i - 1 \in Q\}$ are neighbours in $\mathcal{G}_{\text{gar}}$.

Proof. We prove that some permutation $\varphi \in \mathfrak{S}_n^{++}$ satisfies $\mathbf{d}_{\text{tw}}(\varphi) = P$ and $\mathbf{d}_{\text{tw}}(\varphi^{-1}) = Q$ for all subsets P and Q of $\{0, \ldots, n-1\}$ such that $|P| = |Q|$. Note that P and Q play symmetric roles.

Let ψ be the transposition $(0 \leftrightarrow 1)$, and let $M : \mathbf{A}^+ \mapsto \mathbf{A}^+$ be the isomorphism of monoids such that $M : \sigma_{i+1} \mapsto \sigma_{\psi(i)+1}$. Since ϕ_Δ induces an isomorphism of the Coxeter graph of D_n, we know that either $\phi_\Delta = M$ or $\phi_\Delta = \mathbf{Id}$. Hence, in both cases, there exists an integer $\epsilon \in \{0, 1\}$ such that $\phi_\Delta = M^\epsilon$.

Consequently, Lemma 6.1 proves that, if there exists a permutation $\varphi \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\text{tw}}(\varphi) = P$ and $\mathbf{d}_{\text{tw}}(\varphi^{-1}) = Q$, there also exist permutations $\mu, \nu \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\text{tw}}(\mu) = \psi(P)$, $\mathbf{d}_{\text{tw}}(\mu^{-1}) = \psi(Q)$, $\mathbf{d}_{\text{tw}}(\nu) = \overline{\psi^\epsilon(P)}$, $\mathbf{d}_{\text{tw}}(\nu^{-1}) = \overline{Q}$. We proceed now to a disjunction of cases.

- If $0 \notin P \cup Q$, then Lemma 6.3 proves that there exists a permutation $\theta \in \mathfrak{S}_n$ such that $\mathbf{d}_{>0}(\theta) = P$ and $\mathbf{d}_{>0}(\theta^{-1}) = Q$. Hence, the permutation $\varphi \in \mathfrak{S}_n^{++}$ such that $\varphi : i \mapsto \theta(i)$ if $i \geqslant 1$ satisfies $\mathbf{d}_{\mathrm{tw}}(\varphi) = P$ and $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = Q$.
- If $1 \notin P \cup Q$, then we just proved that there exists a permutation $\theta \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\mathrm{tw}}(\theta) = \psi(P)$ and $\mathbf{d}_{\mathrm{tw}}(\theta^{-1}) = \psi(Q)$. Consequently, there also exists a permutation $\varphi \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\mathrm{tw}}(\varphi) = P$ and $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = Q$.
- If $0 \in P \cap Q$, since P and Q play symmetric roles, we assume that $1 \in P$. It follows that $0 \notin \overline{P} \cup \overline{\psi(P)} \cup \overline{Q}$, hence that $0 \notin \overline{\psi^{\epsilon}(P)} \cup \overline{Q}$. Hence, we just proved that there exists a permutation $\theta \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\mathrm{tw}}(\theta) = \overline{P}$ and $\mathbf{d}_{\mathrm{tw}}(\theta^{-1}) = \overline{Q}$. Consequently, there also exists a permutation $\varphi \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\mathrm{tw}}(\varphi) = P$ and $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = Q$.
- If $1 \in P \cap Q$, then we just proved that there exists a permutation $\theta \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\mathrm{tw}}(\theta) = \psi(P)$ and $\mathbf{d}_{\mathrm{tw}}(\theta^{-1}) = \psi(Q)$. Consequently, there also exists a permutation $\varphi \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\mathrm{tw}}(\varphi) = P$ and $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = Q$.

Therefore, we can focus on the case where 0 and 1 both belong to the symmetric difference $P \Delta Q = (P \backslash Q) \cup (Q \backslash P)$. Since P and Q play symmetric roles, we assume henceforth that $0 \in P \backslash Q$ and that either $1 \in P \backslash Q$ or $1 \in Q \backslash P$.

In addition, let k be the cardinality of both sets P and Q. Then, let $p_1 < \ldots < p_k$ be the elements of the set $\{i : 1 \leqslant i \leqslant n, i-1 \in P\}$ and let $\overline{p}_1 < \ldots < \overline{p}_k$ be the elements of the set $\{i : 1 \leqslant i \leqslant n, i-1 \notin P\}$. Likewise, let $q_1 < \ldots < q_k$ be the elements of the set $\{i : 1 \leqslant i \leqslant n, i-1 \in Q\}$ and let $\overline{q}_1 < \ldots < \overline{q}_k$ be the elements of the set $\{i : 1 \leqslant i \leqslant n, i-1 \notin Q\}$.

- If $1 \in P \backslash Q$ or if $k \in 2\mathbb{Z}$, then one checks easily that the permutation $\varphi \in \mathfrak{S}_n^{++}$ such that
$$\begin{aligned} \varphi : 1 &\longmapsto (-1)^{k+1} q_1 \\ p_i &\longmapsto -q_i \text{ if } 2 \leqslant i \leqslant k \\ \overline{p}_i &\longmapsto \overline{q}_i \text{ if } 1 \leqslant i \leqslant n-k. \end{aligned}$$
satisfies $\mathbf{d}_{\mathrm{tw}}(\varphi) = P$ and $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = Q$.
- If $1 \in Q \backslash P$ and $k = 1$, then $P = \{0\}$ and $Q = \{1\}$, hence the permutation $\varphi \in \mathfrak{S}_n^{++}$ such that $\varphi = (1 \to -3 \to -1 \to -3 \to 1)(2 \leftrightarrow -2)$ satisfies $\mathbf{d}_{\mathrm{tw}}(\varphi) = P$ and $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = Q$.
- If $1 \in Q \backslash P$, $k \in 2\mathbb{Z} + 1$ and $k \geqslant 3$, then one checks easily that the permutation $\varphi \in \mathfrak{S}_n^{++}$ such that
$$\begin{aligned} \varphi : p_1 &\longmapsto -\overline{q}_2,\ 2 \longmapsto -q_1,\ \overline{p}_2 \longmapsto 1 \\ p_i &\longmapsto -q_i \text{ if } 2 \leqslant i \leqslant k \\ \overline{p}_i &\longmapsto \overline{q}_i \text{ if } 3 \leqslant i \leqslant n-k. \end{aligned}$$
satisfies $\mathbf{d}_{\mathrm{tw}}(\varphi) = P$ and $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = Q$.

□

Lemma 6.14.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type with Coxeter group D_n, and let $\mathcal{G}_{\mathrm{gar}}$ be the bilateral automaton of $\mathbf{A}^+$. Consider the set $P := (2\mathbb{Z}+1) \cap \{1, \ldots, n-1\}$. For each

integer $k \in \{1, \ldots, n-1\}$, there exists a neighbour Q of the set $\{\sigma_i : i-1 \in P\}$ in $\mathcal{G}_{\text{gar}}$ such that $|Q| = k$.

Proof. Let $m := \lfloor n/2 \rfloor$. We treat two cases separately, depending on whether $k \geqslant n-2$ or $k = n-1$.

- If $k \leqslant n-2$, then Lemma 6.4 already proves that there exists a permutation $\varphi \in \mathfrak{S}_n$ such that $|\mathbf{d}_{>0}(\varphi)| = k$ and $\mathbf{d}_{>0}(\varphi^{-1}) = P$. Hence, the permutation $\theta \in \mathfrak{S}_n^{++}$ such that $\theta : i \mapsto \varphi(i)$ if $i \geqslant 1$ satisfies $|\mathbf{d}_{\text{tw}}(\theta)| = k$ and $\mathbf{d}_{\text{tw}}(\theta^{-1}) = P$.
- If $k = n-1$, consider the permutation $\theta \in \mathfrak{S}_n^{\pm}$ such that
$$\begin{aligned} \theta : 1 &\longmapsto (-1)^{m+1}2 \\ i &\longmapsto -2i \text{ if } 2 \leqslant i \leqslant m \\ i &\longmapsto 2(n-i)+1 \text{ if } m+1 \leqslant i \leqslant n. \end{aligned}$$
One checks easily that $\mathbf{d}_{\geqslant 0}(\theta) = \{0, \ldots, m-1, m+1, \ldots, n-1\}$, whence $|\mathbf{d}_{\geqslant 0}(\theta)| = k$, and that $\mathbf{d}_{\geqslant 0}(\theta^{-1}) = P$

□

Lemma 6.15.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of spherical type with Coxeter group D_n, and let $\mathcal{G}_{\text{gar}}$ be the bilateral automaton of $\mathbf{A}^+$. If $n \geqslant 8$, then the sets $\{\sigma_n\}$ and $\{\sigma_1, \ldots, \sigma_{n-1}\}$ are at distance at least 4 from each other in $\mathcal{G}_{\text{gar}}$.

Proof. Let a be some element of $\{1, \ldots, n-1\}$ and let $\varphi \in \mathfrak{S}_n^{++}$ be some permutation such that $\mathbf{d}_{\text{tw}}(\varphi) = \{a\}$. In addition, let κ be the cardinality of the set $\{j \in \{a+1, \ldots, n\} : \varphi^{-1}(j) < 0\}$. We have $-\varphi(2) < \varphi(1) < \varphi(2) < \ldots < \varphi(a)$, $\varphi(a+1) < \ldots < \varphi(a+\kappa) < 0 < \varphi(a+\kappa+1) < \ldots < \varphi(n)$, and $(-1)^\kappa \varphi(1) > 0$.

Hence, let j be a positive element of $\mathbf{d}_{\text{tw}}(\varphi^{-1})$. Let k and ℓ be non-negative integers such that $j = \pm\varphi(k)$ and $j+1 = \pm\varphi(\ell)$.

- If $\varphi^{-1}(j) < -1$, then $0 > \varphi(k) = -j > -(j+1) = \varphi(\ell)$. Hence, either $\ell = 1$ and $a+1 \leqslant k \leqslant a+\kappa$, or $a+1 \leqslant \ell < k \leqslant a+\kappa$. In both cases, we have $\ell < k$, hence $\varphi^{-1}(j) = -k < -\ell = \varphi^{-1}(j+1)$, which is impossible. Therefore, we have $\varphi^{-1}(j) \geqslant -1$.
- If $k \leqslant a$ and $\varphi^{-1}(j+1) > 0$, then $\varphi^{-1}(j) > \varphi^{-1}(j+1) = \ell \geqslant 0$. Hence, $\varphi^{-1}(j) = k$ and $a \geqslant k > \ell \geqslant 0$, and thus $j = \varphi(k) > \varphi(\ell) = j+1$, which is impossible. Therefore, if $k \leqslant a$, we have $\varphi^{-1}(j+1) < 0$.
- If $\ell \geqslant a+\kappa+1$, then $\varphi(\ell) > 0$, hence $n \geqslant k > \ell \geqslant a+1$ and $\varphi(k) = j$. This implies that $j = \varphi(k) > \varphi(\ell) = j+1$, which is impossible. Therefore, we have $\ell \leqslant a+\kappa$.

This shows that neither the set $\{j \in \{1, \ldots, n-1\} : \varphi^{-1}(j) < -1 \text{ or } (-1 \leqslant \varphi^{-1}(j) \leqslant a \text{ and } 0 \leqslant \varphi^{-1}(j+1))\}$ nor the set $\{j \in \{1, \ldots, n-1\} : a+\kappa+1 \leqslant \varphi^{-1}(j+1)\}$ intersects $\mathbf{d}_{\geqslant 0}(\varphi^{-1})$.

The set $\{j \in \{1, \ldots, n-1\} : \varphi^{-1}(j) \leqslant a\}$ is of cardinality at least $a+\kappa-1$, hence $\{j \in \{1, \ldots, n-1\} : \varphi^{-1}(j) < -1 \text{ or } (-1 \leqslant \varphi^{-1}(j) \leqslant a \text{ and } 0 \leqslant \varphi^{-1}(j+1))\}$ is of cardinality at least $\max\{\kappa-1, a-2\}$. In addition, the set $\{j \in \{1,\ldots,n-1\} : a+\kappa+1 \leqslant \varphi^{-1}(j+1)\}$ is of cardinality at least $n-a-\kappa-1$. This shows that

$$\begin{aligned}|\mathbf{d}_{\mathrm{tw}}(\varphi^{-1})| &\leqslant \min\{n+1-\kappa, n+2-a, a+\kappa+1\} \\ &\leqslant \frac{(n+1-\kappa)+(n+2-a)+(a+\kappa+1)}{3} = \frac{2n+4}{3} < n-1.\end{aligned}$$

Moreover, let ψ be the permutation $(0 \leftrightarrow 1)$, and let $M : \sigma_{i+1} \mapsto \sigma_{\psi(i)+1}$ be the induced automorphism of monoid of $\mathbf{A}^+$. If $\varphi \in \mathfrak{S}_n^{++}$ is a permutation such that $\mathbf{d}_{\mathrm{tw}}(\varphi) = \{0\}$, then Lemma 6.1 proves that there exists a permutation $\theta \in \mathfrak{S}_n^{++}$ such that $\mathbf{d}_{\mathrm{tw}}(\theta) = \{1\}$ and $\mathbf{d}_{\mathrm{tw}}(\theta^{-1}) = \psi(\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}))$, whence $|\mathbf{d}_{\mathrm{tw}}(\varphi^{-1})| < n-1$.

Now, let us investigate the possible values of $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1})$ when $\mathbf{d}_{\mathrm{tw}}(\varphi) = \{n-1\}$. Two cases are possible.

1. There exists an integer $k \in \{2,\ldots,n\}$ such that $\varphi : 1 \mapsto \pm 1$, $\varphi : i \mapsto i$ if $2 \leqslant i < k$, $\varphi : i \mapsto i+1$ if $k \leqslant i < n$ and $\varphi : n \mapsto \pm k$. In this case, one checks easily that $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = \{0,1\}$ (if $k = 2$ and $\varphi(n) = -k$), $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = \{k-1\}$ (if $3 \leqslant k$ and $\varphi(n) = -k$) or $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = \{k\}$ (if $\varphi(n) = k$, and therefore $2 \leqslant k \leqslant n-1$).
2. We have $\varphi : 1 \mapsto \pm 2$, $\varphi : i \mapsto i+1$ if $2 \leqslant i < n$ and $\varphi : n \mapsto \pm 1$. In this case, one checks easily that $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = \{0\}$ (if $\varphi(n) = -1$) or $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1}) = \{1\}$ (if $\varphi(n) = 1$).

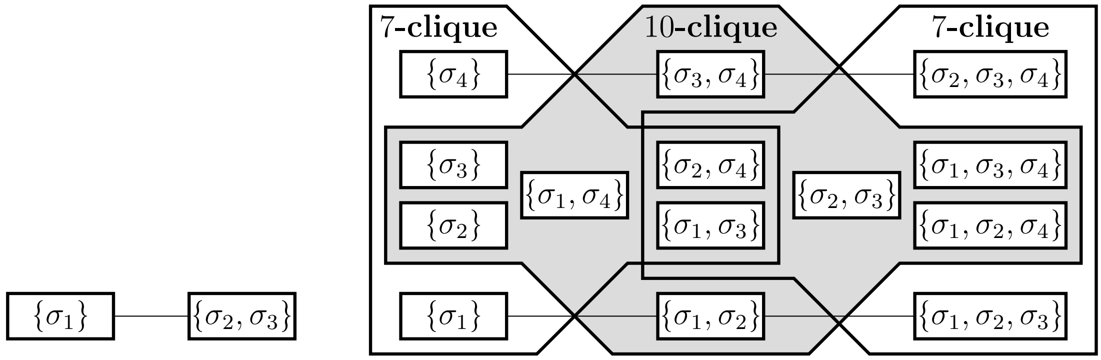
It follows that the neighbours of $\{\sigma_n\}$ in $\mathcal{G}_{\mathrm{gar}}$ are $\{\sigma_1, \sigma_2\}$ and sets of cardinality 1.

Finally, let us investigate the possible values of $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1})$ when $\mathbf{d}_{\mathrm{tw}}(\varphi) = \{0,1\}$. We mimic here the study of the case $\mathbf{d}_{\mathrm{tw}}(\varphi) = \{a\}$ with $1 \leqslant a \leqslant n-1$. Let $b \in \{2,\ldots,n\}$ be the integer such that $|\varphi(1)| < |\varphi(2)|$ and $\varphi(2) < \ldots < \varphi(b) < 0 < \varphi(b+1) < \ldots < \varphi(n)$. In addition, let j be a positive element of $\mathbf{d}_{\mathrm{tw}}(\varphi^{-1})$, and let k and ℓ be non-negative integers such that $j = \pm\varphi(k)$ and $j+1 = \pm\varphi(\ell)$. It comes quickly that $\varphi^{-1}(j+1) \leqslant b$ and that $\varphi^{-1}(j) \geqslant -1$, hence that $|\mathbf{d}_{\mathrm{tw}}(\varphi^{-1})| \leqslant \min\{b+1, n+1-b\} \leqslant \frac{n+2}{2} < n-2$.

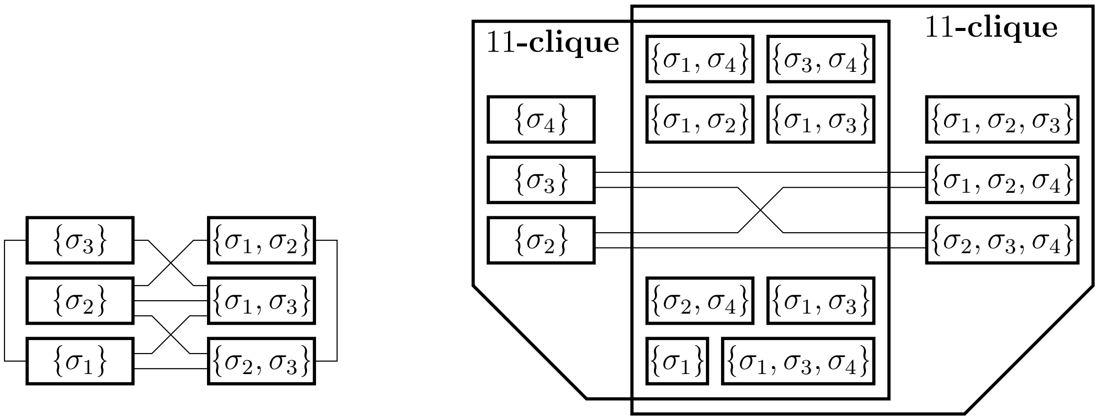
Using Lemma 6.1, it follows that the neighbours of $\{\sigma_1,\ldots,\sigma_{n-1}\}$ are $\{\sigma_3,\ldots,\sigma_n\}$ and sets of cardinality $n-1$, and that the neighbours or $\{\sigma_3,\ldots,\sigma_n\}$ have cardinality at least 3. Consequently, the nodes at distance 2 of $\{\sigma_n\}$ are sets of cardinality at most $n-2$, excluding the set $\{\sigma_3,\ldots,\sigma_n\}$. This proves that $\{\sigma_n\}$ and $\{\sigma_1,\ldots,\sigma_{n-1}\}$ are at distance at least 4 from each other. □

Proposition 6.16.
For each integer $n \geqslant 8$, the bilateral Garside automaton $\mathcal{G}_{\mathrm{gar}}$ of the monoid $\mathbf{A}^+$ with Coxeter group D_n has diameter 4.

Proof. Consider the set $P := (2\mathbb{Z}+1) \cap \{1,\ldots,n-1\}$. Combining Lemmas 6.13 and 6.14 proves that the eccentricity of $\{\sigma_i : i-1 \in P\}$ in $\mathcal{G}_{\mathrm{gar}}$ is at most 2, from which it follows that the diameter of $\mathcal{G}_{\mathrm{gar}}$ is at most 4. Then, Lemma 6.15 proves that this diameter is at least 4, which completes the proof. □



Automaton $\mathcal{G}_{\text{gar}}$ of $\mathbf{A}^+$ ($\mathbf{W} = I_2(a)$) Automaton $\mathcal{G}_{\text{gar}}$ of $\mathbf{A}^+$ ($\mathbf{W} = F_4$)



Automaton $\mathcal{G}_{\text{gar}}$ of $\mathbf{A}^+$ ($\mathbf{W} = H_3$) Automaton $\mathcal{G}_{\text{gar}}$ of $\mathbf{A}^+$ ($\mathbf{W} = H_4$)

Figure 6.17 – Bilateral Garside automata of $\mathbf{A}^+$ (when $\mathbf{W} = I_2(a)$, $\mathbf{W} = F_4$ and $\mathbf{W} = H_n$)

6.4 Exceptional Cases

Having treated the cases where the Coxeter group $\mathbf{W}$ of the monoid $\mathbf{A}^+$ is of type A_n, B_n or D_n, we already know that there exists an upper bound for the diameters of *all* bilateral Garside automata of all Artin–Tits monoids of spherical type. However, we push our analysis further, and compute the exact value of this diameter when $\mathbf{W}$ is of type E_n, F_4, H_n or $I_2(a)$.

Figure 6.17 presents the bilateral Garside automata of the monoid $\mathbf{A}^+$ when $\mathbf{W} = I_2(a)$, F_4 and H_n, where the labelling of the edges has been omitted and loops or multiple edges are not represented. It shows that the graphs $\mathcal{G}_{\text{gar}}$ have respective diameters 1, 2 and 2 when $\mathbf{W} = I_2(a)$, F_4 and H_n. Brute-force computations show that the graph $\mathcal{G}_{\text{gar}}$ (not shown in Fig. 6.17) has diameter 3 when $\mathbf{W} = E_6$.

While plain brute force, by computing the order $\leqslant_\ell$ on the group $\mathbf{W}$, is easy to carry when $\mathbf{W} = E_6$, F_4, H_3, H_4 or $I_2(a)$, it is not so when $\mathbf{W} = E_7$ or $\mathbf{W} = E_8$. Indeed, these two groups have rather large cardinalities (respectively 2903040 and 696729600), and storing data about *all* the elements of $\mathbf{W}$ raises space issues.

let $\mathbf{L}_0 := \{\mathbf{1}\}$, $\mathbf{L}_1 := \{\sigma_1, \ldots, \sigma_n\}$ and $k = 1$
let $\mathbf{left}(\sigma_i) := \{\sigma_i\}$ and $\mathbf{right}(\sigma_i) := \{\sigma_i\}$
let $\mathcal{G}^*_{\mathrm{gar}}$ be the graph whose nodes are subsets of $\{\sigma_1, \ldots, \sigma_n\}$ and without arcs
while $\mathbf{L}_k \neq \varnothing$ **do**
  let $\mathbf{L}_{k+1} := \varnothing$
  for each element $\mathbf{a}$ of $\mathbf{L}_k$ and each generator σ_i such that neither $\mathbf{a}\sigma_i^{-1}$ and $\mathbf{a}\sigma_i$ was created **do**
    create a new element $\mathbf{b}$ and add it to $\mathbf{L}_{k+1}$
    for each generator σ_j such that $\mathbf{a} \geqslant_r \Delta_{\{\sigma_i,\sigma_j\}}\sigma_i^{-1}$ **do**
      let $\mathbf{c}_j := \mathbf{a}\left(\Delta_{\{\sigma_i,\sigma_j\}}\sigma_i^{-1}\right)^{-1}\left(\Delta_{\{\sigma_i,\sigma_j\}}\sigma_j^{-1}\right)$
      store the relation $\mathbf{c}_j\sigma_j = \mathbf{b}$
      add the elements of $\mathbf{left}(\mathbf{c}_j)$ to $\mathbf{left}(\mathbf{b})$, and add σ_j to $\mathbf{right}(\mathbf{b})$
    end
    add the arcs $(\mathbf{left}(\mathbf{b}), \mathbf{right}(\mathbf{b}))$ and $(\overline{\mathbf{left}(\mathbf{b})}, \overline{\mathbf{right}(\mathbf{b})})$ to $\mathcal{G}_{\mathrm{gar}}$
  end
  for each element $\mathbf{d}$ of $\mathbf{L}_{k-2}$ **do**
    delete the element $\mathbf{d}$, the sets $\mathbf{left}(\mathbf{d})$ and $\mathbf{right}(\mathbf{d})$, and all the multiplication relations of the form $\mathbf{d}\sigma_i = \mathbf{e}$
  end
  let $\mathbf{C}$ be the connected component of $\{\sigma_1\}$ in $\mathcal{G}^*_{\mathrm{gar}}$
  display the cardinality of $\mathbf{C}$ and the diameter of $\mathbf{C}$
  $k := k + 1$
end
output: the diameter of $\mathbf{C}$

Algorithm 6.18: Computing the diameter of $\mathcal{G}_{\mathrm{gar}}$ when $\mathbf{W} = E_7$ or E_8

If $\mathbf{W} = E_7$, we alleviate this problem by using Algorithm 6.18. Indeed, instead of storing simultaneously the sets $\mathbf{left}(\mathbf{a})$ and $\mathbf{right}(\mathbf{a})$ and the relations of the type $\mathbf{a}\sigma_i = \mathbf{b}$ for all the elements $\mathbf{a}$ and $\mathbf{b}$ of $\mathbf{W}$, we do so only for elements of sets $\bigcup_{i=0}^{3} \mathbf{L}_{k+i}$, where $\mathbf{L}_k$ is the set of simple braids of Artin length k. Such sets are of cardinality at most 521120 (for $k = 30$), which is much smaller than the 2903040 elements of E_7 itself.

However, if $\mathbf{W} = E_8$, this refined approach fails, because even the sets $\mathbf{L}_k$ have huge cardinalities. Nevertheless, a partial execution of Algorithm 6.18 up to the Artin length $k = 13$ already proves that the bilateral graph contains some connected subgraph $\mathbf{C}$ such that $|\mathbf{C}| = 254$, and such that $\mathbf{C}$ has diameter 3. Hence, each node of $\mathcal{G}_{\mathrm{gar}}$ belongs to $\mathbf{C}$, and since $\mathbf{C}$ is a subgraph of $\mathcal{G}_{\mathrm{gar}}$, the graph $\mathcal{G}_{\mathrm{gar}}$ itself must have diameter at most 3. Therefore, it remains to prove that $\mathcal{G}_{\mathrm{gar}}$ has diameter at least 3, which we do by showing that the vertices $\{\sigma_8\}$ and $\{\sigma_1, \ldots, \sigma_7\}$ have no common neighbour in $\mathcal{G}_{\mathrm{gar}}$.

Generating all the elements $\mathbf{a}$ of $\mathbf{W}$ such that $\mathbf{left}(\mathbf{a}) = \{\sigma_8\}$ seems to be difficult if we follow the tracks of Algorithm 6.18. Hence, we take a geometric point of view. Indeed,

$\mathbf{W}$ is isomorphic to the group $\mathcal{O}(\mathbf{R}_8)$ of isometries of the set

$$\mathbf{R}_8 := \left\{(e_1,\ldots,e_8) \in (2\mathbb{Z})^8 : \sum_{i=1}^{8} e_i^2 = 8\right\} \cup \left\{(e_1,\ldots,e_8) \in \mathbb{Z}^8 : \prod_{i=1}^{8} e_i = 1\right\}.$$

Note that each element $\mathbf{e}$ of the set $\mathbf{R}_8$ has an Euclidian norm $|\mathbf{e}| = 8$. Then, consider the *primitive roots* $\mathbf{r}_1,\ldots,\mathbf{r}_8$, all of which belong to the set $\mathbf{R}^8$, and the *positive vector* $\mathbf{p}$, defined as

$$\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \\ \mathbf{r}_5 \\ \mathbf{r}_6 \\ \mathbf{r}_7 \\ \mathbf{r}_8 \\ \mathbf{p} \end{pmatrix} := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \\ -12 & -11 & -10 & 9 & 8 & 7 & 6 & 5 \end{pmatrix}.$$

In addition, consider the associated reflections $\mathbf{s}_i : z \mapsto z - \frac{1}{4}(z \cdot \mathbf{r}_i)\mathbf{r}_i$, for $i \in \{1,\ldots,8\}$. We have $\mathbf{r}_i \cdot \mathbf{r}_j = 0$ if $m_{i,j} = 2$ and $\mathbf{r}_i \cdot \mathbf{r}_j = -\frac{1}{2}|\mathbf{r}_i|\,|\mathbf{r}_j|$ if $m_{i,j} = 3$. Therefore, there exists an (unique) isomorphism of groups $\varphi : \mathbf{W} \mapsto \mathcal{O}(\mathbf{R}_8)$ such that $\varphi : \sigma_i \mapsto \mathbf{s}_i$. Furthermore, we check easily that $\mathbf{r}_i \cdot \mathbf{p} > 0$ for all $i \in \{1,\ldots,8\}$, whence the name “positive” given to the vector $\mathbf{p}$. In addition, let us prove that $\mathbf{e} \cdot \mathbf{p} \neq 0$ for all $\mathbf{e} \in \mathbf{P}_8$.

Let $(e_1,\ldots,e_8)$ and $(p_1,\ldots,p_8)$ be the respective entries of $\mathbf{e}$ and $\mathbf{p}$. First, if we have $\mathbf{e} \in (2\mathbb{Z})^8$ and $\sum_{i=1}^{8} e_i^2 = 8$, then two entries of $\mathbf{e}$ are non-zero. Since no two entries of $\mathbf{p}$ have the same absolute value, it follows that $\mathbf{e} \cdot \mathbf{p} \neq 0$. Second, if $\mathbf{e} \in \left\{(e_1,\ldots,e_8) \in \mathbb{Z}^8 : \prod_{i=1}^{8} e_i = 1\right\}$, Since $\prod_{i=1}^{8} e_i p_i < 0$, the vector $(e_1p_1,\ldots,e_8p_8)$ has an odd number of negative entries, and therefore not as many positive entries as negative entries. Hence, we have $|\mathbf{e} \cdot \mathbf{p}| \geqslant 5+6+7+8+9-10-11-12 = 2$. This proves that $\mathbf{p} \cdot \mathbf{e} \neq 0$ for all $\mathbf{e} \in \mathbf{R}_8$.

Then, we use a direct characterisation of the left and right sets [62, page 14].

Lemma 6.19.
Let $\mathbf{W}$ be the Coxeter group of type E_8. Let $\mathbf{a}$ be an element of $\mathbf{W}$, let $\varphi(\mathbf{a})$ be its image in $\mathcal{O}(\mathbf{R}_8)$. In addition, let $\mathbf{r}_1,\ldots,\mathbf{r}_8$ be elements of $\mathbf{R}_8$, with associated reflections $\mathbf{s}_i$, such that the mapping $\varphi : \sigma_i \mapsto \mathbf{s}_i$ induces an isomorphism of groups between $\mathbf{W}$ and $\mathcal{O}(\mathbf{R}_8)$. Finally, let $\mathbf{p}$ be some vector such that $\mathbf{r}_i \cdot \mathbf{p} > 0$ for all $i \in \{1,\ldots,8\}$ and $\mathbf{e} \cdot \mathbf{p} \neq 0$ for all $\mathbf{e} \in \mathbf{R}_8$. We have

$$\mathbf{left}(\mathbf{a}) = \{\sigma_i : \mathbf{p} \cdot \varphi(\mathbf{a})(\mathbf{r}_i) < 0\} \text{ and } \mathbf{right}(\mathbf{a}) = \{\sigma_i : \mathbf{p} \cdot \varphi(\mathbf{a})^{-1}(\mathbf{r}_i) < 0\}.$$

Hence, we generate inductively the set of isometries $\iota \in \mathcal{O}(\mathbf{R}_8)$ such that $\{\sigma_8\} = \{\sigma_i : \mathbf{p} \cdot \iota(\mathbf{r}_i) < 0\}$, by choosing step by step the possible values of $\iota(\mathbf{r}_i)$ for $i \in \{1,\ldots,8\}$. In practice, this generation is very efficient, and the set of isometries considered has a small cardinality. Then, for each such isometry, it is easy to compute the set $\{\sigma_i : \mathbf{p}\cdot\iota^{-1}(\mathbf{r}_i) < 0\}$. Thereby, we obtain the set $\Sigma := \{P : (\{\sigma_8\}, P) \text{ is an arc of } \mathcal{G}_{\text{gar}}\}$. Moreover, note that $(P, \{\sigma_1, \ldots, \sigma_7\})$ is an arc of $\mathcal{G}_{\text{gar}}$ if and only if $\overline{P} \in \Sigma$. Our computation of Σ proves that this never happens, which completes the proof that the graph $\mathcal{G}_{\text{gar}}$ has diameter 3 when $\mathbf{W}$ is of type E_8.

This concludes our systematic analysis of the diameter of the bilateral Garside automaton.

Theorem 6.20.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid with finite Coxeter type. Let $\mathbf{W}$ be the Coxeter group associated with $\mathbf{A}^+$, and let $\mathcal{G}_{\text{gar}}$ be the bilateral Garside automaton of $\mathbf{A}^+$. The diameter of the graph $\mathcal{G}_{\text{gar}}$ is:

- $\mathfrak{D} = 1$ *if* $\mathbf{W} = I_2(a)$ *(with* $a \geqslant 3$*);*
- $\mathfrak{D} = 2$ *if* $\mathbf{W} = F_4$*,* H_3 *or* H_4*;*
- $\mathfrak{D} = 3$ *if* $\mathbf{W} = A_3$*,* B_3*,* B_4*,* D_4 *or* E_n *(with* $6 \leqslant n \leqslant 8$*);*
- $\mathfrak{D} = 4$ *if* $\mathbf{W} = A_n$ *(with* $n \geqslant 4$*),* B_n *(with* $n \geqslant 5$*) or* D_n *(with* $n \geqslant 6$*).*

Chapter 7

Building Uniform Measures on Braids

Résumé

Nous introduisons et étudions une nouvelle notion de mesure probabilité multiplicative sur les mots finis et infinis dans les monoïdes d'Artin–Tits de type FC irréductibles. Nous classifions et paramétrisons les mesures de probabilité multiplicatives, en particulier la sous-famille des mesures uniformes. Nous construisons les mesures multiplicatives et uniformes en termes de processus de Markov et de forme normale de Garside. Nous prouvons également que les mesures de probabilité uniformes usuelles sur les sphères, dans les monoïdes d'Artin–Tits de type FC, convergent vers notre notion de mesure de probabilité uniforme (au sens faible). Enfin, nous montrons comment utiliser les graphes pondérés conditionnés pour en appliquer des théorèmes centraux limites aux mesures uniformes sur les sphères dans les monoïdes d'Artin–Tits de type FC.

Le contenu de ce chapitre provient d'un travail en cours de rédaction, en collaboration avec Samy Abbes, Sébastien Gouëzel et Jean Mairesse.

Abstract

We introduce and study the new notion of multiplicative probability distributions on finite and infinite words in irreducible Artin–Tits monoids of FC type. We classify and parametrise multiplicative probability distributions, with a special emphasis on the sub-family of *uniform* distributions. We provide explicit constructions of multiplicative and uniform distributions in terms of Markov processes and of Garside normal form. We also prove that "standard" uniform probability distributions on spheres in Artin–Tits monoids of FC type converge weakly toward our notion of uniform probability distribution. Finally, we show how to use the framework of conditioned weighted graphs to derive central limit theorems for distributions on spheres in Artin–Tits monoids of FC type.

The content of this chapter is the result of a paper in progress, written in collaboration with Samy Abbes, Sébastien Gouëzel and Jean Mairesse.

Chapter 7 is devoted to the construction of *uniform measures* on Artin–Tits monoids of FC type. Such uniform measures have already been studied for heap monoids [2], and we develop here analogous arguments in the context of Artin–Tits monoids of FC type. Positive heaps can be represented by their Cartier-Foata normal form. The Cartier-Foata normal form is regular, hence there exists a (minimal) automaton that recognises this normal form. Consequently, we may identify positive heaps with finite paths in this minimal automaton.

This identification between positive heaps and finite paths led to the construction of uniform measures [2], which are similar to the Parry measure [72, 79] and to the Patterson-Sullivan measure [65, 81, 87]. The Parry measure is the measure of maximal entropy of a sofic subshift, and the Patterson-Sullivan measure is also a uniform measure on the border at infinity of some geometric groups, whose proof of existence is not constructive in general.

The identification between positive elements and finite paths in an automaton $\mathcal{A}$ that recognises the left Garside normal form is easily extended to all Artin–Tits monoids of FC type. However, in the case of irreducible Artin–Tits groups of spherical type, the automaton $\mathcal{A}$ is not strongly connected, which prevents using directly the Perron-Frobenius theory [85] and complicates things. Hence, this chapter consists mainly in proving that the automaton $\mathcal{A}$ is strongly connected for all irreducible Artin–Tits monoids of FC type that are not Artin–Tits monoid of spherical type, and in showing how to generalise the work of [2] to irreducible Artin–Tits monoids of spherical type.

Our construction of the uniform measure on Artin–Tits monoids of FC type, which we identify with a weak limit of uniform measures on spheres of finite radius, leads to a wide range of convergence results, for instance on the leftmost letter of left Garside normal words of braids chosen uniformly at random among the set of braids of length k. Moreover, we also extend results of Hennion and Hervé [61] to the framework of Artin–Tits monoids of spherical type, and derive finer convergence results about *Garside-additive functions* and additive functions in Artin–Tits monoids of FC type, including central limit theorems.

Most notions presented in Chapter 7 are borrowed from the above-mentioned literature [61, 65, 72, 79, 81, 85, 87] and adapted to the framework of Artin–Tits monoids of FC type. This leads to proving some important, original results (Propositions 7.62 and 7.85, and Theorems 7.39 and 7.84).

7.1 Uniform Measures on Artin–Tits Monoids of FC Type

We study here an Artin–Tits monoid of FC type $\mathbf{A}^+$ generated by $n \geqslant 2$ generators $\sigma_1, \ldots, \sigma_n$, with smallest two-way Garside family $\mathbf{S}$. In Section 7.1.3, we focus most specifically on irreducible Artin–Tits monoids of FC type, i.e. monoids that are not an abelian product of Artin–Tits monoids.

7.1.1 Algebraic Generating Function and Möbius Transforms

In Chapter 4, we considered the *geometric generating functions* of braid monoids. Here, we focus on their most standard, algebraic counterpart. Henceforth, we will only refer to the *generating function* of an Artin–Tits monoid $\mathbf{A}^+$ (which will mostly have FC type), the word *algebraic* being left implicit.

Definition 7.1 (Algebraic generating function and Möbius polynomial)**.**
Let $\mathbf{A}^+$ *be an Artin–Tits monoid. The* algebraic generating function *of the monoid* $\mathbf{A}^+$ *is defined as the function* $\mathcal{G}_\mathbf{A} : z \mapsto \sum_{\mathbf{a} \in \mathbf{A}^+} z^{\lambda(\mathbf{a})}$, *where* λ *is the length function on* $\mathbf{A}^+$.

The Möbius polynomial *of* $\mathbf{A}^+$ *is defined by* $\mathcal{H}_\mathbf{A} : z \mapsto \sum_{I \in \mathcal{P}} (-1)^{|I|} z^{\lambda(\Delta_I)}$, *where* $\mathcal{P}$ *denotes the set* $\{I \subseteq \{\sigma_1, \ldots, \sigma_n\} : \mathbf{LCM}_{\leqslant_\ell}(I)$ *exists*$\}$ *and* $\Delta_I := \mathbf{LCM}_{\leqslant_\ell}(I)$ *for all* $I \in \mathcal{P}$.

Standard results [4, 21] outline the link between the generating function and the Möbius polynomial of $\mathbf{A}^+$.

Proposition 7.2.
Let $\mathbf{A}^+$ *be an Artin–Tits monoid. The generating function* $\mathcal{G}_\mathbf{A}$ *and the Möbius polynomial* $\mathcal{H}_\mathbf{A}$ *of the monoid* $\mathbf{A}^+$ *are inverses of each other, i.e.* $\mathcal{G}_\mathbf{A}(z)\mathcal{H}_\mathbf{A}(z) = 1$.

Proof. A direct computation, using the change of variable $\mathbf{b} := \Delta_I \mathbf{a}$, shows that

$$\begin{aligned}
\mathcal{G}_\mathbf{A}(z)\mathcal{H}_\mathbf{A}(z) &= \sum_{\mathbf{a} \in \mathbf{A}^+} \sum_{I \in \mathcal{P}} (-1)^{|I|} z^{\lambda(\Delta_I) + \lambda(\mathbf{a})} = \sum_{\mathbf{a} \in \mathbf{A}^+} \sum_{I \in \mathcal{P}} (-1)^{|I|} z^{\lambda(\Delta_I \mathbf{a})} \\
&= \sum_{\mathbf{b} \in \mathbf{A}^+} \sum_{I \subseteq \mathbf{left}(\mathbf{b})} (-1)^{|I|} z^{\lambda(\mathbf{b})} = \sum_{\mathbf{b} \in \mathbf{A}^+} \mathbf{1}_{\mathbf{left}(\mathbf{b}) = \varnothing} z^{\lambda(\mathbf{b})} = 1.
\end{aligned}$$

□

Efficient algorithms allow us to compute recursively the Möbius polynomial when $\mathbf{A}^+$ is a (reducible or not) Artin–Tits monoid of spherical type (see [21] in the case of braid monoids and [5] in the general case).

Proposition 7.3.
Let $\mathbf{A}^+$ *be an Artin–Tits monoid. If* $\mathbf{A}^+$ *is a direct product of two non-trivial Artin–Tits monoids* $\mathbf{B}^+$ *and* $\mathbf{C}^+$, *then* $\mathcal{H}_\mathbf{A}(z) = \mathcal{H}_\mathbf{B}(z)\mathcal{H}_\mathbf{C}(z)$. *In addition, if the Coxeter group* $\mathbf{W}$ *of* $\mathbf{A}^+$ *is irreducible, several cases are possible, according to the classification of Theorem 2.26, and are enumerated below.*

- *If* $\mathbf{W}$ *is of type* A_n, *let* $\overline{\mathcal{H}}_{a,n}(z)$ *be the Möbius polynomial of* $\mathbf{A}^+$, *and let us define the polynomials* $\overline{\mathcal{H}}_{a,-1}(z) = \overline{\mathcal{H}}_{a,0}(z) = 1$. *We have*

$$\overline{\mathcal{H}}_{a,n}(z) = \sum_{k=0}^{n} (-1)^k z^{k(k+1)/2} \overline{\mathcal{H}}_{a,n-k-1}(z).$$

- *If* $\mathbf{W}$ *is of type* B_n*, let* $\overline{\mathcal{H}}_{b,n}(z)$ *be the Möbius polynomial of* $\mathbf{A}^+$*, and let us define the polynomial* $\overline{\mathcal{H}}_{b,0}(z) = 1$*. We have*

$$\overline{\mathcal{H}}_{b,n}(z) = \sum_{k=0}^{n-1} (-1)^k z^{k(k+1)/2} \overline{\mathcal{H}}_{b,n-k-1}(z) + (-1)^n z^{n^2}.$$

- *If* $\mathbf{W}$ *is of type* D_n*, let* $\overline{\mathcal{H}}_{d,n}(z)$ *be the Möbius polynomial of* $\mathbf{A}^+$*, and let us define the polynomial* $\overline{\mathcal{H}}_{d,2}(z) = 1 - 2z + z^2$*. We have*

$$\overline{\mathcal{H}}_{d,n} = \sum_{k=0}^{n-3} (-1)^{k(k+1)/2} \overline{\mathcal{H}}_{d,n-k-1}(z) + (-1)^n \left(z^{(n-2)(n-1)/2} - 2z^{(n-1)n/2} + z^{(n-1)n} \right).$$

- *If* $\mathbf{W}$ *is of exceptional type, then* $\mathcal{H}_{\mathbf{A}}(z)$ *is as presented in Fig. 7.4, according to the type of* $\mathbf{W}$*.*

$\mathbf{W}$	$\mathcal{H}_{\mathbf{A}}(z)$
E_6	$1-6z+10z^2-10z^4+5z^5-4z^6+3z^7+4z^{10}-2z^{11}+z^{12}-z^{15}-2z^{20}+z^{36}$
E_7	$1-7z+15z^2-5z^3-16z^4+12z^5-3z^6+8z^7-3z^8-3z^9+6z^{10}-5z^{11}+z^{12}-3z^{15}+z^{16}-2z^{20}+2z^{21}+z^{30}+z^{36}-z^{63}$
E_8	$1-8z+21z^2-14z^3-21z^4+28z^5-7z^6+12z^7-8z^8-10z^9+10z^{10}-12z^{11}+7z^{12}+2z^{13}-z^{14}-3z^{15}+2z^{16}-2z^{20}+6z^{21}-z^{22}-z^{23}-z^{28}+z^{30}+z^{36}-z^{37}-z^{42}-z^{63}+z^{120}$
F_4	$1-4z+3z^2+2z^3-z^4-2z^9+z^{24}$
H_3	$1-3z+z^2+z^3+z^5-z^{15}$
H_4	$1-4z+3z^2+2z^3-z^4+z^5-2z^6-z^{15}+z^{60}$
$I_2(a)$	$1-2z+z^a$

Figure 7.4 – Computing Möbius polynomials in irreducible Artin–Tits monoids

Proof. The computation of $\mathcal{H}_{\mathbf{A}}(z)$ when $\mathbf{W}$ is of exceptional case can be performed by brute-force. Hence, we focus on the case where $\mathbf{W}$ is of type A_n: the cases where $\mathbf{W}$ is of type B_n or D_n are analogous.

If $\mathbf{W}$ is of type A_n, for each set $I \in \mathcal{P}$, let $k(I)$ be the largest integer such that $\{\sigma_{n+1-k(I)}, \ldots, \sigma_n\} \subseteq I$, and let us denote by J the set $I \backslash \{\sigma_{n+1-k(I)}, \ldots, \sigma_n\}$. In addition, for all $i \leqslant n$, let $\mathcal{P}_i := 2^{\{\sigma_1,\ldots,\sigma_{n-1-i}\}}$. The set J must belong to $\mathcal{P}_{k(I)}$, hence $\Delta_I = \Delta_J \Delta_{\{\sigma_{n+1-k(I)},\ldots,\sigma_n\}}$. This proves that $\lambda(\Delta_I) = \lambda(\Delta_J) + k(I)(k(I)+1)/2$. Hence, we compute

$$\begin{aligned}
\overline{\mathcal{H}}_{a,n}(z) &= \sum_{I \in \mathcal{P}} (-1)^{|I|} z^{\lambda(\Delta_I)} = \sum_{i=0}^{n} \sum_{I \in \mathcal{P}} \mathbf{1}_{k(I)=i} (-1)^{|I|} z^{\lambda(\Delta_I)} \\
&= \sum_{i=0}^{n} \sum_{J \in \mathcal{P}_i} (-1)^{|J|+i} z^{\lambda(\Delta_J)+i(i+1)/2} = \sum_{i=0}^{n} (-1)^i z^{i(i+1)/2} \sum_{J \in \mathcal{P}_i} (-1)^{|J|} z^{\lambda(\Delta_J)} \\
&= \sum_{i=0}^{n} (-1)^i z^{i(i+1)/2} \overline{\mathcal{H}}_{a,n-1-i}(z).
\end{aligned}$$

□

In addition, the radius of convergence of the generating function $\mathcal{G}_{\mathbf{A}}(z)$ enjoys many properties, such as the following one, which we will then rephrase in a stronger form in Corollary 7.56.

Proposition 7.5.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid with $n \geqslant 2$ generators. Let $p_{\mathbf{A}}$ be the radius of convergence of the generating function $\mathcal{G}_{\mathbf{A}}(z)$. The real number $p_{\mathbf{A}}$ belongs to the half-open interval $[1/n, 2/3)$ and is a root of the Möbius polynomial $\mathcal{H}_{\mathbf{A}}$ of minimal modulus, i.e. $|\rho| \geqslant p_{\mathbf{A}}^+$ for all roots ρ of $\mathcal{H}_{\mathbf{A}}$.

Proof. First, let $\mathbf{B}^+$ be the free monoid with n generators. The monoid $\mathbf{A}^+$ is a quotient monoid of $\mathbf{B}^+$, and the monoid $\mathbb{Z}_{\geqslant 0}$ is a quotient monoid of $\mathbf{A}^+$, whence $p_{\mathbf{B}} \leqslant p_{\mathbf{A}} \leqslant p_{\mathbb{Z}_{\geqslant 0}}$, where $p_{\mathbf{A}}$, $p_{\mathbf{B}}$ and $p_{\mathbb{Z}_{\geqslant 0}}$ are the respective radii of convergence of the generating functions $\mathcal{G}_{\mathbf{A}}(z)$, $\mathcal{G}_{\mathbf{B}}(z)$ and $\mathcal{G}_{\mathbb{Z}_{\geqslant 0}}(z)$. Since $\mathcal{G}_{\mathbf{B}}(z) = \frac{1}{1-nz}$ and $\mathcal{G}_{\mathbb{Z}_{\geqslant 0}}(z) = \frac{1}{1-z}$, it follows that $\frac{1}{n} = p_{\mathbf{B}} \leqslant p_{\mathbf{A}} \leqslant p_{\mathbb{Z}_{\geqslant 0}} = 1$.

The equality $\mathcal{G}_{\mathbf{A}}(z)\mathcal{H}_{\mathbf{A}}(z) = 1$ holds on the open (complex) disk of convergence of $\mathcal{G}_{\mathbf{A}}(z)$, which means that we have $\mathcal{H}_{\mathbf{A}}(z) \neq 0$ whenever $|z| < p_{\mathbf{A}}$ and that $\mathcal{G}_{\mathbf{A}}(z)$ is the rational fraction $\mathcal{H}_{\mathbf{A}}(z)^{-1}$. Since the terms of the generating function $\mathcal{G}_{\mathbf{A}}(z)$ are non-negative, it follows that $p_{\mathbf{A}} = \sup\{z \in \mathbb{R}_{\geqslant 0} : \mathcal{G}(z) < +\infty\}$, and therefore that $p_{\mathbf{A}}$ is a pole of $\mathcal{G}_{\mathbf{A}}$, i.e. a root of $\mathcal{H}_{\mathbf{A}}$. Hence, $p_{\mathbf{A}}$ is a root of $\mathcal{H}_{\mathbf{A}}$ of minimal modulus.

Finally, the monoid $\mathbf{A}^+$ must contain some 2-generator irreducible submonoid, which we call $\mathbf{C}^+$ and The monoid $\mathbf{C}^+$ is either a dihedral monoid with Coxeter group $I_2(a)$ for some $a \geqslant 3$, or a free monoid, with Coxeter group $I_2(a)$ for $a = +\infty$. Again, we have $p_{\mathbf{A}} \leqslant p_{\mathbf{C}}$, where $p_{\mathbf{C}}$ is the radius of convergence of $\mathcal{G}_{\mathbf{C}}(z)$, i.e. a root of $\mathcal{H}_{\mathbf{C}}(z)$ of minimal modulus. Since $\mathcal{H}_{\mathbf{C}}(0) = 1$ and

$$\mathcal{H}_{\mathbf{C}}\left(\frac{2}{3}\right) = 1 - \frac{4}{3} + \left(\frac{2}{3}\right)^a \leqslant 1 - \frac{4}{3} + \left(\frac{2}{3}\right)^3 = -\frac{1}{27},$$

it follows that $p_{\mathbf{C}} < \frac{2}{3}$, which completes the proof. □

Moreover, (algebraic) generating functions and Möbius polynomials extend easily to the framework of *valuations*, i.e. monoid morphisms from $\mathbf{A}^+$ to $\mathbb{C}\backslash\{0\}$.

Definition 7.6 (Valuation)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid with n generators. Let $(r_i)_{1 \leqslant i \leqslant n}$ be a collection of complex numbers. We call valuation *of parameters (r_i) the (unique) multiplicative function $r : \mathbf{A}^+ \mapsto \mathbb{C}$ such that $r(\sigma_i) = r_i$ for all $i \in \{1, \ldots, n\}$.*

If each number r_i is a non-negative real number, we say that r is non-negative. *If each number r_i is positive, we even say that r is* positive.

Henceforth, we identify each valuation $r : \mathbf{A}^+ \mapsto \mathbb{C}$ with the collection $(r_1, \ldots, r_n)$. Note that functions $\mathbf{x} \mapsto z^{\lambda(\mathbf{x})}$, which appear implicitly in the definitions of the generating function and of the Möbius polynomial of $\mathbf{A}^+$, are specific valuations. Hence, we denote them by z^λ, so that $z^\lambda : \mathbf{x} \mapsto z^{\lambda(\mathbf{x})}$. In particular, multiplying the tuple $(r_1, \ldots, r_n)$ by a constant θ amounts to multiplying the valuation r by the valuation θ^λ.

In addition, we decide to focus only on positive valuations. Indeed, if $r_i = 0$ for some $i \in \{1, \ldots, n\}$, studying the valuation r on the monoid $\mathbf{A}^+$ amounts to studying its restriction to the submonoid generated by $\{\sigma_j : i \neq j\}$. If $\mathbf{A}^+$ is a braid monoid $\mathbf{B}_n^+$, then the braid relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ imply that $r_i r_{i+1}(r_{i+1} - r_i) = 0$. Since $r_i > 0$ for all i, we must have $r_1 = \ldots = r_n$.

In general, we must have $r_i = r_j$ whenever $m_{i,j}$ is odd, due the relation $[\sigma_i\sigma_j]^{m_{i,j}} = [\sigma_j\sigma_i]^{m_{i,j}}$. Hence, let $\mathbf{G}$ be the Coxeter graph of the monoid $\mathbf{A}^+$, and consider the subgraph $\mathbf{G}^1$ obtained by deleting all the edges (i, j) of $\mathbf{G}$ such that $m_{i,j} \in 2\mathbb{Z}$. The equality $r_i = r_j$ must hold whenever i and j belong to the same connected component of $\mathbf{G}^1$, and may not hold if i and j belong to distinct connected component of $\mathbf{G}^1$.

For instance, we must have $r_1 = \ldots = r_n$ if $\mathbf{A}^+$ is an Artin–Tits monoid with Coxeter group of type A_n, D_n, E_n, H_n or $I_2(a)$ with a odd. On the contrary, if the Coxeter group of $\mathbf{A}^+$ is of type B_n, then we only have $r_2 = r_3 = \ldots = r_n$, but the equality $r_1 = r_2$ does not necessarily hold. Likewise, if $\mathbf{A}^+$ is a heap monoid, then the graphs $\mathbf{G}$ and $\mathbf{G}^1$ are equal, and therefore we may choose $r_1, \ldots, r_n$ independently.

This discussion leads to the notion of valuation manifold, whose elements are the tuples $(r_1, \ldots, r_n)$ associated with some positive valuation $r : \mathbf{A}^+ \mapsto \mathbb{C}$.

Definition 7.7 (Valuation manifold).
Let $\mathbf{A}^+$ be an Artin–Tits monoid with n generators. The valuation manifold *of $\mathbf{A}^+$ is defined as the set*

$$\mathcal{V}_\mathbf{A} := \{(r_1, \ldots, r_n) \in (0, +\infty)^n : \forall i, j \in \{1, \ldots, n\}, m_{i,j} \in 2\mathbb{Z} + 1 \Rightarrow r_i = r_j\}.$$

Definition 7.8 (Multivariate generating function and Möbius polynomial).
Let $\mathbf{A}^+$ be an Artin–Tits monoid. The multivariate algebraic generating function *of the monoid $\mathbf{A}^+$ is defined as the function $\mathcal{G}_\mathbf{A} : r \mapsto \sum_{\mathbf{a} \in \mathbf{A}^+} r(\mathbf{a})$, where $r : \mathbf{A}^+ \mapsto \mathbb{C}$ is a valuation.*

The multivariate Möbius polynomial *of $\mathbf{A}^+$ is defined by $\mathcal{H}_\mathbf{A} : r \mapsto \sum_{I \in \mathcal{P}} (-1)^{|I|} r(\Delta_I)$.*

These are indeed generalisations of the standard generating function and Möbius polynomial, through the identification $\mathcal{G}_\mathbf{A}(z) = \mathcal{G}_\mathbf{A}(z^\lambda)$ and $\mathcal{H}_\mathbf{A}(z) = \mathcal{H}_\mathbf{A}(z^\lambda)$. Furthermore, let $C_1, \ldots, C_k$ be the connected components of the graph $\mathbf{G}^1$ defined above, and let $\mathbf{t}_1, \ldots, \mathbf{t}_k$ be the elements of $\mathbb{R}^n$ such that $\mathbf{t}_j := (\mathbf{1}_{i \in C_j})_{1 \leqslant i \leqslant n}$ for all $j \in \{1, \ldots, k\}$. The valuation manifold is the cone generated positively by the family $\mathbf{t}_1, \ldots, \mathbf{t}_k$, i.e. $\mathcal{V}_\mathbf{A} = \{x_1\mathbf{t}_1 + \ldots + x_k\mathbf{t}_k : 0 < x_1, \ldots, x_k\}$.

In particular, when restricting our study to positive valuations, we can identify each valuation r with the tuple $(x_1, \ldots, x_k)$ such that $(r_1, \ldots, r_n) = \sum_{j=1}^{k} x_j \mathbf{t}_j$. Hence, we may view $\mathcal{G}_\mathbf{A}$ and $\mathcal{H}_\mathbf{A}$ as actual multivariate functions, with (positive) variables $(x_1, \ldots, x_k)$.

Proposition 7.9.
Let $\mathbf{A}^+$ be an Artin–Tits monoid. The multivariate generating function $\mathcal{G}_\mathbf{A}$ and the multivariate Möbius polynomial $\mathcal{H}_\mathbf{A}$ of the monoid $\mathbf{A}^+$ are inverses of each other, i.e. $\mathcal{G}_\mathbf{A}(r)\mathcal{H}_\mathbf{A}(r) = 1$.

Proof. A direct computation, using the change of variable $\mathbf{b} := \Delta_I \mathbf{a}$, shows that

$$\begin{aligned} \mathcal{G}_\mathbf{A}(r)\mathcal{H}_\mathbf{A}(r) &= \sum_{\mathbf{a} \in \mathbf{A}^+} \sum_{I \in \mathcal{P}} (-1)^{|I|} r(\Delta_I) r(\mathbf{a}) = \sum_{\mathbf{a} \in \mathbf{A}^+} \sum_{I \in \mathcal{P}} (-1)^{|I|} r(\Delta_I \mathbf{a}) \\ &= \sum_{\mathbf{b} \in \mathbf{A}^+} \sum_{I \subseteq \mathbf{left}(\mathbf{b})} (-1)^{|I|} r(\mathbf{b}) = \sum_{\mathbf{b} \in \mathbf{A}^+} \mathbf{1}_{\mathbf{left}(\mathbf{b}) = \varnothing} r(\mathbf{b}) = 1. \end{aligned}$$

□

Proposition 7.9 has the flavour of inclusion-exclusion principles. Pushing further towards this direction naturally leads to the notion of Möbius transform of functions [83, 86].

Definition 7.10 (Möbius transform and inverse Möbius transform)**.**
Consider some function $f : \mathbf{A}^+ \mapsto \mathbb{C}$. We define the Möbius transform *of f as the function $\mathbf{M}f : \mathbf{x} \mapsto \sum_{I \in \mathcal{P}} (-1)^{|I|} f(\mathbf{x}\Delta_I)$.*

In addition, if f is L^1, i.e. if $\sum_{\mathbf{a} \in \mathbf{A}^+} |f(\mathbf{a})| < +\infty$, then we define the inverse Möbius transform *of f as the function $\overline{\mathbf{M}}f : \mathbf{x} \mapsto \sum_{\mathbf{y} \in \mathbf{A}^+} f(\mathbf{xy})$.*

Proposition 7.11.
Let $\mathbf{A}^+$ be an Artin–Tits monoid and let $f : \mathbf{A}^+ \mapsto \mathbb{C}$ be an L^1 function. The Möbius transform $\mathbf{M}f$ is also an L^1 function, and we have $\mathbf{M}(\overline{\mathbf{M}}f) = \overline{\mathbf{M}}(\mathbf{M}f) = f$.

Proof. First, since $\mathbf{M}f$ is a finite sum of L^1 functions, it is also an L^1 function. Hence, for all $\mathbf{a} \in \mathbf{A}^+$, and using the changes of variables $\mathbf{c} := \mathbf{b}\Delta_I$ and $\mathbf{d} := \Delta_I \mathbf{b}$, we have

$$\begin{aligned} \mathbf{M}(\overline{\mathbf{M}}f)(\mathbf{a}) &= \sum_{\mathbf{b} \in \mathbf{A}^+} \sum_{I \in \mathcal{P}} (-1)^{|I|} f(\mathbf{ab}\Delta_I) = \sum_{\mathbf{c} \in \mathbf{A}^+} \sum_{I \subseteq \mathbf{right}(\mathbf{c})} (-1)^{|I|} f(\mathbf{ac}) \\ &= \sum_{\mathbf{c} \in \mathbf{A}^+} \mathbf{1}_{\mathbf{right}(\mathbf{c}) = \varnothing} f(\mathbf{ac}) = f(\mathbf{a}) \text{ and} \\ \overline{\mathbf{M}}(\mathbf{M}f)(\mathbf{a}) &= \sum_{\mathbf{b} \in \mathbf{A}^+} \sum_{I \in \mathcal{P}} (-1)^{|I|} f(\mathbf{a}\Delta_I \mathbf{b}) = \sum_{\mathbf{d} \in \mathbf{A}^+} \sum_{I \subseteq \mathbf{left}(\mathbf{d})} (-1)^{|I|} f(\mathbf{ad}) \\ &= \sum_{\mathbf{d} \in \mathbf{A}^+} \mathbf{1}_{\mathbf{left}(\mathbf{d}) = \varnothing} f(\mathbf{ad}) = f(\mathbf{a}). \end{aligned}$$

□

One shortcoming of the above-defined Möbius transform is that its inverse is only defined on L^1 functions. Hence, in the specific case of monoids of FC type, we introduce a variant of Möbius transforms, which involves the modified length $\|\mathbf{x}\|_\gamma := |\mathbf{NF}_\ell(\mathbf{x})| + \mathbf{1}_{\mathbf{x}=\mathbf{1}}$ and modified functions $f_k : \mathbf{x} \mapsto \mathbf{1}_{\|\mathbf{x}\|_\gamma \leqslant k} f(\mathbf{x})$. The corrective term $\mathbf{1}_{\mathbf{x}=\mathbf{1}}$ in the definition of $\|\mathbf{x}\|_\gamma$ is meant to provide the equivalence $\|\mathbf{x}\|_\gamma = 1 \Leftrightarrow \mathbf{x} \in \mathbf{S}$.

Definition 7.12 (Graded Möbius transform and inverse graded Möbius transform)**.** *Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type and let $f : \mathbf{A}^+ \mapsto \mathbb{C}$ be a function. We define the* graded Möbius transform *of f as the function $\mathbf{M}_\gamma f : \mathbf{x} \mapsto (\mathbf{M} f_{\|\mathbf{x}\|_\gamma})(\mathbf{x})$, and we define the* inverse graded Möbius transform *of f as the function $\overline{\mathbf{M}}_\gamma f : \mathbf{x} \mapsto (\overline{\mathbf{M}} f_{\|\mathbf{x}\|_\gamma})(\mathbf{x})$, where $\mathbf{M}$ and $\overline{\mathbf{M}}$ are the (standard) Möbius transform and inverse Möbius transform in the sense of Definition 7.10.*

Proposition 7.13.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. The graded Möbius transform $\mathbf{M}_\gamma$ and the inverse graded Möbius transform $\overline{\mathbf{M}}_\gamma$ are inverse (linear) bijections of the set $\{f : \mathbf{A}^+ \mapsto \mathbb{C}\}$.

Proof. Let $f : \mathbf{A}^+ \mapsto \mathbb{C}$ be a function, let $\mathbf{a}$ be some element of $\mathbf{A}^+$, and let k be some integer.

We first show that $(\mathbf{M}_\gamma f)_k = \mathbf{M}_\gamma(f_k)$ and $(\overline{\mathbf{M}}_\gamma f)_k = \overline{\mathbf{M}}_\gamma(f_k)$. Indeed, we have

$$\begin{aligned}
(\mathbf{M}_\gamma f_k)(\mathbf{a}) &= (\mathbf{M}(f_k)_{\|\mathbf{a}\|_\gamma})(\mathbf{a}) = (\mathbf{M} f_{\|\mathbf{a}\|_\gamma})(\mathbf{a}) = (\mathbf{M}_\gamma f)(\mathbf{a}) = (\mathbf{M}_\gamma f)_k(\mathbf{a}) \text{ if } \|\mathbf{a}\|_\gamma \leqslant k;\\
(\mathbf{M}_\gamma f_k)(\mathbf{a}) &= (\mathbf{M}(f_k)_{\|\mathbf{a}\|_\gamma})(\mathbf{a}) = (\mathbf{M} f_k)(\mathbf{a}) = 0 = (\mathbf{M}_\gamma f)_k(\mathbf{a}) \text{ if } \|\mathbf{a}\|_\gamma > k;\\
(\overline{\mathbf{M}}_\gamma f_k)(\mathbf{a}) &= (\overline{\mathbf{M}}(f_k)_{\|\mathbf{a}\|_\gamma})(\mathbf{a}) = (\overline{\mathbf{M}} f_{\|\mathbf{a}\|_\gamma})(\mathbf{a}) = (\overline{\mathbf{M}}_\gamma f)(\mathbf{a}) = (\overline{\mathbf{M}}_\gamma f)_k(\mathbf{a}) \text{ if } \|\mathbf{a}\|_\gamma \leqslant k;\\
(\overline{\mathbf{M}}_\gamma f_k)(\mathbf{a}) &= (\overline{\mathbf{M}}(f_k)_{\|\mathbf{a}\|_\gamma})(\mathbf{a}) = (\overline{\mathbf{M}} f_k)(\mathbf{a}) = 0 = (\overline{\mathbf{M}}_\gamma f)_k(\mathbf{a}) \text{ if } \|\mathbf{a}\|_\gamma > k.
\end{aligned}$$

In addition, $\mathbf{a} \mapsto \|\mathbf{a}\|_\gamma$ is non-decreasing for the ordering $\leqslant_\ell$, i.e. $\|\mathbf{a}\|_\gamma \leqslant \|\mathbf{ab}\|_\gamma$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{A}^+$. It follows that

$$\begin{aligned}
(\overline{\mathbf{M}}_\gamma(\mathbf{M}_\gamma f))(\mathbf{a}) &= (\overline{\mathbf{M}}(\mathbf{M}_\gamma f)_{\|\mathbf{a}\|_\gamma})(\mathbf{a}) = (\overline{\mathbf{M}}(\mathbf{M}_\gamma f_{\|\mathbf{a}\|_\gamma}))(\mathbf{a}) = \sum_{\mathbf{b} \in \mathbf{A}^+} (\mathbf{M}_\gamma f_{\|\mathbf{a}\|_\gamma})(\mathbf{ab})\\
&= \sum_{\mathbf{b} \in \mathbf{A}^+} (\mathbf{M}(f_{\|\mathbf{a}\|_\gamma})_{\|\mathbf{ab}\|_\gamma})(\mathbf{ab}) = \sum_{\mathbf{b} \in \mathbf{A}^+} (\mathbf{M} f_{\|\mathbf{a}\|_\gamma})(\mathbf{ab})\\
&= (\overline{\mathbf{M}}(\mathbf{M} f_{\|\mathbf{a}\|_\gamma}))(\mathbf{a}) = f_{\|\mathbf{a}\|_\gamma}(\mathbf{a}) = f(\mathbf{a}) \text{ and}\\
(\mathbf{M}_\gamma(\overline{\mathbf{M}}_\gamma f))(\mathbf{a}) &= (\mathbf{M}(\overline{\mathbf{M}}_\gamma f)_{\|\mathbf{a}\|_\gamma})(\mathbf{a}) = (\mathbf{M}(\overline{\mathbf{M}}_\gamma f_{\|\mathbf{a}\|_\gamma}))(\mathbf{a}) = \sum_{I \in \mathcal{P}} (-1)^{|I|} (\overline{\mathbf{M}}_\gamma f_{\|\mathbf{a}\|_\gamma})(\mathbf{a}\Delta_I)\\
&= \sum_{I \in \mathcal{P}} (-1)^{|I|} (\overline{\mathbf{M}}(f_{\|\mathbf{a}\|_\gamma})_{\|\mathbf{a}\Delta_I\|_\gamma})(\mathbf{a}\Delta_I) = \sum_{I \in \mathcal{P}} (-1)^{|I|} (\overline{\mathbf{M}} f_{\|\mathbf{a}\|_\gamma})(\mathbf{a}\Delta_I)\\
&= (\mathbf{M}(\overline{\mathbf{M}} f_{\|\mathbf{a}\|_\gamma}))(\mathbf{a}) = f_{\|\mathbf{a}\|_\gamma}(\mathbf{a}) = f(\mathbf{a}).
\end{aligned}$$

□

7.1.2 Extended Artin–Tits Monoid and Finite Measures

We wish to embed $\mathbf{A}^+$ into a compact metric space $\overline{\mathbf{A}}^+$ in a way that would also preserve the partial order $\leqslant_\ell$. A standard construction for this is performed into the framework of projective systems [17].

Definition & Proposition 7.14 (Extended Artin–Tits monoid)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. Consider the sets Ω_k, for $k \geqslant 0$, and maps $\theta_{k,m} : \Omega_m \mapsto \Omega_k$, for $m \geqslant k \geqslant 0$, defined by:

$$\begin{aligned} \Omega_k &= \{\mathbf{x} \in \mathbf{A}^+ : \|\mathbf{x}\| \leqslant k\} \\ \theta_{k,m} &: \mathbf{x} \mapsto \alpha_\ell^k(\mathbf{x}), \end{aligned}$$

where $\alpha_\ell^k : \mathbf{x} \mapsto \mathbf{LCM}_{\leqslant_\ell}(\{\mathbf{y} \in \mathbf{S}^k : \mathbf{y} \leqslant_\ell \mathbf{x}\})$.

We have $\theta_{k,k} = \mathbf{Id}_{\Omega_k}$ and $\theta_{k,m} \circ \theta_{m,n} = \theta_{k,n}$ for all $k \leqslant m \leqslant n$. Hence, we say that the families $(\Omega_k)_{k\geqslant 0}$ and $(\theta_{k,m})_{m\geqslant k\geqslant 0}$ form a projective system*.*

We call extended Artin–Tits monoid *the* projective limit

$$\overline{\mathbf{A}}^+ := \left\{ (\mathbf{x}_k)_{k\geqslant 0} \in \prod_{k\geqslant 0} \Omega_k : \forall k, m \geqslant 0, m \geqslant k \Rightarrow \theta_{k,m}(\mathbf{x}_m) = \mathbf{x}_k \right\}.$$

We equip the set $\overline{\mathbf{A}}^+$ with the partial order, which we denote by $\leqslant_\ell$, inherited from the product partial order on $\prod_{k\geqslant 0}\Omega_k$, and with a collection of order-preserving mappings $\theta_{k,\infty} : \overline{\mathbf{A}^+} \mapsto \Omega_k$ such that $\theta_{k,\infty} : (\mathbf{x}_m)_{m\geqslant 0} \mapsto \mathbf{x}_k$.

Lemma 7.15.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. The mapping $\iota : \mathbf{A}^+ \mapsto \overline{\mathbf{A}}^+$ defined by $\iota : \mathbf{x} \mapsto (\alpha_\ell^k(\mathbf{x}))_{k\geqslant 0}$ is an embedding of the ordered set $\mathbf{A}^+$ into $\overline{\mathbf{A}}^+$.

Proof. Consider two elements $\mathbf{x}$ and $\mathbf{y}$ of $\mathbf{A}^+$, and let $k := \max\{\|\mathbf{x}\|, \|\mathbf{y}\|\}$. Since $\mathbf{x} = \alpha_\ell^k(\mathbf{x})$ and $\mathbf{y} = \alpha_\ell^k(\mathbf{y})$, it follows that

$$\begin{aligned} &\mathbf{x} \leqslant_\ell \mathbf{y} \Rightarrow (\forall m \geqslant 0, \alpha_\ell^m(\mathbf{x}) \leqslant_\ell \alpha_\ell^m(\mathbf{y})) \Rightarrow \iota(\mathbf{x}) \leqslant_\ell \iota(\mathbf{y}); \\ &\iota(\mathbf{x}) \leqslant \iota(\mathbf{y}) \Rightarrow \alpha_\ell^k(\mathbf{x}) \leqslant_\ell \alpha_\ell^k(\mathbf{y}) \Rightarrow \mathbf{x} \leqslant_\ell \mathbf{y}. \end{aligned}$$

This completes the proof. □

Definition 7.16 (Finite and infinite elements)**.**
We identify henceforth the Artin–Tits monoid of FC type $\mathbf{A}^+$ with its embedding $\iota(\mathbf{A}^+)$ into the set $\overline{\mathbf{A}}^+$. Hence, we call finite elements *of $\overline{\mathbf{A}}^+$ the elements of $\mathbf{A}^+$, and* infinite elements *of $\overline{\mathbf{A}}^+$ the elements of the set $\partial\mathbf{A}^+ := \overline{\mathbf{A}}^+ \backslash \mathbf{A}^+$.*

From this first definition follow already some properties on the ordered set $(\overline{\mathbf{A}}^+, \leqslant_\ell)$.

Lemma 7.17.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. The set $\overline{\mathbf{A}}^+$, equipped with the order $\leqslant_\ell$, is a complete lower semilattice, i.e. arbitrary non-empty subsets of $\overline{\mathbf{A}}^+$ admit a largest common divisor. Furthermore, each subset of $\overline{\mathbf{A}}^+$ that has some common multiple has a least common multiple.

Proof. Let S be some non-empty subset of $\overline{\mathbf{A}}^+$. We first show that S admits a largest common divisor. Indeed, each set Ω_k is a finite lower semilattice, hence is a complete lower semilattice. Hence, consider the sequence $(\mathbf{x}_k)_{k\geqslant 0}$ such that $\mathbf{x}_k = \mathbf{GCD}_{\leqslant_\ell}(\theta_{k,\infty}(S))$. For all integers $\ell \geqslant k \geqslant 0$, we have

$$\theta_{k,\ell}(\mathbf{x}_\ell) = \mathbf{GCD}_{\leqslant_\ell}(\theta_{k,\ell} \circ \theta_{\ell,\infty}(S)) = \mathbf{GCD}_{\leqslant_\ell}(\theta_{k,\infty}(S)) = \mathbf{x}_k,$$

hence $(\mathbf{x}_k)_{k\geqslant 0}$ belongs to $\overline{\mathbf{A}}^+$. It follows immediately that $(\mathbf{x}_k)_{k\geqslant 0} = \mathbf{GCD}_{\leqslant_\ell}(S)$.

Consequently, the set S also admits a least common multiple, which is

$$\mathbf{LCM}_{\leqslant_\ell}(S) := \mathbf{GCD}_{\leqslant_\ell}(\{\zeta \in \overline{\mathbf{A}}^+ : \forall \mathbf{x} \in S, \mathbf{x} \leqslant_\ell \zeta\}).$$

□

In particular, if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, then $\overline{\mathbf{A}}^+$ is a complete lattice. We also extend the notion of left Garside normal form to the extended Artin–Tits monoid $\overline{\mathbf{A}}^+$.

Definition 7.18 (Extended Garside normal form)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. For each integer $k \geqslant 1$, consider the k-th extension mapping $\Theta_k : \overline{\mathbf{A}}^+ \mapsto \mathbf{A}^+$*, which we define by $\Theta_k : \mathbf{x} \mapsto \theta_{k-1,\infty}(\mathbf{x})^{-1}\theta_{k,\infty}(\mathbf{x})$.*

We call extended Garside normal form *of $\mathbf{x}$, and denote by $\mathbf{NF}^\omega_\ell(\mathbf{x})$, the right-infinite word $\Theta_1(\mathbf{x}) \cdot \Theta_2(\mathbf{x}) \cdot \ldots$*

Proposition 7.19.
Let $\mathbf{x}$ be an element of an Artin–Tits monoid of FC type $\mathbf{A}^+$. The (extended) Garside words $\mathbf{NF}_\ell(\mathbf{x})$ and $\mathbf{NF}^\omega_\ell(\mathbf{x})$ satisfy the relation $\mathbf{NF}^\omega_\ell(\mathbf{x}) = \mathbf{NF}_\ell(\mathbf{x}) \cdot \mathbf{1} \cdot \mathbf{1} \cdot \ldots$

Proof. Let us write down the letters of the words $\mathbf{NF}_\ell(\mathbf{x})$ and $\mathbf{NF}^\omega_\ell(\mathbf{x})$, e.g. $\mathbf{NF}_\ell(\mathbf{x}) = x_1 \cdot \ldots \cdot x_k$ and $\mathbf{NF}^\omega_\ell(\mathbf{x}) = z_1 \cdot z_2 \cdot \ldots$, where $k = \|\mathbf{x}\|$. For all integers $i \geqslant 1$, Corollary 2.51 proves that $x_1 \ldots x_{\min\{k,i\}} = z_1 \ldots z_i$. Hence, an immediate induction on i proves Proposition 7.19.
□

Corollary 7.20.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type and let $\overline{\mathbf{A}}^+$ be the associated extended Artin–Tits monoid of FC type. The extended Garside normal form $\mathbf{z} \mapsto \mathbf{NF}^\omega_\ell(\mathbf{z})$ is injective.

Furthermore, the extended Garside words are the right-infinite words $\underline{\mathbf{z}} = z_1 \cdot z_2 \cdot \ldots$ whose letters belong to $\mathbf{S}$ and satisfy the relations $z_i \longrightarrow z_{i+1}$ for all $i \geqslant 1$. In addition, $\underline{\mathbf{z}}$ is the extended Garside normal form of a finite element if and only if there exists some integer $k \geqslant 1$ such that $z_i = \mathbf{1}$ for all $i \geqslant k$, and in this case we have $\underline{\mathbf{z}} = \mathbf{NF}^\omega_\ell(z_1 \ldots z_{k-1})$.

Proof. It comes immediately that the extended normal form is injective, i.e. that

$$\mathbf{NF}_\ell^\omega(\mathbf{x}) = \mathbf{NF}_\ell^\omega(\mathbf{y}) \Leftrightarrow \mathbf{x} = \mathbf{y}.$$

Then, Lemma 2.40 and Proposition 7.19 prove that the extended Garside normal forms of finite elements of $\overline{\mathbf{A}}^+$ are words $\underline{\mathbf{z}} := z_1 \cdot z_2 \cdot \ldots$ such that $z_1 \longrightarrow z_2 \longrightarrow \ldots$ such that $z_k = \mathbf{1}$ for some integer $k \geqslant 1$, and it comes immediately that $\underline{\mathbf{z}} = \mathbf{NF}_\ell^\omega(z_1 \ldots z_{k-1})$.

Moreover, for all elements $\mathbf{x} = (\mathbf{x}_k)_{k \geqslant 0}$ of $\partial\mathbf{A}$ and all integers $k \geqslant 0$, we have $\|\mathbf{x}_k\| \leqslant k$ and therefore Proposition 7.19 proves that $\Theta_1(\mathbf{x}) \cdot \ldots \cdot \Theta_k(\mathbf{x}) \cdot \mathbf{1} \cdot \ldots = \mathbf{NF}_\ell^\omega(\mathbf{x}_k)$, hence that $\Theta_1(\mathbf{x}) \longrightarrow \ldots \longrightarrow \Theta_k(\mathbf{x})$. Since $\mathbf{x}$ must be infinite, it follows immediately that $\Theta_k(\mathbf{x}) \neq \mathbf{1}$ for all $k \geqslant 1$.

Finally, if $\underline{\mathbf{z}} := z_1 \cdot z_2 \cdot \ldots$ is a right-infinite word such that $z_1 \longrightarrow z_2 \longrightarrow \ldots$ and $z_k \neq \mathbf{1}$ for all $k \geqslant 1$, then Corollary 2.51 proves that the sequence $(\mathbf{x}_k)_{k \geqslant 0}$, where $\mathbf{x}_k := z_1 \ldots z_k$, is an element of $\overline{\mathbf{A}}^+$. This element cannot belong to $\mathbf{A}^+$ itself, hence it belongs to $\partial\mathbf{A}$, which completes the proof. □

Hence, the extended Garside normal form $\mathbf{NF}_\ell^\omega : \overline{\mathbf{A}}^+ \mapsto \{z_1 \cdot z_2 \cdot \ldots \in \mathbf{S}^\omega : \forall i \geqslant 1, z_i \longrightarrow z_{i+1}\}$ is a bijection. It maps finite elements to the words that contain the letter $\mathbf{1}$, and infinite elements to the words that do not contain the letter $\mathbf{1}$. In addition, the set $\{\mathbf{NF}_\ell^\omega(\mathbf{z}) : \mathbf{z} \in \overline{\mathbf{A}}^+\}$ embeds naturally as a subset of the product set $\mathbf{S}^\omega := \{(\mathbf{z}_i)_{i \geqslant 1} : \forall i \geqslant 1, \mathbf{z}_i \in \mathbf{S}\}$, equipped with the product topology, i.e. the coarsest topology for which the mappings $(\mathbf{z}_k)_{k \geqslant 1} \mapsto \mathbf{z}_\ell$ are continuous, for all integers $\ell \geqslant 1$.

It is natural to look for a topology on $\overline{\mathbf{A}}^+$ such that $\mathbf{NF}_\ell^\omega$ would be a homeomorphism, and therefore we focus on the projective topology [17]. In addition, we provide $\overline{\mathbf{A}}^+$ with the associated Borel σ-algebra, and $\partial\mathbf{A}^+$ with the induced σ-algebra.

Definition 7.21 (Projective topology on $\overline{\mathbf{A}}^+$)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. Since each set $\Omega_k = \{\mathbf{x} \in \mathbf{A}^+ : \|x\| \leqslant k\}$ is finite, let us equip it with the discrete topology (i.e. the topology in which all sets are open).

The projective topology *on the projective limit $\overline{\mathbf{A}}^+$ is the coarsest topology for which the mappings $\theta_{k,\infty} : \overline{\mathbf{A}}^+ \mapsto \Omega_k$ are continuous.*

Proposition 7.22.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. The extended Garside normal form $\mathbf{NF}_\ell^\omega : \overline{\mathbf{A}}^+ \mapsto \{\mathbf{NF}_\ell^\omega(\mathbf{z}) : \mathbf{z} \in \overline{\mathbf{A}}^+\}$ is a homeomorphism.

Proof. By definition, the product topology on $\{\mathbf{NF}_\ell^\omega(\zeta) : \zeta \in \overline{\mathbf{A}}^+\}$ is generated by the open sets $\mathfrak{B}_1(k, \mathbf{a}) := \{\underline{\mathbf{z}} : z_k = \mathbf{a}\}$, for all integers $k \geqslant 1$ and all strong elements $\mathbf{a} \in \mathbf{S}$. Likewise, the projective topology on $\overline{\mathbf{A}}^+$ is generated by the open sets $\mathfrak{B}_2(k, \mathbf{b}) := \{(\mathbf{x}_i)_{i \geqslant 0} : \mathbf{x}_k = \mathbf{b}\}$, for all integers $k \geqslant 0$ and all elements $\mathbf{b}$ of Ω_k.

Hence, let k be some integer, let $\mathbf{a}$ be some strong element of $\mathbf{A}^+$, and let $\mathbf{b}$ be some element of Ω_k. In addition, let $z_1 \cdot z_2 \cdot \ldots$ be the extended Garside normal form of $\mathbf{b}$. The direct image of $\mathfrak{B}_2(k, \mathbf{b})$ by $\mathbf{NF}_\ell^\omega$ is the open subset $\bigcap_{i=1}^k \mathfrak{B}_1(i, z_i)$ of $\{\mathbf{NF}_\ell^\omega(\zeta) : \zeta \in \overline{\mathbf{A}}^+\}$. Conversely, the inverse image of $\mathfrak{B}_1(k, \mathbf{a})$ by $\mathbf{NF}_\ell^\omega$ is the open subset $\bigcup_{\mathbf{b} \in \Omega_k(\mathbf{a})} \mathfrak{B}_2(k, \mathbf{b})$ of $\overline{\mathbf{A}}^+$, where the set $\Omega_k(\mathbf{a})$ is defined as $\Omega_k(\mathbf{a}) := \{\mathbf{b} \in \Omega_k : \mathbf{b} = \alpha_\ell^{k-1}(\mathbf{b})\mathbf{a}\}$. This completes the proof. □

Definition 7.23 (Elementary cylinders).
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. For each finite element $\mathbf{x} \in \mathbf{A}^+$, we call full elementary cylinder *the set $\Uparrow \mathbf{x} := \{\mathbf{y} \in \overline{\mathbf{A}}^+ : \mathbf{x} \leqslant_\ell \mathbf{y}\}$ and* elementary cylinder *the set $\uparrow \mathbf{x} := (\Uparrow \mathbf{x}) \cap \partial\mathbf{A}^+$. In addition, for each (finite or not) element $\mathbf{y} \in \overline{\mathbf{A}}^+$, we call* downward elementary cylinder *the set $\Downarrow \mathbf{y} := \{\mathbf{z} \in \overline{\mathbf{A}}^+ : \mathbf{z} \leqslant_\ell \mathbf{y}\}$.*

Proposition 7.24.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. The set $\overline{\mathbf{A}}^+$ is a metric, compact set, and the set $\partial\mathbf{A}^+$ is a closed subset of $\overline{\mathbf{A}}^+$. In addition, consider some elements $\mathbf{x} \in \mathbf{A}^+$ and $\mathbf{y} \in \overline{\mathbf{A}}^+$. The sets $\{\mathbf{x}\}$ and $\Uparrow \mathbf{x}$ are both open and closed in $\overline{\mathbf{A}}^+$, the set $\Downarrow \mathbf{y}$ is closed in $\overline{\mathbf{A}}^+$, and the set $\uparrow \mathbf{x}$ is both open and closed in $\partial\mathbf{A}^+$.

Proof. First, observe that *all* projective limits of finite sets are metric, compact sets (see [17] for details). Alternatively, this result also follows from the fact that $\{\mathbf{NF}_\ell^\omega(\mathbf{z}) : \mathbf{z} \in \overline{\mathbf{A}}^+\}$ is a closed subset of the metric, compact set $\mathbf{S}^\omega$.

Then, consider two elements $\mathbf{X} := (\mathbf{x}_k)_{k \geqslant 0}$ and $\mathbf{Y} := (\mathbf{y}_k)_{k \geqslant 0}$ of $\overline{\mathbf{A}}^+$. Recall that $\mathbf{X} \leqslant_\ell \mathbf{Y}$ if and only if $\mathbf{x}_k \leqslant_\ell \mathbf{y}_k$ for all $k \geqslant 0$. Equivalently, if $\mathbf{X}$ belongs to $\mathbf{A}^+$, we have $\mathbf{X} \leqslant_\ell \mathbf{Y} \Leftrightarrow \mathbf{X} \leqslant_\ell \mathbf{y}_{\|\mathbf{X}\|}$.

Moreover, by construction, each set $\mathfrak{B}_2(k, \mathbf{b})$, where $\mathbf{b} \in \Omega_k$, is both open and closed in $\overline{\mathbf{A}}^+$. Hence, let $\mathbf{x}$ be an element of $\mathbf{A}^+$. The set $\Uparrow \mathbf{x} = \bigcup_{\mathbf{y} \in \Omega_{\|\mathbf{x}\|} : \mathbf{x} \leqslant_\ell \mathbf{y}} \mathfrak{B}_2(\|\mathbf{x}\|, \mathbf{y})$ is both open and closed in $\overline{\mathbf{A}}^+$, and therefore the set $\uparrow \mathbf{x} = (\Uparrow \mathbf{x}) \cap \partial\mathbf{A}^+$ is both open and closed in $\partial\mathbf{A}^+$. Furthermore, the singleton set $\{\mathbf{x}\} = \Uparrow \mathbf{x} \setminus \bigcup_{i=1}^n \Uparrow (\mathbf{x}\sigma_i)$ is itself open and closed in $\overline{\mathbf{A}}^+$.

Finally, the sets $\overline{\mathbf{A}}^+ \setminus \Downarrow \zeta = \bigcup_{\mathbf{x} \in \mathbf{A}^+ : \mathbf{x} \notin \Downarrow \zeta} \Uparrow \mathbf{x}$ and $\mathbf{A}^+ = \bigcup_{\mathbf{x} \in \mathbf{A}^+} \{\mathbf{x}\}$ are unions of open sets, hence are open subsets of $\overline{\mathbf{A}}^+$. It follows that their respective complements $\Downarrow \zeta$ and $\partial\mathbf{A}^+$ are closed subsets of $\overline{\mathbf{A}}^+$. □

Elementary cylinders will play a crucial role in the subsequent parts of this chapter. In addition, they are closely related to the closed and open balls $\mathfrak{B}_2(k, \mathbf{b}) := \{(\mathbf{x}_i)_{i \geqslant 0} : \mathbf{x}_k = \mathbf{b}\}$.

Proposition 7.25.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type, let ν be a finite measure on $\overline{\mathbf{A}}^+$, and let

$f : \mathbf{A}^+ \mapsto \mathbb{C}$ be the function such that $f : \mathbf{x} \mapsto \nu(\Uparrow \mathbf{x})$. For all finite elements $\mathbf{x} \in \mathbf{A}^+$, we have $\nu(\mathfrak{B}_2(\|\mathbf{x}\|_\gamma, \mathbf{x})) = (\mathbf{M}_\gamma f)(\mathbf{x})$.

In particular, the measure ν is entirely determined by the values $f(\mathbf{x})$, for $\mathbf{x} \in \mathbf{A}^+$.

Proof. Consider the function $g : \mathbf{x} \mapsto \nu(\mathfrak{B}_2(\|\mathbf{x}\|_\gamma, \mathbf{x}))$. The full elementary cylinder $\Uparrow \mathbf{x}$ is the disjoint union of the sets $\mathfrak{B}_2(\|\mathbf{x}\|_\gamma, \mathbf{xy})$ for all the elements $\mathbf{y} \in \mathbf{A}^+$ such that $\|\mathbf{xy}\|_\gamma \leqslant \|\mathbf{x}\|_\gamma$, i.e. such that $\|\mathbf{xy}\|_\gamma = \|\mathbf{x}\|_\gamma$. It follows that $f(\mathbf{x}) = (\overline{\mathbf{M}}_\gamma g)(\mathbf{x})$, and therefore that $g = \mathbf{M}_\gamma f$.

Moreover, since the open sets $\varnothing$ and $\mathfrak{B}_2(\|\mathbf{x}\|_\gamma, \mathbf{x})$, for $\mathbf{x} \in \mathbf{A}^+$, form a π-system (i.e. a family closed under finite intersection) and generate the projective topology on $\mathbf{A}^+$, it follows that ν is entirely determined by the values $\nu(\mathfrak{B}_2(\|\mathbf{x}\|_\gamma, \mathbf{x}))$, for $\mathbf{x} \in \mathbf{A}^+$, and therefore by f. □

Bifinite domains [1, 82] provide us with an alternative proof of Proposition 7.25.

Definition & Proposition 7.26 (Bifinite domain and Lawson topology).
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. Consider the sets Ω_k, the maps $\theta_{k,\ell} : \Omega_\ell \mapsto \Omega_k$, as well as the natural embeddings $\vartheta_{k,\ell} : \Omega_k \mapsto \Omega_\ell$, for $\ell \geqslant k \geqslant 0$. Recall that $\overline{\mathbf{A}}^+$ was defined as the projective limit of the projective system $(\Omega_k)_{k\geqslant 0}, (\theta_{k,\ell})_{\ell\geqslant k\geqslant 0}$.

For $\ell \geqslant k \geqslant 0$, both maps $\theta_{k,\ell}$ and $\vartheta_{k,\ell}$ are order-preserving (for the order $\leqslant_\ell$). In addition, we have $\theta_{k,\ell} \circ \vartheta_{k,\ell} = \mathbf{Id}_{\Omega_k}$, and $\vartheta_{k,\ell} \circ \theta_{k,\ell} \leqslant_\ell \mathbf{Id}_{\Omega_\ell}$. Hence, we say that the pair $(\theta_{k,\ell}, \vartheta_{k,\ell})$ is a projection-embedding pair*, and we say that $\overline{\mathbf{A}}^+$ is a* bifinite domain*.*

Finally, let us define the Lawson topology *on $\overline{\mathbf{A}}^+$ as the topology generated by the open sets $\Uparrow \mathbf{x}$, for $\mathbf{x} \in \mathbf{A}^+$, and $\overline{\mathbf{A}}^+ \setminus \Downarrow \zeta$, for $\zeta \in \overline{\mathbf{A}}^+$.*

Proposition 7.27.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. The projective topology and the Lawson topology coincide on $\overline{\mathbf{A}}^+$. In addition, every finite Borel measure on $\overline{\mathbf{A}}^+$ is entirely determined by its values on the collection of full elementary cylinders, and every finite Borel measure on $\partial\mathbf{A}^+$ is entirely determined by its values on the collection of elementary cylinders.

Proof. Proposition 7.27 follows from general topological results about bifinite domains. More precisely, the first part of Proposition 7.27 follows directly from [1, Theorem 1], while the second part follows from [1, Theorem 2]. □

7.1.3 Uniform Measures on Extended Monoids

Definition 7.28 (Uniform measure).
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type, and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be some non-negative

valuation. We call uniform measure *of parameter r on $\overline{\mathbf{A}}^+$ a probability measure ν_r on $\overline{\mathbf{A}}^+$ such that $\forall \mathbf{x} \in \mathbf{A}^+, \nu_r(\Uparrow \mathbf{x}) = r(\mathbf{x})$. If r is positive, then we say that ν_r is* positive.

Moreover, if r is of the form q^λ for some non-negative real number q (i.e. if $r : \mathbf{x} \mapsto q^\lambda(\mathbf{x})$), then we say that ν_r is the uniform measure of parameter q *and also denote it by ν_q.*

Finally, if ν_r is a uniform measure such that $\nu_r(\mathbf{A}^+) = 0$, then we say that ν_r is a Bernoulli *measure.*

Observe that, since r is multiplicative, so is the associated uniform measure, i.e. $\nu_r(\Uparrow (\mathbf{xy})) = \nu_r(\Uparrow \mathbf{x})\nu_r(\Uparrow \mathbf{y})$. Conversely, if ν is a multiplicative measure on $\overline{\mathbf{A}}^+$, then the function $\mathbf{x} \mapsto \nu(\Uparrow \mathbf{x})$ is a non-negative valuation which is entirely determined by the values $\nu_i := \nu(\Uparrow \sigma_i)$.

Aiming to characterise uniform measures, we mention now two uniqueness and existence results.

Lemma 7.29.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be some positive valuation. The uniform measure ν_r of parameter r is unique, if it exists. Moreover, for all $\mathbf{x} \in \mathbf{A}^+$, we have $\nu_r(\{\mathbf{x}\}) = \mathcal{H}_\mathbf{A}(r)r(\mathbf{x})$.

Proof. Let us assume that there exist uniform measures μ_r and ν_r of parameter r. We have $\mu_r(\Uparrow \mathbf{x}) = r(\mathbf{x}) = \nu_r(\Uparrow \mathbf{x})$ for all $\mathbf{x} \in \mathbf{A}^+$, and therefore Proposition 7.25 shows that $\mu_r = \nu_r$.

Then, the inclusion-exclusion principle shows that

$$\nu_r(\{\mathbf{x}\}) = \sum_{I \in \mathcal{P}} (-1)^{|I|} \nu_r(\Uparrow (\mathbf{x}\Delta_I)) = \sum_{I \in \mathcal{P}} (-1)^{|I|} r(\mathbf{x}) r(\Delta_I) = \mathcal{H}_\mathbf{A}(r) r(\mathbf{x}),$$

which completes the proof. □

Definition 7.30 (Convergence manifold and limit convergence manifold)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type with n generators and let $\mathcal{V}_\mathbf{A}$ be the valuation manifold of $\mathbf{A}^+$. The convergence manifold *and the* limit convergence manifold *of $\mathbf{A}^+$ are defined as the sets*

$$\mathcal{R}_\mathbf{A} := \{(r_1, \ldots, r_n) \in \mathcal{V}_\mathbf{A} : \mathcal{G}_\mathbf{A}(r) < +\infty\} \text{ and } \mathcal{R}_\mathbf{A}^\partial := \partial \mathcal{R}_\mathbf{A} \cap \mathcal{V}_\mathbf{A}.$$

By extension, and identifying the valuation r with the tuple $\widehat{\mathbf{r}} := (r_1, \ldots, r_n)$, we will also write $r \in \mathcal{R}_\mathbf{A}$ and $r \in \mathcal{R}_\mathbf{A}^\partial$ as shorthand notations for $\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A}$ and $\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A}^\partial$.

For all tuples $\widehat{\mathbf{x}} := (x_1, \ldots, x_n)$ and $\widehat{\mathbf{y}} := (y_1, \ldots, y_n)$ of real numbers, let us write $\widehat{\mathbf{x}} \leqslant \widehat{\mathbf{y}}$ if $\forall i \in \{1, \ldots, n\}, x_i \leqslant y_i$, and $\widehat{\mathbf{x}} < \widehat{\mathbf{y}}$ if $\forall i \in \{1, \ldots, n\}, x_i < y_i$. In addition, if z is a real number, we extend the relations $\leqslant$ and $<$ by identifying z with the tuple $\widehat{\mathbf{z}} := (z, \ldots, z)$, i.e. we write $\widehat{\mathbf{x}} \leqslant z$, $\widehat{\mathbf{x}} < z$, $z \leqslant \widehat{\mathbf{y}}$ and $z < \widehat{\mathbf{y}}$ if $\widehat{\mathbf{x}} \leqslant \widehat{\mathbf{z}}$, $\widehat{\mathbf{z}} < \widehat{\mathbf{z}}$, $\widehat{\mathbf{z}} \leqslant \widehat{\mathbf{y}}$ and $\widehat{\mathbf{z}} < \widehat{\mathbf{y}}$ respectively.

Lemma 7.31.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. Consider two elements $\widehat{\mathbf{q}}$ and $\widehat{\mathbf{r}}$ of the valuation manifold $\mathcal{V}_\mathbf{A}$ of $\mathbf{A}^+$. If $\widehat{\mathbf{q}}$ belongs to $\mathcal{R}_\mathbf{A}$ and if $\widehat{\mathbf{r}} \leqslant \widehat{\mathbf{q}}$, then $\widehat{\mathbf{r}}$ belongs to $\mathcal{R}_\mathbf{A}$.

Proof. Let q and r be the valuations associated with the tuples $\widehat{\mathbf{q}}$ and $\widehat{\mathbf{r}}$. For all $\mathbf{x} \in \mathbf{A}^+$, we necessarily have $0 \leqslant r(\mathbf{x}) \leqslant q(\mathbf{x})$. Since the series $\mathcal{G}_\mathbf{A}(q)$ converges, so does the series $\mathcal{G}_\mathbf{A}(r)$, i.e. $r \in \mathcal{R}_\mathbf{A}$. □

Lemma 7.32.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators. Let $p_\mathbf{A}$ be the smallest positive root of the Möbius polynomial $\mathcal{H}_\mathbf{A}$, and let $\widehat{\mathbf{r}}$ be some element of $\mathcal{V}_\mathbf{A}$. If $\widehat{\mathbf{r}} < p_\mathbf{A}$, then $\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A}$, and if $\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A}$, then $\widehat{\mathbf{r}} < 1$.

Proof. Without loss of generality, we assume that $r_1 = \max\{r_1, \ldots, r_n\}$. First, if $r_1 < p_\mathbf{A}$, then $\mathcal{G}_\mathbf{A}(r_1) = \mathcal{G}_\mathbf{A}(r_1^\lambda)$ converges, i.e. $(r_1, \ldots, r_1) \in \mathcal{R}_\mathbf{A}$, hence Lemma 7.31 proves that $\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A}$. Second, if $r_1 \geqslant 1$, then $\mathcal{G}_\mathbf{A}(r) \geqslant \sum_{k \geqslant 0} r(\sigma_1^k) = \sum_{k \geqslant 0} r_1^k = +\infty$. □

Proposition 7.33.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators, and let $\widehat{\mathbf{r}}$ be some element of the valuation manifold $\mathcal{V}_\mathbf{A}$ of $\mathbf{A}^+$. There exists a positive real number $\varphi(\widehat{\mathbf{r}})$ such that, for all $x > 0$, $x\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A} \Leftrightarrow x < \varphi(\widehat{\mathbf{r}})$ and $x\widehat{\mathbf{r}} \in \mathcal{R}^\partial_\mathbf{A} \Leftrightarrow x = \varphi(\widehat{\mathbf{r}})$. Furthermore, the mapping $\varphi : \widehat{\mathbf{r}} \mapsto \varphi(\widehat{\mathbf{r}})$ is continuous.

Proof. Without loss of generality, let us first assume that $\max\{r_1, \ldots, r_n\} = 1$, and consider the set $\mathbf{R} := \{x \in (0, +\infty) : x\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A}\}$. Lemma 7.32 proves that $(0, p_\mathbf{A}) \subseteq \mathbf{R} \subseteq (0, 1)$. Now, let r be the valuation associated with the tuple $\widehat{\mathbf{r}}$, and consider the functions $G_\mathbf{A} : x \mapsto \mathcal{G}_\mathbf{A}(x^\lambda r)$ and $H_\mathbf{A} : x \mapsto \mathcal{H}_\mathbf{A}(x^\lambda r)$. Since $H_\mathbf{A}$ is a polynomial and since the equality $G_\mathbf{A}(x) H_\mathbf{A}(x) = 1$ holds when $x \in \mathbf{R}$, it follows that $G_\mathbf{A}$ is the rational function $H_\mathbf{A}^{-1}$. Consequently, if $\varphi(\widehat{\mathbf{r}})$ denotes the smallest positive root of $H_\mathbf{A}$, i.e. the smallest positive pole of $G_\mathbf{A}$, we have $\mathbf{R} = (0, \varphi(\widehat{\mathbf{r}}))$, and $\varphi(\widehat{\mathbf{r}})\widehat{\mathbf{r}} \in \mathcal{R}^\partial_\mathbf{A}$.

Then, consider some $x > \varphi(\widehat{\mathbf{r}})$, and consider a sequence $\widehat{\mathbf{q}}^1, \widehat{\mathbf{q}}^2, \ldots$ of elements of $\mathcal{V}_\mathbf{A}$ that converges towards $x\widehat{\mathbf{r}}$. Since $x\widehat{\mathbf{r}} > \varphi(\widehat{\mathbf{r}})\widehat{\mathbf{r}}$, there exists some $k \geqslant 0$ such that $\widehat{\mathbf{q}}^k > \varphi(\widehat{\mathbf{r}})\widehat{\mathbf{r}}$. Since $\varphi(\widehat{\mathbf{r}})\widehat{\mathbf{r}} \notin \mathcal{R}_\mathbf{A}$, Lemma 7.31 proves that $\widehat{\mathbf{q}}^k \notin \mathcal{R}_\mathbf{A}$. This proves that no sequence of elements of $\mathcal{R}_\mathbf{A}$ converges towards $x\widehat{\mathbf{r}}$, i.e. that $x\widehat{\mathbf{r}} \notin \mathcal{R}^\partial_\mathbf{A}$. It follows, for all $x \in (0, +\infty)$, that $x\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A} \Leftrightarrow x < \varphi(\widehat{\mathbf{r}})$ and $x \in \mathcal{R}^\partial_\mathbf{A} \Leftrightarrow x = \varphi(\widehat{\mathbf{r}})$.

Finally, let us prove that φ is continuous, and consider some element $\widehat{\mathbf{r}} := (r_1, \ldots, r_n)$ of $\mathcal{V}_\mathbf{A}$. Then, consider some real number $\varepsilon \in (0, 1/2)$, and some element $\widehat{\mathbf{q}}$ of $\mathcal{V}_\mathbf{A}$ such that $|r_i - q_i| \leqslant \varepsilon r_i$ for all $i \in \{1, \ldots, n\}$. It follows that $\varphi(\widehat{\mathbf{r}})\widehat{\mathbf{r}} \leqslant \frac{\varphi(\widehat{\mathbf{r}})}{1-\varepsilon}\widehat{\mathbf{q}}$, and therefore Lemma 7.31 proves that $\varphi(\widehat{\mathbf{q}}) \leqslant \frac{1}{1-\varepsilon}\varphi(\widehat{\mathbf{r}})$. Moreover, since $(1-\varepsilon) r_i \leqslant q_i$, it follows that $|r_i - q_i| \leqslant \frac{\varepsilon}{1-\varepsilon} q_i$, and therefore that $\varphi(\widehat{\mathbf{r}}) \leqslant \frac{1-\varepsilon}{1-2\varepsilon}\varphi(\widehat{\mathbf{q}})$. This proves that $\varphi : \widehat{\mathbf{r}} \mapsto \varphi(\widehat{\mathbf{r}})$ is continuous. □

Corollary 7.34.
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of FC type with at least* 2 *generators. The limit convergence manifold of* $\mathbf{A}^+$ *is homeomorphic to an open simplex of dimension* $\dim(\mathcal{V}_{\mathbf{A}}) - 1$*, and the convergence manifold of* $\mathbf{A}^+$ *is homeomorphic to an open ball of dimension* $\dim(\mathcal{V}_{\mathbf{A}})$.

More precisely, the convergence manifold of $\mathbf{A}^+$ *is the connected component of the manifold* $\{\widehat{\mathbf{r}} \in \mathcal{V}_{\mathbf{a}} : \mathcal{H}_{\mathbf{A}}(\widehat{\mathbf{r}}) > 0\}$ *whose boundary (in* $\mathbb{R}^n$*) contains the point* $(0, \ldots, 0)$.

Proof. Consider the sets $\mathcal{V}_{\mathbf{A}}^{<1} := \{\widehat{\mathbf{r}} \in \mathcal{V}_{\mathbf{A}} : \widehat{\mathbf{r}} < 1\}$ and $\mathcal{V}_{\mathbf{A}}^{1} := \partial\mathcal{V}_{\mathbf{A}}^{<1} \cap \mathcal{V}_{\mathbf{A}} = \{\widehat{\mathbf{r}} \in \mathcal{V}_{\mathbf{A}} : \max\{r_1, \ldots, r_n\} = 1\}$, which are respectively homeomorphic to an open ball of dimension $\dim(\mathcal{V}_{\mathbf{A}})$ and to an open simplex of dimension $\dim(\mathcal{V}_{\mathbf{A}}) - 1$. We claim that the mapping $\Phi : \widehat{\mathbf{r}} \mapsto \varphi(\widehat{\mathbf{r}})\widehat{\mathbf{r}}$ induces a homeomorphism from $\mathcal{V}_{\mathbf{A}}^{1}$ to $\mathcal{R}_{\mathbf{A}}^{\partial}$. First, Proposition 7.33 proves that $\{\Phi(\widehat{\mathbf{r}})\} = \mathcal{R}_{\mathbf{A}}^{\partial} \cap \{x\widehat{\mathbf{r}} : x \in (0, +\infty)\}$ for all $\widehat{\mathbf{r}} \in \mathcal{V}_{\mathbf{A}}$. It follows that Φ induces a bijection from $\mathcal{V}_{\mathbf{A}}^{1}$ to $\mathcal{R}_{\mathbf{A}}^{\partial}$.

Moreover, Proposition 7.33 also proves that Φ is continuous, and Lemma 7.32 proves that $p_{\mathbf{A}} \leqslant \varphi(\widehat{\mathbf{r}}) \leqslant 1$ for all $\widehat{\mathbf{r}} \in \mathcal{V}_{\mathbf{A}}^{1}$. In addition, for each element $\widehat{\mathbf{r}}$ of $\mathcal{V}_{\mathbf{A}}$, let $\psi(\widehat{\mathbf{r}}) := \max\{r_1, \ldots, r_n\}$. The mapping $\Psi : \widehat{\mathbf{r}} \mapsto \psi(\widehat{\mathbf{r}})^{-1}\widehat{\mathbf{r}}$ induces a mapping from $\mathcal{R}_{\mathbf{A}}^{\partial}$ to $\mathcal{V}_{\mathbf{A}}^{1}$, which is the inverse bijection of Φ. Moreover, we know that ψ and ψ^{-1} are continuous on $\mathcal{V}_{\mathbf{A}}$, so that Ψ is continuous too. Consequently, the mapping $\Phi = \Psi^{-1}$ is indeed a homeomorphism from $\mathcal{V}_{\mathbf{A}}^{1}$ to $\mathcal{R}_{\mathbf{A}}^{\partial}$.

In addition, it follows immediately that the mapping $\widehat{\mathbf{r}} \mapsto \psi(\widehat{\mathbf{r}})\Phi(\widehat{\mathbf{r}})$ induces a bijection from $\mathcal{V}_{\mathbf{A}}^{<1}$ to $\mathcal{R}_{\mathbf{A}}$. Since both this mapping and its inverse are continuous, it follows that $\mathcal{R}_{\mathbf{A}}$ is homeomorphic with $\mathcal{V}_{\mathbf{A}}^{<1}$, i.e. with an open ball of dimension $\dim(\mathcal{V}_{\mathbf{A}})$. Finally, let $\mathbf{C}$ be the connected component of the manifold $\{\widehat{\mathbf{r}} \in \mathcal{V}_{\mathbf{a}} : \mathcal{H}_{\mathbf{A}}(\widehat{\mathbf{r}}) > 0\}$ whose boundary (in $\mathbb{R}^n$) contains the point $(0, \ldots, 0)$. We mentioned in the proof of Proposition 7.33 that $\mathcal{H}_{\mathbf{A}} > 0$ on $\mathcal{R}_{\mathbf{A}}$. Since $\mathcal{R}_{\mathbf{A}}$ is connected and non-empty (because $\mathcal{R}_{\mathbf{A}}$ is homeomorphic to a ball), it follows that $\mathcal{R}_{\mathbf{A}}$ is a non-empty subset of $\mathbf{C}$. Moreover, we also mentioned that $\mathcal{H}_{\mathbf{A}} = 0$ on $\mathcal{R}_{\mathbf{A}}^{\partial}$, which proves that $\partial\mathcal{R}_{\mathbf{A}} \cap \mathbf{C} = \varnothing$. It follows that $\mathcal{R}_{\mathbf{A}} = \mathbf{C}$, which completes the proof. □

In what follows, for all positive valuations $r : \mathbf{A}^+ \mapsto \mathbb{C}$, we denote by $\varphi(r)$ the unique positive real number such that the valuation $\varphi(r)^{\lambda}r : \mathbf{x} \mapsto \varphi(r)^{\lambda(\mathbf{x})}r(\mathbf{x})$ belongs to $\mathcal{R}_{\mathbf{A}}^{\partial}$, and we denote by $\Phi(r)$ the valuation $\varphi(r)^{\lambda}r$.

Theorem 7.35.
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of FC type with at least* 2 *generators, and let* $r : \mathbf{A}^+ \mapsto \mathbb{C}$ *be a valuation. If* $r \in \mathcal{R}_{\mathbf{A}}$*, then there exists a (unique) uniform measure* ν_r *of parameter* r *on* $\overline{\mathbf{A}}^+$*, whose support is* $\mathbf{A}^+$*. If* $r \in \mathcal{R}_{\mathbf{A}}^{\partial}$*, then there exists a (unique) uniform measure* ν_r *of parameter* r *on* $\overline{\mathbf{A}}^+$*, whose support is contained into* $\partial\mathbf{A}^+$*.*

Proof. If $r \in \mathcal{R}_{\mathbf{A}}$, then the generating function $\mathcal{G}_{\mathbf{A}}(r)$ is (absolutely) convergent, and the equality $\mathcal{G}_{\mathbf{A}}(r)\mathcal{H}_{\mathbf{A}}(r) = 1$ holds in the field of real numbers. It follows that $\mathcal{H}_{\mathbf{A}}(r) > 0$, and therefore that the measure ν_r such that $\nu_r(\{\mathbf{x}\}) = \mathcal{H}_{\mathbf{A}}(r)r(\mathbf{x})$ and $\nu_r(\partial\mathbf{A}^+) = 0$ is well-defined. Hence, consider some finite braid $\mathbf{x} \in \mathbf{A}^+$. Since the support of ν_r is $\mathbf{A}^+$, we have

$$\nu_r(\Uparrow \mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{A}^+} \nu_r(\{\mathbf{xy}\}) = \sum_{\mathbf{y}\in\mathbf{A}^+} r(\mathbf{x})r(\mathbf{y})\mathcal{H}_{\mathbf{A}}(r) = r(\mathbf{x})\mathcal{G}_{\mathbf{A}}(r)\mathcal{H}_{\mathbf{A}}(r) = r(\mathbf{x}),$$

which proves that ν_r is uniform with parameter r.

Then, if $r \in \mathcal{R}^{\partial}_{\mathbf{A}}$, let $(q^j)_{j\geqslant 1}$ be a sequence of valuations in $\mathcal{R}_{\mathbf{A}}$ such that $q^j_k \to r_k$ for all $k \in \{1,\ldots,n\}$. Proposition 7.24 proves that $\overline{\mathbf{A}}^+$ is metric and compact, and therefore we may assume that the sequence (ν_{q^j}) converges weakly, towards a limit that we call ν_r.

Let $\mathbf{x} \in \mathbf{A}^+$ be some finite braid. Proposition 7.24 also states that $\Uparrow \mathbf{x}$ is both open and closed in $\overline{\mathbf{A}}^+$, and therefore $\Uparrow \mathbf{x}$ has an empty topological boundary. Hence, the Portemanteau theorem [12] implies that $\lim_{j\to\infty} \nu_{q^j}(\Uparrow \mathbf{x}) = \nu_r(\Uparrow \mathbf{x})$. This proves that

$$\nu_r(\Uparrow \mathbf{x}) = \lim_{j\to\infty} q^j(\mathbf{x}) = r(\mathbf{x}),$$

which means that ν_r is uniform with parameter r and completes the proof. □

In addition, if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type (i.e. if the lower semilattice $\overline{\mathbf{A}}^+$ admits a supremum), then there also exists a uniform measure ν_r associated with the valuation $r : \mathbf{x} \mapsto 1$. This measure degenerate, and is the Dirac measure at $\Delta^\omega = (\Delta^k)_{k\geqslant 0}$. It coincides with the naive limit of the random walk on the monoid $\mathbf{A}^+$, such as studied in Chapter 5.

Henceforth, we cast aside both the case of the valuation $r : \mathbf{x} \mapsto \mathbf{1}_{\mathbf{x}=\mathbf{1}}$, which leads to the uniform measure $\nu_r = \delta_{\mathbf{1}}$, and the valuation $r : \mathbf{x} \mapsto 1$, which leads to the uniform measure $\nu_r = \delta_{\Delta^\omega}$ when $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type

Conversely, let us prove that the above-constructed uniform measures are the only existing uniform measures. In order to do so, we need to use Perron-Frobenius theory (see [85, Chapter 1] for details).

Definition 7.36 (Positive matrix and primitive matrix).
Let M be a real square matrix. If all the entries of M are non-negative real numbers, we say that M is non-negative*. If they are positive real numbers, we say that M is* positive*.*

Moreover, if M is a non-negative matrix such that, for some integer $k \geqslant 0$, the matrices $(M^\ell)_{\ell\geqslant k}$ are all positive, then we say that M is primitive*.*

Theorem 7.37 (Perron).
Let M be a primitive matrix. There exists a positive real number ρ, one row vector $\mathbf{l}$ with positive entries, and one column vector $\mathbf{r}$ with positive entries, such that:

- *ρ is a* simple *eigenvalue of M, for the left eigenvector $\mathbf{l}$ and the right eigenvector $\mathbf{r}$ (i.e. $\mathbf{l} \cdot M = \rho\,\mathbf{l}$ and $M \cdot \mathbf{r} = \rho\,\mathbf{r}$);*

- *for all eigenvalues λ of M, if $\rho \neq \lambda$, then $|\rho| > |\lambda|$;*
- *each left eigenvector of M with non-negative entries is a (positive) multiple of $\mathbf{l}$, and each right eigenvector of M with non-negative entries is a (positive) multiple of $\mathbf{r}$.*

We say that ρ is the Perron *eigenvalue of M, and that $\mathbf{l}$ and $\mathbf{r}$ are* Perron *eigenvectors of M.*

In addition, if $\mathbf{l}$ and $\mathbf{r}$ are normalised so that $\mathbf{l} \cdot \mathbf{r} = 1$, then, for all row vectors $\mathbf{g}$ and column vectors $\mathbf{h}$ with non-negative entries, the equivalence relations

$$M^k \sim \rho^k \mathbf{r} \cdot \mathbf{l};\ \mathbf{g} \cdot M^k \sim \rho^k \mathbf{g} \cdot \mathbf{r} \cdot \mathbf{l};\ M^k \cdot \mathbf{h} \sim \rho^k \mathbf{r} \cdot \mathbf{l} \cdot \mathbf{h} \text{ and } \mathbf{g} \cdot M^k \cdot \mathbf{h} \sim \rho^k (\mathbf{g} \cdot \mathbf{r})(\mathbf{l} \cdot \mathbf{h})$$

hold when $k \to +\infty$. Furthermore, the spectrum of the matrix $N := M - \mathbf{r} \cdot \mathbf{l}$ is $\{0\} \cup \mathrm{sp}(M) \backslash \{\rho\}$.

Finally, for all matrices N such that both N and $M - N$ are non-negative, and for all eigenvalues λ of N, we have

- *if $\rho \neq \lambda$, then $|\rho| > |\lambda|$;*
- *if $\rho = \lambda$, then $M = N$.*

Aiming to apply Theorem 7.37 to combinatorially meaningful matrices, we first prove some strong connexity results.

Definition 7.38 (Essential left Garside acceptor graph).
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators. Let $\mathbf{S}$ be the smallest two-way Garside family of $\mathbf{A}^+$. We call essential elements *of $\mathbf{A}^+$ the elements of the set $\mathcal{E} := \mathbf{S} \backslash \{\mathbf{1}, \Delta\}$ if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, or $\mathcal{E} := \mathbf{S} \backslash \{\mathbf{1}\}$ otherwise (i.e. if Δ does not exist), and call* essential Garside set *the set $\mathcal{E}$ itself.*

We also call essential left Garside acceptor graph *the oriented labelled graph $\mathcal{G}^{\mathbf{left}}_{\mathrm{gar}} := (V, E)$ defined by*

$$\begin{aligned} V &= \{\overline{\mathbf{right}}(\mathbf{a}) : \mathbf{a} \in \mathcal{E}\}, \\ E &= \{(P, Q) \in V \times V : \exists \mathbf{a} \in \mathcal{E}, \mathbf{left}(\mathbf{a}) \subseteq P \text{ and } \overline{\mathbf{right}}(\mathbf{a}) = Q\}, \end{aligned}$$

and such that each essential element $\mathbf{a} \in \mathcal{E}$ labels the edges (P, Q) such that $\mathbf{left}(\mathbf{a}) \subseteq P$ and $\overline{\mathbf{right}}(\mathbf{a}) = Q$.

Theorem 7.39.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators, and let $\mathcal{G}^{\mathbf{left}}_{\mathrm{gar}}$ be the essential left Garside acceptor graph of $\mathbf{A}^+$. The paths in $\mathcal{G}^{\mathbf{left}}_{\mathrm{gar}}$ are precisely the Δ-free left Garside words if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, or the left Garside words otherwise. Furthermore, the graph $\mathcal{G}^{\mathbf{left}}_{\mathrm{gar}}$ is strongly connected and contains loops and, if $\mathbf{A}^+$ is not an Artin–Tits monoid of spherical type, it contains the state $\{\sigma_1, \ldots, \sigma_n\}$.

Proof. First, if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, observe that $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$ is the oriented labelled graph induced by the left Garside acceptor automaton, from which the node $\{\sigma_1, \ldots, \sigma_n\}$ has been deleted. Hence, the paths in $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$ are precisely the Δ-free left Garside words. Furthermore, the bilateral Garside automaton $\mathcal{G}_{\mathrm{gar}}$, which we introduced in Definition 5.18, is obtained by deleting edges from $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$. Since Proposition 5.21 states that $\mathcal{G}_{\mathrm{gar}}$ is strongly connected and contains loops, so must $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$ be connected and contain loops.

We focus henceforth on the case where $\mathbf{A}^+$ is *not* an Artin–Tits monoid of spherical type, i.e. when its Coxeter diagram contains an edge $\circ{=}\circ$. Once again, $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$ is the oriented labelled graph induced by the left Garside acceptor automaton, hence the paths in $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$ are precisely the left Garside words. It remains to show that $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$ contains some loop and is strongly connected, which we do by proving that $\{\sigma_1, \ldots, \sigma_n\}$ is a vertex of $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$ that is accessible from each vertex of $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$.

Let $\mathbf{G}$ be the Coxeter diagram of the monoid $\mathbf{A}^+$, and let us denote by $d_{\mathbf{G}}$ the distance in the graph $\mathbf{G}$, by using either edges $\circ{=}\circ$ or $\circ{-}\circ$. Henceforth, since we must handle both the essential left Garside acceptor graph (V, E) and the Coxeter diagram $\mathbf{G}$, the word "vertex" will always refer to vertices of $\mathbf{G}$. Furthermore, for avoiding too heavy notations, we will identify the sets $A \in \{1, \ldots, n\}$ with $\{\sigma_a : a \in A\}$.

In addition, for all sets $S \subseteq \{1, \ldots, n\}$, we will denote by $S^{\circ=\circ}$ the set $\{i \in \{1, \ldots, n\} : \exists j \in S, m_{i,j} = \infty\}$, i.e. the set of vertices that are linked to S via an edge $\circ{=}\circ$, and by $S^{\circ=\circ *}$ the set of vertices accessible from S via edges $\circ{=}\circ$ (including the vertices of S), i.e. the smallest set T such that $S \cup T^{\circ=\circ} \subseteq T$. Similarly, we will denote by $S^{\circ-\circ}$ the set $\{i \in \{1, \ldots, n\} : \exists j \in S, m_{i,j} \geqslant 3\}$, i.e. the set of vertices that are linked to S via an edge $\circ{-}\circ$, and by $S^{\circ-\circ *}$ the set of vertices accessible from S via edges $\circ{-}\circ$ (including the vertices of S), i.e. the smallest set T such that $S \cup T^{\circ-\circ} \subseteq T$.

We already note that, if S is $\circ{=}\circ$-independent, then there exists a strong element Δ_S, whence $\mathbf{left}(\Delta_S) = S$ and $\overline{\mathbf{right}}(\Delta_S) = S \cup S^{\circ=\circ}$. Now, let $P \subseteq \{1, \ldots, n\}$ be an element of V. Since $P = \overline{\mathbf{right}}(\mathbf{a})$ for some non-trivial element $\mathbf{a}$, we already know that P is not empty.

First, if $P \subsetneq P^{\circ=\circ *}$, consider some vertices $i \in P$ and $j \in \{1, \ldots, n\} \backslash P$ such that $m_{i,j} = \infty$. Then, let $\overline{P}$ be a maximal $\circ{=}\circ$-independent subset of P such that $i \in \overline{P}$. We have $\mathbf{left}(\Delta_{\overline{P}}) = \overline{P} \subseteq P$ and $\mathbf{right}(\Delta_{\overline{P}}) = \overline{P} \cup \overline{P}^{\circ=\circ} \supseteq P \cup \{j\}$. Consequently, an immediate induction shows that the set $P^{\circ=\circ *}$ is accessible from P in $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$. In particular, the set $P^{\circ=\circ *}$ is accessible from itself in $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$ in one step, which means that it belongs to a loop.

Note that this part of the proof already implies Theorem 7.39 in the case of irreducible heap monoids.

Second, if $P \subsetneq P^{\circ-\circ *}$, let X be a maximal connected $\circ{=}\circ$-independent subset of $\mathbf{G}$ such that $\varnothing \subsetneq X \cap P \subsetneq X$. Observe that $|X| \geqslant 2$. Moreover, since $\mathbf{G}$ contains some edge $\circ{=}\circ$, the set X cannot contain all the vertices of $\mathbf{G}$, hence it contains some element x

that is linked to some edge $\circ\!\!=\!\!\circ$, i.e. such that $x \in (X^{\circ\!=\!\circ})^{\circ\!=\!\circ}$.

Let $\mathbf{B}^+$ be the submonoid of $\mathbf{A}^+$ generated by the set $\{\sigma_i : i \in X\}$. According to Proposition 5.21, there exists a sequence of simple braids $\mathbf{b}_1, \ldots, \mathbf{b}_u$ in $\mathbf{B}^+$ such that $\mathbf{left}(\mathbf{b}_1) \subseteq X \cap P$, $\mathbf{b}_1 \longrightarrow \ldots \longrightarrow \mathbf{b}_u$ and $\mathbf{right}(\mathbf{b}_u) = X \backslash \{x\}$.

Then, let $\mathbf{b}_{u+1} := \sigma_x^{-1} \Delta_X$ and $\mathbf{b}_{u+2} := \Delta_X \sigma_x^{-1}$. Note that $\mathbf{b}_{u+2}$ is the "mirror" of $\mathbf{b}_{u+1}$, i.e. is obtained by reversing the order of the factors of $\mathbf{b}_{u+1}$. In addition, let y be some neighbour of x in X. Since $m_{x,y} \geqslant 3$ and $[\sigma_x \sigma_y]^{m_{x,y}} = \Delta_{\{x,y\}} \leqslant_\ell \Delta_X = \sigma_x \mathbf{b}_{u+1}$, it follows that σ_x is a factor of $\mathbf{b}_{u+1}$, and also of $\mathbf{b}_{u+2}$. Hence, we have $\mathbf{b}_u \longrightarrow \mathbf{b}_{u+1} \longrightarrow \mathbf{b}_{u+2}$ and $\overline{\mathbf{right}}(\mathbf{b}_{u+2}) \supseteq X^{\circ\!=\!\circ} \cup X \backslash \{x\}$.

Now, observe that $\{j : d_{\mathbf{G}}(X, j) = 1\} = X^{\circ\!=\!\circ}$. Then, let $\overline{P}$ be a maximal $\circ\!\!=\!\!\circ$-independent subset of $\{j \in P : d_{\mathbf{G}}(j, X) \geqslant 2\}$. Since $\Delta_{\overline{P}}$ commutes with $\mathbf{b}_1, \ldots, \mathbf{b}_{u+2}$, the elements $\Delta_{\overline{P}} \mathbf{b}_1, \ldots, \Delta_{\overline{P}} \mathbf{b}_{u+2}$ are strong elements such that

- $\mathbf{left}(\Delta_{\overline{P}} \mathbf{b}_1) = \overline{P} \cup \mathbf{left}(\mathbf{b}_1) \subseteq P$;
- $\Delta_{\overline{P}} \mathbf{b}_1 \longrightarrow \ldots \longrightarrow \Delta_{\overline{P}} \mathbf{b}_{u+2}$;
- $\overline{\mathbf{right}}(\Delta_{\overline{P}} \mathbf{b}_{u+2}) = \overline{P} \cup \overline{P}^{\circ\!=\!\circ} \cup \overline{\mathbf{right}}(\mathbf{b}_{u+2}) \supseteq P \cup X^{\circ\!=\!\circ} \cup X \backslash \{x\}$.

It follows that both sets $Q := \overline{\mathbf{right}}(\Delta_{\overline{P}} \mathbf{b}_{u+2})$ and $Q^{\circ\!=\!\circ *}$ are accessible from P in $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$, and that $P \cup X \subseteq Q \cup \{x\} \subseteq Q \cup (X^{\circ\!=\!\circ})^{\circ\!=\!\circ} \subseteq Q^{\circ\!=\!\circ *}$. Consequently, an immediate induction shows that the set $P^{\circ\!-\!\circ *}$ is accessible from P in $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$.

Hence, let S be the smallest subset of $\mathbf{G}$ such that $P \subseteq S = S^{\circ\!=\!\circ *} = S^{\circ\!-\!\circ *}$. The two above points show that S is accessible from P in $\mathcal{G}_{\mathrm{gar}}^{\mathbf{left}}$. Moreover, by construction, S is a union of maximal connected components of $\mathbf{G}$. Since $\mathbf{G}$ itself is connected, it follows that $S = \{\sigma_1, \ldots, \sigma_n\}$, which completes the proof. $\square$

Definition 7.40 (Garside matrix and expanded Garside matrix)**.**
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of FC type with at least* 2 *generators, with two-way Garside family* $\mathbf{S}$ *and essential Garside set* $\mathcal{E}$. *In addition, let* $r : \mathbf{A}^+ \mapsto \mathbb{C}$ *be a positive valuation. We call* Garside matrix *of parameter* r *the matrix* $M := (M_{\mathbf{x},\mathbf{y}})$, *indexed by pairs of braids* $\mathbf{x}, \mathbf{y} \in \mathcal{E}$, *and defined by* $M_{\mathbf{x},\mathbf{y}} = \mathbf{1}_{\mathbf{x} \longrightarrow \mathbf{y}} r(\mathbf{y})$.

We also call expanded Garside matrix *of parameter* r *the matrix* $N := (N_{(i,\mathbf{x}),(j,\mathbf{y})})$, *where* $\mathbf{x}, \mathbf{y} \in \mathbf{S} \backslash \{\mathbf{1}\}$, $1 \leqslant i \leqslant \lambda(\mathbf{x})$ *and* $1 \leqslant j \leqslant \lambda(\mathbf{y})$, *and defined by*

$$\begin{aligned} N_{(i,\mathbf{x}),(j,\mathbf{y})} &= 1 \text{ if } \mathbf{x} = \mathbf{y} \text{ and } i + 1 = j \\ &= r(\mathbf{y}) \text{ if } \mathbf{x} \longrightarrow \mathbf{y}, i = \lambda(\mathbf{x}) \text{ and } j = 1 \\ &= 0 \text{ otherwise.} \end{aligned}$$

Finally, we call essential Garside matrix *of parameter* r *the restriction of* N *to indices* $(i, \mathbf{x})$ *such that* $\mathbf{x} \in \mathcal{E}$.

In particular, if $\mathbf{A}^+$ is *not* an Artin–Tits monoid of spherical type, then its expanded Garside matrices and its essential Garside matrices are identical. The expanded Garside matrix and the essential Garside matrix are analogous to constructions already used successfully for the analysis of heaps [71].

Lemma 7.41.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators, and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation. The Garside matrix of parameter r and the essential Garside matrix of parameter r are primitive.

Proof. The non-negativity of these two matrices is immediate, and their primitiveness follows from Theorem 7.39. In particular, since the essential Garside graph of $\mathbf{A}^+$ is connected and contains loops, let $\mathfrak{D}$ be its diameter. It comes immediately that, between any two states of the graph, there exists a path of length exactly $2\mathfrak{D} + 1$. □

Definition 7.42 (Successor Garside set)**.**
Consider some element $\mathbf{x}$ *of an Artin–Tits monoid of FC type* $\mathbf{A}^+$*. We call* successor Garside set *of* $\mathbf{x}$*, and denote by* $\mathbf{S}(\mathbf{x})$*, the set* $\{\mathbf{y} \in \mathbf{S} : \Theta_{\|\mathbf{x}\|_\gamma}(\mathbf{x}) \longrightarrow \mathbf{y}\}$*, i.e. the set* $\{\mathbf{1}\}$ *if* $\mathbf{x} = \mathbf{1}$*, or* $\{\mathbf{y} \in \mathbf{S} : \mathbf{NF}_\ell(\mathbf{xy}) = \mathbf{NF}_\ell(\mathbf{x}) \cdot \mathbf{y}\}$ *if* $\mathbf{x} \neq \mathbf{1}$*.*

Lemma 7.43.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type, let $f : \mathbf{A}^+ \mapsto \mathbb{C}$ be a valuation, and let $g : \mathbf{x} \mapsto \sum_{\mathbf{y} \in \mathbf{S}(\mathbf{x})} (\mathbf{M}_\gamma f)(\mathbf{y})$. For all $\mathbf{x} \in \mathbf{A}^+$, we have $(\mathbf{M}_\gamma f)(\mathbf{x}) = f(\mathbf{x}) g(\mathbf{x})$.

Proof. First, if $\mathbf{x} = \mathbf{1}$, then $\mathbf{S}(\mathbf{x}) = \{\mathbf{1}\}$ and $f(\mathbf{1}) = 1$, whence the equality $(\mathbf{M}_\gamma f)(\mathbf{1}) = f(\mathbf{1})g(\mathbf{1})$. Henceforth, we assume that $\mathbf{x}$ is non-trivial, and denote by $x_1 \cdot \ldots \cdot x_k$ be the left Garside normal form of $\mathbf{x}$, with $k = \|\mathbf{x}\| = \|\mathbf{x}\|_\gamma$.

Consider the two functions $F, G : 2^{\{\sigma_1,\ldots,\sigma_n\}} \mapsto \mathbb{C}$ defined by

$$F : A \mapsto \sum_{I \in \mathcal{P}} \mathbf{1}_{\{I \cap A = \varnothing\}} (-1)^{|I|} f(\Delta_I) \text{ and } G : A \mapsto \sum_{\mathbf{y} \in \mathbf{S}} \mathbf{1}_{\{\mathbf{left}(\mathbf{y}) \subseteq A\}} (\mathbf{M}_\gamma f)(\mathbf{y}).$$

We first prove that $F = G$. Indeed, for each $I \in \mathcal{P}$ and for each $\mathbf{y} \in \mathbf{S}$, we have $I \subseteq \mathbf{left}(\mathbf{y}) \Leftrightarrow \Delta_I \leqslant_\ell \mathbf{y}$, whence

$$f(\Delta_I) = (\overline{\mathbf{M}}_\gamma(\mathbf{M}_\gamma f))(\Delta_I) = \sum_{\mathbf{y} \in \mathbf{A}^+} \mathbf{1}_{\|\mathbf{y}\|_\gamma = 1} \mathbf{1}_{\Delta_I \leqslant_\ell \mathbf{y}} (\mathbf{M}_\gamma f)(\mathbf{y}) = \sum_{\mathbf{y} \in \mathbf{S}} \mathbf{1}_{I \subseteq \mathbf{left}(\mathbf{y})} (\mathbf{M}_\gamma f)(\mathbf{y});$$

$$F(A) = \sum_{\mathbf{y} \in \mathbf{S}} \sum_{I \in \mathcal{P}} \mathbf{1}_{\{I \cap A = \varnothing\}} \mathbf{1}_{I \subseteq \mathbf{left}(\mathbf{y})} (-1)^{|I|} (\mathbf{M}_\gamma f)(\mathbf{y}) = \sum_{\mathbf{y} \in \mathbf{S}} \mathbf{1}_{\mathbf{left}(\mathbf{y}) \subseteq A} (\mathbf{M}_\gamma f)(\mathbf{y}) = G(A).$$

According to Lemma 2.99, for all elements Δ_I, we have $\|\mathbf{x}\Delta_I\| \leqslant k \Leftrightarrow \overline{\mathbf{right}}(x_k) \cap I = \varnothing$. Since f is multiplicative, it follows that

$$(\mathbf{M}_\gamma f)(\mathbf{x}) = (\mathbf{M} f_k)(\mathbf{x}) = \sum_{I \in \mathcal{P}} (-1)^{|I|} \mathbf{1}_{\|\mathbf{x}\Delta_I\| \leqslant k} f(\mathbf{x}\Delta_I) = f(\mathbf{x}) F(\overline{\mathbf{right}}(x_k)) \text{ and that}$$

$$g(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{S}(\mathbf{x})} (\mathbf{M}_\gamma f)(\mathbf{y}) = \sum_{\mathbf{y} \in \mathcal{S}} \mathbf{1}_{\mathbf{left}(\mathbf{y}) \subseteq \overline{\mathbf{right}}(x_k)} (\mathbf{M}_\gamma f)(\mathbf{y}) = G(\overline{\mathbf{right}}(x_k)),$$

which completes the proof. □

Proposition 7.44.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators, let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation associated with a uniform measure ν_r of parameter r such that $\nu_r(\mathbf{A}^+) = 0$, and let M be the Garside matrix of parameter r. The Perron eigenvalue of M is 1, and the right Perron eigenvectors of M are multiples of the vector $\mathbf{g} = (\mathbf{g}_\mathbf{x})_{\mathbf{x} \in \mathcal{E}}$ such that $\mathbf{g}_\mathbf{x} := g(\mathbf{x})$, where $g : \mathbf{x} \mapsto \sum_{\mathbf{y} \in \mathbf{S}(\mathbf{x})} (\mathbf{M}_\gamma r)(\mathbf{y})$.

Proof. We first prove that $\mathbf{g}$ is non-zero and has non-negative entries. Recall that the relation $\mathbf{M}_\gamma r = rg$ holds on $\mathbf{S}$. Therefore, for all $\mathbf{x} \in \mathcal{E}$, Proposition 7.25 shows that $\mathbf{g}_\mathbf{x} = g(\mathbf{x}) = r(\mathbf{x})(\mathbf{M}_\gamma r)(\mathbf{x}) = r(\mathbf{x})\nu_r(\mathfrak{B}_2(1, \mathbf{x})) \geqslant 0$. In addition, if $\mathbf{g}$ is zero, then we have $(\mathbf{M}_\gamma r)(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{E}$, and therefore

$$\begin{aligned}
1 &= r(\mathbf{1}) = (\overline{\mathbf{M}}_\gamma(\mathbf{M}_\gamma r))(\mathbf{1}) = (\mathbf{M}_\gamma r)(\mathbf{1}) + (\mathbf{M}_\gamma r)(\Delta) \\
&= \nu_r(\mathfrak{B}_2(1, \mathbf{1})) + \nu_r(\mathfrak{B}_2(1, \Delta)) = \nu_r(\{\mathbf{1}\}) + \nu_r(\Uparrow \Delta) = r(\Delta) \text{ if } \Delta \text{ exists, or} \\
1 &= r(\mathbf{1}) = (\overline{\mathbf{M}}_\gamma(\mathbf{M}_\gamma r))(\mathbf{1}) = (\mathbf{M}_\gamma r)(\mathbf{1}) = \nu_r(\mathfrak{B}_2(1, \mathbf{1})) = \nu_r(\{\mathbf{1}\}) = 0 \text{ otherwise.}
\end{aligned}$$

This proves that Δ exists, and that $r(\Delta) = 1$. Since we must have $0 < r_i \leqslant 1$ for all $i \in \{1, \ldots, n\}$, it follows that r is the constant valuation $\mathbf{x} \mapsto 1$, which we decided above to cast aside.

Therefore, we have proven that $\mathbf{g}$ is non-zero, and we focus on proving that $M \cdot \mathbf{g} = \mathbf{g}$. Proposition 7.25 shows that $g(\mathbf{1}) = (\mathbf{M}_\gamma r)(\mathbf{1}) = \nu_r(\mathfrak{B}_2(1, \mathbf{1})) = \nu_r(\{\mathbf{1}\}) = 0$. It follows that

$$(M \cdot \mathbf{g})_\mathbf{x} = \sum_{\mathbf{y} \in \mathbf{S}(\mathbf{x})} \mathbf{1}_{\mathbf{y} \neq \mathbf{1}} r(\mathbf{y}) g(\mathbf{y}) = \sum_{\mathbf{y} \in \mathbf{S}(\mathbf{x})} r(\mathbf{y}) g(\mathbf{y}) = \sum_{\mathbf{y} \in \mathbf{S}(\mathbf{x})} (\mathbf{M}_\gamma r)(\mathbf{y}) = g(\mathbf{x}) = \mathbf{g}_\mathbf{x}.$$

□

Proposition 7.45.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators, and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation associated with a uniform measure ν_r of parameter r. The valuation r belongs to the set $\mathcal{R}_\mathbf{A} \cup \mathcal{R}_\mathbf{A}^\partial$.

Proof. Consider the tuple $\widehat{\mathbf{r}} := (r_1, \ldots, r_n)$ and let us assume, for the sake of contradiction, that $\widehat{\mathbf{r}} \notin \mathcal{R}_\mathbf{A} \cup \mathcal{R}_\mathbf{A}^\partial$, i.e. that $\varphi(\widehat{\mathbf{r}}) < 1$, where φ was defined in Proposition 7.33.

First, Lemma 7.29 proves that $\nu_r(\{\mathbf{x}\}) = \mathcal{H}_\mathbf{A}(r) r(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{A}^+$. This proves that $\mathcal{H}_\mathbf{A}(r) \geqslant 0$. Moreover, if $\mathcal{H}_\mathbf{A}(r) > 0$, then

$$1 = \nu_r(\Uparrow \mathbf{1}) \geqslant \nu_r(T) = \mathcal{H}_\mathbf{A}(r) \sum_{\mathbf{x} \in T} r(\mathbf{x})$$

for all finite sets $T \subseteq \mathbf{A}^+$. It follows that the series $\sum_{\mathbf{x} \in \mathbf{A}^+} r(\mathbf{x})$ is convergent, i.e. that $\widehat{\mathbf{r}} \in \mathcal{R}_\mathbf{A}$, which is false.

Hence, we know that $\mathcal{H}_\mathbf{A}(r) = 0$ and that $\nu_r(\mathbf{A}^+) = 0$, i.e. that ν_r is concentrated at infinity. Then, Proposition 7.44 proves that both the Garside matrices M and N, with respective parameters r and $\Psi(r) = \varphi(r)^\lambda\, r$, are primitive matrices with Perron eigenvalue 1. Since $\varphi(r)^{\lambda(\mathbf{x})}\, r(\mathbf{x}) < r(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{A}^+ \backslash \{\mathbf{1}\}$, it follows that $M \neq N$, and that both N and $M - N$ are non-negative, which contradicts Theorem 7.37. Therefore, our initial assumption was false, which proves Proposition 7.45. □

Adding Theorem 7.35 and Proposition 7.45, we obtain the following result.

Theorem 7.46.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators. The positive uniform measures on $\overline{\mathbf{A}}^+$ are:

- *the measures ν_r with support equal to $\mathbf{A}^+$, with parameter $r \in \mathcal{R}_\mathbf{A}$, that are defined by $\nu_q : \{\mathbf{x}\} \mapsto \mathcal{H}_\mathbf{A}(r) r(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{A}^+$;*
- *the measures ν_r with support in $\partial \mathbf{A}^+$, with parameter $r \in \mathcal{R}^\partial_\mathbf{A}$;*
- *the Dirac measure at Δ^ω, with parameter 1, if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type.*

Whenever we consider below a function f with domain in $\overline{\mathbf{A}}^+$ and a probability measure μ on $\overline{\mathbf{A}}^+$, we denote by $f(\mu)$ the distribution of the random variable $f(\mathbf{x})$ when $\mathbf{x}$ is drawn at random according to the measure μ. If $(f_k)_{k \geqslant 0}$ is a random process, we will also denote by $(f_k(\mu))_{k \geqslant 0}$ the distribution of the random variable $(f_k(\mathbf{x}))_{k \geqslant 0}$ when $\mathbf{x}$ is drawn at random according to the measure μ.

A remarkable feature of uniform measures ν_r is that the process $(\Theta_k(\nu_r))_{k \geqslant 1}$ of *extension mappings*, which we introduced in Definition 7.18, is realised by a Markov chain.

Definition 7.47 (Markov Garside matrix)**.**
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators, let ν_r be a positive uniform measure on $\mathbf{A}^+$, and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be the parameter of ν_r. In addition, let us denote by $\mathfrak{C}$ the set $\mathbf{S}$ (if $r \in \mathcal{R}_\mathbf{A}$) or $\mathbf{S}\backslash\{\mathbf{1}\}$ (if $r \in \mathcal{R}^\partial_\mathbf{A}$).

The Markov Garside matrix *of r is the matrix $P = (P_{\mathbf{u},\mathbf{v}})$, where $\mathbf{u}, \mathbf{v} \in \mathfrak{C}$, which is defined by*

$$P_{\mathbf{1},\mathbf{v}} := \mathbf{1}_{\mathbf{1}=\mathbf{v}} \text{ (if } q \neq p_\mathbf{A}\text{), and } P_{\mathbf{u},\mathbf{v}} := \mathbf{1}_{\mathbf{v} \in \mathbf{S}(\mathbf{u})} r(\mathbf{u}) \frac{(\mathbf{M}_\gamma r)(\mathbf{v})}{(\mathbf{M}_\gamma r)(\mathbf{u})} \text{ if } \mathbf{u} \neq \mathbf{1}.$$

Theorem 7.48.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators. Let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation, with $r \in \mathcal{R}_\mathbf{A} \cup \mathcal{R}^\partial_\mathbf{A}$, and let $\mathbf{x}$ be an extended braid chosen according to the uniform measure ν_r.

The process $(\Theta_k(\nu_r))_{k \geqslant 1}$ is a Markov chain that takes its values in $\mathfrak{C}$. Its initial distribution coincides with $\mathbf{M}_\gamma r$ on $\mathfrak{C}$, and its transition matrix is the Markov Garside matrix of r.

Proof. We first show that $(\mathbf{M}_\gamma r)(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathfrak{C}$. Indeed:

- if $r \in \mathcal{R}_{\mathbf{A}}$, then Proposition 7.25 and Lemma 7.29 indicate that $(\mathbf{M}_\gamma r)(\mathbf{x}) = \nu_r(\mathfrak{B}_2(1,\mathbf{x})) \geqslant \nu_r(\{\mathbf{x}\}) = \mathcal{H}_{\mathbf{A}}(r)r(\mathbf{x}) > 0$;
- if $r \in \mathcal{R}^{\partial}_{\mathbf{A}}$, then Theorem 7.37, Lemma 7.43 and Proposition 7.44 indicate that $g(\mathbf{x}) > 0$, where g is the function, defined in Lemma 7.43, such that $(\mathbf{M}_\gamma r)(\mathbf{x}) = r(\mathbf{x})g(\mathbf{x})$.

Hence, $(\mathbf{M}_\gamma r)(\mathbf{x}) > 0$, and the Markov Garside matrix of r is well-defined.

If $r \in \mathcal{R}^{\partial}_{\mathbf{A}}$, then we know that $\nu_r(\Theta_k = \mathbf{1}) \leqslant \nu_r(\mathbf{A}^+) = 0$ for all $k \geqslant 0$, hence the random variable Θ_k indeed takes its values in $\mathfrak{C}$. Now, consider some tuple $(x_1,\ldots,x_k)$ of elements of $\mathfrak{C}$. We prove now that the real numbers $\delta := \nu_r(\Theta_1 = x_1,\ldots,\Theta_k = x_k)$ and $\delta' := (\mathbf{M}_\gamma r)(x_1)P_{x_1,x_2}\ldots P_{x_{k-1},x_k}$ are equal.

First, observe that we can focus on the case where $x_1 \longrightarrow \ldots \longrightarrow x_k$. Indeed, in the contrary case, we have $\delta = \delta' = 0$. Now, consider the element $\mathbf{x} = x_1x_2\ldots x_k$. By construction, we have $\Theta_i(\mathbf{x}) = x_i$, hence $\delta = \nu_r(\theta_{k,\infty} = \mathbf{x}) = \nu_r(\mathfrak{B}_2(k,\mathbf{x})) = (\mathbf{M}_\gamma r)(\mathbf{x})$. Moreover, note that $\|\mathbf{x}\| < k$ if $x_k = \mathbf{1}$ (and therefore $r \in \mathcal{R}_{\mathbf{A}}$), and that $\|\mathbf{x}\| = k$ if $x_k \neq \mathbf{1}$. It follows that

$$\begin{aligned}
\delta &= \nu_r(\{\mathbf{x}\}) = \mathcal{H}_{\mathbf{A}}(r)r(\mathbf{x}) = (\mathbf{M}_\gamma r)(\mathbf{1})\ r(\mathbf{x}) \\
&= r(\mathbf{x})r(x_k)^{-1}(\mathbf{M}_\gamma r)(x_k) \text{ if } \|\mathbf{x}\| < k; \\
\delta &= (\mathbf{M}_\gamma r)(\mathbf{x}) = r(\mathbf{x})g(\mathbf{x}) = r(\mathbf{x}) \sum_{\mathbf{y}\in\mathcal{S}(x_k)} (\mathbf{M}_\gamma r)(\mathbf{y}) \\
&= r(\mathbf{x})r(x_k)^{-1}(\mathbf{M}_\gamma r)(x_k) \text{ if } \|\mathbf{x}\| = k.
\end{aligned}$$

Moreover, note that

$$\delta' = (\mathbf{M}_\gamma r)(x_1)\prod_{i=1}^{k-1} r(x_i)\frac{(\mathbf{M}_\gamma r)(x_{i+1})}{(\mathbf{M}_\gamma r)(x_i)} = r(\mathbf{x})r(x_k)^{-1}(\mathbf{M}_\gamma r)(x_k) = \delta.$$

This proves that $(\Theta_k)_{k\geqslant 1}$ is indeed a Markov chain with the specified initial distribution and transition matrix. □

In particular, Theorem 7.48 leads to a characterisation of $\mathcal{R}^{\partial}_{\mathbf{A}}$ in terms of polynomial inequalities, thanks to the notion of *Möbius valuation.*

Definition 7.49 (Möbius valuation).
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type, and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a valuation. We say that r is Möbius *if $(M_\gamma r)(\mathbf{1}) = 0$ and $(M_\gamma r)(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{S}\backslash\{\mathbf{1}\}$.*

Proposition 7.50.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators, and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation. The valuation r is Möbius if and only if it belongs to $\mathcal{R}^{\partial}_{\mathbf{A}}$.

Proof. First, if $r \in \mathcal{R}^{\partial}_{\mathbf{A}}$, then the uniform measure ν_r of parameter r satisfies $\nu_r(\mathbf{A}^+) = 0$. Then, let $g : \mathbf{x} \mapsto \sum_{\mathbf{y}\in\mathbf{S}(\mathbf{x})}(\mathbf{M}_\gamma r)(\mathbf{y})$. Proposition 7.44 proves that $(\mathbf{M}_\gamma r)(\mathbf{x}) = r(\mathbf{x})g(\mathbf{x}) >$

0 for all $\mathbf{x} \in \mathcal{E}$. Hence, it remains to prove that $(\mathbf{M}_\gamma r)(\mathbf{1}) = 0$ and that $(\mathbf{M}_\gamma r)(\Delta) > 0$ if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type. The first equality comes from Proposition 7.25, which proves that $(\mathbf{M}_\gamma r)(\Delta) = \nu_r(\mathfrak{B}_2(1,\mathbf{1})) = \nu_r(\{\mathbf{1}\}) \leqslant \nu_r(\mathbf{A}^+) = 0$, and the second is due to the fact that $(\mathbf{M}_\gamma r)(\Delta) = r(\Delta) > 0$. Hence, if $r \in \mathcal{R}^\partial_{\mathbf{A}}$, it follows that r is Möbius.

Conversely, if r is Möbius, let us build a Bernoulli measure μ with parameter r, using the recipe provided by Theorem 7.48. Consider the matrix $P = (P_{\mathbf{x},\mathbf{y}})$, where $\mathbf{x}, \mathbf{y} \in \mathbf{S}\backslash\{\mathbf{1}\}$, that is defined by

$$P_{\mathbf{x},\mathbf{y}} := \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} r(\mathbf{x}) \frac{(\mathbf{M}_\gamma r)(\mathbf{y})}{(\mathbf{M}_\gamma r)(\mathbf{x})}.$$

Note that, since r is Möbius, the matrix P is well-defined. Furthermore, for all $\mathbf{x} \in \mathbf{S}\backslash\{\mathbf{1}\}$, and since r is Möbius, we also have

$$\begin{aligned}\sum_{\mathbf{y}\in\mathbf{S}\backslash\{\mathbf{1}\}} P_{\mathbf{x},\mathbf{y}} &= \frac{r(\mathbf{x})}{(\mathbf{M}_\gamma r)(\mathbf{x})} \sum_{\mathbf{y}\in\mathbf{S}\backslash\{\mathbf{1}\}} \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} (\mathbf{M}_\gamma r)(\mathbf{y}) \\ &= \frac{r(\mathbf{x})}{(\mathbf{M}_\gamma r)(\mathbf{x})} \sum_{\mathbf{y}\in\mathbf{S}} \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} (\mathbf{M}_\gamma r)(\mathbf{y}) = \frac{r(\mathbf{x})g(\mathbf{x})}{(\mathbf{M}_\gamma r)(\mathbf{x})} = 1,\end{aligned}$$

which proves that P is a stochastic matrix.

In addition, let ι be the measure on $\mathbf{S}\backslash\{\mathbf{1}\}$ such that $\iota : \mathbf{x} \mapsto (\mathbf{M}_\gamma r)(\mathbf{x})$. Since $\mathbf{A}^+$ is the disjoint union of the open balls $\mathfrak{B}_2(1,\mathbf{x})$ for $\mathbf{x} \in \mathbf{S}$ and since $\iota(\mathbf{1}) = 0$, it follows that

$$\iota(\mathbf{S}\backslash\{\mathbf{1}\}) = \iota(\mathbf{S}) = \sum_{\mathbf{x}\in\mathbf{S}} \iota(\mathbf{x}) = (\overline{\mathbf{M}}_\gamma \iota)(\mathbf{1}) = r(\mathbf{1}) = 1,$$

which means that ι is a probability distribution.

Hence, consider the Markov chain $(X_k)_{k\geqslant 1}$ of non-trivial strong elements (i.e. elements of $\mathbf{S}\backslash\{\mathbf{1}\}$) with initial distribution ι and with transition matrix P. Due to Proposition 7.22, which states that $\overline{\mathbf{A}}^+$ is homeomorphic to $\mathbf{NF}^\omega_\ell(\overline{\mathbf{A}}^+)$, there exists a probability measure μ on $\overline{\mathbf{A}}^+$ such that $\mu : S \mapsto \mathbb{P}[X_1 \cdot X_2 \cdot \ldots \in \mathbf{NF}^\omega_\ell(S)]$. It immediately follows that $\mu(\mathbf{A}^+) = \mathbb{P}[\exists k \geqslant 1 : X_k = \mathbf{1}] = 0$, hence that μ is supported by $\partial\mathbf{A}^+$. Hence, consider the function $s : \mathbf{A}^+ \mapsto \mathbb{C}$ such that $s : \mathbf{x} \mapsto \mu(\Uparrow \mathbf{x}) = \mu(\uparrow \mathbf{x})$. It remains to prove that $r = s$.

First, observe that $0 = \mu(\{\mathbf{1}\}) = \mu(\mathfrak{B}_2(1,\mathbf{1})) = (\mathbf{M}_\gamma s)(\mathbf{1})$. Then, consider some non-trivial element $\mathbf{x}$ of $\mathbf{A}^+$, with left Garside normal form $x_1 \cdot \ldots \cdot x_k$. Let $F : A \mapsto \sum_{I\in\mathcal{P}} \mathbf{1}_{\{I\cap A=\varnothing\}}(-1)^{|I|} r(\Delta_I)$. Since $x_1 \longrightarrow \ldots \longrightarrow x_k$, and due to the relation $(\mathbf{M}_\gamma r)(\mathbf{x}) = r(\mathbf{x})F(\overline{\mathbf{right}}(x_k))$ demonstrated in the proof of Lemma 7.43, it follows that

$$\begin{aligned}(\mathbf{M}_\gamma r)(\mathbf{x}) &= r(\mathbf{x})F(\overline{\mathbf{right}}(x_k)) = r(x_1 \ldots x_{k-1}) \cdot (\mathbf{M}_\gamma r)(x_k) \\ &= (\mathbf{M}_\gamma r)(x_1) \prod_{i=1}^{k-1} r(x_i) \frac{(\mathbf{M}_\gamma r)(x_{i+1})}{(\mathbf{M}_\gamma r)(x_i)} = \iota(x_1) \prod_{i=1}^{k-1} P_{x_i,x_{i+1}} \\ &= \mathbb{P}[X_1 = x_1, \ldots, X_k = x_k] = \mu(\mathfrak{B}_2(k,\mathbf{x})) = (\mathbf{M}_\gamma s)(\mathbf{x}).\end{aligned}$$

This means that $\mathbf{M}_\gamma r = \mathbf{M}_\gamma s$, and therefore that $r = s$, which completes the proof. □

Furthermore, while Corollary 7.34 proves that $\mathcal{R}^{\partial}_{\mathbf{A}}$ is homeomorphic to an open simplex for all Artin–Tits monoids of FC type $\mathbf{A}$, its proof refers to a homeomorphism Φ whose expression is not simple in general, since it involves looking for roots of polynomials of arbitrary degree. However, if $\mathbf{A}^+$ is a heap monoid, then these polynomials have degree at most 1 in each of their variables, which leads to simple parametrisations of the convergence manifold and of the limit convergence manifold.

Theorem 7.51.
Let $\mathbf{A}^+$ be an irreducible heap monoid with $n \geqslant 2$ generators. There exists rational functions $f_1, \ldots, f_{n-1}$ such that each function f_i has i variables, is positive (and has no pole) on $(0,1)^i$, and such that the manifolds $\mathcal{R}_{\mathbf{A}}$ and $\mathcal{R}^{\partial}_{\mathbf{A}}$ are given by the equations

$$\begin{aligned}\mathcal{R}_{\mathbf{A}} &= \{(x_1, f_1(x_1)x_2, \ldots, f_n(x_1, \ldots, x_{n-1})x_n) : 0 < x_1, \ldots, x_n < 1\};\\ \mathcal{R}^{\partial}_{\mathbf{A}} &= \{(x_1, f_1(x_1)x_2, \ldots, f_n(x_1, \ldots, x_{n-1})x_n) : 0 < x_1, \ldots, x_{n-1} < 1 \text{ and } x_n = 1\}.\end{aligned}$$

Proof. We prove the first part of Theorem 7.51, i.e. the equation about $\mathcal{R}_{\mathbf{A}}$, by induction on n. First, the result is immediate for $n = 1$. Then, if $n \geqslant 2$, consider the set $N := \{i : 1 \leqslant i \leqslant n-1 \text{ and } m_{i,n} \neq 0\}$. For all sets $S \subseteq \{1, \ldots, n-1\}$, let $\mathbf{A}^+_S$ denote the submonoid of $\mathbf{A}^+$ generated by $\{\sigma_i : i \in S\}$. We may view the Möbius polynomial of $\mathbf{A}^+_S$ as the polynomial $\mathcal{H}_S(x_1, \ldots, x_{n-1}) := \mathcal{H}_{\mathbf{A}}(x_1^S, \ldots, x_n^S)$ in $n-1$ variables, where $x_i^S := \mathbf{1}_{i \in S} x_i$.

For all tuples $(x_1, \ldots, x_n) \in \mathcal{R}_{\mathbf{A}}$, both series $\mathcal{G}_{\mathbf{A}}(x_1, \ldots, x_n)$ and $\mathcal{G}_{\mathbf{A}}(x_1^S, \ldots, x_n^S)$ are convergent, which proves that $(x_i)_{i \in S} \in \mathcal{R}_{\mathbf{A}_S}$, hence that $\mathcal{H}_S(x_1, \ldots, x_{n-1}) > 0$. Moreover, it comes immediately that $\mathcal{H}_{\mathbf{A}} = \mathcal{H}_{\{1,\ldots,n-1\}} - x_n \mathcal{H}_N$. Now, consider rational functions $f_1, \ldots, f_{n-1}$, such that each function f_i has i variables, is positive and has no pole on $(0,1)^i$, and such that the convergence manifold of $\mathbf{A}^+_{\{1,\ldots,n-1\}}$ is

$$\mathcal{R}_{\mathbf{A}_{\{1,\ldots,n-1\}}} = \{(y_1, \ldots, y_{n-1}) : 0 < x_1, \ldots, x_{n-1} < 1\},$$

where $y_i := f_{i-1}(x_1, \ldots, x_{i-1}) x_i$ for all $i \in \{1, \ldots, n-1\}$. Then, we define

$$f_n(x_1, \ldots, x_{n-1}) := \frac{\mathcal{H}_{\{1,\ldots,n-1\}}(y_1, \ldots, y_{n-1})}{\mathcal{H}_N(y_1, \ldots, y_{n-1})}$$

and $y_n = f_n(x_1, \ldots, x_{n-1}) x_n$

When $0 < x_1, \ldots, x_{n-1} < 1$, we proved above that $\mathcal{H}_{\{1,\ldots,n-1\}}(y_1, \ldots, y_{n-1})$ and $\mathcal{H}_N(y_1, \ldots, y_{n-1})$ are well-defined and positive. This proves that f_n has no pole inside the set $(0,1)^{n-1}$, on which it is positive.

We know that $\mathcal{H}_{\mathbf{A}}(y_1, \ldots, y_n) > 0$ when $0 < x_1, \ldots, x_{n-1}$ and $x_n = 0$. Hence, it follows from Corollary 7.34 that the set $\{(y_1, \ldots, y_n) : 0 < x_1, \ldots, x_n < 1\}$ is a subset of $\mathcal{R}_{\mathbf{A}}$. Furthermore, since $\mathcal{R}_{\mathbf{A}_{\{1,\ldots,n-1\}}} = \{(y_1, \ldots, y_{n-1}) : 0 < x_1, \ldots, x_{n-1} < 1\}$, we know that $\mathcal{R}_{\mathbf{A}} \subseteq \{(y_1, \ldots, y_n) : 0 < x_1, \ldots, x_{n-1} < 1 \text{ and } 0 < x_n\}$. Finally, when $0 < x_1, \ldots, x_{n-1} < 1$, observe that $\mathcal{H}_{\mathbf{A}}(y_1, \ldots, y_n) > 0 \Leftrightarrow x_n < 1$. Hence, it follows that $\mathcal{R}_{\mathbf{A}} = \{(y_1, \ldots, y_n) : 0 < x_1, \ldots, x_n < 1\}$, which completes our induction and proves the first part of Theorem 7.51.

It remains to prove that $\mathcal{R}_{\mathbf{A}}^{\partial} = \{(y_1,\ldots,y_n) : 0 < x_1,\ldots,x_{n-1} < 1 \text{ and } x_n = 1\}$. First, it is clear that

$$\begin{aligned} \mathcal{R}_{\mathbf{A}}^{\partial} &\supseteq \{(y_1,\ldots,y_n) : 0 < x_1,\ldots,x_{n-1} < 1 \text{ and } x_n = 1\}; \\ &\subseteq \{(y_1,\ldots,y_n) : 0 < x_1,\ldots,x_n \leqslant 1 \text{ and } \max\{x_1,\ldots,x_n\} = 1\}. \end{aligned}$$

Hence, consider some tuple $(x_1,\ldots,x_n)$ such that $(y_1,\ldots,y_n) \in \mathcal{R}_{\mathbf{A}}^{\partial}$, and let us assume that $x_i = 1$ for some integer $i \leqslant n-1$, which we assume to be minimal. In addition, let $y : \mathbf{A}^+ \mapsto \mathbb{C}$ be the Möbius valuation such that $y_i = y(\sigma_i)$, and let ν_y be the associated Bernoulli measure.

It follows immediately that $\mathcal{H}_{\{1,\ldots,i\}}(y_1,\ldots,y_n) = 0$, which proves, using an inclusion-exclusion formula, that $\nu_y\left(\bigcup_{j=1}^{i} \uparrow \sigma_j\right) = 1 - \nu_y(\{\mathbf{1}\}) = 1$. However, y is Möbius, which means that $\nu_y(\mathfrak{B}_2(1,\sigma_n)) = (\mathbf{M}_\gamma y)(\sigma_n) > 0$, although $\mathfrak{B}_2(1,\sigma_n) \cap \bigcup_{j=1}^{i} \uparrow \sigma_j = \varnothing$. This contradiction proves that our assumption was false, which completes the proof. □

7.1.4 Uniform Measures on Spheres

We relate now the above-defined uniform measures with *standard* uniform measures on finite "spheres" of the form $\{\mathbf{x} \in \mathbf{A}^+ : \lambda(\mathbf{x}) = k\}$, i.e. on elements conditioned by their Artin length.

Definition 7.52 (Uniform distribution on spheres)**.**
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type. For each integer $k \geqslant 0$, let $\mathbf{A}^+(k)$ denote the set of elements of $\mathbf{A}^+$ whose length is k, i.e. $\mathbf{A}^+(k) := \{\mathbf{x} \in \mathbf{A}^+ : \lambda(\mathbf{x}) = k\}$. In addition, let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation. We define the uniform distribution of parameter r *on $\mathbf{A}^+(k)$ as the probability distribution μ_k on $\overline{\mathbf{A}}^+$ such that: $\mu_k : \mathbf{x} \mapsto \mathbf{1}_{\mathbf{x}\in\mathbf{A}^+(k)} \frac{r(\mathbf{x})}{\sum_{\mathbf{y}\in\mathbf{A}^+(k)} r(\mathbf{y})}$.*

Lemma 7.53.
Consider some real number ρ, as well as some real sequence $(u_n)_{n\geqslant 0}$ such that $u_n \sim \rho^n$ when $n \to +\infty$. In addition, let ℓ be some non-negative integer, and let $f_\ell : z \mapsto \sum_{k\geqslant 0} k^\ell z^k$. Finally, consider the sequence $(v_n)_{n\geqslant 0}$ defined by $v_n := \sum_{k=0}^{n} (n-k)^\ell u_k$ and the generating function $U(z)$ defined by $U : z \mapsto \sum_{n\geqslant 0} u_n z^n$.

- *If $\rho > 1$, then $v_n \sim \rho^n f_\ell(\rho)$ when $n \to +\infty$;*
- *If $\rho > 0$, then $U(z) \sim \frac{1}{1-\rho z}$ when $z \to \rho^{-1}$ (with $0 \leqslant z < \rho^{-1}$).*

Proof. Let ε be a positive real number, and let κ be some integer such that $(1-\varepsilon)\rho^n \leqslant u_n \leqslant (1+\varepsilon)\rho^n$ whenever $n \geqslant \kappa$. In addition, let $A := \sum_{k=0}^{\kappa} |u_k|$. If $\rho > 1$, it comes immediately that

$$\begin{aligned} \rho^{-n} v_n &\leqslant (1+\varepsilon) \sum_{k=0}^{n-\kappa} k^\ell \rho^{-k} + A\rho^{-n} \to (1+\varepsilon) f_\ell(\rho) \\ &\geqslant (1-\varepsilon) \sum_{k=0}^{n-\kappa} k^\ell \rho^{-k} - A\rho^{-n} \to (1-\varepsilon) f_\ell(\rho) \end{aligned}$$

when $n \to +\infty$. Choosing ε to be arbitrarily small, it comes that $v_n \sim \rho^n f_\ell(\rho)$ when $n \to +\infty$.

Moreover, when $0 \leqslant z < \rho^{-1}$, we also have

$$\begin{aligned}(1-\rho z)U(z) &\leqslant (1+\varepsilon)(\rho z)^\kappa + (1-\rho z)A \to 1+\varepsilon \\ &\geqslant (1-\varepsilon)(\rho z)^\kappa - (1-\rho z)A \to 1-\varepsilon\end{aligned}$$

when $z \to \rho^{-1}$. Choosing ε to be arbitrarily small, it comes that $U(z) \sim \frac{1}{1-\rho z}$ when $z \to \rho^{-1}$ (with $0 \leqslant z < \rho^{-1}$). □

Corollary 7.54.
Consider some real number ρ, some integer $\ell \geqslant 0$, as well as some real sequence $(u_n)_{n\geqslant 0}$ such that $u_n \sim \rho^n$ when $n \to +\infty$. In addition, consider the sequence $(v_n)_{n\geqslant 0}$ defined by $v_n := \sum_{k=0}^{\lfloor n/a \rfloor} k^\ell u_{n-ak}$. If $\rho > 1$, then $v_n \sim \rho^n f_\ell(\rho^a)$ when $n \to +\infty$.

Proof. Let $\mathbf{t}$ be some element of the set $\{0,\ldots,a-1\}$. Consider the sequences $(\overline{u}_n)$ and $(\overline{v}_n)$ defined by $\overline{u}_n := \rho^{-\mathbf{t}} u_{\mathbf{t}+an}$ and $\overline{v}_n := \rho^{-\mathbf{t}} v_{\mathbf{t}+an}$. It comes immediately that $\overline{u}_n \sim \rho^{an}$ and that $\overline{v}_n = \sum_{k=0}^{n} (n-k)^\ell \overline{u}_k$. Hence, Lemma 7.53 proves that $\overline{v}_n \sim \rho^{an} f_\ell(\rho^a)$, i.e. that $v_{\mathbf{t}+an} \sim \rho^{\mathbf{t}+an} f_\ell(\rho^a)$. Since this relation holds for all $\mathbf{t}$, Corollary 7.54 follows. □

Proposition 7.55.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators. Let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a valuation in $\mathcal{R}_{\mathbf{A}}^{\partial}$, and let M be the essential Garside matrix of $\mathbf{A}^+$ of parameter r. The Perron eigenvalue of M is 1. In addition, consider the polynomial $H_{\mathbf{A},r} : z \mapsto \mathcal{H}_{\mathbf{A}}(z^\lambda r)$. We have $\sum_{\mathbf{x}\in\mathbf{A}^+(k)} r(\mathbf{x}) \to -\frac{1}{H'_{\mathbf{A},r}(1)}$ when $k \to +\infty$.

Proof. Let us denote by $\mathbf{B}^+$ the set of Δ-free braids (if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type) or of elements of $\mathbf{A}^+$ (if $\mathbf{A}^+$ is not an Artin–Tits monoid of spherical type). In addition, let us denote by $\mathbf{B}^+(k)$ the set $\mathbf{A}^+(k) \cap \mathbf{B}^+$. Finally, for all finite sets $S \subseteq \mathbf{A}^+$, we denote by $r(S)$ the sum $\sum_{\mathbf{x}\in S} r(\mathbf{x})$.

First, let us assume that $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, and consider the function $\mathbf{I} : \mathbf{A}^+ \mapsto \mathbb{Z}_{\geqslant 0} \times \mathbf{B}^+$ such that $\mathbf{I} : \mathbf{x} \mapsto (\inf(\mathbf{x}), \Delta^{-\inf(\mathbf{x})}\mathbf{x})$, where we recall that $\inf(\mathbf{x}) := \max\{k : \Delta^k \leqslant \mathbf{x}\}$. The function $\mathbf{I}$ induces a bijection from $\mathbf{A}^+(k)$ to the set $\{(i,\mathbf{y}) \in \mathbb{Z}_{\geqslant 0} \times \mathbf{B}^+ : k = i\lambda(\Delta) + \lambda(\mathbf{y})\}$, which proves the equalities

$$r(\mathbf{A}^+(k)) = \sum_{i=0}^{\lfloor k/\lambda(\Delta) \rfloor} r(\Delta)^i r(\mathbf{B}^+(k-\lambda(\Delta)i));$$

$$(1-(z^\lambda r)(\Delta)) \left(\sum_{k\geqslant 0} r(\mathbf{A}^+(k)) z^k\right) = \sum_{k\geqslant 0} r(\mathbf{B}^+(k)) z^k.$$

Since 1 is the smallest positive pole and the radius of convergence of the rational series $G_{\mathbf{A},r} : z \mapsto H_{\mathbf{A},r}(z)^{-1} = \sum_{k\geqslant 0} r(\mathbf{A}^+(k)) z^k$, and since $z \mapsto 1-(z^\lambda r)(\Delta)$ has one unique root at $r(\Delta)^{-1/\lambda(\Delta)} > 1$, it follows that 1 is also the convergence radius of the rational series $\sum_{k \geqslant 0} r(\mathbf{B}^+(k))$.

Hence, regardless of whether $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, we know that 1 is the convergence radius of $\sum_{k \geqslant 0} r(\mathbf{B}^+(k))$. Then, consider the row vector $\mathbf{g} = (\mathbf{g}_{(i,\mathbf{x})})$ and the column vector $\mathbf{h} = (\mathbf{h}_{(i,\mathbf{x})})$ defined by $\mathbf{g}_{(i,\mathbf{x})} := \mathbf{1}_{i=1} r(\mathbf{x})$ and $\mathbf{h}_{(i,\mathbf{x})} := \mathbf{1}_{i=\lambda(\mathbf{x})}$. It comes immediately that $r(\mathbf{B}^+(k)) = \mathbf{g} \cdot M^{k-1} \cdot \mathbf{h}$ for all integers $k \geqslant 1$.

Lemma 7.41 proves that M is primitive. Let ρ be its Perron eigenvalue, and let $\mathbf{u}$ and $\mathbf{v}$ be left and right Perron eigenvectors of M such that $\mathbf{u} \cdot \mathbf{v} = 1$. Theorem 7.37 proves that $r(\mathbf{B}^+(k)) = \mathbf{g} \cdot M^{k-1} \cdot \mathbf{h} \sim \rho^k L$ when $k \to +\infty$, where $L := \rho^{-1}(\mathbf{g} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{h})$ is necessarily positive, since both $\mathbf{u}$ and $\mathbf{v}$ have positive entries, while $\mathbf{g}$ and $\mathbf{h}$ are non-zero vectors with non-negative entries. This proves that $\rho = 1$, which was the first statement of Proposition 7.55.

In particular, if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, let $\zeta := r(\Delta)^{-1/\lambda(\Delta)}$. We can apply Corollary 7.54 to the sequences $u_k = \frac{\zeta^k}{L} r(\mathbf{B}^+(k))$ and $v_k = \frac{\zeta^k}{L} r(\mathbf{A}^+(k))$, which proves that $r(\mathbf{A}^+(k)) \sim \frac{L}{1-\zeta^{-\lambda(\Delta)}} = \frac{L}{1-r(\Delta)}$. Hence, regardless of whether $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, there exists a positive real number $\mathbf{R}$ such that $r(\mathbf{A}^+(k)) \to \mathbf{R}$. Consequently, Lemma 7.53 proves that $G_{\mathbf{A},r}(z) \sim \frac{\lambda}{1-z}$ when $z \to 0$ (with $0 \leqslant z < 1$). It follows that

$$H_{\mathbf{A},r}(z) = G_{\mathbf{A},r}(z)^{-1} \sim \frac{1-z}{\mathbf{R}},$$

and therefore that $H'_{\mathbf{A},r}(1) = -\frac{1}{\mathbf{R}}$, i.e. $\mathbf{R} = -\frac{1}{H'_{\mathbf{A},r}(1)}$. This completes the proof. □

Corollary 7.56.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators, and let $p_{\mathbf{A}}$ be the smallest positive root of the (univariate) Möbius polynomial $\mathcal{H}_{\mathbf{A}}$. We have $|\mathbf{A}^+(k)| \sim -\frac{p_{\mathbf{A}}^{-k-1}}{\mathcal{H}'_{\mathbf{A}}(p_{\mathbf{A}})}$ when $k \to +\infty$. Moreover, each root of the polynomial $\frac{\mathcal{H}_{\mathbf{A}}(X)}{X-p_{\mathbf{A}}}$ is of modulus strictly greater than $p_{\mathbf{A}}$.

Proof. According to Theorem 7.46, we know that the valuation $r : \mathbf{x} \mapsto p_{\mathbf{A}}^{\lambda(\mathbf{x})}$ belongs to $\mathcal{R}_{\mathbf{A}}^{\partial}$. Consequently, Proposition 7.55 indicates that $\sum_{\mathbf{x} \in \mathbf{A}^+(k)} r(\mathbf{x}) \to -\frac{1}{H'_{\mathbf{A},r}(1)}$, where $H_{\mathbf{A},r}(x) := \mathcal{H}_{\mathbf{A}}(p_{\mathbf{A}} x)$. Since $r(\mathbf{x}) = p_{\mathbf{A}}^k$ for all $\mathbf{x} \in \mathbf{A}^+(k)$, it already follows that

$$|\mathbf{A}^+(k)| = p_{\mathbf{A}}^{-k} \sum_{\mathbf{x} \in \mathbf{A}^+(k)} r(\mathbf{x}) \sim -\frac{p_{\mathbf{A}}^{-k}}{H'_{\mathbf{A},r}(1)} = -\frac{p_{\mathbf{A}}^{-k-1}}{\mathcal{H}'_{\mathbf{A}}(p_{\mathbf{A}})}$$

when $k \to +\infty$.

Second, let M be the essential Garside matrix of $\mathbf{A}^+$ of parameter r. Proposition 7.55 states that the Perron eigenvalue of M is 1. Let $\mathbf{B}^+$ be the set of Δ-free braids (if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type) or of elements of $\mathbf{A}^+$ (if $\mathbf{A}^+$ is not an Artin–Tits monoid of spherical type). In addition, let $\mathbf{u}$ and $\mathbf{v}$ be left and right Perron eigenvectors of M such that $\mathbf{u} \cdot \mathbf{v} = 1$, and let N be the matrix $M - \mathbf{v} \cdot \mathbf{u}$. Theorem 7.37 proves that the spectrum of N is $\{0\} \cup \mathrm{sp}(M)\backslash\{1\}$. Hence, there exists a positive constant $\kappa \in (0,1)$ such that $|\lambda| \leqslant \kappa$ for all eigenvalues λ of N. Without loss of generality, we may even assume that $\kappa^{1/2} \geqslant p_{\mathbf{A}}$.

In addition, recall the row vector $\mathbf{g} = (\mathbf{g}_{(i,\mathbf{x})})$ and the column vector $\mathbf{h} = (\mathbf{h}_{(i,\mathbf{x})})$ defined by $\mathbf{g}_{(i,\mathbf{x})} := \mathbf{1}_{i=1} r(\mathbf{x})$ and $\mathbf{h}_{(i,\mathbf{x})} := \mathbf{1}_{i=\lambda(\mathbf{x})}$, and such that $r(\mathbf{B}^+(k)) = \mathbf{g} \cdot M^{k-1} \cdot \mathbf{h}$ for all integers $k \geqslant 1$. Moreover, observe that $(\mathbf{v} \cdot \mathbf{u})^2 = \mathbf{v} \cdot \mathbf{u}$ and that

$$\begin{aligned} N \cdot (\mathbf{v} \cdot \mathbf{u}) &= (M - \mathbf{v} \cdot \mathbf{u}) \cdot \mathbf{v} \cdot \mathbf{u} = (M \cdot \mathbf{v}) \cdot \mathbf{u} - \mathbf{v} \cdot (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} = 0; \\ (\mathbf{v} \cdot \mathbf{u}) \cdot N &= \mathbf{v} \cdot \mathbf{u} \cdot (M - \mathbf{v} \cdot \mathbf{u}) = \mathbf{v} \cdot (\mathbf{u} \cdot M) - \mathbf{v} \cdot (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} = 0. \end{aligned}$$

It follows, whenever $k \geqslant 1$, that $r(\mathbf{B}^+(k)) = \mathbf{g} \cdot (N + \mathbf{v} \cdot \mathbf{u})^{k-1} \cdot \mathbf{h} = \mathbf{g} \cdot N^{k-1} \cdot \mathbf{h} + (\mathbf{g} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{h})$.

Since $|\lambda| \leqslant \kappa < 1$ for all eigenvalues λ of N, it follows that $r(\mathbf{B}^+(k)) = (\mathbf{g} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{h}) + \mathcal{O}(\kappa^{k/2})$. Consequently, if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, since $r : \mathbf{x} \mapsto p_{\mathbf{A}}^{\lambda(\mathbf{x})}$ and $\kappa^{1/2} \geqslant p_{\mathbf{A}}$, we also find that

$$r(\mathbf{A}^+(k)) = \sum_{i=0}^{\lfloor k/\lambda(\Delta) \rfloor} r(\Delta)^i r(\mathbf{B}^+(k - \lambda(\Delta) i)) = \frac{(\mathbf{g} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{h})}{1 - r(\Delta)} + \mathcal{O}(\kappa^{k/2}).$$

Hence, setting $\mathbf{R} := \frac{(\mathbf{g} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{h})}{1 - r(\Delta)}$ if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, or $\mathbf{R} := (\mathbf{g} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{h})$ otherwise, we have $r(\mathbf{A}^+(k)) = \mathbf{R} + \mathcal{O}(\kappa^{k/2})$, i.e. $|\mathbf{A}^+(k)| = \mathbf{R} p_{\mathbf{A}}^{-k} + \mathcal{O}(p_{\mathbf{A}}^{-k} \kappa^{k/2})$. It follows that

$$\begin{aligned} \frac{z - p_{\mathbf{A}}}{\mathcal{H}_{\mathbf{A}}(z)} &= (z - p_{\mathbf{A}}) \mathcal{G}_{\mathbf{A}}(z) = (z - p_{\mathbf{A}}) \sum_{k \geqslant 0} |\mathbf{A}^+(k)| z^k \\ &= -p_{\mathbf{A}} + \sum_{k \geqslant 0} \left(|\mathbf{A}^+(k)| - p_{\mathbf{A}} |\mathbf{A}^+(k+1)| \right) z^{k+1} \\ &= -p_{\mathbf{A}} + \sum_{k \geqslant 0} \mathcal{O}(p_{\mathbf{A}}^{-k} \kappa^{k/2}) z^{k+1} \end{aligned}$$

has a radius of convergence at least $\frac{p_{\mathbf{A}}}{\kappa^{1/2}}$, i.e. that the roots of $\frac{\mathcal{H}_{\mathbf{A}}(z)}{z - p_{\mathbf{A}}}$ has a modulus at least $\frac{p_{\mathbf{A}}}{\kappa^{1/2}}$, which completes the proof. □

Theorem 7.57.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators. Let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation, and let $\Phi(r)$ be the associated valuation in $\mathcal{R}_{\mathbf{A}}^{\partial}$ (i.e. the only element of $\mathcal{R}_{\mathbf{A}}^{\partial}$ of the form $\mathbf{x} \mapsto q^{\lambda(\mathbf{x})} r(\mathbf{x})$).

For all integers $k \geqslant 0$, let μ_k be the uniform distribution with parameter r on the sphere $\mathbf{A}^+(k)$. The uniform measure $\nu_{\Phi(r)}$ with parameter $\Phi(r)$ is the weak limit of the sequence $(\mu_k)_{k \geqslant 0}$.

Proof. Since $\mathbf{A}^+(k)$ is a subset of the monoid $\mathbf{A}^+$, it is identified with a subset of $\overline{\mathbf{A}}^+$, and therefore we identify μ_k with a discrete probability distribution on $\overline{\mathbf{A}}^+$. In addition, let $\phi(r)$ be the positive real number such that $\Phi(r) = \phi(r)^\lambda r$.

For all $\mathbf{x} \in \mathbf{A}^+$ and all $k \geqslant \lambda(\mathbf{x})$, we have $\mathbf{x}\ \mathbf{A}^+(k - \lambda(\mathbf{x})) = (\Uparrow \mathbf{x}) \cap \mathbf{A}^+(k)$, which shows that $\mu_k(\Uparrow \mathbf{x}) = \frac{r(\mathbf{x})r(\mathbf{A}^+(k-\lambda(\mathbf{x})))}{r(\mathbf{A}^+(k))}$. It follows that

$$r(\mathbf{A}^+(k)) = \phi(r)^{-k}\Phi(r)(\mathbf{A}^+(k)) \sim -\frac{1}{H'_{\mathbf{A},\Phi(r)}(1)}\phi(r)^{-k} \text{ and}$$
$$\mu_k(\Uparrow \mathbf{x}) \to \phi(r)^{\lambda(\mathbf{x})}r(\mathbf{x}) = \Phi(r)(\mathbf{x}) = \nu_{\Phi(r)}(\Uparrow \mathbf{x})$$

when $k \to +\infty$.

The end of the proof is similar to that of Theorem 7.35. Since $\overline{\mathbf{A}}^+$ is metric and compact, there exists an increasing sequence $(u_j)_{j\geqslant 1}$ such that $(\mu_{u_j})_{j\geqslant 1}$ is weakly convergent. Let μ_∞ be the weak limit of $(\mu_{u_j})_{j\geqslant 1}$. For each braid $\mathbf{x} \in \mathbf{A}^+$, the set $\Uparrow \mathbf{x}$ has an empty topological boundary, hence the Portemanteau theorem [12] implies that $\mu_\infty(\Uparrow \mathbf{x})) = \lim_{j\to\infty} \mu_{u_j}(\Uparrow \mathbf{x}) = \nu_{\Phi(r)}(\Uparrow \mathbf{x})$, and Proposition 7.25 proves that $\mu_\infty = \nu_{\Phi(r)}$. Since this equality holds for all limits of all weakly convergent subsequences of $(\mu_k)_{k\geqslant 0}$, and since $\overline{\mathbf{A}}^+$ is metric and compact, it follows that the sequence $(\mu_k)_{k\geqslant 0}$ is itself (weakly) convergent towards $\nu_{\Phi(r)}$. □

7.1.5 Applications to Artin–Tits Monoids of Spherical Type

Theorem 7.57, along with the above study of the uniform measure, leads to a wide range of results. We focus here on the specific case of the constant valuation $r : \mathbf{x} \mapsto 1$, and we first rephrase some of the above results.

Proposition 7.58.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, and let $p_\mathbf{A}$ be the smallest positive root of the Möbius polynomial $\mathcal{H}_\mathbf{A}(z)$. The "standard" uniform distributions on the spheres $\mathbf{A}^+(k)$ converge weakly towards the distribution $\nu_{p_\mathbf{A}}$ with parameter $p_\mathbf{A}$.

Proof. According to Proposition 7.33 and to Theorem 7.57, the sequence $(\mu_k)_{k\geqslant 1}$ of uniform distributions of parameter r converges weakly towards the distribution $\nu_{\Phi(r)}$, where $\Phi(r)$ is the valuation $\phi(r)^\lambda r$ such that $\phi(r)$ is the smallest positive root of the polynomial $H_{\mathbf{A},r}(z) = \mathcal{H}_\mathbf{A}(z)$. This proves that $\phi(r) = p_\mathbf{A}$, and completes the proof. □

Corollary 7.59.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators. Let $n \geqslant 0$ be an integer, let $\theta_{n,\infty} : \overline{\mathbf{A}}^+ \mapsto \Omega_n$ be the projection defined in Definition 7.14, and let $\nu_{p_\mathbf{A}}$ be the uniform measure at infinity. The distribution $\theta_{n,\infty}(\mu_k)$ converges towards $\theta_{n,\infty}(\nu_{p_\mathbf{A}})$ when $k \to +\infty$.

Proof. By construction of the projective topology, the mapping $\theta_{n,\infty}$ is continuous. Hence, Theorem 7.57 completes the proof. □

Corollary 7.60.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators, and let $\inf(\mathbf{x})$ *denote the* infimum *of a braid* $\mathbf{x}$*, i.e. the number of braids* Δ *that arise in the Garside normal forms of* $\mathbf{x}$*. The distribution* $\inf(\nu_{p_\mathbf{A}})$ *follows a geometric law of parameter* $p_\mathbf{A}^{\lambda(\Delta)}$*, and all the moments of* $\inf(\mu_k)$ *converge towards the associated moments of* $\inf(\nu_{p_\mathbf{A}})$*, when* $k \to +\infty$*.*

Proof. The function $\inf : \overline{\mathbf{A}}^+ \mapsto \mathbb{Z}$ is well-defined and continuous almost everywhere for the measure $\nu_{p_\mathbf{A}}$: the only point where it is not well-defined is Δ^ω. Hence, Theorem 7.57 already proves the first part of Corollary 7.60. It remains to prove that $\mathbb{E}_{\mu_k}[\inf^\ell] \to \mathbb{E}_{\nu_{p_\mathbf{A}}}[\inf^\ell]$ for all integers $\ell \geqslant 0$. Note that this result is not so immediate, since inf is not bounded on $\overline{\mathbf{A}}^+$.

Hence, we mimic the proof of Proposition 7.55. First, by using again the bijection $\mathbf{I} : \mathbf{x} \mapsto (\inf(\mathbf{x}), \Delta^{-\inf(\mathbf{x})}\mathbf{x})$ that we have introduced in this proof, we obtain the equalities

$$|\mathbf{A}^+(k)| = \sum_{i=0}^{\lfloor k/\lambda(\Delta) \rfloor} |\mathbf{B}^+(k-\lambda(\Delta)i)| \text{ and } |\mathbf{A}^+(k)|\mathbb{E}_{\mu_k}[\inf^\ell] = \sum_{i=0}^{\lfloor k/\lambda(\Delta) \rfloor} i^\ell |\mathbf{B}^+(k-\lambda(\Delta)i)|.$$

Moreover, we also proved that there exists a real number $\mathbf{R} > 0$ such that $|\mathbf{B}^+(k)| \sim \mathbf{R} p_\mathbf{A}^{-k}$. Hence, Corollary 7.54 proves that

$$|\mathbf{A}^+(k)| \sim \mathbf{R} p_\mathbf{A}^{-k} f_0(p_\mathbf{A}^{\lambda(\Delta)}) \text{ and } |\mathbf{A}^+(k)|\mathbb{E}_{\mu_k}[\inf^\ell] \sim \mathbf{R} p_\mathbf{A}^{-k} f_\ell(p_\mathbf{A}^{\lambda(\Delta)}),$$

which implies that $\mathbb{E}_{\mu_k}[\inf] \to \frac{f_\ell(p_\mathbf{A}^{\lambda(\Delta)})}{f_0(p_\mathbf{A}^{\lambda(\Delta)})} = \mathbb{E}_{\nu_{p_\mathbf{A}}}[\inf^\ell]$. □

We continue by focusing on the *stable region conjecture* [55, Conjecture 3.1], which its authors formulate on the basis of a thorough experimental analysis.

Conjecture 7.61.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of spherical type with at least 2 generators For all braids $\mathbf{x} \in \mathbf{A}^+$, let $\lambda_i(\mathbf{x})$ denote the i-th leftmost letter of the extended Garside normal form $\mathbf{NF}_\ell^\omega(\mathbf{x})$ that occurs after the rightmost letter Δ, i.e. $\lambda_i(\mathbf{x}) := \Theta_{\inf(\mathbf{x})+i}(\mathbf{x})$. It is conjectured that:

1. *for each integer $i \geqslant 1$, the sequence $(\lambda_i(\mu_k))_{k \geqslant 1}$ is convergent when $k \to \infty$.*
2. *there exists a probability measure λ_∞ on $\mathcal{S}^\circ$ and a constant K such that, for all $i \geqslant K$, $\lambda_i(\mu_k) \to \lambda_\infty$ when $k \to +\infty$.*

Proposition 7.62.
The first statement of Conjecture 7.61 holds, and the second statement of Conjecture 7.61 holds if and only if $\mathbf{A}^+$ is a dihedral monoid, i.e. if $\mathbf{W} = I_2(a)$ for some $a \geqslant 3$.

In general, the following weaker statement holds: the distribution of $\lim_{k \to +\infty} \lambda_i(\mu_k)$ converges when $i \to +\infty$.

Proof. We first prove the first part of Conjecture 7.61. Let Δ^ω be the largest element of $\overline{\mathbf{A}}^+$, i.e. $\Delta^\omega := (\Delta_k)_{k\geqslant 0}$. For each integer $i \geqslant 1$, we extend λ_i by continuity to a function $\lambda_i : \overline{\mathbf{A}}^+\backslash\{\Delta^\omega\} \mapsto \mathcal{S}$. Since $\{\Delta^\omega\}$ has measure 0 for all the distributions μ_k as well as for $\nu_{p_\mathbf{A}}$, and due to Theorem 7.57, the sequence $(\lambda_i(\mu_k))_{k\geqslant 1}$ converges towards $\lambda_i(\nu_{p_\mathbf{A}})$.

Now, let us focus on the second part of Conjecture 7.61. According to Theorem 7.48, the Markov process $(\lambda_i(\nu_{p_\mathbf{A}}))_{i\geqslant 1}$ follows the same law as the Markov process $(\Theta_i(\nu_{p_\mathbf{A}}))_{i\geqslant 1})$ *conditioned* to satisfy the relation $\Theta_1 \neq \Delta$. Hence, consider the valuation $r : \mathbf{x} \mapsto p_\mathbf{A}^{\lambda(\mathbf{x})}$, and let P be the Markov Garside matrix of r, i.e. the transition matrix of $(\Theta_i(\nu_{p_\mathbf{A}}))_{i\geqslant 1})$.

In addition, let $\overline{P}$ be the restriction of P to indices in $\mathcal{S}^\circ = \mathcal{S}\backslash\{\mathbf{1}, \Delta\}$. It follows from Proposition 5.21 or, alternatively, from Theorem 7.39, that $\overline{P}$ is primitive. Moreover, Theorem 7.48 proves that our latter *conditioned* Markov process has transition matrix $\overline{P}$ and initial distribution $\iota : \mathbf{x} \mapsto \frac{(M_\gamma r)(\mathbf{x})}{1-(M_\gamma r)(\Delta)}$ for all $\mathbf{x} \in \mathcal{S}^\circ$.

Therefore, the second part of Conjecture 7.61 amounts to saying that $\mathbf{d}\cdot\overline{P}^k = \mathbf{d}\cdot\overline{P}^{k+1}$ for some integer $k \geqslant 0$, where $\mathbf{d}$ is the vector defined by $\mathbf{x_x} := \iota(\mathbf{x})$. This would mean that $\mathbf{d}$ is an eigenvector of $\overline{P}^{k+1}$ and, since $\mathbf{d}$ has non-negative entries and $\overline{P}$ is primitive, that $\mathbf{d}$ is a left Perron eigenvector of $\overline{P}$, i.e. that $\mathbf{d} = \mathbf{d}\cdot\overline{P}$.

Denoting by $\mathbf{K}$ the positive constant $\frac{1}{1-(M_\gamma r)(\Delta)}$, we compute, for all $\mathbf{y} \in \mathcal{S}^\circ$, that

$$(\mathbf{d}\cdot\overline{P})_\mathbf{y} = K \sum_{\mathbf{x}\in\mathcal{S}^\circ} \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} p_\mathbf{A}^{\lambda(\mathbf{x})} (\mathbf{M}_\gamma r)(\mathbf{y}) = \sum_{\mathbf{x}\in\mathcal{S}^\circ} \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} p_\mathbf{A}^{\lambda(\mathbf{x})} \mathbf{x}_y.$$

Since $\mathbf{d_y} > 0$, it follows that $\mathbf{d}\cdot\overline{P}^k = \mathbf{d}\cdot\overline{P}^{k+1}$ if and only if the equality $\sum_{\mathbf{x}\in\mathcal{S}^\circ} \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} p_\mathbf{A}^{\lambda(\mathbf{x})}$ holds for all $\mathbf{y} \in \mathcal{S}^\circ$.

If $\mathbf{A}^+$ is not a dihedral monoid, i.e. if it is generated by a family $\{\sigma_1, \ldots, \sigma_n\}$ such that $n \geqslant 3$, then the set $\{\mathbf{x} \in \mathcal{S}^\circ : \mathbf{x} \longrightarrow \Delta_{\{\sigma_1,\sigma_2\}}\}$ is a strict subset of $\{\mathbf{x} \in \mathcal{S}^\circ : \mathbf{x} \longrightarrow \sigma_1\}$, and therefore the above equality cannot hold simultaneously for both elements σ_1 and $\Delta_{\{\sigma_1,\sigma_2\}}$ of $\mathcal{S}^\circ$. However, if $\mathbf{A}^+$ is a dihedral monoid, with $\mathbf{W} = I_2(a)$, then we check easily that

$$\sum_{\mathbf{x}\in\mathcal{S}^\circ} \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} p_\mathbf{A}^{\lambda(\mathbf{x})} = \sum_{i=1}^{a-1} p_\mathbf{A}^i = \frac{p_\mathbf{A} - p_\mathbf{A}^a}{1 - p_\mathbf{A}} = 1 + \frac{\mathcal{H}_\mathbf{A}(p_\mathbf{A})}{1 - p_\mathbf{A}} = 1$$

for all braids $\mathbf{y} \in \mathcal{S}^\circ$, which proves that the second part of Conjecture 7.61 holds if and only if $\mathbf{A}^+$ is a dihedral monoid.

Finally, observe that, in all cases, and when $i \to +\infty$, the distribution of the *conditioned* random process $(\Theta_i(\nu_{p_\mathbf{A}}))_{i\geqslant 1})$ converges towards the stationary measure $\pi_{\overline{P}}$ of $\overline{P}$. This completes the proof of Proposition 7.62. □

However, finer convergence results, such as variants of the central limit theorem, are not provided by Theorem 7.57. They will be the focal point of Section 7.2.

7.2 Asymptotics and Conditioned Weighted Graphs

7.2.1 General Framework

Definition 7.63 (Conditioned weighted graph).
Let M be a primitive matrix of size $n \times n$, where $n \geqslant 2$. We may interpret the matrix M as an oriented, labelled graph, with vertices $\{1, \ldots, n\}$ and arcs (i, j) labelled by $M_{i,j}$ whenever $M_{i,j} > 0$.

Let $\mathbf{w}^{\mathbf{in}}$ and $\mathbf{w}^{\mathbf{out}}$ be two non-zero vectors of length n. We call them respectively input weight *and* output weight*. We say that the triple $\mathcal{G} := (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ is a* conditioned weighted graph*.*

In addition, we define the transpose *of $\mathcal{G}$ as the triple $\mathcal{G}^\top := (M^\top, \mathbf{w}^{\mathbf{out}}, \mathbf{w}^{\mathbf{in}})$, where $M^\top$ is the transpose of the matrix M.*

Definition 7.64 (Weight of a path).
Let $\mathcal{G} := (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be a conditioned weighted graph. A path *in $\mathcal{G}$ is a non-empty, finite sequence $\underline{\mathbf{p}} := p_1 \cdot \ldots \cdot p_k$, whose terms belong to $\{1, \ldots, n\}$. We denote by $\mathcal{P}_\mathcal{G}$ the set of paths in $\mathcal{G}$, by $|\underline{\mathbf{p}}|$ the length of the path $\underline{\mathbf{p}}$ (here, we have $|\underline{\mathbf{p}}| = k$), and by $\mathcal{P}_\mathcal{G}(k)$ the set of paths of length k in $\mathcal{G}$.*

The weight *of $\underline{\mathbf{p}}$ is defined as*

$$w(\underline{\mathbf{p}}) := \mathbf{w}^{\mathbf{in}}_{p_1} M_{p_1,p_2} \ldots M_{p_{k-1},p_k} \mathbf{w}^{\mathbf{out}}_{p_k}.$$

Moreover, since M is primitive and since $\mathbf{w}^{\mathbf{in}}$ and $\mathbf{w}^{\mathbf{out}}$ are non-zero, the set $\{|\underline{\mathbf{p}}| : \underline{\mathbf{p}}$ is a path such that $w(\underline{\mathbf{p}}) > 0\}$ is cofinite, which justifies the following definition.

Definition 7.65 (Uniform distribution on paths).
Let $\mathcal{G} = (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be a conditioned weighted graph. For each integer $k \geqslant 1$, let $\mathcal{P}_\mathcal{G}(k)$ denote the set of paths of length k in $\mathcal{G}$, i.e. $\mathcal{P}_\mathcal{G}(k) := \{1, \ldots, n\}^k$.

Let $w(\mathcal{P}_\mathcal{G}(k)) := \sum_{\underline{\mathbf{q}} \in \mathcal{P}_\mathcal{G}(k)} w(\underline{\mathbf{q}})$ be the cumulated weight of all paths of length k. If $w(\mathcal{P}_\mathcal{G}(k)) > 0$, which happens if k is large enough, then we define the uniform distribution *on $\mathcal{P}_\mathcal{G}(k)$ as the probability distribution μ_k on $\mathcal{P}_\mathcal{G}(k)$ such that: $\mu_k : \underline{\mathbf{p}} \mapsto \frac{w(\underline{\mathbf{p}})}{w(\mathcal{P}_\mathcal{G}(k))}$.*

Like Artin–Tits monoids of FC type, families of paths of increasing lengths give rise to a notion of projective limit.

Definition & Proposition 7.66 (Extended paths).
Let $\mathcal{G} = (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be a conditioned weighted graph. Consider the sets Γ_k, for $k \geqslant 0$, and maps $\xi_{k,\ell} : \Gamma_\ell \mapsto \Gamma_k$, for $\ell \geqslant k \geqslant 0$, defined by:

$$\Gamma_k = \bigcup_{i=1}^{k} \mathcal{P}_\mathcal{G}(i) \text{ and } \xi_{k,\ell} : p_1 \cdot \ldots \cdot p_i \mapsto p_1 \cdot \ldots \cdot p_{\min\{k,i\}}.$$

We have $\xi_{k,k} = \mathbf{Id}_{\Gamma_k}$ *and* $\xi_{k,\ell} \circ \xi_{\ell,m} = \xi_{k,m}$ *for all* $k \leqslant \ell \leqslant m$*. Hence, the families* $(\Gamma_k)_{k \geqslant 1}$ *and* $(\xi_{k,\ell})_{\ell \geqslant k \geqslant 0}$ *form a projective system.*

We call extended paths *the elements of the projective limit*

$$\overline{\mathcal{P}}_{\mathcal{G}} := \left\{ (\underline{\mathbf{p}}^k)_{k \geqslant 1} \in \prod_{k \geqslant 1} \Gamma_k : \forall k, \ell \geqslant 0, \ell \geqslant k \Rightarrow \xi_{k,\ell}(\underline{\mathbf{p}}^\ell) = \underline{\mathbf{p}}^k \right\}.$$

We equip each (finite) set Γ_k *with the discrete topology, from which the projective limit* $\overline{\mathcal{P}}_{\mathcal{G}}$ *inherits a* projective topology.

Note that, as mentioned in the proof of Proposition 7.24, the set $\overline{\mathcal{P}}_{\mathcal{G}}$ is metric and compact for the projective topology (see [17]). However, in general, the collection of probability measures $(\mu_k)_{k \geqslant 1}$ attached to the sets Γ_k do not satisfy consistency relations $\mu_k = \xi_{k,\ell}(\mu_\ell)$, and therefore do not form a *projective system of probability measures*, as shown below.

Example 7.67.
Let $\mathcal{G} = (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be the conditioned weighted graph defined by

$$M := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \mathbf{w}^{\mathbf{in}} = \mathbf{w}^{\mathbf{out}} = \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

Straightforward computations show that $w(\mathcal{P}_{\mathcal{G}}(1)) = 5$ and $w(\mathcal{P}_{\mathcal{G}}(2)) = 9$, whence

$$\mu_1(\{1\}) = \frac{1}{5} \neq \frac{1}{3} = \mu_2(\{1 \cdot 1, 1 \cdot 2\}) = \mu_2(\xi_{1,2}^{-1}(\{1\})).$$

However, in this case, weak limits of probability measures are still an adequate substitute to projective limits of probability measures. Indeed, each set $\mathcal{P}_{\mathcal{G}}(k)$ is naturally embedded into Γ_k and $\overline{\mathcal{P}}_{\mathcal{G}}$. Hence, we identify the distribution μ_k with a discrete probability measure on the set $\overline{\mathcal{P}}_{\mathcal{G}}$ equipped with its Borel σ-algebra.

Mimicking Section 7.1, we consider the mappings $\xi_{k,\infty} : \overline{\mathcal{P}}_{\mathcal{G}} \mapsto \Gamma_k$ and $\Xi_k : \overline{\mathcal{P}}_{\mathcal{G}} \mapsto \{1, \ldots, n\}$ defined by:

$$\xi_{k,\infty} : (\underline{\mathbf{p}}^i)_{i \geqslant 0} \mapsto \underline{\mathbf{p}}^k,\ \Xi_1 = \xi_{1,\infty} \text{ and } \xi_{k,\infty}(\mathbf{x}) \cdot \Xi_{k+1}(\mathbf{x}) := \xi_{k+1,\infty}(\mathbf{x}) \text{ for all } \mathbf{x} \in \overline{\mathcal{P}}_{\mathcal{G}}.$$

In other words, Ξ_k maps the extended path $\mathbf{x}$ to its $(k+1)$-th leftmost letter, or to the empty word if $|\mathbf{x}| < k$.

Theorem 7.68.
Let $\mathcal{G} = (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ *be a conditioned weighted graph. Let* ρ *be the Perron eigenvalue of* M*, and let* $\mathbf{l}$ *and* $\mathbf{r}$ *be Perron eigenvectors, with positive entries, and such that* $\mathbf{l} \cdot \mathbf{r} = 1$*.*

The sequence $(\mu_k)_{k \geqslant 1}$ *converges weakly towards a probability measure* μ_∞ *on* $\overline{\mathcal{P}}_{\mathcal{G}}$*, which is concentrated on the set of* infinite paths $\partial \mathcal{P}_{\mathcal{G}} := \overline{\mathcal{P}}_{\mathcal{G}} \backslash \mathcal{P}_{\mathcal{G}}$*. In addition, the random process*

$(\Xi_k(\mu_\infty))_{k\geqslant 1}$ *is an ergodic Markov chain, whose initial distribution* ι *and transition matrix* P *are respectively defined by:*

$$\iota : x \mapsto \frac{\mathbf{w}^{\mathbf{in}}_x \mathbf{r}_x}{\mathbf{w}^{\mathbf{in}} \cdot \mathbf{r}} \text{ and } P_{x,y} := \frac{M_{x,y}\mathbf{r}_y}{\rho \mathbf{r}_x}.$$

The stationary measure of the Markov chain $(\Xi_k(\mu_\infty))_{k\geqslant 1}$ *is given by* $x \mapsto \mathbf{l}_x \mathbf{r}_x$.

Proof. Let κ be an integer such that, for all $k \geqslant \kappa$, the matrix M^k is positive, and therefore the measure μ_k is well-defined. The projective topology on $\overline{\mathcal{P}}_{\mathcal{G}}$ is generated by the (simultaneously closed and) open sets $\Uparrow \underline{\mathbf{x}} := \{\underline{\mathbf{y}} \in \overline{\mathcal{P}}_{\mathcal{G}} : \xi_{|\underline{\mathbf{x}}|,\infty}(\underline{\mathbf{y}}) = \underline{\mathbf{x}}\}$, for all paths $\underline{\mathbf{x}} \in \mathcal{P}_{\mathcal{G}}$.

Hence, let $\underline{\mathbf{x}} = x_1 \cdot \ldots \cdot x_j$ be a path of length j. Let δ be the (column) vector defined by $\delta_z := \mathbf{1}_{z=x_j}$, and let us consider the real number $\tilde{w}(\underline{\mathbf{x}}) := \mathbf{w}^{\mathbf{in}}_{x_1} M_{x_1,x_1} \ldots M_{x_{j-1},x_j}$. Note that both $\mathbf{w}^{\mathbf{in}} \cdot \mathbf{r}$ and $\mathbf{l} \cdot \mathbf{w}^{\mathbf{out}}$ are positive real numbers.

Therefore, Theorem 7.37 proves that, whenever $k \geqslant \max\{\kappa, j+1\}$, we have:

$$\begin{aligned} \mu_k(\Uparrow \underline{\mathbf{x}}) &= \frac{1}{w(\mathcal{P}_{\mathcal{G}}(k))} \sum_{x_{j+1},\ldots,x_k} \mathbf{w}^{\mathbf{in}}_{x_1} \prod_{i=1}^{k-1} M_{x_i,x_{i+1}} \mathbf{w}^{\mathbf{out}}_{x_k} = \tilde{w}(\underline{\mathbf{x}}) \frac{\delta \cdot M^{k-j-1} \cdot \mathbf{w}^{\mathbf{out}}}{\mathbf{w}^{\mathbf{in}} \cdot M^{k-1} \cdot \mathbf{w}^{\mathbf{out}}} \\ &\sim \tilde{w}(\underline{\mathbf{x}}) \frac{\rho^{k-j-1}(\delta \cdot \mathbf{r})(\mathbf{l} \cdot \mathbf{w}^{\mathbf{out}})}{\rho^{k-1}(\mathbf{w}^{\mathbf{in}} \cdot \mathbf{r})(\mathbf{l} \cdot \mathbf{w}^{\mathbf{out}})} = \tilde{w}(\underline{\mathbf{x}}) \frac{\rho^{-j}\, \mathbf{r}_{x_j}}{\mathbf{w}^{\mathbf{in}} \cdot \mathbf{r}}. \end{aligned}$$

Hence, since $\overline{\mathcal{P}}_{\mathcal{G}}$ is a metric, compact set, let μ_∞ be a probability measure towards which some subsequence of $(\mu_k)_{k\geqslant 1}$ converges weakly. We necessarily have $\mu_\infty(\Uparrow \underline{\mathbf{x}}) = \frac{\tilde{w}(\underline{\mathbf{x}})\rho^{-j}\mathbf{r}_{x_j}}{\mathbf{w}^{\mathbf{in}} \cdot \mathbf{r}}$. Furthermore, along with the empty set, the family $\Uparrow \underline{\mathbf{x}}$, for $\underline{\mathbf{x}} \in \mathcal{P}_{\mathcal{G}}$, forms a π-system, and therefore two finite measures on $\overline{\mathcal{P}}_{\mathcal{G}}$ coincide if and only if they coincide on those sets. This proves that the sequence $(\mu_k)_{k\geqslant 1}$ itself converges weakly towards μ_∞.

Moreover, since $\mu_k(\{\underline{\mathbf{x}}\}) = 0$ whenever $k > |\underline{\mathbf{x}}|$, it follows that $\mu_k(\{\underline{\mathbf{x}}\}) \to 0$ when $k \to +\infty$, and therefore that $\mu_\infty(\{\underline{\mathbf{x}}\}) = 0$. This equality holds for all finite paths, which shows that $\mu_\infty(\mathcal{P}_{\mathcal{G}}) = 0$, i.e. that μ_∞ is concentrated on $\partial\mathcal{P}_{\mathcal{G}}$.

In addition, observe that

$$\begin{aligned} \mathbb{P}_{\mu_\infty}(\Xi_1 = x_1, \ldots, \Xi_j = x_j) &= \mu_\infty(\Uparrow \underline{\mathbf{x}}) = \tilde{w}(\underline{\mathbf{x}}) \frac{\rho^{-j}\, \mathbf{r}_{x_j}}{\mathbf{w}^{\mathbf{in}} \cdot \mathbf{r}} \\ &= \frac{\mathbf{w}^{\mathbf{in}}_{x_1} \mathbf{r}_{x_1}}{\mathbf{w}^{\mathbf{in}} \cdot \mathbf{r}} \prod_{i=1}^{j-1} \frac{M_{x_i,x_{i+1}} \mathbf{r}_{x_{i+1}}}{\rho\, \mathbf{r}_{x_i}} = \iota(x_1) \prod_{i=1}^{j-1} P_{x_i,x_{i+1}}, \end{aligned}$$

which proves that $(\Xi_k(\mu_\infty))_{k\geqslant 1}$ is indeed a Markov chain with initial distribution ι and transition matrix P.

Finally, since M is primitive, the matrix P is the irreducible transition matrix of a Markov chain. Hence, the Markov chain $(\Xi_k(\mu_\infty))_{k\geqslant 1}$ is ergodic, and converges towards the (unique) stationary measure of P. Furthermore, let π be the (row) vector such that $\pi_x := \mathbf{l}_x \mathbf{r}_x$. Observe that $\sum_x \pi_x = \mathbf{l} \cdot \mathbf{r} = 1$ and that, for all $y \in \{1, \ldots, n\}$, we have

$$(\pi \cdot P)_y := \sum_x \pi_x P_{x,y} = \rho^{-1} \sum_x \mathbf{l}_x M_{x,y} \mathbf{r}_y = \rho^{-1} (\mathbf{l} \cdot M)_y \mathbf{r}_y = \mathbf{l}_y \mathbf{r}_y = \pi_y.$$

This proves that π is invariant under P, i.e. that $y \mapsto \pi_y$ is indeed the stationary measure of P, and completes the proof. □

Definition 7.69 (Uniform measure at infinity)**.**
Let $\mathcal{G} = (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be a conditioned weighted graph. We call uniform measure at infinity *of $\mathcal{G}$ the weak limit μ_∞ of the sequence $(\mu_k)_{k \geqslant 1}$.*

The transformation applied on M in order to get the matrix P was first introduced by Parry [69, 72, 79] in its construction of a stationary Markov chain reaching the maximum entropy. It also has the same form as the transition matrix of the *survival process* of a discrete time, absorbing Markov chain [30, 33] with finitely many states.

Definition 7.70 (Survival process)**.**
Let $P = (P_{x,y})_{0 \leqslant x,y \leqslant n}$ be a stochastic matrix, i.e. a matrix such that $P_{x,y} \geqslant 0$ and $\sum_z P_{x,z} = 1$ for all $x, y \in \{0, \ldots, n\}$, whose restriction to entries in $\{1, \ldots, n\}$ is primitive, and such that $P_{0,0} = 1$. We say that 0 is an absorbing state *of P.*

Let $(Y_i)_{i \geqslant 1}$ be the Markov chain with initial distribution $x \mapsto \mathbf{1}_{x=1}$ and transition matrix P. We say that $(Y_i)_{i \geqslant 1}$ is a finite absorbing Markov chain*, which* survives after k steps *if $Y_k \neq 0$. The* survival process *of $(Y_i)_{i \geqslant 1}$, if it exists, is the process $(X_i)_{i \geqslant 1}$ such that*

$$\mathbb{P}[X_1 = x_1, \ldots, X_j = x_j] = \lim_{k \to +\infty} \mathbb{P}[Y_1 = x_1, \ldots, Y_j = x_j \mid Y_k \neq 0].$$

Proposition 7.71.
Let $(Y_i)_{i \geqslant 1}$ be a finite absorbing Markov chain with transition matrix P, and let M be the restriction of P to the states $\{1, \ldots, n\}$. In addition, let $\mathcal{G} = (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be the conditioned weighted graph defined by $\mathbf{w}^{\mathbf{in}}_x := \mathbf{1}_{x=1}$ and $\mathbf{w}^{\mathbf{out}}_x := 1$. For all $k \geqslant 1$, let μ_k be the uniform measure on $\mathcal{P}_{\mathcal{G}}(k)$, if such a measure exists, and let μ_∞ be the uniform measure at infinity of $\mathcal{G}$.

For all paths $x_1 \cdot \ldots \cdot x_j$, we have

$$\begin{aligned} \mathbb{P}[Y_1 = x_1, \ldots, Y_j = x_j \mid Y_k \neq 0] &= \mathbb{P}_{\mu_k}[\Theta_1 = x_1, \ldots, \Theta_j = x_j]; \\ \lim_{k \to +\infty} \mathbb{P}[Y_1 = x_1, \ldots, Y_j = x_j \mid Y_k \neq 0] &= \mathbb{P}_{\mu_\infty}[\Theta_1 = x_1, \ldots, \Theta_j = x_j]. \end{aligned}$$

In addition, the lack of symmetry between the initial and the final weights in the statement of Theorem 7.68 might seem surprising at first. This lack of symmetry comes from the fact that we considered weak limits obtained when focusing on the *first* elements of paths. Hence, analogous results hold when focusing on the *last* elements of paths.

Corollary 7.72.
Let $\mathcal{G} = (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be a conditioned weighted graph. Let ρ be the Perron eigenvalue of M, and let $\mathbf{l}$ and $\mathbf{r}$ be Perron eigenvectors, with positive entries, and such that $\mathbf{l} \cdot \mathbf{r} = 1$.

For all integers $k \geqslant j \geqslant 0$, let consider the mapping $\tilde{\Xi}_{k,j} = \Xi_{k-j}$. For all finite paths, $x_1 \cdot \ldots \cdot x_\ell$, we have

$$\mathbb{P}_{\mu_k}[\tilde{\Xi}_{k,1} = x_1, \ldots, \tilde{\Xi}_{k,\ell} = x_\ell] \mapsto \mathbb{P}[\tilde{X}_1 = x_1, \ldots, \tilde{X}_\ell = x_\ell],$$

where $(\tilde{X}_i)_{i \geqslant 1}$ is the Markov chain associated to the uniform measure at infinity of the transpose $\mathcal{G}^\top$, i.e. the ergodic Markov chain, whose initial distribution $\tilde{\imath}$ and transition matrix $\tilde{P}$ are respectively defined by:

$$\tilde{\imath} : x \mapsto \frac{\mathbf{w}^{\mathbf{out}}_x \mathbf{l}_x}{\mathbf{w}^{\mathbf{out}} \cdot \mathbf{l}} \text{ and } \tilde{P}_{x,y} := \frac{M_{y,x}\mathbf{l}_y}{\rho \mathbf{l}_x}.$$

The stationary measure of the Markov chain $(\tilde{X}_i)_{i \geqslant 1}$ is given by $x \mapsto \mathbf{l}_x \mathbf{r}_x$.

7.2.2 Concentration Theorems and Generalisations

We focus now on answering the questions raised at the end of Section 7.1.4.

Definition 7.73 (Ergodic mean)**.**
Let $\mathcal{G} := (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be a conditioned weighted graph, whose matrix M is of size $n \times n$. Consider a function $f : \{1, \ldots, n\} \mapsto \mathbb{C}$, which we call cost function*. For all integers $k \geqslant 0$, the* ergodic mean *$\langle f \rangle$ of f along a path $\underline{\mathbf{x}} = x_1 \cdot \ldots \cdot x_k$ of length k is defined by:*

$$\langle f \rangle : \underline{\mathbf{x}} \mapsto \frac{1}{k} \sum_{i=1}^{k} f(x_i).$$

Hence, the ergodic mean $\langle f \rangle$ is a random variable defined on the set $\mathcal{P}_\mathcal{G}$, and we may look for convergences in law of the distributions $\langle f \rangle(\mu_k)$. In order to do so, we will need to use a result from perturbation theory [61, 66].

Theorem 7.74.
Let P be a primitive matrix, with Perron eigenvalue ρ and Perron eigenvectors $\mathbf{l}$ and $\mathbf{r}$, such that $\mathbf{l} \cdot \mathbf{r} = 1$. In addition, let $\| \cdot \|$ denote the L^2 norm on vectors, and let $u \mapsto P(u)$ be an analytic perturbation of P, such that $P = P(0)$.

There exists real numbers $\varepsilon > 0$ and $\lambda \in (0, 1)$, as well as analytic perturbations $u \mapsto \rho(u)$, $u \mapsto \mathbf{l}(u)$ and $u \mapsto \mathbf{r}(u)$ of $\rho = \rho(0)$, $\mathbf{l} = \mathbf{l}(0)$ and $\mathbf{r} = \mathbf{r}(0)$ such that, whenever $|u| \leqslant \varepsilon$:

- *$\rho(u)$ is a simple eigenvalue of $P(u)$, with associated eigenvectors $\mathbf{l}(u)$ and $\mathbf{r}(u)$;*
- *$\mathbf{l}(u) \cdot \mathbf{r}(u) = \mathbf{l} \cdot \mathbf{r}(u) = 1$;*
- *the matrix $E(u) := P(u) - \rho(u)\mathbf{r}(u) \cdot \mathbf{l}(u)$ is such that $\|E(u) \cdot \mathbf{x}\| \leqslant \lambda \rho(u) \|\mathbf{x}\|$ for all vectors $\mathbf{x}$.*

From Theorem 7.74 follows a first result about the convergence of $\langle f\rangle(\mu_k)$.

Theorem 7.75.
Let $\mathcal{G} := (M, \mathbf{w^{in}}, \mathbf{w^{out}})$ be a conditioned weighted graph, whose matrix M is of size $n \times n$, and let $f : \{1, \ldots, n\} \mapsto \mathbb{C}$ be a cost function. In addition, let $\mathbf{m}$ be the stationary measure associated with the uniform measure at infinity of $\mathcal{G}$, and let μ_k be the uniform measure on $\mathcal{P}_{\mathcal{G}}(k)$.

The sequence of ergodic means $(\langle f\rangle(\mu_k))_{k \geqslant 0}$ converges in law towards the Dirac measure at γ_f, where $\gamma_f := f(\mathbf{m}) = \sum_{j=1}^{n} \mathbf{m}(j) f(j)$.

Proof. Using characteristic functions, it is enough to prove that the convergence relations $\mathbb{E}_{\mu_k}[\exp(\mathbf{i}\ t\ \langle f\rangle)] \to \exp(\mathbf{i}\ t\ \gamma_f)$ hold for all real numbers t.

Consider the variable $u := \frac{t}{k}$. Let $\mathbf{l}$ and $\mathbf{r}$ be left and right Perron eigenvectors of M, such that $\mathbf{l} \cdot \mathbf{r} = 1$, and let ω be the Perron eigenvalue of M. In addition, let P be the transition matrix of the random process $(\Xi_k(\mu_\infty))_{k \geqslant 1}$, i.e. the matrix defined by $P_{x,y} := \frac{M_{x,y}\mathbf{r}_y}{\omega \mathbf{r}_x}$, and let $\mathbf{L}$ and $\mathbf{R}$ be left and right Perron eigenvectors of P. According to Theorem 7.68, we choose $\mathbf{L}_i := \mathbf{l}_i \mathbf{r}_i$ and $\mathbf{R}_i := 1$ for all $i \in \{1, \ldots, n\}$, which provides us with the equality $\mathbf{L} \cdot \mathbf{R} = 1$.

Then, for all u, consider a *modified* matrix $P(u)$ and *modified* input and output weights $\overline{\mathbf{w}}^{\mathbf{in}}$, $\overline{\mathbf{w}}^{\mathbf{in}}(u)$ and $\overline{\mathbf{w}}^{\mathbf{out}}$ defined by:

$$D := \mathbf{Diag}(f(j))_{1 \leqslant j \leqslant n},\ P(u) := P \cdot \exp(\mathbf{i}\ u\ D),$$
$$\overline{\mathbf{w}}^{\mathbf{in}}_{\mathbf{x}} := \mathbf{w}^{\mathbf{in}}_{\mathbf{x}} \mathbf{r}_{\mathbf{x}},\ \overline{\mathbf{w}}^{\mathbf{in}}(u) := \overline{\mathbf{w}}^{\mathbf{in}} \cdot \exp(\mathbf{i}\ u\ D) \text{ and } \mathbf{w}^{\mathbf{out}}_{\mathbf{x}} := \frac{\mathbf{w}^{\mathbf{out}}_{\mathbf{x}}}{\mathbf{r}_{\mathbf{x}}}.$$

Since $P(u)$ is an analytic perturbation of $P = P(0)$, let us apply Theorem 7.74, and reuse its notations: we denote by $\rho(u)$, $\mathbf{L}(u)$ and $\mathbf{R}(u)$ the Perron eigenvalue and the left and right Perron eigenvectors of $P(u)$. In particular, note that the Perron eigenvalue of P is $\rho := \rho(0) = 1$.

For all $u \in \mathbb{R}$ such that $|u| \leqslant \varepsilon$, we have

$$\mathbf{L}(u) \cdot E(u) = \mathbf{L}(u) \cdot P(u) - \rho(u)\mathbf{L}(u) \cdot \mathbf{R}(u) \cdot \mathbf{L}(u) = 0,$$

and therefore

$$\begin{aligned} P(u)^k &= (\rho(u)\mathbf{R}(u) \cdot \mathbf{L}(u) + E(u))^k = \rho(u)^k(\mathbf{R}(u) \cdot \mathbf{L}(u))^k + E(u)^k \\ &= \rho(u)^k \mathbf{R}(u) \cdot \mathbf{L}(u) + E(u)^k \end{aligned}$$

for all integers $k \geqslant 0$.

Moreover, note that the scalar products $\overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot \mathbf{R}(u)$ and $\mathbf{L}(u) \cdot \overline{\mathbf{w}}^{\mathbf{out}}$ converge respectively towards the positive real numbers $\overline{\mathbf{w}}^{\mathbf{in}} \cdot \mathbf{R}$ and $\mathbf{L} \cdot \overline{\mathbf{w}}^{\mathbf{out}}$ when $u \to 0$. Meanwhile, observe that $|\overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot E(u)^k \cdot \overline{\mathbf{w}}^{\mathbf{out}}| \leqslant \rho(u)^k \lambda^k \|\overline{\mathbf{w}}^{\mathbf{in}}(u)\| \, \|\overline{\mathbf{w}}^{\mathbf{out}}\|$. Therefore, if ε is small enough, and whenever $|u| \leqslant \varepsilon$, we can ensure that the inequalities

$$|\overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot \mathbf{R}(u)| \geqslant \frac{1}{2}|\overline{\mathbf{w}}^{\mathbf{in}} \cdot \mathbf{R}|, \; |\mathbf{L}(u) \cdot \overline{\mathbf{w}}^{\mathbf{out}}| \geqslant \frac{1}{2}|\mathbf{L} \cdot \overline{\mathbf{w}}^{\mathbf{out}}| \text{ and } \|\overline{\mathbf{w}}^{\mathbf{in}}(u)\| \leqslant 2\|\overline{\mathbf{w}}^{\mathbf{in}}\|$$

simultaneously hold. It follows that

$$\begin{aligned}
\overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot P(u)^k \cdot \overline{\mathbf{w}}^{\mathbf{out}} &= \rho(u)^k(\overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot \mathbf{R}(u))(\mathbf{L}(u) \cdot \mathbf{w}^{\mathbf{out}}) + \overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot E(u)^k \cdot \overline{\mathbf{w}}^{\mathbf{out}} \\
&\sim \rho(u)^k(\overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot \mathbf{R}(u))(\mathbf{L}(u) \cdot \overline{\mathbf{w}}^{\mathbf{out}})
\end{aligned}$$

uniformly in $u \in (-\varepsilon, \varepsilon)$, when $k \to +\infty$. Hence, we compute, for all integers $k \geqslant 1$, that

$$\begin{aligned}
\mathbb{E}_{\mu_k}(\exp(\mathbf{i}\ t\ \langle f \rangle)) &= \frac{\sum_{x_1,\ldots,x_k} \mathbf{w}^{\mathbf{in}}_{x_1} \exp(\mathbf{i}\ u\ f(x_1)) \prod_{j=1}^{k-1} M_{x_j,x_{j+1}} \exp(\mathbf{i}\ u\ f(x_{j+1}))\mathbf{w}^{\mathbf{out}}_{x_k}}{w(\mathcal{P}_{\mathcal{G}}(k))} \\
&= \frac{\omega^{k-1}\overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot P(u)^{k-1} \cdot \overline{\mathbf{w}}^{\mathbf{out}}}{\omega^{k-1}\overline{\mathbf{w}}^{\mathbf{in}} \cdot P^{k-1} \cdot \overline{\mathbf{w}}^{\mathbf{out}}} \\
&\sim \rho(u)^{k-1}\frac{(\overline{\mathbf{w}}^{\mathbf{in}}(u) \cdot \mathbf{R}(u))(\mathbf{L}(u) \cdot \overline{\mathbf{w}}^{\mathbf{out}})}{(\overline{\mathbf{w}}^{\mathbf{in}} \cdot \mathbf{R})(\mathbf{L} \cdot \overline{\mathbf{w}}^{\mathbf{out}})} \sim \rho(u)^{k-1}
\end{aligned}$$

when $k \to +\infty$. Moreover, since $u = \frac{t}{k}$, we have

$$\rho(u)^{k-1} = (\rho + u\rho'(0) + \mathcal{O}(k^{-2}))^{k-1} \to \exp(t\rho'(0))$$

when $k \to +\infty$. Hence, it remains to evaluate the derivative $\rho'(0)$.

We proceed by deriving the equality $P(u) \cdot \mathbf{R}(u) = \rho(u)\mathbf{R}(u)$, obtaining $P'(0) \cdot \mathbf{R} + P \cdot \mathbf{R}'(0) = \rho'(0)\mathbf{R} + \rho\mathbf{R}'(0)$. Then, multiplying both members to the left by $\mathbf{L}$, and since $\mathbf{L} \cdot P \cdot \mathbf{R}'(0) = \rho\mathbf{L} \cdot \mathbf{R}'(0)$, it follows that

$$\begin{aligned}
\rho'(0) &= \rho'(0)\mathbf{L} \cdot \mathbf{R} = \mathbf{L} \cdot P'(0) \cdot \mathbf{R} = \mathbf{i}\ \mathbf{L} \cdot P \cdot D \cdot \mathbf{R} \\
&= \mathbf{i}\ \mathbf{L} \cdot D \cdot \mathbf{R} = \mathbf{i}\sum_{j=1}^{n} f(j)\mathbf{L}_j\mathbf{R}_j = \mathbf{i}\gamma_f.
\end{aligned}$$

It follows that $\mathbb{E}_{\mu_k}(\exp(\mathbf{i}\ t\ \langle f \rangle)) \to \exp(\mathbf{i}\ t\ \gamma_f)$, which completes the proof. □

Pushing this analysis further, we obtain a Central limit theorem.

Theorem 7.76.
Let $\mathcal{G} := (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be a conditioned weighted graph, whose matrix M is primitive and of size $n \times n$, let $f : \{1, \ldots, n\} \mapsto \mathbb{C}$ be a cost function. In addition, let μ_k be the uniform measure on $\mathcal{P}_{\mathcal{G}}(k)$, and let γ_f be defined as in Theorem 7.75.

If $\gamma_f = 0$, then there exists a non-negative real σ^2 such that the sequence $(\sqrt{k}\langle f \rangle(\mu_k))_{k\geqslant 1}$ converges in law towards a normal law $\mathcal{N}(0, \sigma^2)$ when $k \to +\infty$. Moreover, we have $\sigma^2 = 0$ if and only if there exists a function $\overline{g} : \{1, \ldots, n\} \mapsto \mathbb{C}$ such that $f(j) = \overline{g}(i) - \overline{g}(j)$ for all pairs (i, j) such that $M_{i,j} > 0$.

Proof. Let us keep the notations used in the proof of Theorem 7.75, and let us consider the variable $v := \sqrt{k}u = \frac{t}{\sqrt{k}}$, and $\sigma^2 := -\frac{\rho''(0)}{2}$. Since $\rho'(0) = \mathbf{i}\gamma_f = 0$, it follows that

$$\mathbb{E}_{\mu_k}(\exp(\mathbf{i}\ t\ \sqrt{k}\ \langle f \rangle)) \sim \rho(v)^k = \left(1 + v^2 \frac{\rho''(0)}{2} + o(k^{-3/2})\right)^k \to \exp(-\sigma^2 t^2)$$

when $k \to +\infty$. Since $\underline{\mathbf{x}} \mapsto \exp(\mathbf{i}\ t\ \sqrt{k}\ \langle f \rangle(\underline{\mathbf{x}}))$ takes it values in $\{z \in \mathbb{C} : |z| = 1\}$, it follows that $|\mathbb{E}_{\mu_k}(\exp(\mathbf{i}\ t\ \sqrt{k}\ \langle f \rangle))| \leqslant 1$ for all k, hence that $|\exp(-\sigma^2 t^2)| \leqslant 1$. Since this holds for all $t \in \mathbb{R}$, it follows that $\sigma^2 \geqslant 0$.

Then, let Δ be the diagonal matrix such that $\Delta \cdot \mathbf{R} := D \cdot \mathbf{R} - \mathbf{i}\ \mathbf{R}'(0)$, and let $\overline{\Delta}$ be the diagonal matrix such that $\overline{\Delta} \cdot \mathbf{R} := P \cdot \Delta \cdot \mathbf{R}$.

Moreover, deriving the equality $P(v) \cdot \mathbf{R}(v) = \rho(v)\mathbf{R}(v)$, and recalling that $\rho = 1$ and $\rho'(0) = 0$, we obtain

$$\mathbf{R}'(0) = P'(0) \cdot \mathbf{R} + P \cdot \mathbf{R}'(0) = \mathbf{i}\ P \cdot D \cdot \mathbf{R} + P \cdot \mathbf{R}'(0) = \mathbf{i}\ P \cdot \Delta \cdot \mathbf{R} = \mathbf{i}\ \overline{\Delta} \cdot \mathbf{R}.$$

It follows that $\overline{\Delta} \cdot \mathbf{R} = -\mathbf{i}\ \mathbf{R}'(0) = (\Delta - D) \cdot \mathbf{R}$. Since D, Δ and $\overline{\Delta}$ are diagonal matrices, and since $\mathbf{R}$ has positive coefficients, it follows that $\overline{\Delta} = \Delta - D$.

Then, deriving twice the equality $P(v) \cdot \mathbf{R}(v) = \rho(v)\mathbf{R}(v)$, we obtain

$$\begin{aligned} \rho''(0)\mathbf{R} &= P''(0) \cdot \mathbf{R} + 2P'(0) \cdot \mathbf{R}'(0) + P \cdot \mathbf{R}''(0) - \mathbf{R}''(0) \\ &= -P \cdot D^2 \cdot \mathbf{R} + 2\mathbf{i}\ P \cdot D \cdot \mathbf{R}'(0) + P \cdot \mathbf{R}''(0) - \mathbf{R}''(0). \end{aligned}$$

Multiplying this equality to the left by the vector $\mathbf{L}$, we obtain

$$\begin{aligned} \rho''(0) &= -\mathbf{L} \cdot D^2 \cdot \mathbf{R} + 2\mathbf{i}\ \mathbf{L} \cdot D \cdot \mathbf{R}'(0) = -\mathbf{L} \cdot D^2 \cdot \mathbf{R} - 2\mathbf{L} \cdot D \cdot \overline{\Delta} \cdot \mathbf{R} \\ &= -\mathbf{L} \cdot D \cdot (D + 2\overline{\Delta}) \cdot \mathbf{R} = -\mathbf{L} \cdot (\Delta - \overline{\Delta}) \cdot (\Delta + \overline{\Delta}) \cdot \mathbf{R} = \mathbf{L} \cdot (\overline{\Delta}^2 - \Delta^2) \cdot \mathbf{R}. \end{aligned}$$

Since $\nu : i \mapsto \mathbf{L}_i$ is the invariant probability of P, let $(x_1, x_2, \ldots)$ be a Markov chain with initial distribution ν and transition matrix P. Moreover, consider the functions $g, \overline{g} : \{1, \ldots, n\} \mapsto \mathbb{C}$ defined by $g : i \mapsto \Delta_{i,i}$ and $\overline{g} : i \mapsto \overline{\Delta}_{i,i}$. It follows immediately that $\overline{g}(i) = \overline{\Delta}_{i,i} = \mathbb{E}[g(x_2) \mid x_1 = i]$ for all integers $i \in \{1, \ldots, n\}$.

By construction, for all $i \in \{1, \ldots, n\}$, we have $\mathbb{E}[g(x_2)^2 \mid x_1 = i] \geqslant \overline{g}(i)^2$, with equality if and only if $g(x_2) = \overline{g}(x_1)$ almost surely when $x_1 = i$. It follows that $\mathbb{E}_\nu[g(x_2)^2 - g(x_1)^2] \geqslant 0$, with equality if and only if $g(x_2) = \overline{g}(x_1)$ almost surely, Hence, we compute

$$\rho''(0) = \mathbf{L} \cdot (\overline{\Delta}^2 - \Delta^2) \cdot \mathbf{R} = \sum_{i=1}^{n} \mathbb{E}_\nu[\overline{g}(x_1)^2] - \mathbb{E}_\nu[g(x_2)^2] = \mathbb{E}_\nu[\overline{g}(x_1)^2 - g(x_2)^2].$$

Besides proving (again) that $\sigma^2 := -\frac{\rho''(0)}{2}$ must be non-negative, this equality also proves that $\sigma^2 = 0$ if and only if $g(x_2) = \overline{g}(x_1)$ almost surely. Using the equality $\Delta = D + \overline{\Delta}$, i.e. $g = f + \overline{g}$, this amounts to saying that $f(x_2) = \overline{g}(x_1) - \overline{g}(x_2)$ almost surely.

Conversely, if there exists a function $\overline{g} : \{1, \ldots, n\} \mapsto \mathbb{C}$ such that $f(j) = \overline{g}(i) - \overline{g}(j)$ for all pairs (i, j) such that $M_{i,j} > 0$, it follows immediately that $\langle f \rangle(\underline{\mathbf{p}}) = 0$ whenever $\underline{\mathbf{p}}$ is a cycle, and therefore that $\left||\underline{\mathbf{p}}|\langle f \rangle(\underline{\mathbf{p}})\right| \leqslant \|f\|_1 = \sum_{i=1}^{n} |f(i)|$ for all paths $\underline{\mathbf{p}}$. This proves that $\sqrt{k}\,|\langle f \rangle(\mu_k)| \leqslant \frac{\|f\|_1}{\sqrt{k}}$ converges in law towards a Dirac measure at 0, i.e. the normal law $\mathcal{N}(0, 0)$. □

In addition, similar results hold if we release slightly the assumption that M is primitive, as follows.

Definition 7.77 (Semi-primitive matrix)**.**
Let $M = (M_{x,y})_{1 \leqslant x,y \leqslant n}$ be a square matrix with non-negative coefficients, and let x and y be two indices of M. Considering the matrix M as a labelled oriented graph, we say that y is accessible *from x if there exists an integer $i > 0$ such that $M^i_{x,y} > 0$. We also say that x is* essential *if it is accessible from each index of M.*

Let us assume that there exists a partition S_0, S_1 of $\{1, \ldots, n\}$ and an integer $\kappa \geqslant 1$ such that:

- *S_0 is the set of essential indices of M, is non-empty, and the restriction $(M_{x,y})_{x,y \in S_0}$ of M to indices in S_0 is primitive; we denote its Perron eigenvalue by ρ;*
- *each eigenvalue of the restriction $(M_{x,y})_{x,y \in S_1}$ of M to indices in S_1 has a modulus less than ρ.*

We say that M is semi-primitive*, and ρ is called the* Perron eigenvalue *of M*

It is from the word *essential* that we derived the locutions "essential set" and "essential element" for denoting the set $\mathcal{E}$ and its elements.

Moreover, it comes immediately that the spectrum of M is the union of the spectra of the two submatrices $(M_{x,y})_{x,y \in S_0}$ and $(M_{x,y})_{x,y \in S_1}$, and that ρ is the (unique) eigenvalue of M with maximal modulus, with left and right eigenvectors $\mathbf{l}$ and $\mathbf{r}$ whose entries are positive on S_0 and zero on S_1. In this slightly broader framework, most of the above definitions and theorems remain unchanged. In particular, Theorems 7.75 and 7.76 can be rephrased as follows.

Theorem 7.78.
Let $\mathcal{G} := (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ be a conditioned weighted graph, whose matrix M is a semi-primitive matrix of size $n \times n$, and let $f : \{1, \ldots, n\} \mapsto \mathbb{C}$ be a cost function. In addition, let $\mathbf{m}$ be the (unique) stationary measure associated with the uniform measure at infinity of $\mathcal{G}$, and let μ_k be the uniform measure on $\mathcal{P}_{\mathcal{G}}(k)$. Finally, let us assume that there exists essential indices i_1 and i_2 of M such that $\mathbf{w}^{\mathbf{in}}_{i_1} > 0$ and $\mathbf{w}^{\mathbf{out}}_{i_2} > 0$.

The sequence of ergodic means $(\langle f \rangle(\mu_k))_{k \geqslant 0}$ converges in law towards the Dirac measure at γ_f, where $\gamma_f := f(\mathbf{m}) = \sum_{j=1}^{n} \mathbf{m}(j) f(j)$. In addition, there exists a non-negative real σ^2 such that the sequence $(\sqrt{k}(\langle f \rangle(\mu_k) - \gamma_f))_{k \geqslant 1}$ converges in law towards a normal law $\mathcal{N}(0, \sigma^2)$ when $k \to +\infty$. We have $\sigma^2 = 0$ if and only if there exists a function $\overline{g} : \{1, \ldots, n\}$ such that $f(j) = \overline{g}(j) - \overline{g}(i)$ for all pairs (i, j) of essential indices such that $M_{i,j} > 0$.

7.2.3 Asymptotics in Artin–Tits Monoids of FC Type

We proceed now to applying the above results to the framework of uniform distributions in Artin–Tits monoids of FC type. Henceforth, we will always consider an Artin–Tits monoid of FC type $\mathbf{A}^+$ and and a Möbius valuation $r : \mathbf{A}^+ \mapsto \mathbb{C}$. Hence, in this context, there exists one Bernoulli measure ν_r such that $\nu_r(\uparrow \mathbf{x}) = r(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{A}^+$ and $\nu_r(\mathbf{A}^+) = 0$.

In particular, we focus on the asymptotic behaviour of *Garside-additive functions.*

Definition 7.79 (Garside-additive function).
et $\mathbf{A}^+$ *be an Artin–Tits monoid of FC type. A function* $f : \mathbf{A}^+ \mapsto \mathbb{C}$ *is said to be* Garside-additive *if, for all left Garside words* $x_1 \cdot \ldots \cdot x_k$*, we have* $f(x_1) + \ldots + f(x_k) = f(x_1 \ldots x_k)$.

Garside-additive functions are generalisations of additive functions such as the Garside length, but also of non-additive functions such as the length of the Garside normal forms, i.e. $\mathbf{x} \mapsto \|\mathbf{x}\|$.

Aiming to apply the above results to the framework of Garside-additive functions, we first introduce meaningful combinatorial structures that will bridge the gap between conditioned weighted graphs and Artin–Tits monoids of FC type.

Lemma 7.80.
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of FC type with at least* 2 *generators and let* $r : \mathbf{A}^+ \mapsto \mathbb{C}$ *be a positive valuation. The expanded Garside matrix of parameter* r *is semi-primitive if* $\mathbf{A}^+$ *is an Artin–Tits monoid of spherical type, and is primitive otherwise.*

Proof. If $\mathbf{A}^+$ is not an Artin–Tits monoid of spherical type, then its expanded Garside matrix or parameter r is equal to is essential Garside matrix or parameter r, hence is primitive. Therefore, we focus on the case where $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type.

Let M be the expanded Garside matrix of parameter r. The non-negativity of M is immediate, and its essential states are the pairs $(i, \mathbf{x})$ such that $\mathbf{x} \in \mathcal{E}$. In particular, the restriction of M to essential states is if $\mathbf{A}^+$ is primitive, as shown in Lemma 7.41, and the Perron eigenvalue of this restriction is $p_{\mathbf{A}}^{-1}$, as mentioned in Proposition 7.55. The set of its non-essential states is $S_1 := \{(i, \Delta) : 1 \leqslant i \leqslant \lambda(\Delta)\}$, and the restriction of M to indices in S_1 is a permutation matrix, hence its eigenvalues have modulus $1 < p_{\mathbf{A}}^{-1}$, which completes the proof. □

Definition 7.81 (Expanded Garside graph).
Let $\mathbf{A}^+$ *be an irreducible Artin–Tits monoid of FC type with at least* 2 *generators and let* $r : \mathbf{A}^+ \mapsto \mathbb{C}$ *be a positive valuation. We call* expanded Garside graph *of parameter* r *the conditioned weighted graph* $\mathcal{G} = (M, \mathbf{w}^{\mathbf{in}}, \mathbf{w}^{\mathbf{out}})$ *such that* M *is the expanded Garside matrix of parameter* r*, and whose weights* $\mathbf{w}^{\mathbf{in}}$ *and* $\mathbf{w}^{\mathbf{out}}$ *are defined by* $\mathbf{w}^{\mathbf{in}}_{(i,\mathbf{x})} := \mathbf{1}_{i=1} r(\mathbf{x})$ *and* $\mathbf{w}^{\mathbf{out}}_{(i,\mathbf{x})} := \mathbf{1}_{i=\lambda(\mathbf{x})}$.

Paths of positive weight and of length k in $\mathcal{P}_{\mathcal{G}}$ are in bijection with elements of $\mathbf{A}^+(k)$, and each element $\mathbf{x}$ is associated to a unique word $\underline{\mathbf{x}}$ of positive weight, which satisfies $w(\underline{\mathbf{x}}) = r(\mathbf{x})$. Therefore, the uniform distribution μ_k on paths of length k corresponds to the uniform distribution on $\mathbf{A}^+(k)$, and there is a natural isomorphism between the projective limits $\overline{\mathbf{A}}^+$ and $\overline{\mathcal{P}}_{\mathcal{G}}$. Theorems 7.57 and 7.68 both provide us with weak limits fot the sequence of uniform distributions. Hence, these limits must coincide, i.e. the uniform measure at infinity for braids corresponds to the uniform measure at infinity of the expanded Garside graph.

However, the two families of elementary cylinders $\Uparrow \mathbf{x}$ (for $\mathbf{x} \in \mathbf{A}^+$) and $\Uparrow \underline{\mathbf{x}}$ (for $\underline{\mathbf{x}} \in \mathcal{P}_{\mathcal{G}}$) do not correspond to each other. Instead, the elementary cylinder $\Uparrow \underline{\mathbf{x}}$ corresponds to the open set $\mathfrak{B}_2(\|\mathbf{x}\|_\gamma, \mathbf{x})$ of $\overline{\mathbf{A}}^+$. Hence, Proposition 7.25 proves that $\mu_\infty(\Uparrow \mathbf{x}) = r(\mathbf{x})$ in $\overline{\mathbf{A}}^+$ and that $\mu_\infty(\Uparrow \underline{\mathbf{x}}) = (\mathbf{M}_\gamma r)(\mathbf{x})$ in $\overline{P}_{\mathcal{G}}$.

Lemma 7.82.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation. Let P be the Markov Garside matrix of r. The matrix P is semi-primitive if $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type, and P is primitive otherwise. In both cases, its essential indices are the elements of $\mathcal{E}$.

In addition, let M be the Garside matrix of parameter r, and let $\mathbf{g}$ and $\mathbf{h}$ be right and left Perron eigenvectors of M, such that $\mathbf{h} \cdot \mathbf{g} = 1$. The invariant probability of P is the distribution π defined by $\pi : \mathbf{x} \mapsto \mathbf{h}_\mathbf{x} \mathbf{g}_\mathbf{x}$ for all $\mathbf{x} \in \mathcal{E}$, and $\pi : \mathbf{x} \mapsto 0$ for all $\mathbf{x} \notin \mathcal{E}$.

Proof. The case where $\mathbf{A}^+$ is not an Artin–Tits monoid of spherical type is immediate. Hence, we focus on the case where $\mathbf{A}^+$ is an Artin–Tits monoid of spherical type. First, it comes immediately that the essential indices of P are the elements of $\mathcal{E}$, and that the restriction of P to indices in $\mathcal{E}$ is a primitive stochastic matrix. Hence, Δ is the only non-essential index of P, and since $P_{\Delta,\sigma_1} > 0$, it follows that the restriction of P to the index Δ is the 1×1 matrix $(P_{\Delta,\Delta})$, with only eigenvalue $P_{\Delta,\Delta} < 1$. This proves the first part of Lemma 7.82.

Let us now prove the second part, which has the flavour of Theorem 7.68. First, since $\mathbf{h} \cdot \mathbf{g} = 1$, it follows that π is a probability distribution. In addition, abusing notation and denoting by π the column vector such that $\pi_\mathbf{x} := \mathbf{h}_\mathbf{x} \mathbf{g}_\mathbf{x}$, and since Proposition 7.44 states that we can define $\mathbf{g}$ by the relations $\mathbf{g}_\mathbf{x} = g(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}(\mathbf{x})} (\mathbf{M}_\gamma r)(\mathbf{y}) = \frac{(\mathbf{M}_\gamma r)(\mathbf{x})}{r(\mathbf{x})}$, we compute that

$$
\begin{aligned}
(\pi \cdot P)_\mathbf{y} &= \sum_{\mathbf{x} \in \mathcal{E}} \mathbf{1}_{\mathbf{x} \longrightarrow \mathbf{y}} \mathbf{h}_x g(\mathbf{x}) r(\mathbf{x}) \frac{(\mathbf{M}_\gamma r)(\mathbf{y})}{(\mathbf{M}_\gamma r)(\mathbf{x})} = (\mathbf{M}_\gamma r)(\mathbf{y}) \sum_{\mathbf{x} \in \mathcal{E}} \mathbf{1}_{\mathbf{x} \longrightarrow \mathbf{y}} \mathbf{h}_x \\
&= \frac{(\mathbf{M}_\gamma r)(\mathbf{y})}{r(\mathbf{y})} (\mathbf{h} \cdot M)_\mathbf{y} = \mathbf{g}_\mathbf{y} \mathbf{h}_\mathbf{y} = \pi_\mathbf{y}
\end{aligned}
$$

for all $\mathbf{y} \in \mathcal{E}$. By construction, we have $(\pi \cdot P)_\mathbf{y} = 0 = \pi_\mathbf{y}$ for all $\mathbf{y} \notin \mathcal{E}$, which proves that π is the (unique) stationary measure of P and completes the proof. $\square$

Lemma 7.83.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators and let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a positive valuation. Let M be the Garside matrix of parameter r, and let N be the essential Garside matrix of parameter r. In addition, let $\mathbf{g} = (\mathbf{g_x})_{\mathbf{x}\in\mathcal{E}}$ and $\mathbf{h} = (\mathbf{h_x})_{\mathbf{x}\in\mathcal{E}}$ be right and left Perron eigenvectors of M.

The vectors $\overline{\mathbf{g}} = (\overline{\mathbf{g}}_{(i,\mathbf{x})})$ and $\overline{\mathbf{h}} = (\overline{\mathbf{h}}_{(i,\mathbf{x})})$, whose indices range over the set $\{(i,\mathbf{x}) : \mathbf{x} \in \mathcal{E}, 1 \leqslant i \leqslant \lambda(\mathbf{x})\}$, and defined by $\overline{\mathbf{g}}_{(i,\mathbf{x})} := \mathbf{g_x}$ and $\overline{\mathbf{h}}_{(i,\mathbf{x})} := \mathbf{g_x}$ are right and left Perron eigenvectors of N, whose Perron eigenvalue is 1.

Proof. By construction, both vectors $\overline{\mathbf{g}}$ and $\overline{\mathbf{h}}$ have positive entries. Moreover, Proposition 7.44 proves that M has Perron eigenvalue 1, i.e. that $M \cdot \mathbf{g} = \mathbf{g}$ and that $\mathbf{h} \cdot M = \mathbf{h}$. For all cliques $\mathbf{y} \in \mathcal{E}$, we compute that

$$\begin{aligned}
(N \cdot \overline{\mathbf{g}})_{(1,\mathbf{y})} &= \sum_{\mathbf{x}\in\mathcal{E}} \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} r(\mathbf{y})\mathbf{g_y} = (M \cdot \mathbf{g})_{\mathbf{y}} = \mathbf{g_y} = \overline{\mathbf{g}}_{(1,\mathbf{y})}, \text{ and} \\
(N \cdot \overline{\mathbf{g}})_{(i,\mathbf{y})} &= \overline{\mathbf{g}}_{(i-1,\mathbf{y})} = \overline{\mathbf{g}}_{(i,\mathbf{y})} \text{ if } i \geqslant 2; \\
(\overline{\mathbf{h}} \cdot N)_{(1,\mathbf{y})} &= \sum_{\mathbf{x}\in\mathcal{E}} \mathbf{h_x} \mathbf{1}_{\mathbf{x}\longrightarrow\mathbf{y}} r(\mathbf{y}) = (\mathbf{h} \cdot M)_{\mathbf{y}} = \mathbf{h_y} = \overline{\mathbf{h}}_{(1,\mathbf{y})}, \text{ and} \\
(\overline{\mathbf{h}} \cdot N)_{(i,\mathbf{y})} &= \overline{\mathbf{h}}_{(i-1,\mathbf{y})} = \overline{\mathbf{h}}_{(i,\mathbf{y})} \text{ if } i \geqslant 2.
\end{aligned}$$

This proves that $\overline{\mathbf{g}}$ and $\overline{\mathbf{h}}$ are Perron eigenvectors of N, whose Perron eigenvalue must be 1. $\square$

Theorem 7.84.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators. Let $r : \mathbf{A}^+ \mapsto \mathbb{C}$ be a Möbius valuation let $\kappa : \mathbf{A}^+ \mapsto \mathbb{C}$ be a Garside-additive function, and let μ_k denote the uniform probability distribution of parameter r on the set $\mathbf{A}^+(k)$. In addition, let P be the Markov Garside matrix of r, and let π be the stationary distribution associated with P.

The random variables $\frac{1}{k}\kappa(\mu_k)$ converge in law towards the Dirac measure at $\overline{\kappa} := \frac{\kappa(\pi)}{\lambda(\pi)}$. Furthermore, there exists a non-negative real number σ^2 such that the convergence in law

$$\sqrt{k}\left(\frac{1}{k}\kappa(\mu_k) - \overline{\kappa}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\sigma^2)$$

holds when $k \to +\infty$, and $\sigma^2 = 0$ if and only if $\kappa(\mathbf{x}) = \overline{\kappa}\lambda(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{E}$.

Proof. Let M be the Garside matrix of parameter r, and let N be the extended Garside matrix of parameter r. In addition, consider the conditioned weighted graph $\mathcal{G} = (N, \mathbf{w^{in}}, \mathbf{w^{out}})$ such that $\mathbf{w}^{\mathbf{in}}_{(i,\mathbf{x})} := \mathbf{1}_{i=1} r(\mathbf{x})$ and $\mathbf{w}^{\mathbf{out}}_{(i,\mathbf{x})} := \mathbf{1}_{i=\lambda(\mathbf{x})}$. Let ν_k be the uniform distribution on $\mathcal{P}_{\mathcal{G}}(k)$, i.e. on paths of length k. By construction of $\mathcal{G}$, the distribution ν_k corresponds to the uniform probability distribution μ_k of parameter r.

Moreover, consider the function $F_\kappa : (i,\mathbf{x}) \mapsto \kappa(\mathbf{x})\mathbf{1}_{i=\lambda(\mathbf{x})}$, and let ℓ be some positive integer. For all elements $\mathbf{y}$ of $\mathbf{A}^+$ or Artin length ℓ, let $\underline{\mathbf{p}} := (i_1, p_1) \cdot \ldots \cdot (i_\ell, p_\ell)$ be the (unique) associated path in $\mathcal{G}$ of length ℓ and with positive weight. By construction, we have $\sum_{j=1}^{\ell} F_\kappa(i_j, p_j) = \kappa(\mathbf{y})$, and therefore $\langle F_\kappa \rangle(\underline{\mathbf{p}}) = \frac{\kappa(\mathbf{y})}{\ell}$. It follows that $\langle F_\kappa \rangle(\nu_\ell) = \frac{1}{\ell}\kappa(\mu_\ell)$.

Then, let $\mathbf{g}$ and $\mathbf{h}$ be right and left Perron eigenvectors of M, and let $\overline{\mathbf{g}}$ and $\overline{\mathbf{h}}$ be the vectors defined in Lemma 7.83. Lemma 7.82 states that $\pi(\mathbf{x}) = \mathbf{g_x h_x}$ for all $\mathbf{x} \in \mathcal{E}$, and Lemma 7.83 states that $\overline{\mathbf{g}}$ and $\overline{\mathbf{h}}$ are right and left Perron eigenvectors of N. Then, Theorem 7.68 states that the invariant probability measure θ of N is defined by $\theta(i, \mathbf{x}) := \frac{\overline{\mathbf{g}}_{(i,\mathbf{x})} \overline{\mathbf{h}}_{(i,\mathbf{x})}}{\overline{\mathbf{h}} \cdot \overline{\mathbf{g}}} = \frac{\mathbf{g_x h_x}}{\overline{\mathbf{h}} \cdot \overline{\mathbf{g}}}$.

Consequently, Theorem 7.75 proves that the ergodic means $\frac{1}{k}\kappa(\mu_k)\langle F_\kappa \rangle(\nu_k)$ converge in law towards the Dirac measure at $\overline{\kappa} := \sum_{\mathbf{x}\in\mathcal{E}} \sum_{i=1}^{\lambda(\mathbf{x})} \theta(i,\mathbf{x}) F_\kappa(i,\mathbf{x})$, and Theorem 7.76 proves that there exists a non-negative real σ^2 such that $(\sqrt{k}(\frac{1}{k}\kappa(\mu_k) - \overline{\kappa})_{k\geqslant 1}$ converges in law towards a normal law $\mathcal{N}(0, \sigma^2)$ when $k \to +\infty$, with $\sigma^2 = 0$ if $\kappa = \overline{\kappa}\lambda$ on $\mathcal{E}$ and $\sigma^2 > 0$ otherwise.

Finally, one computes that $\overline{\mathbf{h}} \cdot \overline{\mathbf{g}} = \sum_{\mathbf{x}\in\mathcal{E}} \sum_{i=1}^{\lambda(\mathbf{x})} \mathbf{h_x g_x} = \lambda(\pi)$, whence

$$\overline{\kappa} = \sum_{\mathbf{x}\in\mathcal{E}} \sum_{i=1}^{\lambda(\mathbf{x})} \theta(i,\mathbf{x}) F_\kappa(i,\mathbf{x}) = \sum_{\mathbf{x}\in\mathcal{E}} \theta(\lambda(\mathbf{x}),\mathbf{x})\kappa(\mathbf{x}) = \frac{\sum_{\mathbf{x}\in\mathcal{E}} \mathbf{g_x h_x}\kappa(\mathbf{x})}{\overline{\mathbf{h}} \cdot \overline{\mathbf{g}}} = \frac{\kappa(\pi)}{\lambda(\pi)},$$

which completes the proof. □

It remains to apply Theorem 7.84 to various Garside-additive functions. First, as mentioned above, Theorem 7.84 directly applies to *additive* cost functions. However, we can also derive theorems that do not concern *directly* Garside-additive functions, such as the *mean length of the Garside normal forms.*

Proposition 7.85.
Let $\mathbf{A}^+$ be an irreducible Artin–Tits monoid of FC type with at least 2 generators. Consider the positive valuation $s : \mathbf{x} \mapsto 1$, such that $\Phi(s) = p_{\mathbf{A}}^{\lambda}$, i.e. $\Phi(s) : \mathbf{x} \mapsto p_{\mathbf{A}}^{\lambda(\mathbf{x})}$. Let μ_k denote the "standard" uniform probability distribution on the set $\mathbf{A}^+(k)$ of elements of length k. In addition, let P be the Markov Garside matrix of $\Phi(s)$, and let π be the stationary distribution associated with P.

Finally, for all elements $\mathbf{x} \in \mathbf{A}^+$, let us denote by $\Lambda(\mathbf{x})$ the ratio $\frac{\lambda(\mathbf{x})}{\|\mathbf{x}\|_\gamma}$. The random variables $\Lambda(\mu_k)$ converge towards the Dirac measure at $\lambda(\pi) = \sum_{\mathbf{x}\in\mathcal{E}} \pi(\mathbf{x})\lambda(\mathbf{x})$. Furthermore, there exists a positive *real number σ^2 such that the convergence in law*

$$\sqrt{k}(\Lambda(\mu_k) - \lambda(\pi)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$$

holds when $k \to +\infty$.

Proof. Since the function $\Gamma : \mathbf{x} \mapsto \|\mathbf{x}\|_\gamma$ is Garside-additive, Theorem 7.84 proves that $\Lambda^{-1}(\mu_k) = \frac{1}{k}\Gamma(\mu_k)$ converges towards the Dirac measure at $\frac{\Gamma(\pi)}{\lambda(\pi)} = \frac{1}{\lambda(\pi)}$, and that

$$\sqrt{k}(\Lambda^{-1}(\mu_k) - \lambda(\pi)^{-1})$$

converges towards a normal distribution $\mathcal{N}(0,\tau^2)$, for some non-negative real number τ^2, with $\tau^2 = 0$ if and only if $\lambda(\pi) = \lambda(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{E}$, which is not the case.

It follows that $\Lambda(\mu_k)$ converges towards the Dirac measure at $\lambda(\pi)$. Hence, for all real numbers $a < b$, it follows that

$$\begin{aligned}
\mathbb{P}_{\mu_k}[a \leqslant \sqrt{k}(\Lambda - \lambda(\pi)) \leqslant b] &= \mathbb{P}_{\mu_k}[-\frac{b}{\Lambda\lambda(\pi)} \leqslant \sqrt{k}(\Lambda^{-1} - \lambda(\pi)^{-1}) \leqslant -\frac{a}{\Lambda\lambda(\pi)}] \\
&\to \mathbb{P}_{\mu_k}[-\lambda(\pi)^{-2}b \leqslant \sqrt{k}(\Lambda^{-1} - \lambda(\pi)^{-1}) \leqslant -\lambda(\pi)^{-2}a] \\
&\to \mathbb{P}_{\mathcal{N}(0,\tau^2)}[-\lambda(\pi)^{-2}b \leqslant z \leqslant -\lambda(\pi)^{-2}a] \\
&\to \mathbb{P}_{\mathcal{N}(0,\tau^2\lambda(\pi)^{-4})}[a \leqslant z \leqslant b]
\end{aligned}$$

when $k \to +\infty$. This proves that $\sqrt{k}(\Lambda(\mu_k) - \lambda(\pi)) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\tau^2\lambda(\pi)^{-4})$, which completes the proof. □

In particular, in the framework of heap monoids, the study of the ratio Λ has an interpretation in terms of *speed-up*. If generators σ_i correspond to elementary computations, then two generators are independent if the associated computations can be performed in parallel. The length $\lambda(\mathbf{x})$ corresponds thus to the *sequential computation time*, whereas the height $\|\mathbf{x}\|$ corresponds to the *parallel execution time* of a computational process. Hence, Proposition 7.85 states that the speed-up converges (in distribution) and obeys a Central limit theorem.

7.3 Computations in $\mathbf{B}_n^+$ and $\mathcal{M}_n^+$

We focus here on specific cases of the above framework, namely the case of the monoids $\mathbf{A}^+ = \mathbf{B}_3^+$, $\mathbf{B}_4^+$, $\mathcal{M}_3^+$ and $\mathcal{M}_4^+$, choosing the trivial valuation $r : \mathbf{x} \mapsto 1$ and its associated Möbius valuation $s : \mathbf{x} \mapsto p_{\mathbf{A}}^{\lambda(\mathbf{x})}$. We compute explicitly the Markov Garside matrix P of s, its invariant probability measure π, and the limit $\overline{\kappa}$ of the random variables $\frac{1}{k}\kappa(\mu_k)$, when κ is a Garside-additive function. In particular, we also compute the limit $\lambda(\pi)$ of the random variables $\Lambda(\mu_k)$, where $\Lambda : \mathbf{x} \mapsto \frac{\lambda(\mathbf{x})}{\|\mathbf{x}\|_\gamma}$.

Furthermore, when $\mathbf{A}^+ = \mathcal{M}_3^+$ or $\mathcal{M}_4^+$, we also compute simple expressions of $\overline{\kappa}$ when κ is an *additive* function and of the limit convergence manifold $\mathcal{R}_{\mathbf{A}}^{\partial}$ of $\mathbf{A}$. We do not perform these additional computations when $\mathbf{A}^+ = \mathbf{B}_3^+$ or $\mathbf{B}_4^+$, since in this case all additive functions are multiples of the Artin length λ, and $\mathcal{R}_{\mathbf{A}}^{\partial}$ is the singleton set $\{(p_{\mathbf{A}},\ldots,p_{\mathbf{A}})\}$.

7.3.1 Computations in $\mathbf{B}_3^+$

We focus here on the monoid $\mathbf{B}_3^+ := \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle^+$, whose Garside element is $\Delta_3 := \sigma_1\sigma_2\sigma_1$ and whose two-way Garside family is the set $\mathbf{S} := \{\mathbf{1}, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \Delta_3\}$

of the divisors of Δ_3.

The Hasse diagram of the lattice $(\mathbf{S}, \leqslant_\ell)$ is represented in Fig. 7.86.

$$\begin{tikzcd}[column sep=small]
 & \Delta_3 & \\
\sigma_1\sigma_2 \arrow[ur, dash] & & \sigma_2\sigma_1 \arrow[ul, dash] \\
\sigma_1 \arrow[u, dash] & & \sigma_2 \arrow[u, dash] \\
 & \mathbf{1} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Figure 7.86 – Hasse diagram of the lattice $(\mathbf{S}, \leqslant_\ell)$ in $\mathbf{B}_3^+$

Hence, the Möbius polynomial of $\mathbf{B}_3^+$ is

$$\mathcal{H}_{\mathbf{B}_3}(z) = 1 - 2z + z^3 = (z-1)\left(z - \frac{\sqrt{5}-1}{2}\right)\left(z - \frac{\sqrt{5}+1}{2}\right),$$

with smallest positive root $p_3 = \frac{\sqrt{5}-1}{2} \approx 0.618$. In particular, the Möbius valuation associated with the trivial valuation $\mathbf{x} \mapsto 1$ is the valuation $s : \mathbf{x} \mapsto p_3^{\lambda(\mathbf{x})}$.

Using the equality $1 - p_3 - p_3^2 = 0$, we provide in Fig. 7.87 expressions of the graded Möbius transform of s, of the Markov Garside matrix P of s, and of the invariant probability measure π of P. Functions are represented by column vectors indexed by $\mathbf{S}\backslash\{\mathbf{1}\}$, and the entries of the vectors and matrices are indexed by the braids σ_1, σ_2, $\sigma_1\sigma_2$, $\sigma_2\sigma_1$ and Δ, in this order.

$$\mathbf{M}_\gamma s = \begin{bmatrix} 2p_3-1 \\ 2p_3-1 \\ 2-3p_3 \\ 2-3p_3 \\ 2p_3-1 \end{bmatrix}; P = \begin{bmatrix} p_3 & 0 & 1-p_3 & 0 & 0 \\ 0 & p_3 & 0 & 1-p_3 & 0 \\ 0 & p_3 & 0 & 1-p_3 & 0 \\ p_3 & 0 & 1-p_3 & 0 & 0 \\ 2p_3-1 & 2p_3-1 & 2-3p_3 & 2-3p_3 & 2p_3-1 \end{bmatrix}; \pi = \frac{1}{2}\begin{bmatrix} p_3 \\ p_3 \\ 1-p_3 \\ 1-p_3 \\ 0 \end{bmatrix}.$$

Figure 7.87 – Möbius transform $\mathbf{M}_\gamma s$, Markov Garside matrix P and its invariant probability measure π (in $\mathbf{B}_3^+$)

Hence, the random variables $\Lambda(\mu_k)$ converge (in distribution) towards the Dirac measure at $\lambda(\pi)$ and, for all Garside-additive functions $\kappa : \mathbf{A}^+ \mapsto \mathbb{C}$, the random variables $\frac{1}{k}\kappa(\mu_k)$ converge towards the Dirac measure at $\overline{\kappa}$, where

$$\lambda(\pi) = 2 - p_3 = \frac{5-\sqrt{5}}{2} \approx 1.382 \text{ and}$$
$$\overline{\kappa} = \frac{\kappa(\pi)}{\lambda(\pi)} = \frac{\kappa(\sigma_1)+\kappa(\sigma_2)}{2\sqrt{5}} + \frac{\kappa(\sigma_1\sigma_2)+\kappa(\sigma_2\sigma_1)}{5+\sqrt{5}}.$$

7.3.2 Computations in $\mathbf{B}_4^+$

We focus here on the monoid

$$\mathbf{B}_4^+ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_1\sigma_3 = \sigma_3\sigma_1\rangle^+,$$

whose Garside element is $\Delta_4 = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$ and whose two-way Garside family is the set

$$\begin{aligned}\mathbf{S} = \{&\mathbf{1}, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_1\sigma_3, \sigma_2\sigma_1, \sigma_2\sigma_3, \sigma_3\sigma_2, \sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_3,\\ &\sigma_1\sigma_3\sigma_2, \sigma_2\sigma_1\sigma_3, \sigma_2\sigma_3\sigma_2, \sigma_3\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\sigma_3, \sigma_1\sigma_3\sigma_2\sigma_1, \sigma_1\sigma_3\sigma_2\sigma_3,\\ &\sigma_2\sigma_1\sigma_3\sigma_2, \sigma_2\sigma_3\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2, \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3, \sigma_2\sigma_1\sigma_3\sigma_2\sigma_1, \Delta_4\}\end{aligned}$$

of the divisors of Δ_4.

The Hasse diagram of the lattice $(\mathbf{S}, \leqslant_\ell)$ is already represented in Fig. 2.92, and we draw it again below, in Fig. 7.88.

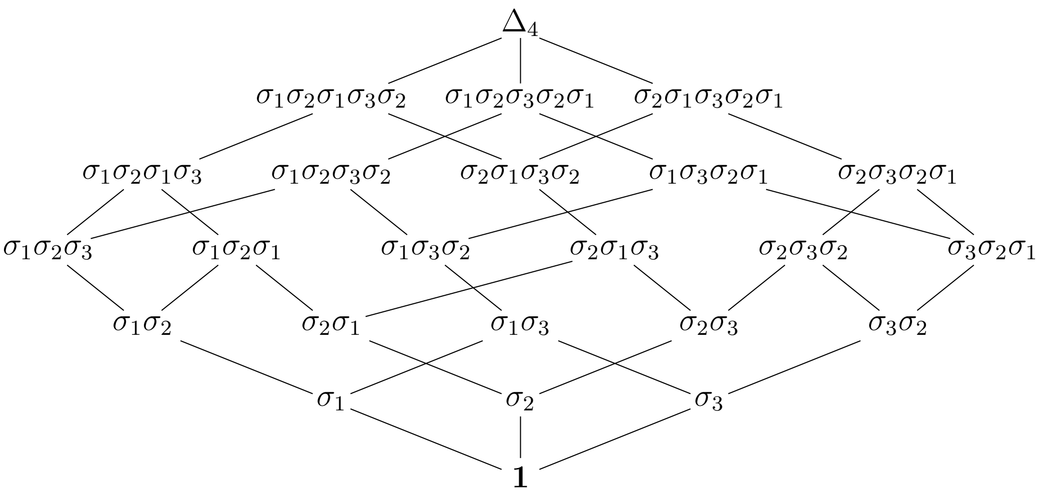


Figure 7.88 – Hasse diagram of the lattice $(\mathbf{S}, \leqslant_\ell)$ in $\mathbf{B}_4^+$

Hence, the Möbius polynomial of $\mathbf{B}_4^+$ is

$$\mathcal{H}_{\mathbf{B}_4}(z) = 1 - 3z + z^2 + 2z^3 - z^6 = -(z-1)(z^5 + z^4 + z^3 - z^2 - 2z + 1),$$

with smallest positive root $p_4 \approx 0.479$. In particular, the Möbius valuation associated with the trivial valuation $\mathbf{x} \mapsto 1$ is the valuation $s : \mathbf{x} \mapsto p_4^{\lambda(\mathbf{x})}$.

Using the equality $1 - 2p_4 - p_4^2 + p_4^3 + p_4^4 + p_4^5$, we provide in Figures 7.89 and 7.90 expressions of the graded Möbius transform of s, of the Markov Garside matrix P of s, of the invariant probability measure π of P, and of auxiliary polynomials $A_1, \ldots, A_{20}$. These polynomials are coefficients of the matrix P, and we decided to represent them as auxiliary data so that the matrix P (of size 23×23) could fit on one page. Functions are represented by row vectors indexed by $\mathbf{S}\backslash\{\mathbf{1}\}$, and the entries of the vectors and matrices are indexed by elements of $\mathbf{S}\backslash\{\mathbf{1}\}$ in the Short-Lex order (i.e. the order in which they are enumerated in the above expression of the set $\mathbf{S}$).

$$
P = \begin{bmatrix}
p_4 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & p_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_4 & 0 & 0 & 0 & A_2 & A_2 & 0 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 & p_4^4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_4 & 0 & 0 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & p_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_4 & 0 & 0 & 0 & A_2 & A_2 & 0 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 & p_4^4 & 0 & 0 & 0 & 0 & 0 \\
A_4 & 0 & A_4 & A_5 & p_4^2 & 0 & 0 & A_5 & 0 & A_6 & A_7 & 0 & 0 & A_6 & 0 & p_4^4 & p_4^4 & 0 & 0 & 0 & A_8 & 0 & 0 \\
p_4 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & p_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_4 & 0 & 0 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & p_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_4 & 0 & 0 & 0 & A_2 & A_2 & 0 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 & p_4^4 & 0 & 0 & 0 & 0 & 0 \\
A_4 & A_9 & 0 & A_5 & 0 & A_{10} & A_{10} & 0 & p_4^3 & A_6 & 0 & p_4^3 & 0 & 0 & p_4^4 & 0 & 0 & A_{11} & 0 & A_8 & 0 & 0 & 0 \\
0 & 0 & p_4 & 0 & 0 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & p_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_4 & 0 & 0 & 0 & A_2 & A_2 & 0 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 & p_4^4 & 0 & 0 & 0 & 0 & 0 \\
A_4 & 0 & A_4 & A_5 & p_4^2 & 0 & 0 & A_5 & 0 & A_6 & A_7 & 0 & 0 & A_6 & 0 & p_4^4 & p_4^4 & 0 & 0 & 0 & A_8 & 0 & 0 \\
0 & A_9 & A_4 & 0 & 0 & A_{10} & A_{10} & A_5 & 0 & 0 & 0 & p_4^3 & p_4^3 & A_6 & 0 & 0 & 0 & A_{11} & p_4^4 & 0 & 0 & A_8 & 0 \\
p_4 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & p_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_4 & 0 & A_4 & A_5 & p_4^2 & 0 & 0 & A_5 & 0 & A_6 & A_7 & 0 & 0 & A_6 & 0 & p_4^4 & p_4^4 & 0 & 0 & 0 & A_8 & 0 & 0 \\
0 & A_9 & A_4 & 0 & 0 & A_{10} & A_{10} & A_5 & 0 & 0 & 0 & p_4^3 & p_4^3 & A_6 & 0 & 0 & 0 & A_{11} & p_4^4 & 0 & 0 & A_8 & 0 \\
A_4 & A_9 & 0 & A_5 & 0 & A_{10} & A_{10} & 0 & p_4^3 & A_6 & 0 & p_4^3 & 0 & 0 & p_4^4 & 0 & 0 & A_{11} & 0 & A_8 & 0 & 0 & 0 \\
0 & p_4 & 0 & 0 & 0 & A_2 & A_2 & 0 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 & p_4^4 & 0 & 0 & 0 & 0 & 0 \\
A_4 & 0 & A_4 & A_5 & p_4^2 & 0 & 0 & A_5 & 0 & A_6 & A_7 & 0 & 0 & A_6 & 0 & p_4^4 & p_4^4 & 0 & 0 & 0 & A_8 & 0 & 0 \\
0 & A_9 & A_4 & 0 & 0 & A_{10} & A_{10} & A_5 & 0 & 0 & 0 & p_4^3 & p_4^3 & A_6 & 0 & 0 & 0 & A_{11} & p_4^4 & 0 & 0 & A_8 & 0 \\
A_4 & 0 & A_4 & A_5 & p_4^2 & 0 & 0 & A_5 & 0 & A_6 & A_7 & 0 & 0 & A_6 & 0 & p_4^4 & p_4^4 & 0 & 0 & 0 & A_8 & 0 & 0 \\
A_4 & A_9 & 0 & A_5 & 0 & A_{10} & A_{10} & 0 & p_4^3 & A_6 & 0 & p_4^3 & 0 & 0 & p_4^4 & 0 & 0 & A_{11} & 0 & A_8 & 0 & 0 & 0 \\
A_{12} & A_{13} & A_{12} & A_{14} & A_5 & A_{15} & A_{15} & A_{14} & A_7 & A_{16} & A_{17} & A_7 & A_7 & A_{16} & A_{11} & A_{11} & A_{11} & A_{18} & A_{11} & A_{19} & A_{19} & A_{19} & A_{20}
\end{bmatrix}.
$$

Figure 7.89 – Markov Garside matrix P (in $\mathbf{B}_4^+$)

Hence, the random variables $\Lambda(\mu_k)$ converge (in distribution) towards the Dirac measure at $\lambda(\pi)$ and, for all Garside-additive functions $\kappa : \mathbf{A}^+ \mapsto \mathbb{C}$, the random variables $\frac{1}{k}\kappa(\mu_k)$ converge towards the Dirac measure at $\overline{\kappa}$, where $\overline{\kappa} = \lambda(\pi)^{-1}\kappa(\pi)$ and

$$\lambda(\pi) = \frac{11086100 - 12019728p_4 + 8470263p_4^2 + 11669838p_4^3 + 5899597p_4^4}{4935059} \approx 1.797.$$

The expressions of $\kappa(\pi)$ and of $\overline{\kappa}$ are not so enlightening and consist in summing 14 different terms (using symmetries in the expression of π that are due to the isomorphism of monoids ϕ_Δ), which makes them hard to read. Moreover, they follow directly from the values of κ on $\mathbf{S}\backslash\{\mathbf{1}, \Delta\}$ and from the expression of the invariant probability measure π. Consequently, we do not write them explicitly here.

Observe that limit $\lambda(\pi)$ is a polynomial in p_4, with rational coefficients. This is not coincidental nor specific to the monoid $\mathbf{B}_4$, as outlined by the following result.

$$\mathbf{M}_\gamma s = \begin{bmatrix} A_{12} \\ A_{13} \\ A_{12} \\ A_{14} \\ A_5 \\ A_{15} \\ A_{15} \\ A_{14} \\ A_7 \\ A_{16} \\ A_{17} \\ A_7 \\ A_7 \\ A_{16} \\ A_{11} \\ A_{11} \\ A_{11} \\ A_{18} \\ A_{11} \\ A_{19} \\ A_{19} \\ A_{19} \\ A_{20} \end{bmatrix}; \pi = \frac{1}{4935059} \begin{bmatrix} 263236+882180p_4-249748p_4^2-671138p_4^3+328026p_4^4 \\ 1636548+83720p_4-2006161p_4^2-838931p_4^3-749927p_4^4 \\ 263236+882180p_4-249748p_4^2-671138p_4^3+328026p_4^4 \\ 146252+1457662p_4-604750p_4^2-1473442p_4^3-1012652p_4^4 \\ -359221+1227568p_4-317678p_4^2+112746p_4^3-377032p_4^4 \\ -285893+1083510p_4+744510p_4^2-929027p_4^3-220414p_4^4 \\ -285893+1083510p_4+744510p_4^2-929027p_4^3-220414p_4^4 \\ 146252+1457662p_4-604750p_4^2-1473442p_4^3-1012652p_4^4 \\ -115150-313535p_4+844758p_4^2+483609p_4^3+427337p_4^4 \\ 999164-2326354p_4-79876p_4^2+2209370p_4^3+421390p_4^4 \\ 866810-2469873p_4+1474043p_4^2-301404p_4^3+920202p_4^4 \\ 1124824-374897p_4-1363027p_4^2-904410p_4^3-1035820p_4^4 \\ -115150-313535p_4+844758p_4^2+483609p_4^3+427337p_4^4 \\ 999164-2326354p_4-79876p_4^2+2209370p_4^3+421390p_4^4 \\ -427337+739524p_4+113802p_4^2+417421p_4^3+56272p_4^4 \\ -489778+1356588p_4-623507p_4^2+360758p_4^3-430424p_4^4 \\ -489778+1356588p_4-623507p_4^2+360758p_4^3-430424p_4^4 \\ 1167230-2245456p_4-595311p_4^2+1214920p_4^3+590027p_4^4 \\ -427337+739524p_4+113802p_4^2+417421p_4^3+56272p_4^4 \\ -56272-314793p_4+795796p_4^2+57530p_4^3+361149p_4^4 \\ 430424-1350626p_4+926164p_4^2-193083p_4^3+791182p_4^4 \\ -56272-314793p_4+795796p_4^2+57530p_4^3+361149p_4^4 \\ 0 \end{bmatrix}; \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \\ A_{15} \\ A_{16} \\ A_{17} \\ A_{18} \\ A_{19} \\ A_{20} \end{bmatrix} = \begin{bmatrix} 1-p_4-p_4^3 \\ 1-p_4-p_4^2-p_4^3-p_4^4 \\ -1+p_4+2p_4^2+2p_4^3+p_4^4 \\ p_4-p_4^2-p_4^3 \\ p_4^2-p_4^3 \\ 1-2p_4-p_4^2+2p_4^3 \\ p_4^3-p_4^4 \\ -1+2p_4+p_4^2-p_4^3-p_4^4 \\ p_4-p_4^2 \\ p_4^2-p_4^3-p_4^4 \\ 1-2p_4-p_4^2+p_4^3+2p_4^4 \\ p_4-2p_4^2+p_4^4 \\ p_4-2p_4^2+p_4^3 \\ p_4^2-2p_4^3+p_4^4 \\ -1+2p_4+2p_4^2-3p_4^3-p_4^4 \\ 1-3p_4+p_4^2+3p_4^3-2p_4^4 \\ -1+2p_4+p_4^2-3p_4^4 \\ 3-7p_4-p_4^2+4p_4^3+3p_4^4 \\ -2+5p_4-3p_4^3-p_4^4 \\ 1-3p_4+p_4^2+2p_4^3 \end{bmatrix}.$$

Figure 7.90 – Möbius transform $\mathbf{M}_\gamma s$, invariant probability π of P and auxiliary data (in $\mathbf{B}_4^+$)

Proposition 7.91.
Let $\mathbf{A}^+$ be an Artin–Tits monoid of FC type with at least 2 generators and let $p_\mathbf{A}$ be the smallest positive root of the Möbius polynomial $\mathcal{H}_\mathbf{A}(z)$. The limit $\lambda(\pi)$, towards which the random variables $\Lambda(\mu_k)$ converge, belongs to the field $\mathbb{Q}[p_\mathbf{A}]$, i.e. can be expressed under the form $Q(p_\mathbf{A})$, where Q is a polynomial with rational coefficients.

Proof. First, note that $p_\mathbf{A}$ is an algebraic number, since it is a root of the integer-valued polynomial $\mathcal{H}_\mathbf{A}$. Consequently, the field generated by $p_\mathbf{A}$ is the set $\mathbb{Q}[p_\mathbf{A}] = \{z \in \mathbb{R} : \exists Q \in \mathbb{Q}[X], z = Q(p_\mathbf{A})\}$. By construction, the valuation s, its Möbius transform $\mathbf{M}_\gamma s$, and therefore the Markov Garside matrix P have entries in the field $\mathbb{Q}[p_\mathbf{A}]$. Consequently, the probability π, which can be seen as the unique normalised vector of the kernel of the matrix $P - \mathbf{Id}$, can be obtained from P by Gaussian elimination, hence its entries also belong to the field $\mathbb{Q}[p_\mathbf{A}]$. Since the function λ is integer-valued, it follows that $\lambda(\pi)$ belongs to $\mathbb{Q}[p_\mathbf{A}]$ too. $\square$

7.3.3 Computations in $\mathcal{M}_3^+$

We focus here on the monoid $\mathcal{M}_3^+ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle^+$, whose two-way Garside family is the set $\mathbf{S} = \{\mathbf{1}, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_3\}$. The value of $\lambda(\pi)$ was already computed in [2, 71], and the parametrisation of $\mathcal{R}^{\partial}_{\mathcal{M}_3}$ was also performed in [2].

The Hasse diagram of the partially ordered set $(\mathbf{S}, \leqslant_\ell)$ is represented in Fig. 7.92.

$$\begin{tikzcd}[column sep=small] & \sigma_1\sigma_3 & \\ \sigma_1 \arrow[ur, dash] & \sigma_2 & \sigma_3 \arrow[ul, dash] \\ & \mathbf{1} \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

Figure 7.92 – Hasse diagram of the partially ordered set $(\mathbf{S}, \leqslant_\ell)$ in $\mathcal{M}_3^+$

Hence, the Möbius polynomial of $\mathcal{M}_3^+$ is

$$\mathcal{H}_{\mathcal{M}_3}(z) = 1 - 3z + z^2 = \left(z - \frac{3-\sqrt{5}}{2}\right)\left(z - \frac{3+\sqrt{5}}{2}\right),$$

with smallest positive root $q_3 = \frac{3-\sqrt{5}}{2} \approx 0.382$. In particular, the Möbius valuation associated with the trivial valuation $\mathbf{x} \mapsto 1$ is the valuation $s : \mathbf{x} \mapsto q_3^{\lambda(\mathbf{x})}$.

$$\mathbf{M}_\gamma s = \begin{bmatrix} 1-2q_3 \\ q_3 \\ 1-2q_3 \\ 3q_3-1 \end{bmatrix}; P = \begin{bmatrix} q_3 & 1-q_3 & 0 & 0 \\ 1-2q_3 & q_3 & 1-2q_3 & 3q_3-1 \\ 0 & 1-q_3 & q_3 & 0 \\ 1-2q_3 & q_3 & 1-2q_3 & 3q_3-1 \end{bmatrix}; \pi = \frac{1}{11}\begin{bmatrix} 2+q_3 \\ 8-7q_3 \\ 2+q_3 \\ 5q_3-1 \end{bmatrix}.$$

Figure 7.93 – Möbius transform $\mathbf{M}_\gamma s$, Markov Garside matrix P and its invariant probability measure π (in $\mathcal{M}_3^+$)

Using the equality $1 - 3q_3 + q_3^2 = 0$, we provide in Fig. 7.93 expressions of the graded Möbius transform of s, of the Markov Garside matrix P of s, and of the invariant probability measure π of P. Functions are represented by column vectors indexed by $\mathbf{S}\backslash\{\mathbf{1}\}$, and the entries of the vectors and matrices are indexed by the elements σ_1, σ_2, σ_3 and $\sigma_1\sigma_3$, in this order.

Hence, the random variables $\Lambda(\mu_k)$ converge (in distribution) towards the Dirac measure at $\lambda(\pi)$ and, for all Garside-additive functions $\kappa : \mathbf{A}^+ \mapsto \mathbb{C}$, the random variables

$\frac{1}{k}\kappa(\mu_k)$ converge towards the Dirac measure at $\overline{\kappa}$, where

$$\begin{aligned}\lambda(\pi) &= \frac{10+5q_3}{11} = \frac{35-5\sqrt{5}}{22} \approx 1.083 \text{ and}\\ \overline{\kappa} &= \frac{\kappa(\pi)}{\lambda(\pi)} = \frac{2(\kappa(\sigma_1)+\kappa(\sigma_3))}{7+\sqrt{5}} + \frac{10\kappa(\sigma_2)}{5+7\sqrt{5}} + \frac{2\kappa(\sigma_1\sigma_3)}{13+5\sqrt{5}}\\ &= \frac{5(\kappa(\sigma_1)+\kappa(\sigma_3))}{10+3\sqrt{5}} + \frac{10\kappa(\sigma_2)}{5+7\sqrt{5}} \text{ if } \kappa \text{ is additive.}\end{aligned}$$

Moreover, applying the algorithm used for proving Theorem 7.51, it follows that the convergence manifold and the limit convergence manifold of $\mathcal{M}_3^+$ are The multivariate Möbius polynomial of $\mathcal{M}_3^+$ is $\mathcal{H}_{\mathcal{M}_3}(x,y,z) = 1-x-y-z+xz = (1-x)(1-z)-y$. The set $\{(x,y,z) \in (0,1)^3 : \mathcal{H}_{\mathcal{M}_3}(x,y,z) > 0\}$ is therefore connected, and, according to Corollary 7.34, is equal to the convergence manifold $\mathcal{R}_{\mathcal{M}_3}$ itself. It follows that

$$\begin{aligned}\mathcal{R}_{\mathcal{M}_3} &= \{(x,y,z) \in (0,1)^3 : \mathcal{H}_{\mathcal{M}_3}(x,y,z) > 0\}\\ &= \{(x,(1-x)y,(1-y)z : 0 < x,y,z < 1\}\\ \mathcal{R}^\partial_{\mathcal{M}_3} &= \{(x,y,z) \in (0,1)^3 : \mathcal{H}_{\mathcal{M}_3}(x,y,z) = 0\}\\ &= \{(1-x,ux,1-u) : 0 < x,u < 1\}.\end{aligned}$$

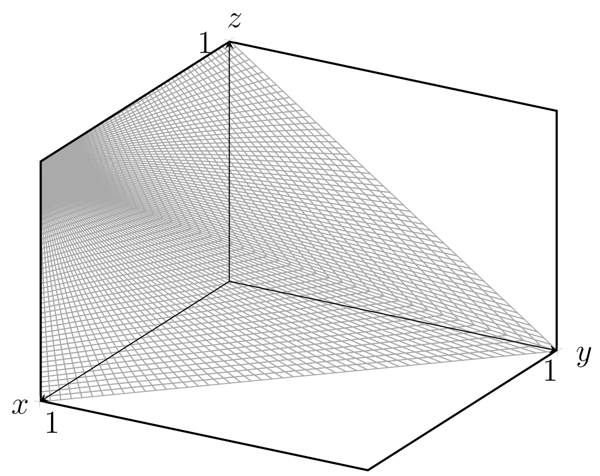


Figure 7.94 – Limit convergence manifold of $\mathcal{M}_3^+$

The latter equation shows that $\mathcal{R}^\partial_{\mathcal{M}_3}$ is in fact a fragment of hyperbolic paraboloid. Figure 7.94 represents the limit convergence manifold $\mathcal{R}^\partial_{\mathcal{M}_3}$ in gray.

7.3.4 Computations in $\mathcal{M}_4^+$

We focus now on the monoid

$$\mathcal{M}_4^+ = \langle \sigma_1,\sigma_2,\sigma_3,\sigma_4 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1, \sigma_1\sigma_4 = \sigma_4\sigma_1, \sigma_2\sigma_4 = \sigma_4\sigma_2\rangle^+,$$

whose two-way Garside family is the set $\mathbf{S} = \{\mathbf{1}, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_1\sigma_3, \sigma_1\sigma_4, \sigma_2\sigma_4\}$. Once again, the value of $\lambda(\pi)$ had already be computed in [71].

The Hasse diagram of the partially ordered set $(\mathbf{S}, \leqslant_\ell)$ is represented in Fig. 7.95.

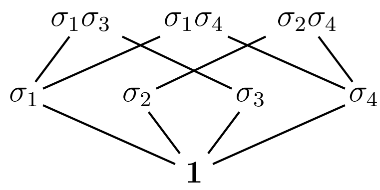


Figure 7.95 – Hasse diagram of the partially ordered set $(\mathbf{S}, \leqslant_\ell)$ in $\mathcal{M}_4^+$

Hence, the Möbius polynomial of $\mathcal{M}_4^+$ is

$$\mathcal{H}_{\mathcal{M}_4}(z) = 1 - 4z + 3z^2 = (z-1)(3z-1),$$

with smallest positive root $q_4 = \frac{1}{3}$. In particular, the Möbius valuation associated with the trivial valuation $\mathbf{x} \mapsto 1$ is the valuation $s : \mathbf{x} \mapsto q_4^{\lambda(\mathbf{x})}$.

We provide in Fig. 7.96 expressions of the graded Möbius transform of s, of the Markov Garside matrix P of s, and of the invariant probability measure π of P. Functions are represented by column vectors indexed by $\mathbf{S}\backslash\{\mathbf{1}\}$, and the entries of the vectors and matrices are indexed by the elements σ_1, σ_2, σ_3, σ_4, $\sigma_1\sigma_3$, $\sigma_1\sigma_4$ and $\sigma_2\sigma_4$, in this order.

$$\mathbf{M}_\gamma s = \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; P = \frac{1}{18}\begin{bmatrix} 6 & 12 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & 6 & 0 & 3 & 0 & 0 \\ 0 & 6 & 6 & 3 & 0 & 0 & 3 \\ 0 & 0 & 12 & 6 & 0 & 0 & 0 \\ 2 & 4 & 4 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 2 & 2 & 2 & 2 \end{bmatrix}; \pi = \frac{1}{57}\begin{bmatrix} 6 \\ 18 \\ 18 \\ 6 \\ 4 \\ 1 \\ 4 \end{bmatrix}.$$

Figure 7.96 – Möbius transform $\mathbf{M}_\gamma s$, Markov Garside matrix P and its invariant probability measure π (in $\mathcal{M}_4^+$)

Hence, the random variables $\Lambda(\mu_k)$ converge (in distribution) towards the Dirac measure at $\lambda(\pi)$ and, for all Garside-additive functions $\kappa : \mathbf{A}^+ \mapsto \mathbb{C}$, the random variables $\frac{1}{k}\kappa(\mu_k)$ converge towards the Dirac measure at $\overline{\kappa}$, where

$$\lambda(\pi) = \frac{22}{19} \approx 1.158 \text{ and}$$

$$\overline{\kappa} = \frac{\kappa(\pi)}{\lambda(\pi)} = \frac{2(\kappa(\sigma_1) + \kappa(\sigma_4))}{19} + \frac{6(\kappa(\sigma_2) + \kappa(\sigma_3))}{19} + \frac{4(\kappa(\sigma_1\sigma_3) + \kappa(\sigma_2\sigma_4))}{57} + \frac{\kappa(\sigma_1\sigma_4)}{57}$$

$$= \frac{11}{57}(\kappa(\sigma_1) + 2\kappa(\sigma_2) + 2\kappa(\sigma_3) + \kappa(\sigma_4)) \text{ if } \kappa \text{ is additive.}$$

Applying the algorithm used for proving Theorem 7.51, we compute the following parametrisations of the convergence manifold and the limit convergence manifold of $\mathcal{M}_3^+$:

$$\begin{aligned}
\mathcal{R}_{\mathcal{M}_3} &= \{(x,(1-x)y,(1-y)z,(1-z)t) : 0 < x,y,z,t < 1\};\\
\mathcal{R}^{\partial}_{\mathcal{M}_3} &= \{(x,(1-x)y,(1-y)z,1-z) : 0 < x,y,z < 1\}.
\end{aligned}$$

7.3.5 Radius of Convergence in $\mathbf{B}_n^+$ and $\mathcal{M}_n^+$

We focus here on studying the radii of convergence of the braid monoids $\mathbf{B}_n^+$ and of dimer models $\mathcal{M}_n^+$, pursuing investigations on the growth of these monoids performed by Xu [94] or Berceanu and Iqbal [10].

Let us focus now on all the monoid

$$\mathcal{M}_n^+ = \langle \sigma_1,\ldots,\sigma_n \mid \forall i,j, |i-j| \geqslant 2 \Rightarrow \sigma_i\sigma_j = \sigma_j\sigma_i\rangle^+,$$

whose smallest two-way Garside family is the set $\mathbf{S} = \{\Delta_I : I \subseteq \{\sigma_1,\ldots,\sigma_n\}$ and $\forall \sigma_i,\sigma_j \in I, i-j \neq \pm 1\}$.

Proposition 7.97.
Let $\mathcal{M}_n^+$ be the dimer model with n generators, and let $\mathcal{H}_{\mathcal{M}_n}(z)$ be the Möbius polynomial of $\mathcal{M}_n^+$. The smallest positive root of the polynomial $\mathcal{H}_{\mathcal{M}_n}(z)$ is $q_n := \frac{1}{4\cos(\frac{\pi}{n+2})^2}$. This root is rational if and only if $n \in \{1,2,4\}$, and is such that $q_n \to \frac{1}{4}$ when $n \to +\infty$.

Proof. An induction analogous to that used for proving Proposition 7.3 shows that the Möbius polynomial of $\mathcal{M}_n^+$ obeys the induction relation $\mathcal{H}_{\mathcal{M}_n}(z) = \mathcal{H}_{\mathcal{M}_{n-1}}(z) - z\mathcal{H}_{\mathcal{M}_{n-2}}(z)$, with $\mathcal{H}_{\mathcal{M}_{-1}}(z) = \mathcal{H}_{\mathcal{M}_0}(z) = 1$. Solving this linear recurrent equation on $\mathcal{H}_{\mathcal{M}_n}(z)$, we find

$$\begin{aligned}
\mathcal{H}_{\mathcal{M}_n}(z) &= \frac{1}{2^{n+2}\sqrt{1-4z}}\left((1+\sqrt{1-4z})^{n+2} - (1-\sqrt{1-4z})^{n+2}\right)\\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{\lfloor (n+1)/2\rfloor}\binom{n+2}{2k+1}(1-4z)^k.
\end{aligned}$$

Hence, let $\theta_{n+2} := \frac{\pi}{n+2}$ and $\omega_{n+2} := \exp(i\theta_{n+2})$. The roots of $\mathcal{H}_{\mathcal{M}_n}$ are the complex numbers $r \in \mathbb{C}\backslash\{\frac{1}{4}\}$ such that $1+\sqrt{1-4r} = (1-\sqrt{1-4r})\omega_{n+2}^{2k}$ for some $k \in \mathbb{Z}$ or, equivalently, such that $r = \frac{1}{4\cos(k\theta_{n+2})^2}$. Consequently, the smallest positive root of $\mathcal{H}_{\mathcal{M}_n}(z)$ is $q_n := \frac{1}{4\cos(\theta_{n+2})^2} = \frac{1}{4\cos(\frac{\pi}{n+2})^2}$, and $q_n \to \frac{1}{4}$ when $n \to +\infty$.

The first values of q_n are $q_1 = 1$, $q_2 = \frac{1}{2}$, $q_3 = \frac{3-\sqrt{5}}{2}$ and $q_4 = \frac{1}{3}$. Then, assume that q_n is rational for some integer $n \geqslant 5$. Let $bX - a$ be its minimal polynomial in $\mathbb{Z}[X]$, so that $q_n = \frac{a}{b}$. Since q_n cancels $\mathcal{H}_{\mathcal{M}_n}(z)$, it follows that $bX - a$ divides $\mathcal{H}_{\mathcal{M}_n}(z)$ in $\mathbb{Z}[X]$, and since $\mathcal{H}_{\mathcal{M}_n}(0) = 1$, it follows that $a = \pm 1$, i.e. that $\frac{1}{q_n} = \pm b$ is an integer. This contradicts the fact that $\frac{1}{4} < q_n < q_4 = \frac{1}{3}$, which proves the irrationality of q_n and completes the proof. $\square$

In addition, using the above equation $\mathcal{H}_{\mathcal{M}_n}(z) = \mathcal{H}_{\mathcal{M}_{n-1}}(z) - z\mathcal{H}_{\mathcal{M}_{n-2}}(z)$, the algorithm used for proving Theorem 7.51 provides us with parametrisations of the convergence manifold and the limit convergence manifold of $\mathcal{M}_n^+$:

$$\begin{aligned} \mathcal{R}_{\mathcal{M}_n} &= \{(x_1, (1-x_1)x_2, \ldots, (1-x_{n-2})x_{n-1}, (1-x_{n-1})x_n) : 0 < x_1, \ldots, x_n < 1\}; \\ \mathcal{R}^\partial_{\mathcal{M}_n} &= \{(x_1, (1-x_1)x_2, \ldots, (1-x_{n-2})x_{n-1}, 1-x_{n-1}) : 0 < x_1, \ldots, x_{n-1} < 1\}. \end{aligned}$$

The situation is analogous in the braid monoid $\mathbf{B}_n^+$, whose smallest two-way Garside family is the set $\mathbf{S} = \{\beta : \beta \leqslant \Delta\}$. However, the Bronfman formula for computing the Möbius polynomials is more complex for braids than for heaps, which explains why our results may seem weaker.

Proposition 7.98.
Let $\mathbf{B}_n^+$ be the n-strand braid monoid, let $\mathcal{H}_{\mathbf{B}_n}(z)$ be the Möbius polynomial of $\mathbf{B}_n^+$, and let p_n be the smallest positive root of the polynomial $\mathcal{H}_{\mathbf{B}_n}(z)$. The sequence $(p_n)_{n\geqslant 1}$ is non-increasing, and converges towards a limit $\mathbf{P}$ such that $\frac{2}{5} > \mathbf{P} > \frac{1}{4}$.

Proof. First, observe that, for all integers $n \geqslant 1$, the monoid $\mathbf{B}_n^+$ is a submonoid of $\mathbf{B}_{n+1}^+$, and therefore that $\mathcal{G}_{\mathbf{B}_{n+1}}(z) \geqslant \mathcal{G}_{\mathbf{B}_n}(z)$, hence that $\mathcal{H}_{\mathbf{B}_{n+1}}(z) \leqslant \mathcal{H}_{\mathbf{B}_n}(z)$, when $0 \leqslant z < \min\{p_n, p_{n+1}\}$. It follows that $p_{n+1} \leqslant p_n$, i.e. that $(p_n)_{n\geqslant 1}$ is non-increasing. Likewise, since $\mathbf{B}_n^+$ is a quotient monoid of the dimer model $\mathcal{M}_{n-1}^+$, it follows that $q_n \leqslant p_{n-1}$, which proves that $\mathbf{P} \geqslant \frac{1}{4}$. However, it is not yet clear whether the two sequences $(p_n)_{n\geqslant 1}$ and $(q_n)_{n\geqslant 1}$ might have the same limit.

Proposition 7.3 shows that the Möbius polynomial of $\mathbf{B}_n^+$ obeys the induction relation $\mathcal{H}_{\mathbf{B}_n}(z) = \sum_{k\geqslant 0}(-1)^k z^{k(k+1)/2}\mathcal{H}_{\mathbf{B}_{n-1}}(z)$, with $\mathcal{H}_{\mathbf{B}_0}(z) = \mathcal{H}_{\mathbf{B}_1}(z) = 1$ and $\mathcal{H}_{\mathbf{B}_n}(z) = 0$ if $n \leqslant -1$. Hence, a simple yet cumbersome computation shows that $\mathcal{H}_{\mathbf{B}_5}\left(\frac{2}{5}\right) < 0$, and therefore that $\mathbf{P} < \frac{2}{5}$.

Now, consider the polynomial $P(z) = 1 - 4z + 4z^3 - 8z^6$. Since

$$P\left(\frac{1}{4}\right) = \frac{31}{512} > 0 > -\frac{143}{729} = P\left(\frac{1}{3}\right),$$

let ρ be the smallest root of P in the interval $\left(\frac{1}{4}, \frac{1}{3}\right)$, and let z be an element of the closed real interval $\left[\frac{1}{4}, \rho\right]$. We prove by induction that $2\mathcal{H}_{\mathbf{B}_{n+1}}(z) \geqslant \mathcal{H}_{\mathbf{B}_n}(z) \geqslant 0$ for all $n \in \mathbb{Z}$. First, the result is immediate when $n \leqslant 0$. Moreover, a straightforward computation shows that

$$\mathcal{H}_{\mathbf{B}_2}(z) = 1 - z = (1-z)\mathcal{H}_{\mathbf{B}_1}(z) \geqslant (1-\rho)\mathcal{H}_{\mathbf{B}_1}(z) \geqslant \frac{2}{3}\mathcal{H}_{\mathbf{B}_1}(z).$$

Hence, let us assume that $n \geqslant 2$. Since $\rho \geqslant z$, note that $z^k \leqslant \rho^k \leqslant 3^{-k} \leqslant \frac{1}{2}$ whenever $k \geqslant 1$. In addition, since $z \geqslant \frac{1}{4}$ and $0 \leqslant \mathcal{H}_{\mathbf{B}_{n-1}}(z) \leqslant \mathcal{H}_{\mathbf{B}_{n-2}}(z)$, observe that

$(\frac{1}{2} - 2z)\mathcal{H}_{\mathbf{B}_{n-1}}(z) \geqslant (\frac{1}{2} - 2z)\mathcal{H}_{\mathbf{B}_{n-2}}(z)$. Finally, since $P(z) \geqslant 0$, It follows that:

$$\begin{aligned}
2\mathcal{H}_{\mathbf{B}_{n+1}}(z) - \mathcal{H}_{\mathbf{B}_n}(z) &= \mathcal{H}_{\mathbf{B}_n}(z) + 2\sum_{k\geqslant 1}(-1)^k z^{k(k+1)/2}\mathcal{H}_{\mathbf{B}_{n-k}}(z) \\
&= \mathcal{H}_{\mathbf{B}_n}(z) - 2z\mathcal{H}_{\mathbf{B}_{n-1}}(z) + 2z^3\mathcal{H}_{\mathbf{B}_{n-2}}(z) - 2z^6\mathcal{H}_{\mathbf{B}_{n-3}}(z) + \\
&\quad \sum_{k\geqslant 2} z^{k(2k+1)}\left(2\mathcal{H}_{\mathbf{B}_{n-2k}}(z) - 2z^{2k+1}\mathcal{H}_{\mathbf{B}_{n-2k-1}}(z)\right) \\
&\geqslant \frac{1-4z}{2}\mathcal{H}_{\mathbf{B}_{n-1}}(z) + 2z^3\mathcal{H}_{\mathbf{B}_{n-2}}(z) - 2z^6\mathcal{H}_{\mathbf{B}_{n-3}}(z) + \\
&\quad \sum_{k\geqslant 2} z^{k(2k+1)}(1 - 2z^{2k+1})\mathcal{H}_{\mathbf{B}_{n-2k-1}}(z) \\
&\geqslant \frac{1-4z+4z^3}{2}\mathcal{H}_{\mathbf{B}_{n-2}}(z) - 2z^6\mathcal{H}_{\mathbf{B}_{n-3}}(z) \\
&\geqslant \frac{P(z)+8z^6}{2}\mathcal{H}_{\mathbf{B}_{n-2}}(z) - 2z^6\mathcal{H}_{\mathbf{B}_{n-3}}(z) \\
&\geqslant 2z^6(2\mathcal{H}_{\mathbf{B}_{n-2}}(z) - \mathcal{H}_{\mathbf{B}_{n-3}}(z)) \geqslant 0,
\end{aligned}$$

which completes the induction.

It follows that $\mathcal{H}_{\mathbf{B}_n}(z) > 0$ for all $n \geqslant 0$, which proves that $p_n \geqslant \rho$, and therefore that $\mathbf{P} \geqslant \rho > \frac{1}{4}$. □

Obtaining further approximations of p_n seems difficult. However, partial results allow us to prove some results and to formulate conjectures, as follows.

Lemma 7.99.
Let $\mathbf{B}_n^+$ be the n-strand braid monoid, let $\mathcal{H}_{\mathbf{B}_n}(z)$ be the Möbius polynomial of $\mathbf{B}_n^+$, let p_n be the smallest positive root of the polynomial $\mathcal{H}_{\mathbf{B}_n}(z)$, and let $\mathbf{P}$ be the limit of the sequence $(p_n)_{n\geqslant 0}$. The inequalities

$$\frac{\mathcal{H}_{\mathbf{B}_{n+2}}(z)}{\mathcal{H}_{\mathbf{B}_{n+1}}(z)} \leqslant \frac{\mathcal{H}_{\mathbf{B}_{n+1}}(z)}{\mathcal{H}_{\mathbf{B}_n}(z)}$$

hold for all integers $n \geqslant 1$ and for all real numbers z such that $0 \leqslant z < \mathbf{P}$.

Proof. Let us write the generating series $\mathcal{G}_{\mathbf{B}_{n+2}}(z)\mathcal{G}_{\mathbf{B}_n}(z)$ and $\mathcal{G}_{\mathbf{B}_{n+1}}(z)^2$ respectively as sums $\sum_{k\geqslant 0} a_{n,k}z^k$ and $\sum_{k\geqslant 0} b_{n,k}z^k$, where $a_{n,k}$ is the cardinality of the set $A_{n,k} := \{(\gamma,\gamma') \in \mathbf{B}_{n+2}^+ \times \mathbf{B}_n^+ : \lambda(\gamma) + \lambda(\gamma') = k\}$ and $b_{n,k}$ is the cardinality of the set $B_{n,k} := \{(\alpha,\beta) \in \mathbf{B}_{n+1}^+ \times \mathbf{B}_{n+1}^+ : \lambda(\alpha) + \lambda(\beta') = k\}$.

Consider the homomorphism of monoids $\phi : \mathbf{B}_{n+1}^+ \mapsto \mathbf{B}_{n+2}^+$ such that $\phi : \sigma_i \mapsto \sigma_{i+1}$. In addition, for all braids $\beta \in \mathbf{B}_{n+1}^+$, let $\delta_{1,n}(\beta)$ denote the greatest element of the set $\{\mathbf{x} \in \langle\sigma_1,\ldots,\sigma_{n-1}\rangle^+ : \mathbf{x} \leqslant_\ell \beta\}$, and let $\delta_{2,n}(\beta)$ denote the braid $\delta_{1,n}^{-1}(\beta)\beta$. Note that $\delta_{1,n}$ induces a mapping from $\mathbf{B}_{n+1}^+$ to $\mathbf{B}_n^+$. We prove now that the function $\Theta : B_{n,k} \mapsto A_{n,k}$ defined by $\Theta : (\alpha,\beta) \mapsto (\alpha\phi(\delta_{2,n}(\beta)),\delta_{1,n}(\beta))$ is injective.

Let us denote by γ the braid $\alpha\phi(\delta_{2,n}(\beta))$ and by γ' the braid $\delta_{1,n}(\beta)$. By construction, we know that $\mathbf{left}(\delta_{2,n}(\beta)) \subseteq \{\sigma_n\}$, i.e. that $\mathbf{left}(\phi(\delta_{2,n}(\beta))) \subseteq \{\sigma_{n+1}\}$ and therefore that $\delta_{1,n+1}(\gamma) \leqslant_\ell \alpha$. Conversely, it is straightforward that $\alpha \leqslant_\ell \delta_{1,n+1}(\gamma)$, which proves that $\alpha = \delta_{1,n+1}(\gamma)$ and that $\beta = \gamma'\alpha^{-1}\gamma$, i.e. that Θ is indeed injective.

It follows that $a_{n,k} \geqslant b_{n,k}$ for all integers n and k, and therefore that $\mathcal{G}_{\mathbf{B}_{n+2}}(z)\mathcal{G}_{\mathbf{B}_n}(z) \geqslant \mathcal{G}_{\mathbf{B}_{n+1}}(z)^2$ for all $z \in [0, p_{n+2})$, thereby proving Lemma 7.99. $\square$

Proposition 7.100.
Let z be an element of the interval $[0, \mathbf{P})$, and let F_z be the generating function $F_z : \theta \mapsto \sum_{k\geqslant 0}(-1)^k z^{k(k+1)/2}\theta^k$, which converges on the entire complex plane.

The equation $1 = \theta F_z(\theta)$ has some complex root. Moreover, let $\Theta := \min\{|\theta| : 1 = \theta F_z(\theta)\}$. We have $1 = \Theta F_z(\Theta)$, and

$$\frac{\mathcal{H}_{\mathbf{B}_{n+2}}(z)}{\mathcal{H}_{\mathbf{B}_{n+1}}(z)} \to \frac{1}{\Theta} \text{ when } n \to +\infty.$$

Proof. Let us define the generating function $S : \theta \mapsto \frac{1}{1-\theta F_z(\theta)}$, and let $\Theta := +\infty$ if the equation $1 = \theta F_z(\theta)$ has no complex root. By construction, we know that Θ is the radius of convergence of S, and that $\Theta > 0$. Let H_n be the coefficients of S, i.e. the real numbers such that $S(\theta) = \sum_{n\geqslant 0} H_n\theta^n$. We also define $H_n := 0$ when $n \leqslant -1$. Observe that $S(\theta) = 1+\theta S(\theta)F_z(\theta)$ when $|\theta| < \Theta$, i.e. that $H_0 = 1$ and $H_n = \sum_{k\geqslant 0}(-1)^k z^{k(k+1)/2} H_{n-k-1}$ when $n \geqslant 1$. It follows immediately that $H_n = \mathcal{H}_{\mathbf{B}_n}(z)$ for all $n \geqslant 0$, and therefore that $H_n > 0$. Moreover, the sequence $\frac{\mathcal{H}_{\mathbf{B}_{n+2}}(z)}{\mathcal{H}_{\mathbf{B}_{n+1}}(z)}$ is non-increasing, hence it has a limit when $n \to +\infty$, and therefore that limit must be $\frac{1}{\Theta}$.

Furthermore, the equality $(1 - \theta F_z(\theta))S(\theta) = 1$ holds on the open complex disk $\{z \in \mathbb{C} : |z| < \Theta\}$. It follows that the function $\theta \mapsto 1 - \theta F_z(\theta)$ is decreasing on the interval $[0, \Theta)$. Moreover, if $\Theta < +\infty$, then $1 - \Theta F_z(\Theta) = 0$, and therefore Θ is the smallest positive root of the equation $1 = \theta F_z(\theta)$. Hence, we finish by proving that $\Theta < +\infty$.

Since $0 < z < \frac{2}{5}$, the inequality

$$\begin{aligned} 2^{2k+2} - 1 - (2^{2k+3} - 1)z^{2k+1} &\geqslant 2^{2k+1} - 2^{2k+3}z^{2k+1} = 2^{2k+1}(1 - 4z^{2k+1}) \\ &\geqslant 2^{2k+1}(1 - 4z^3) \geqslant \frac{93 \cdot 2^{2k+1}}{125} > 0 \end{aligned}$$

holds whenever $k \geqslant 1$. Moreover, consider the functions $f : z \mapsto -2 + 7z - 15z^3 + 31z^6$ and $g : z \mapsto -2 + 7z - 13z^3$. When $0 < z < \frac{2}{5}$, we know that $31z^3 \leqslant \frac{31\cdot 8}{125} = \frac{248}{125} < 2$, whence $f(z) = g(z) + z^3(31z^3 - 2) \leqslant g(z)$. Since the derivative of g is such that $g'(z) = 7 - 39z^2 \geqslant 7 - \frac{39\cdot 4}{25} = \frac{19}{25} > 0$, it follows that $f(z) \leqslant g(z) \leqslant g\left(\frac{2}{5}\right) = -\frac{4}{5^3} < 0$.

Consequently, we observe that

$$\begin{aligned}
2F_z\left(\frac{2}{z}\right) - F_z\left(\frac{1}{z}\right) &= \sum_{k\geqslant 0}(-1)^k z^{k(k-1)/2}(2^{k+1}-1) \\
&= 1 - 3 + 7z - 15z^3 + 31z^6 - \\
&\quad \sum_{k\geqslant 2}\left(2^{2k+2} - 1 - (2^{2k+3}-1)z^{2k+1}\right) z^{2k^2+k} \\
&\leqslant f(z) < 0.
\end{aligned}$$

This proves that $1 - \frac{1}{z}F_z\left(\frac{1}{z}\right) < 1 - \frac{2}{z}F_z\left(\frac{2}{z}\right)$, and therefore that $\theta \mapsto 1 - \theta F_z(\theta)$ is not decreasing on $\left[0, \frac{2}{z}\right)$. It follows that $\Theta < \frac{2}{z}$, which completes the proof. $\square$

These results lead to the following conjecture.

Conjecture 7.101.
Let z be an element of the real interval $[0,1)$, and let F_z be the generating function $F_z : \theta \mapsto \sum_{k\geqslant 0}(-1)^k z^{k(k+1)/2}\theta^k$, which converges on the entire complex plane. In addition, let $\mathbf{B}_n^+$ be the n-strand braid monoid, let $\mathcal{H}_{\mathbf{B}_n}(z)$ be the Möbius polynomial of $\mathbf{B}_n^+$, let p_n be the smallest positive root of the polynomial $\mathcal{H}_{\mathbf{B}_n}(z)$, and let $\mathbf{P}$ be the limit of the sequence $(p_n)_{n\geqslant 0}$.

We conjecture that $\mathbf{P} = \mathbf{Q}$, where

$$\mathbf{Q} := \sup\{z \in [0,1) : \exists\theta \in \mathbb{R}_{>0}, 1 = \theta F_z(\theta) \text{ and } \forall\zeta \in \mathbb{C}, |\zeta| < |\theta| \Rightarrow 1 \neq \zeta F_z(\zeta)\}.$$

Note that Proposition 7.100 already proves that $\mathbf{P} \leqslant \mathbf{Q}$. Hence, it remains to prove the converse inequality $\mathbf{P} \geqslant \mathbf{Q}$, i.e. showing that, whenever the root of $1 - \theta F_z(\theta)$ with smallest modulus is a positive real number, then $\mathcal{H}_{\mathbf{B}_n}(z) > 0$ for all $n \geqslant 0$. This question seems wide open so far.

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