ewath 2016/11/07, 08

$g(x,y,z)=x+4y+9z-1=0$ の下で

$$w=f(x,y,z)=x^2+y^2+z^2$$

(x,y,z) で極大(小)ならば

$$\begin{cases} f_x+\lambda g_x=0 \\ f_y+\lambda g_y=0 \\ f_z+\lambda g_z=0 \\ g=0 \end{cases} \iff \begin{cases} 2x+\lambda\cdot 1=0 & \cdots ① \\ 2y+\lambda\cdot 4=0 & \cdots ② \\ 2z+\lambda\cdot 9=0 & \cdots ③ \\ x+4y+9z-1=0 & \cdots ④ \end{cases}$$

を満たす $\lambda\in\mathbb{R}$ が存在する.

①, ②, ③ から

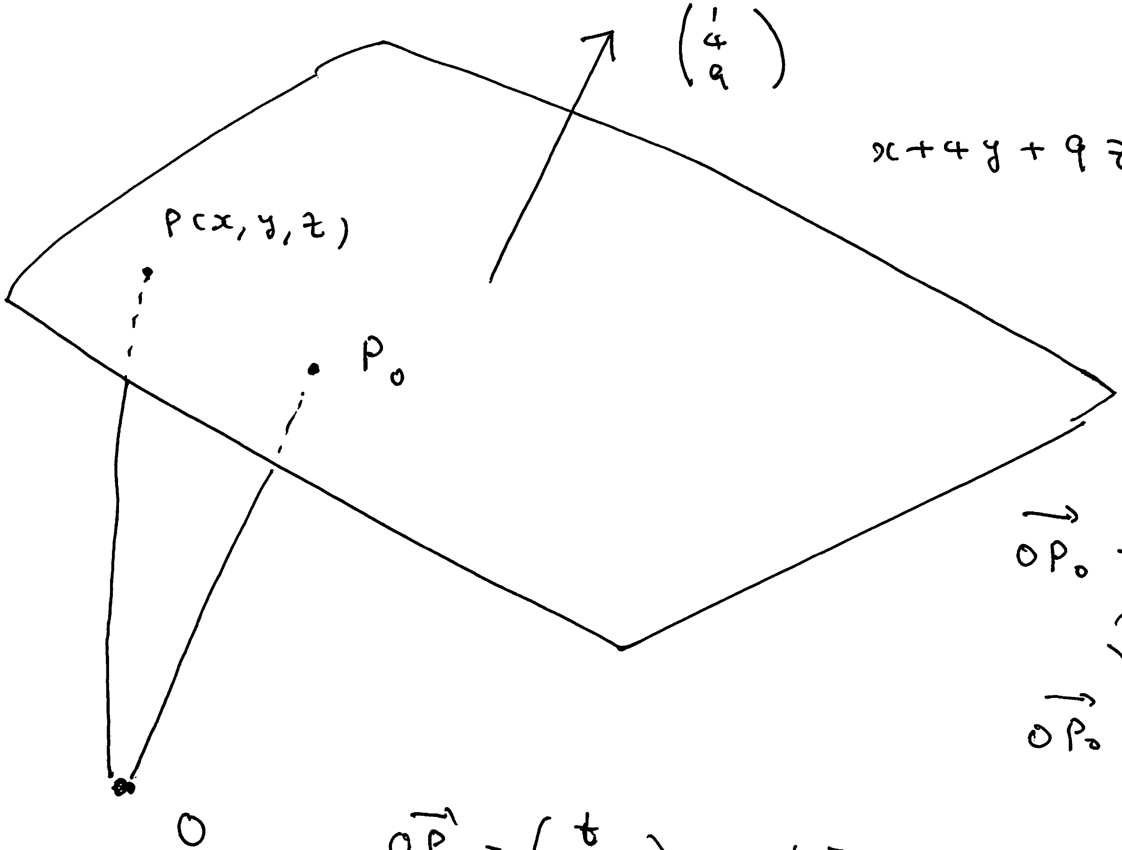
$$x=-\frac{\lambda}{2},\quad y=-2\lambda,\quad z=-\frac{9\lambda}{2} \quad ㊍$$

となる. これを④に代入すると

$$-\frac{\lambda}{2}-8\lambda-\frac{81}{2}\lambda=1$$

よって $\lambda=-\frac{1}{49}$ これを㊍に代入して

$$x=\frac{1}{98},\quad y=\frac{2}{49},\quad z=\frac{9}{98}$$



$x+4y+9z=1$

$\overrightarrow{OP_0} \perp$ 平面 a ことより.

$\Updownarrow$

$\overrightarrow{OP_0} \parallel \begin{pmatrix}1\\4\\9\end{pmatrix}$

$\overrightarrow{OP_0} = \begin{pmatrix}t\\4t\\9t\end{pmatrix}$ とおき

$x+4y+9z-1=0$

に代入して t を求める.

未定乗数法

(x, y, z) で極大(小)

$$\Longrightarrow \begin{cases} f_x(x,y,z) + \lambda g_x(x,y,z) = 0 \\ f_y(\quad) + \lambda g_y(\quad) = 0 \\ f_z(\quad) + \lambda g_z(\quad) = 0 \\ g = 0 \end{cases}$$

を満たす $\lambda \in \mathbb{R}$ が存在する.

$= \nabla(g)(x,y,z)$

$\nabla(f)(x,y,z) + \lambda \nabla(g)(x,y,z) = \vec{0}$

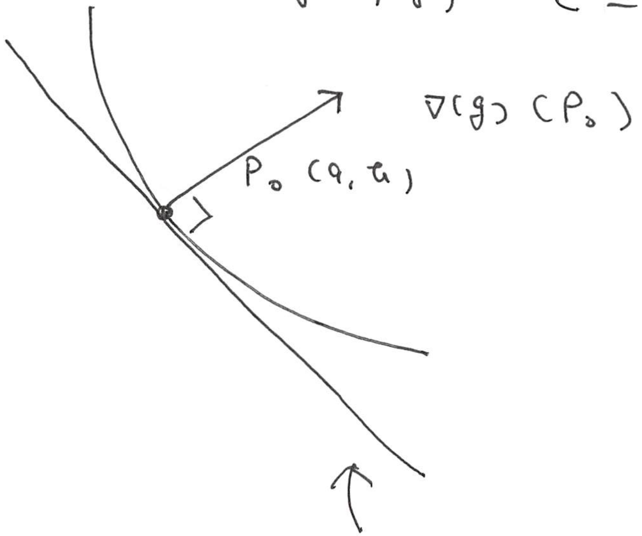
nabla.

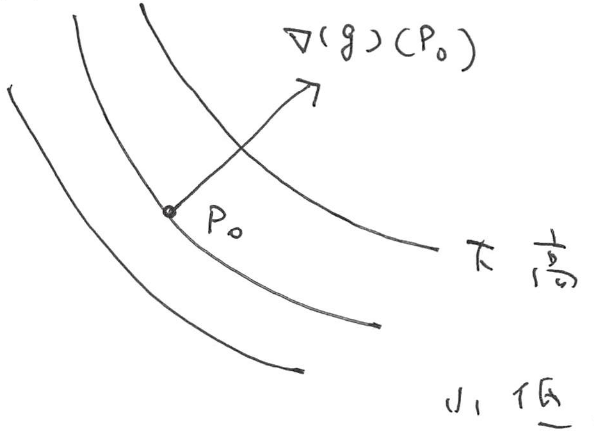
$= \nabla(f)(x,y,z)$

勾配ベクトル

2変数の場合.

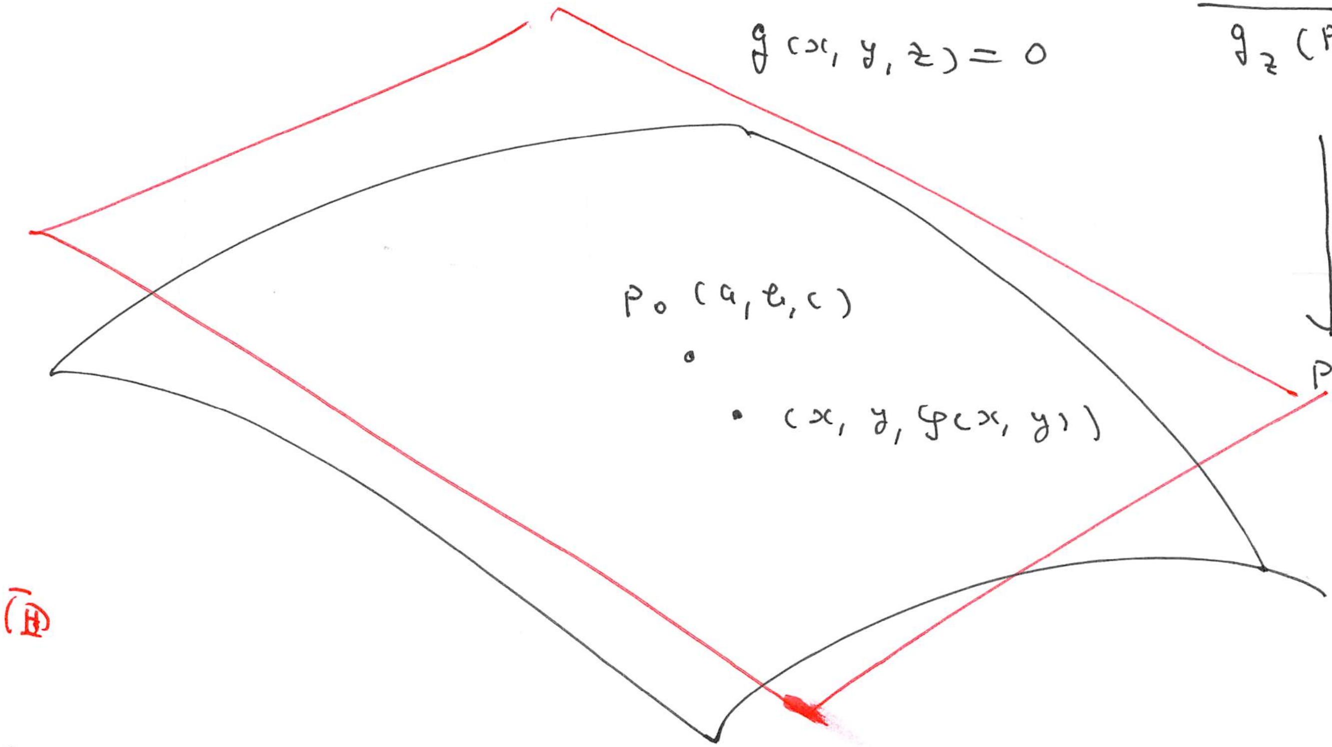
$g(x, y) - c = 0$ 　　等量曲線.





$$g_x(a, b)(x-a) + g_y(a, b)(y-b) = 0$$

3変数の場合.



仮定

$g_z(P_0) \neq 0$ とする

$\downarrow$ 陰関数定理

P_0 の近くで

$z = \varphi(x, y)$

と書ける.

接平面

$$z = \varphi_x(a, b)(x - a) + \varphi_y(a, b)(y - b) + c$$

$\varphi_x(a, b),\ \varphi_y(a, b) = ?$

$\to y$

x で偏微分.

$$g(x, y, \varphi(x, y)) \equiv 0 \longrightarrow$$

$$g_x(x, y, \varphi(x, y)) \cdot 1 + g_y(x, y, \varphi(x, y)) \cdot 0 + g_z(\quad) \cdot \frac{\partial \varphi}{\partial x}(x, y) \equiv 0$$

$x = a,\ y = b$ とすれば $\varphi(a, b) = c$

$$g_x(a, b, c) + \underbrace{g_z(a, b, c)}_{\neq 0} \frac{\partial \varphi}{\partial x}(a, b) = 0$$

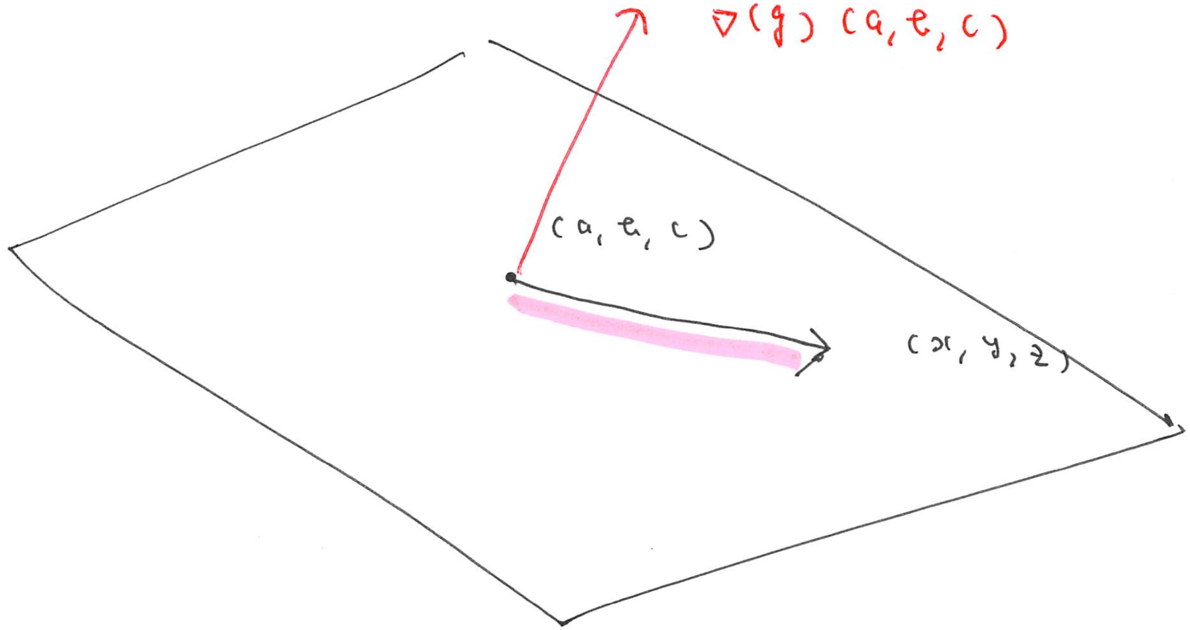
$$\boxed{\frac{\partial \varphi}{\partial x}(a, b) = -\frac{g_x(a, b, c)}{g_z(a, b, c)}}$$

同様に $$\frac{\partial \varphi}{\partial y}(a, b) = -\frac{g_y(a, b, c)}{g_z(a, b, c)}$$

$$z = -\frac{g_x(P_0)}{g_z(P_0)}(x - a) - \frac{g_y(P_0)}{g_z(P_0)}(y - b) + c$$

$$\longrightarrow\ g_x(P_0)(x - a) + g_y(P_0)(y - b) + g_z(P_0)(z - c) = 0$$

$$\begin{pmatrix} g_x(P_0) \\ g_y(P_0) \\ g_z(P_0) \end{pmatrix} \cdot \begin{pmatrix} x - a \\ y - b \\ z - c \end{pmatrix} = 0$$


$\nabla(g)(a, b, c)$
(a, b, c)
(x, y, z)

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} g_x(P_0) \\ g_y(P_0) \\ g_z(P_0) \end{pmatrix} \neq \vec{0} \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

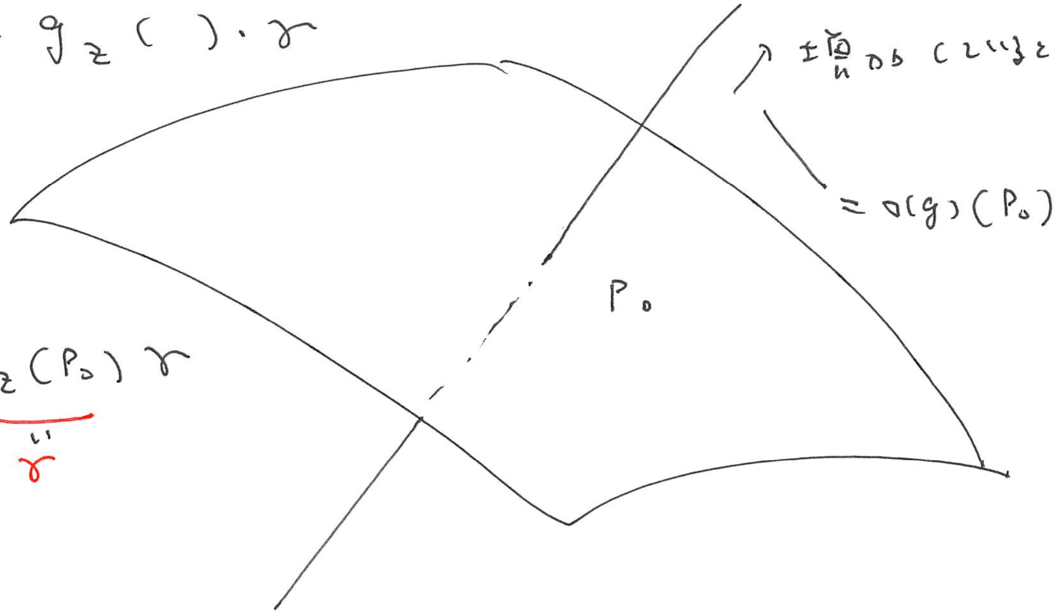
$$G(t) = g(a+\alpha t,\ b+\beta t,\ c+\gamma t)$$

$$G'(t) = g_x(\ \ )\cdot\alpha + g_y(\ \ )\cdot\beta + g_z(\ \ )\cdot\gamma$$

$t=0$ とすれば

$$G'(0) = \overbrace{g_x(P_0)}^{\alpha}\cdot\alpha + \underbrace{g_y(P_0)}_{\beta}\cdot\beta + \underbrace{g_z(P_0)}_{\gamma}\gamma$$

$$= \alpha^2+\beta^2+\gamma^2 > 0$$



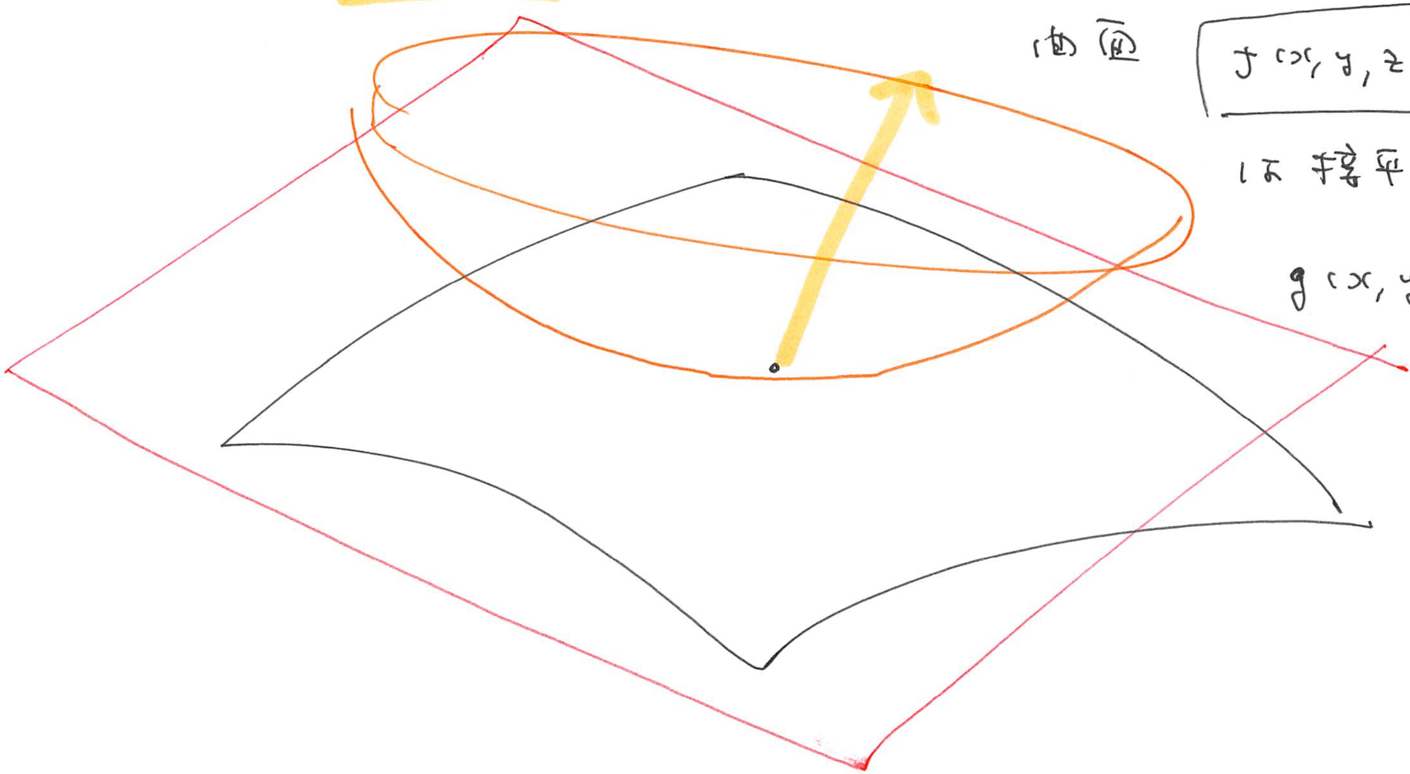
$\nabla(f)(P_0) + \lambda \nabla(g)(P_0) = \vec{0}$ 接点条件.

$\nabla(f)(P_0) = -\lambda \nabla(g)(P_0)$

曲面 $\boxed{f(x, y, z) - f(P_0) = 0}$

は接平面が同じ

$g(x, y, z) = 0$



2制約条件の未定乗数法

$g_1 = x^2 + y^2 + z^2 - 1 = 0$

$g_2 = x - y + z = 0$

$f = 2x - y + 3z$

$\begin{cases} g_1(x, y, z) = 0 \\ g_2(x, y, z) = 0 \end{cases}$ の下で $w = f(x, y, z)$

CTの334pの方法と違う. IT本の方法

$P_0(a, b, c)$ で極大・極小とする.

$g_1(P_0) = g_2(P_0) = 0.$

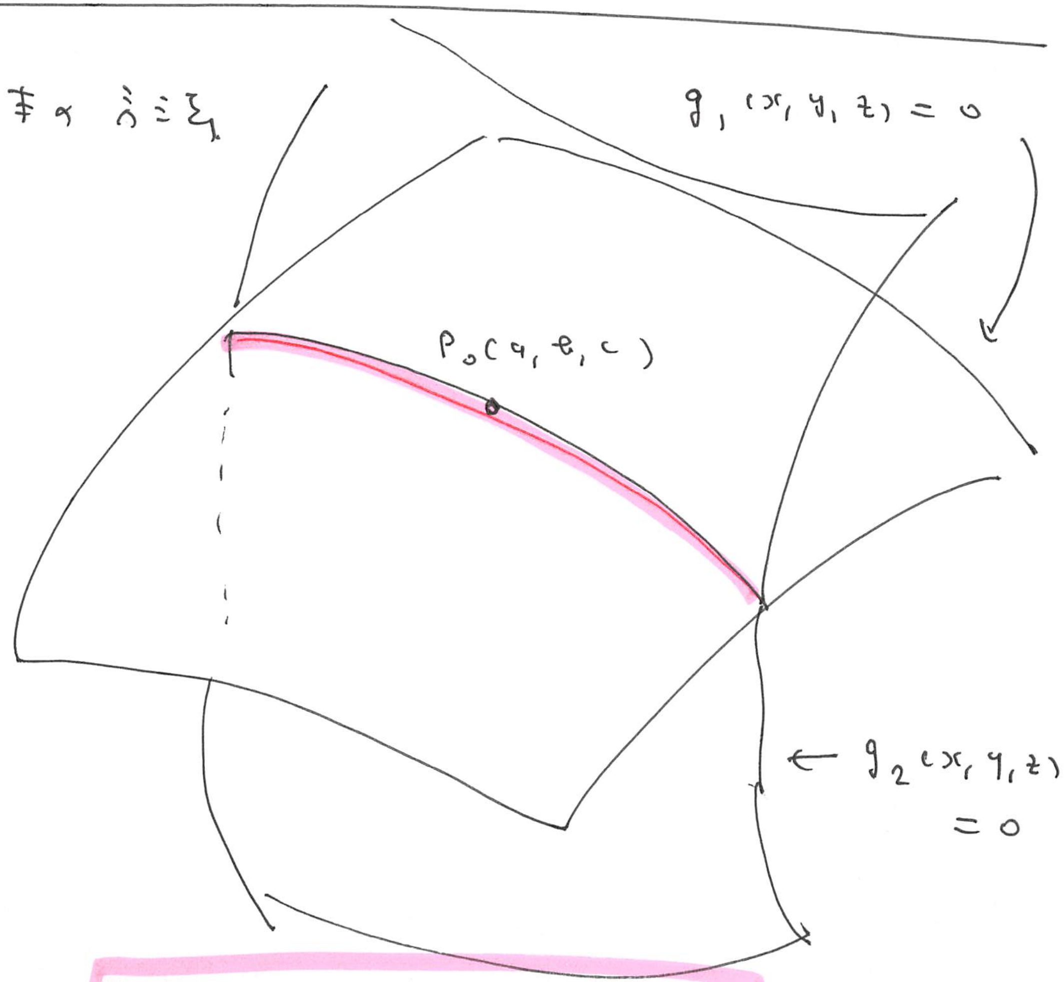
$\nabla(g_1)(P_0) \nparallel \nabla(g_2)(P_0)$

と仮定する.

$$\begin{vmatrix} g_{1x}(P_0) & g_{2x}(P_0) \\ g_{1y}(P_0) & g_{2y}(P_0) \end{vmatrix} \neq 0$$

OR

$$\begin{vmatrix} g_{1x}(\ ) & g_{2x}(\ ) \\ g_{1z}(\ ) & g_{2z}(\ ) \end{vmatrix} \neq 0$$ OR $$\boxed{\begin{vmatrix} g_{1y}(\ ) & g_{2y}(\ ) \\ g_{1z}(\ ) & g_{2z}(\ ) \end{vmatrix} \neq 0}$$



$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\vec{a} \nparallel \vec{b} \iff \left(\lambda \vec{a} + \mu \vec{b} = \vec{0} \Rightarrow \lambda = \mu = 0 \right)$$

$$\iff \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \quad \text{OR} \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \neq 0 \quad \text{OR} \quad \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \neq 0$$

$\Leftarrow$ easy

$\Rightarrow$ difficult

$\Leftarrow$ $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ とする

$$\lambda \vec{a} + \mu \vec{b} = \vec{0} \longleftrightarrow \begin{cases} \lambda a_1 + \mu b_1 = 0 \\ \lambda a_2 + \mu b_2 = 0 \\ \lambda a_3 + \mu b_3 = 0 \end{cases}$$

$$\begin{cases} \lambda a_1 + \mu b_1 = 0 \\ \lambda a_2 + \mu b_2 = 0 \end{cases}$$

$| \; | \neq 0$

クラメルで

$$\lambda = \frac{1}{D} \begin{vmatrix} 0 & b_1 \\ 0 & b_2 \end{vmatrix} = 0$$

$$\mu = \frac{1}{D} \begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix} = 0$$

以下は $\begin{vmatrix} g_{1x}(P_0) & g_{2x}(P_0) \\ g_{1z}(P_0) & g_{2z}(P_0) \end{vmatrix} \neq 0$ のときを考える.

$$\begin{cases} g_1(x, y, z) = 0 \\ g_2(x, y, z) = 0 \end{cases}$$ は P_0 の近くで $y = \varphi(x)$, $z = \psi(x)$ と書ける. $\varphi(a) = b$, $\psi(a) = c$

$\longrightarrow$ 陰関数の定理

$F(t) = f(t, \varphi(t), \psi(t))$ とおく $\longrightarrow$ $F'(a) = 0$ が導かれる.

$$F'(t) = f_x(\quad)\cdot 1 + f_y(\quad)\cdot \varphi'(t) + f_z(\quad)\cdot \psi'(t).$$

$t = a$ を代入

$$0 = F'(a) = f_x(P_0) + f_y(P_0)\varphi'(a) + f_z(P_0)\psi'(a)$$

$$\Updownarrow$$

$$\nabla(f)(P_0) \perp \begin{pmatrix} 1 \\ \varphi'(a) \\ \psi'(a) \end{pmatrix}$$

$$g_{\underset{2}{1}}(t, \varphi(t), \psi(t)) \equiv 0$$

$\longrightarrow$ t で微分

$$g_{\underset{2}{1}x}(\quad)\cdot 1 + g_{\underset{2}{1}y}(\quad)\varphi'(t) + g_{\underset{2}{1}z}(\quad)\psi'(t) \equiv 0$$

$t = a$ を代入

$$\begin{cases} g_{1x}(P_0) + g_{1y}(P_0)\varphi'(a) + g_{1z}(P_0)\psi'(a) = 0 \\ g_{2x}(\quad) + g_{2y}(\quad)\varphi'(a) + g_{2z}(\quad)\psi'(a) = 0 \end{cases}$$

$$\nabla(g_{\underset{2}{1}})(P_0) \perp \begin{pmatrix} 1 \\ \varphi'(a) \\ \psi'(a) \end{pmatrix}$$

$$\left.\begin{aligned} \varphi'(a) &= \cdots \\ \psi'(a) &= \cdots \end{aligned}\right\} \text{解ける}$$

$\longrightarrow$ 来週

$$\nabla(f)(P_0) + \lambda_1 \nabla(g_1)(P_0) + \lambda_2 \nabla(g_2)(P_0) = \vec{0}$$

を満たす λ_1, λ_2 が存在.

$$\begin{cases} x - 2y + z - 1 = 0 \\ x + y - z - 2 = 0 \end{cases}$$

y, z を x で表す（クラメールを使う）