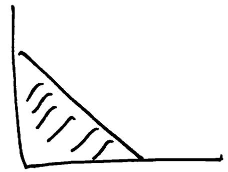
効用関数 $u(x, y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$ $(x, y > 0)$

を $g(x, y) = I - px - qy = 0$ の下で最大化.



$g(x, y) \geqq 0$.
の形で g を決める.

$u_x = \frac{1}{3} x^{-\frac{2}{3}} y^{\frac{1}{3}}$, $u_y = \frac{1}{3} x^{\frac{1}{3}} y^{-\frac{2}{3}}$

$g_x = -p$, $g_y = -q$.

$$\begin{cases} u_x + \lambda g_x = 0 \\ u_y + \lambda g_y = 0 \\ g = 0 \end{cases}$$

を満たす λ が存在.

$$\begin{cases} \frac{1}{3} x^{-\frac{2}{3}} y^{\frac{1}{3}} + \lambda(-p) = 0 & \cdots ① \\ \frac{1}{3} x^{\frac{1}{3}} y^{-\frac{2}{3}} + \lambda(-q) = 0 & \cdots ② \\ I - px - qy = 0 & \cdots ③ \end{cases}$$

を満たす λ が存在する.

① $\frac{1}{3} x^{-\frac{2}{3}} y^{\frac{1}{3}} = \lambda p$ $\cdots ①'$

② $\frac{1}{3} x^{\frac{1}{3}} y^{-\frac{2}{3}} = \lambda q$. $\cdots ②'$

において $\frac{①}{②}$ を考える $\frac{y}{x} = \frac{p}{q}$ となる. 従って $px = yq$ が成立.

③から　$I - px - px = 0$ より　$x = \dfrac{I}{2p}$,　$y = \dfrac{I}{2q}$

需要関数.

<u>このときλ</u>　$= \dfrac{1}{3} \cdot \dfrac{1}{p} \cdot \left(\dfrac{I}{2p}\right)^{-\frac{2}{3}} \left(\dfrac{I}{2q}\right)^{\frac{1}{3}} = \dfrac{1}{3} \cdot \dfrac{2^{\frac{1}{3}}}{I^{\frac{1}{3}} p^{\frac{1}{3}} q^{\frac{1}{3}}}$

$= \dfrac{1}{3} \sqrt[3]{\dfrac{2}{Ipq}}$.　となる.

$\lambda(p, q, I)$　所得の限界効用.

間接効用関数.　$v(p, q, I) = u(x(p, q, I), y(p, q, I))$

$= \left(\dfrac{I}{2p}\right)^{\frac{1}{3}} \left(\dfrac{I}{2q}\right)^{\frac{1}{3}} = 2^{-\frac{2}{3}} I^{\frac{2}{3}} p^{-\frac{1}{3}} q^{-\frac{1}{3}}$

$\dfrac{\partial v}{\partial I} =$

$2^{-\frac{2}{3}} \dfrac{2}{3} I^{-\frac{1}{3}} p^{-\frac{1}{3}} q^{-\frac{1}{3}} = \dfrac{1}{3} 2^{\frac{1}{3}} I^{-\frac{1}{3}} p^{-\frac{1}{3}} q^{-\frac{1}{3}} = \lambda(p, q, I)$

$\boxed{\text{まとめ}\quad \dfrac{\partial v}{\partial I} = \lambda}$

$$\boxed{\frac{\partial v}{\partial p} = -x\frac{\partial v}{\partial I}} \quad \text{Roy の恒等式} \quad = -x\lambda .$$

$$\frac{\partial v}{\partial p} = -\frac{1}{3}\,2^{-\frac{2}{3}}\,I^{\frac{2}{3}}\,p^{-\frac{4}{3}}\,q^{-\frac{1}{3}}$$

$$= -\underbrace{\frac{I}{2p}}_{=\,x(p,q,I)}\cdot\frac{2p}{I}\,\frac{1}{3}\,2^{-\frac{2}{3}}\,I^{\frac{2}{3}}\,p^{-\frac{4}{3}}\,q^{-\frac{1}{3}}$$

$$= -\frac{I}{2p}\cdot\underbrace{\frac{1}{3}\,\frac{2^{\frac{1}{3}}}{I^{\frac{1}{3}}p^{\frac{1}{3}}q^{\frac{1}{3}}}}_{=\,\lambda} = -x(p,q,I)\,\frac{\partial v}{\partial I}(p,q,I)$$

包絡線定理を使わず示す

$u(x,y)$ を　$I-px-qy=0$ の下で最大化.

$(x_0, y_0, \lambda_0, p_0, q_0, I_0)$ において最大化されたとする

未定乗数法

$\longrightarrow$ $\begin{cases} u_x(x,y)+\lambda(-p)=0 \\ u_y(x,y)+\lambda(-q)=0 \\ I-px-qy=0 \end{cases}$ が $x=x_0,\ y=y_0,\ \lambda=\lambda_0$

$p=p_0,\ q=q_0,\ I=I_0$

で成立.

陰関数定理.

$\begin{cases} x=x(p,q,I) \\ y=y(p,q,I) \\ \lambda=\lambda(p,q,I) \end{cases}$ と解ける.　$p\sim p_0,\ q\sim q_0,\ I\sim I_0$.

需要関数

所得の限界効用

$\longrightarrow$ $u_x(x(p,q,I),\ y(p,q,I))+\lambda(p,q,I)(-p)\equiv 0$

$$v = v(p, q, I) = u(x(p, q, I), y(p, q, I))$$

$$\frac{\partial v}{\partial I} = u_x(x(\ ), y(\ )) \cdot \frac{\partial x}{\partial I} + u_y(\qquad) \cdot \frac{\partial y}{\partial I}.$$

$$= \lambda p \frac{\partial x}{\partial I} + \lambda q \frac{\partial y}{\partial I}$$

$$= \lambda \left(\underline{p \frac{\partial x}{\partial I} + q \frac{\partial y}{\partial I}} \right) = \lambda$$

$$\boxed{\frac{\partial v}{\partial I} = \lambda}$$

代入

$$I - p x(p, q, I) - q y(p, q, I) \equiv 0.$$

$\{ \frac{\partial}{\partial I}$ する$\}$ $\quad 1 - p \frac{\partial x}{\partial I} - q \frac{\partial y}{\partial I} = 0 \longrightarrow \underline{p \frac{\partial x}{\partial I} + q \frac{\partial y}{\partial I} = 1}$

Roy の恒等式

$$\boxed{\frac{\partial v}{\partial p} + x \frac{\partial v}{\partial I} \equiv 0.}$$

$$\frac{\partial v}{\partial p} = u_x(x(p, q, I), y(\ )) \cdot \frac{\partial x}{\partial p} + u_y(\qquad) \cdot \frac{\partial y}{\partial p}.$$

$$= \lambda p \frac{\partial x}{\partial p} + \lambda q \frac{\partial y}{\partial p} = \lambda \left(\boxed{p \frac{\partial x}{\partial p} + q \frac{\partial y}{\partial p}} \right) = -\lambda x(p, q, I) = -x \frac{\partial v}{\partial I}$$

$$\boxed{p \frac{\partial x}{\partial p} + q \frac{\partial y}{\partial p}} = -x(p, q, I)$$

$$\frac{\partial}{\partial p} \quad 0 - 1 \cdot x(p, q, I) - p \frac{\partial x}{\partial p} - q \frac{\partial y}{\partial p} = 0$$

II $\quad U(x, y) = \log u(x, y) = \frac{1}{3}\log x + \frac{1}{3}\log y.$

スケール変換

$$u(x, y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$$

$$0 < u_1 < u_2 \iff \log u_1 < \log u_2 \iff U_1 < U_2.$$

$\rightarrow$ $I - px - qy = 0$ の下で U を最大化する.

$$\frac{\partial U}{\partial x} = \frac{1}{3}\cdot\frac{1}{x}, \quad \frac{\partial U}{\partial y} = \frac{1}{3}\cdot\frac{1}{y}$$

$$\begin{cases} \frac{1}{3}\cdot\frac{1}{x} + \lambda(-p) = 0 & \cdots ① \\ \frac{1}{3}\cdot\frac{1}{y} + \lambda(-q) = 0 & \cdots ② \\ I - px - qy = 0 & \cdots ③ \end{cases}$$

① から $\frac{1}{3}\cdot\frac{1}{x} = \lambda p \quad \cdots ①'$ $\qquad \frac{①'}{②'} \quad \frac{y}{x} = \frac{p}{q} \rightarrow$ 従って $xp = yq$

② から $\frac{1}{3}\cdot\frac{1}{y} = \lambda q \quad \cdots ②'$

$$xp = yq = \frac{I}{2} \rightarrow x = \frac{I}{2p}, \; y = \frac{I}{2q}$$

$$
\begin{cases}
g_1(x, y, \lambda, p, q, I) = u_x(x, y) - \lambda p = 0 \\
g_2(\qquad\qquad) = u_y(x, y) - \lambda q = 0 \\
g_3(\qquad\qquad) = I - px - qy = 0.
\end{cases}
$$

$\downarrow$ $\longrightarrow$ 条件

$x = x(p, q, I)$

$y = y(p, q, I)$ と解

$\lambda = \lambda(\qquad)$

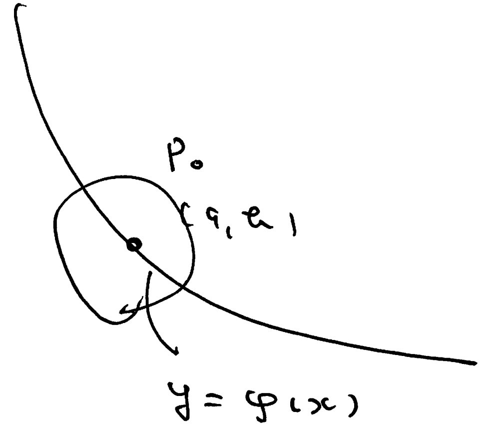
$$
\begin{vmatrix}
\frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial \lambda} \\
\frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial \lambda} \\
\frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial \lambda}
\end{vmatrix} \neq 0. \quad \text{at } (x_0, y_0, \lambda_0, p_0, q_0, I_0)
$$

$$
\parallel
$$

$$
\begin{vmatrix}
u_{xx} & u_{xy} & -p \\
u_{yx} & u_{yy} & -q \\
-p & -q & 0
\end{vmatrix}
$$

一般論

$g(x, y) = 0$ の下で $z = f(x, y)$　$g_y(a, b) \neq 0$



(a, b) で停留点

$F(t) = f(t, \varphi(t))$ において

$F'(a) = 0$

$\longrightarrow$ $\exists \lambda$ に対して

$$\begin{cases} f_x(P_0) + \lambda g_x(P_0) = 0 \\ f_y(P_0) + \lambda g_y(P_0) = 0 \\ g(P_0) = 0 \end{cases}$$

$F''(a) \gtrless 0$?

$F''(a) > 0 \longrightarrow$ 極小

$F''(a) < 0$　　　大

$$F'(t) = f_x(t, \varphi(t)) + f_y(t, \varphi(t)) \cdot \varphi'(t)$$

$$g(t, \varphi(t)) \equiv 0 \longrightarrow g_x(t, \varphi(t)) + g_y(t, \varphi(t))\, \varphi'(t) \equiv 0$$

$$\longrightarrow \varphi'(a) = -\frac{g_x(P_0)}{g_y(P_0)}$$

$$F''(t) = f_{xx}(\ )\cdot 1 + f_{xy}(\ )\, \varphi'(t) +$$

$$\boxed{f_{xy} = f_{yx}}$$

$$\varphi'(t)\left(f_{yx}(\ )\cdot 1 + f_{yy}(\ )\, \varphi'(t) \right)$$

$$+ \varphi''(t)\, f_y(\ )$$

$$= f_{xx} + 2 f_{xy}\, \varphi'(t) + f_{yy}\, \varphi'(t)^2$$

$$+ \varphi''(t) \cdot f_y(\ )$$

$$g_x(t,\varphi(t)) + g_y(t,\varphi(t))\cdot\varphi'(t) \equiv 0$$

$$\to \quad g_{xx}(\ ) + 2g_{xy}(\ )\varphi'(t) + g_{yy}(\ )\varphi'(t)^2$$
$$+ \varphi''(t)\, g_y(\ ) \equiv 0$$

$$\varphi''(a) = -\frac{1}{g_y(P_0)}\left(g_{xx}(P_0) + 2g_{xy}(P_0)\cdot\varphi'(a) + g_{yy}(P_0)\varphi'(a)^2\right)$$

$$F''(a) = f_{xx}(P_0) + 2f_{xy}(P_0)\varphi'(a) + f_{yy}(P_0)\varphi'(a)^2$$
$$\underbrace{\boxed{-\frac{f_y(P_0)}{g_y(P_0)}}}_{=\lambda}\left(g_{xx}(P_0) + 2g_{xy}\varphi'(a) + g_{yy}(P_0)\varphi'(a)^2\right)$$

$$L = f + \lambda g.$$

$$= L_{xx} + 2L_{xy}\varphi'(a) + L_{yy}\varphi'(a)^2$$

$$\boxed{\varphi'(a) = -\frac{g_x(P_0)}{g_y(P_0)}}$$

$$= \frac{1}{g_y(P_0)^2}\left(L_{xx}\, g_y(P_0)^2 - 2L_{xy}\, g_x(P_0)g_y(P_0) + L_{yy}\, g_x(P_0)^2\right)$$

$$\begin{vmatrix} 0 & \alpha & \beta \\ \alpha & A & C \\ \beta & C & B \end{vmatrix} = -\left(A\beta^2 - 2C\alpha\beta + B\alpha^2\right)$$

$$= B$$

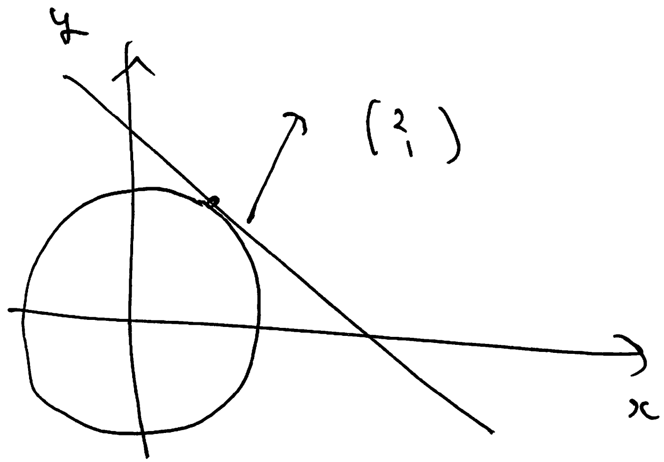
$$F''(a) = -\frac{1}{g_y(P_0)^2} \boxed{\begin{vmatrix} 0 & g_x(P_0) & g_y(P_0) \\ g_x(P_0) & L_{xx}(P_0, \lambda) & L_{xy}(\ \ ) \\ g_y(P_0) & L_{yx}(\ \ ) & L_{yy}(\ \ ) \end{vmatrix}}$$

$B(a, b, \lambda) < 0 \iff F''(a) > 0 \iff (a, b)$ で極小.

$>$ $\qquad$ $<$ $\qquad$ 大.

$g(x, y) = 2x + y - 1 = 0$

$z = f(x, y) = x^2 + y^2$



$$\begin{cases} 2x + \lambda \cdot 2 = 0 \\ 2y + \lambda \cdot 1 = 0 \\ 2x + y - 1 = 0 \end{cases}$$

$x = -\lambda, \quad y = -\frac{1}{2}\lambda$ $\qquad$ $x = \frac{2}{5}, \quad y = \frac{1}{5}$

$-2\lambda - \frac{1}{2}\lambda = 1 \qquad \lambda = -\frac{2}{5}$

$L = x^2 + y^2 + \lambda(2x + y - 1)$

$$B = \begin{vmatrix} 0 & -2 & -1 \\ -2 & 2x & 0 \\ -1 & 0 & 2y \end{vmatrix} = -2x - 8y.$$

$B\left(\frac{2}{5}, \frac{1}{5}, \lambda = -\frac{2}{5}\right) = -\frac{4}{5} - \frac{8}{5} = -\frac{12}{5} < 0$ 極小

$$u(x, y) = \sqrt{xy}.$$

$$\begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} = ?$$

$$\begin{vmatrix} 0 & u_x & u_y \\ u_x & u_{xx} & u_{xy} \\ u_y & u_{yx} & u_{yy} \end{vmatrix} = ?$$