LECTURE 14: LOCAL FUNCTIONAL EQUATION
LECTURE BY ASIF ZAMAN
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NOTES BY DAN DORE AND ASIF ZAMAN

Let F be a local field. We fix $\psi \in \widehat{F}$, $\psi \neq 1$, such that $\psi|_{\mathfrak{p}^d} \cong 1$ for $d \gg 0$ with $\mathfrak{p}$ the maximal ideal (ϖ) of F. Let $q = N(\varpi) = |F/\mathfrak{p}|$, and let v be the valuation. We fix $dx, d^\times x$ Haar measures on $F, F^\times$ respectively with the latter normalized so that $\int_{\mathscr{O}_F^\times} d^\times x = 1$.

We recall the GL_1 theory. Let $\Phi \in C_c^\infty(F)$ be a smooth compactly supported function. Then we may define the Fourier transform

$$\widehat{\Phi}(x) = \int_F \Phi(y)\overline{\psi(xy)}\, dy.$$

This satisfies the Fourier inversion formula $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$. For some character χ on $F^\times = \mathrm{GL}_1(F)$, we define the zeta function

$$\zeta(\Phi, \chi, s) = \int_{F^\times} \Phi(x)\chi(x)|x|^s\, d^\times x$$

and we define the set:

$$\mathcal{Z}(\chi, s) = \left\{\zeta(\Phi, \chi, s) \mid \Phi \in C_c^\infty(F)\right\}.$$

We have the following theorem:

Theorem 1.

(A) (γ-factor) There exists a unique gamma factor $\gamma(\chi, s) \in \mathbf{C}(q^{-s})$ such that

$$\zeta(\widehat{\Phi}, \chi^{-1}, 1-s) = \gamma(\chi, s)\zeta(\Phi, \chi, s)$$

and $\gamma(\chi, s)\gamma(\chi^{-1}, 1-s) = \chi(-1)$.

(B) (L-function) The set $\mathcal{Z}(\chi, s)$ is equal to $L(\chi, s)\mathbf{C}[q^{-s}, q^s]$. Here, we have:

$$L(\chi, s) = \begin{cases} 1 & \chi \text{ ramified} \\ \left(1 - \chi(\varpi)q^{-s}\right)^{-1} & \chi \text{ unramified} \end{cases}$$

(C) (ϵ-factors) The function

$$\epsilon(\chi, s) = \gamma(\chi, s)\frac{L(\chi, s)}{L(\chi^{-1}, 1-s)}$$

satisfies $\epsilon(\chi, s)\epsilon(\chi^{-1}, 1-s) = \chi(-1)$ and $\epsilon(\chi, s) = \epsilon(\chi)q^{(d+f)\left(\frac{1}{2}-s\right)}$ with $|\epsilon(\chi)| = 1$. If χ is unramified then $\epsilon(\chi) = \chi(\varpi)^d$. If χ is ramified then $\epsilon(\chi)$ is a certain Gauss sum depending on χ and ψ. Here f is defined so that the norm of the conductor of χ is q^f.

Remark 2. The ϵ factor acts as the "fudge factor" to make the functional equation hold exactly for the L-function. For more precise formulas of the ϵ-factors and the relevant Gauss sums, see [Go, Equations (233) and (240)] or [BH, p.143]. If χ is unramified then the L-function $L(s,\chi)$ uniquely determines χ. Otherwise, $L(s,\chi)$ gives no information when χ is ramified; in this case, $\epsilon(\chi,s)=\gamma(\chi,s)$ so the ϵ-factor encodes all of the data of χ.

Remark 3. We suppress the dependence on the choice of ψ everywhere. However, while the ϵ and γ factors do depend on the choice of ψ, the L-function is the same for all appropriate ψ.

Now, we will pass to the GL_2 case. Given an irreducible admissible representation π of $\mathrm{GL}_2(F)$, we have the Whittaker and Kirillov models defined by:

$$\mathcal{W}(\pi)=\left\{W\colon G\to\mathbf{C}\mid W\left(\left(\begin{smallmatrix}x & \\ & 1\end{smallmatrix}\right)\cdot g\right)=\psi(x)W(g)\right\},$$

$$\mathcal{K}(\pi)=\left\{\phi\colon F^\times\to\mathbf{C}\mid \pi\left(\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right)\right)\phi(x)=\psi(bx)\phi(ax)\right\}.$$

We have a bijection from $\mathcal{W}(\pi)$ to $\mathcal{K}(\pi)$ defined by sending W to $\phi_W\colon x\mapsto W\left(\left(\begin{smallmatrix}x & 0\\ 0 & 1\end{smallmatrix}\right)\right)$. The "dual" Whittaker functional $\widetilde{W}\in\mathcal{W}(\pi)$ is defined to be the function $g\mapsto W(gw)$ for $w=\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$, the generator of the Weyl group. Recall $C_c^\infty(F^\times)\subseteq\mathcal{K}(\pi)$ always.

Now, define a zeta integral similar to before:

$$Z(W,\chi,s)=\int_{F^\times}\phi_W(x)\chi(x)^{-1}|x|^{2s-1}\,d^\times x$$

The function ϕ_W can be thought of as a test function varying over $W\in\mathcal{W}(\pi)$.

Remark 4. This is defined in vague analogy to the definition given above in the GL_1 case, but the analogy isn't perfect. For example, in the GL_1 case, $\zeta(\Phi,\chi,s)$ is defined for $\Phi\in C_c^\infty(F)$, but here we consider ϕ_W defined on $F^\times$ rather than on F. A closer parallel to the GL_1 theory can be drawn using functions Φ on $M_{2\times 2}(F)$; see [BH, Section 24] for details.

Before we discuss the GL_2 local functional equation, we prove the existence of a Whittaker functional such that it and its dual have Kirillov elements both supported *away from zero*, while also having a non-trivial zeta integral.

Lemma 5. There exists $W\in\mathcal{W}(\pi)$ such that $\phi_W\in C_c^\infty(F^\times)$, $\phi_{\widetilde{W}}\in C_c^\infty(F^\times)$ and $Z(W,\chi,s)\neq 0$.

Proof. From [Go, Lemma 7, page 1.13], there exists non-zero $W\in\mathcal{W}(\pi)$ with $\phi_W\in C_c^\infty(F^\times)$, $\phi_{\widetilde{W}}\in C_c^\infty(F^\times)$ satisfying $\phi_W(xu)=\phi_W(x)\chi(u)$ for all $u\in\mathscr{O}_F^\times$. By direction computation, we verify that the zeta integral does not vanish:

$$\begin{aligned}Z(W,\chi,s)&=\int_{F^\times}\phi_W(x)\chi(x)^{-1}|x|^{2s-1}d^\times x\\&=\sum_{n=-\infty}^{\infty}\int_{\mathscr{O}_F^\times}\phi_W(\varpi^n u)\chi(\varpi^n u)^{-1}|\varpi^n|^{2s-1}d^\times u\\&=\sum_{n=-\infty}^{\infty}\phi_W(\varpi^n)\chi(\varpi^n)^{-1}q^{(2s-1)n}.\end{aligned}$$

Note the sum is actually finite since ϕ_W is compactly supported away from zero. Thus, $Z(W, \chi, s)$ is a non-zero element of $\mathbf{C}[q^{-s}, q^s]$ if and only if $\phi_W \not\equiv 0$. The latter is true as W is non-zero. This finishes Step 2. $\square$

Now, analogous to Theorem 1(A) in the GL_1 case, we have the following theorem:

Theorem 6.

(i) $Z(W, \chi, s)$ converges for $\mathrm{Re}(s) \gg 0$.

(ii) $Z(W, \chi, s)$ admits an analytic continuation to a meromorphic function with at most 2 poles.

(iii) There exists some $\gamma_\pi(\chi, s) \in \mathbf{C}(q^{-s})$ such that:

$$Z(\widetilde{W}, \omega_\pi \chi^{-1}, 1 - s) = \gamma_\pi(\chi, s) Z(W, \chi, s) \tag{1}$$

for all $W \in \mathcal{W}(\pi)$. Also,

$$\gamma_\pi(\chi, s)\gamma_\pi(\chi^{-1}\omega_\pi, 1 - s) = \omega_\pi(-1). \tag{2}$$

Proof. (i) When π is supercuspidal, this part is easy. We have that $\phi_W \in \mathcal{K}(\pi) = C_c^\infty(F^\times)$, so the integral converges due to the compact support *away from zero*. We may split the integral into a *finite* sum according to the valuation of $x \in F^\times$.

Suppose π is not supercuspidal. From an earlier description of the Kirillov model [Go, Section 10], it follows that $\phi_W \in \mathcal{K}(\pi)$ is a sum of terms like $|x|^{1/2}\lambda(x)f(x)$ and $|x|^{1/2}v(x)\lambda(x)f(x)$, where $\lambda \colon F^\times \to \mathbf{C}^\times$ is some character and $f \in C_c^\infty(F)$, with v the valuation. We claim that this implies that the integral converges. Let us consider one such integral. Since f is locally constant near 0, it follows that $f(x) = f(0)$ for $|x| \leqslant q^{-N}$ with $N \geqslant 1$ sufficiently large. Moreover, as f is compactly supported in F, it follows that $f(x) = 0$ for $|x| \geqslant q^M$ with $M \geqslant 1$ sufficiently large. Thus,

$$\int_{F^\times} |x|^{1/2}\lambda(x)f(x) \cdot |x|^{2s-1}\, d^\times x = f(0)\int_{|x| \leqslant q^{-N}} |x|^{2s-1/2}\lambda(x)\, d^\times x + \int_{q^{-N} < |x| \leqslant q^M} (\cdots)$$

The second integral over $q^{-N} < |x| \leqslant q^M$ can be written as a *finite* sum over $m = v(x)$ with $-N \leqslant m \leqslant M$ and therefore it converges for all $s \in \mathbf{C}$. For the first integral, we also divide it according to the valuation $n = -v(x)$ and observe that

$$\int_{|x| \leqslant q^{-N}} |x|^{2s-1/2}\lambda(x)\, d^\times x = \sum_{n \geqslant N} q^{n/2-2ns} \int_{|x|=q^{-n}} \lambda(x) d^\times x$$

For $|x| = q^{-n}$, we may write $x = \varpi^n y$ with $y \in \mathscr{O}_F^\times$. Since λ is a character of $F^\times$, it follows that $|\lambda(y)| \leqslant 1$ and thus $|\lambda(x)| = |\lambda(\varpi)|^n$ for $|x| = q^{-n}$. Moreover, $\int_{|x|=q^{-n}} d^\times x = \int_{\mathscr{O}_F^\times} d^\times x = 1$ by our normalization of the Haar measure. Hence, the above expression is bounded in absolute value by

$$\sum_{n \geqslant N} q^{n/2-2n\,\mathrm{Re}(s)} \int_{|x|=q^{-n}} |\lambda(x)| d^\times x \leqslant \sum_{n \geqslant N} q^{n/2-2n\,\mathrm{Re}(s)} |\lambda(\varpi)|^n.$$

As $|\lambda(\varpi)|$ is some fixed power of q, the above infinite sum converges once $\mathrm{Re}(s) \gg 0$.

(ii) Proof postponed[1]. We will not utilize this result until after Theorem 8.

(iii) This follows from three steps.

Step 1: Show (1) *holds for any* $W \in \mathcal{W}(\pi)$ *with* $\phi_W \in C_c^\infty(F^\times)$.

(We will prove this step last.)

Step 2: Show (2) *holds.* Choose W from Lemma 5. From Steps 1 and 2, we may apply Equation 1 twice to see that:

$$\begin{aligned}\omega_\pi(-1)Z(W,\chi,s) = Z(\breve{\breve{W}},\chi,s) &= \gamma_\pi(\chi^{-1}\omega_\pi, 1-s)Z(\breve{W},\chi^{-1}\omega_\pi,1-s)\\ &= \gamma_\pi(\chi^{-1}\omega_\pi,1-s)\gamma_\pi(\chi,s)Z(W,\chi,s)\end{aligned}$$

Since $Z(W,\chi,s) \not\equiv 0$, we may divide both sides by $Z(W,\chi,s)$ to deduce (2) holds.

Step 3: Show (1) *holds for any* $W \in \mathcal{W}(\pi)$.

For every $W \in \mathcal{W}(\pi)$, there exists $W_1, W_2 \in \mathcal{W}(\pi)$ such that

$$\phi_W = \phi_{W_1} + \phi_{\breve{W}_2} \qquad \text{and} \qquad \phi_{W_1}, \phi_{\breve{W}_2} \in C_c^\infty(F^\times).$$

This follows from the arguments leading to [Go, Section 10, Equation (144)]. Thus, we may apply the functional equation (1) to each of $Z(W_1,\chi,s)$ and $Z(\breve{W}_2,\chi,s)$ and use linearity to deduce (1) for $Z(W,\chi,s)$.

The remainder of the proof is to establish Step 1. First, we make a reduction.

Claim 7. Any $f \in C_c^\infty(F^\times)$ is a linear combination of functions of the form:

$$\lambda(x)\mathbf{1}_{\mathscr{O}_F^\times}(x)$$

with $\lambda\colon F^\times \to \mathbf{C}^\times$ some character and $\mathbf{1}_{\mathscr{O}_F^\times}$ the indicator function on $\mathscr{O}_F^\times$.

Proof of Claim 7: Recall $F^\times \cong \mathbf{Z} \times \mathscr{O}_F^\times$. Since f is compactly supported and locally constant, there exists positive integers $N, M \geqslant 1$ (dependingly only on f) such that for every $x \in F^\times$, $f(x) = f(\varpi^n u)$ for some unique integer $n \in [-N, N]$ and unique u chosen from a fixed set of coset representatives Ω of $\mathscr{O}_F^\times/(1+\varpi^M\mathscr{O}_F^\times)$. Therefore,

$$f(x) = \sum_{-N\leqslant n\leqslant N}\sum_{u\in\Omega} f(\varpi^n u)\mathbf{1}_{u+\varpi^M\mathscr{O}_F^\times}(x\varpi^{-n}).$$

By orthogonality of characters on the finite quotient group $\mathscr{O}_F^\times/(1+\varpi^M\mathscr{O}_F^\times)$, the indicator function $\mathbf{1}_{u+\varpi^M\mathscr{O}_F^\times}(y)$ can be written as a finite linear combination of $\lambda(y)\mathbf{1}_{\mathscr{O}_F^\times}(y)$ where λ is a character of $F^\times$ with conductor at most q^M. This proves the claim. ■

[1]It is not apparent to me that a complete proof is provided in [Go, Section 12]. From the functional equation, $Z(W,\chi,s)$ is meromorphic in $\mathrm{Re}(s) \geqslant A$ and $\mathrm{Re}(s) \leqslant 1-A$ for some large positive A but, without additional work, it is not obvious why it extends to the strip $1-A \leqslant \mathrm{Re}(s) \leqslant A$. Instead, this extra input will follow implicitly from Theorems 8 and 1(B).

Continuing the proof of Step 1, recall we assume $\phi_W \in C_c^\infty(F^\times)$. By Claim 7 and the linearity of $Z(W, \chi, s)$ in W, it suffices to show the functional equation for ϕ_W of the form:

$$\phi_W(x) = \lambda(x)\mathbf{1}_{\mathscr{O}_F^\times}(x)$$

with $\lambda\colon F^\times \to \mathbf{C}^\times$ some arbitrary character. During the course of our computations, it is crucial that the calculated γ factor depends only on π, χ, F and s. In particular, γ should be *independent* of the arbitrary character λ. Now, as λ and χ are characters on $F^\times$, the maps $\lambda|_{\mathscr{O}_F^\times}$ and $\chi|_{\mathscr{O}_F^\times}$ are (necessarily unitary) characters on the compact group $\mathscr{O}_F^\times$. Hence, by orthogonality of characters (for the compact group $\mathscr{O}_F^\times$),

$$Z(W, \chi, s) = \int_{\mathscr{O}_F^\times} \lambda(x)\chi(x)^{-1}\, d^\times x = \begin{cases} 1 & \lambda|_{\mathscr{O}_F^\times} = \chi|_{\mathscr{O}_F^\times} \\ 0 & \text{else} \end{cases} \tag{3}$$

because $\int_{\mathscr{O}_F^\times} d^\times x = 1$ via our choice of normalization.

If $\lambda|_{\mathscr{O}_F^\times} \neq \chi|_{\mathscr{O}_F^\times}$ then one may again verify by a similar computation that

$$Z(\breve{W}, \omega_\pi\chi^{-1}, 1 - s) = 0.$$

Thus, in this case, any choice of $\gamma_\pi(\chi, s)$ would satisfy a functional equation.

Otherwise, we have reduced to the case when

$$\phi_W(x) = \chi(x)\mathbf{1}_{\mathscr{O}_F^\times}(x).$$

In particular, ϕ_W now depends *only* on χ. Therefore, for this ϕ_W, we define

$$\gamma_\pi(\chi, s) := \frac{Z(\breve{W}, \omega_\pi\chi^{-1}, 1 - s)}{Z(W, \chi, s)} = \int_{F^\times} \phi_{\breve{W}}(x)\omega_\pi^{-1}\chi(x)|x|^{1-2s}\, d^\times x$$

with the latter equality following from (3) and the definition of the zeta integral. Evidently, γ_π satisfies the functional equation for this choice of ϕ_W and, since ϕ_W depends only on χ, we see that γ_π depends only on χ and s. Therefore, we've shown that for all $W \in \mathcal{W}(\pi)$ with $\phi_W \in C_c^\infty(F^\times)$ that (1) holds. This completes the proof of Step 1 and hence Theorem 6.

□

Note that the operation $W \mapsto \breve{W}$ is not actually anything to do with a Fourier transform: the duality in the functional equation appearing in Theorem 6 comes from the action of the Weyl group (which of course is trivial in the GL_1 case). In the next theorem, the Fourier transform plays a role .

Theorem 8. Define:

$$L_\pi(\chi, s) = \begin{cases} 1 & \pi \text{ cuspidal} \\ L\left(\chi^{-1}\mu_1, 2s - \frac{1}{2}\right) \cdot L\left(\chi^{-1}\mu_2, 2s - \frac{1}{2}\right) & \pi = \pi_{\mu_1,\mu_2}, \mu_1/\mu_2 \neq |\cdot|, |\cdot|^{-1} \\ L\left(\chi^{-1}\mu_1, 2s - \frac{1}{2}\right) & \pi = \pi_{\mu_1,\mu_2}, \mu_1/\mu_2 = |\cdot| \\ L\left(\chi^{-1}\mu_2, 2s - \frac{1}{2}\right) & \pi = \pi_{\mu_1,\mu_2}, \mu_1/\mu_2 = |\cdot|^{-1} \end{cases}$$

With this definition, we have:

$$\left\{Z(W, \chi, s) \mid W \in \mathcal{W}(\pi)\right\} = L_\pi(\chi, s) \cdot \mathbf{C}[q^{-2s}, q^{2s}]$$

Thus, if π is induced from $\mu_1 \otimes \mu_2$ on $T = B/N \simeq \mathbf{G}_m \times \mathbf{G}_m$, then the L-function is the product of the GL_1 L-functions for μ_1, μ_2. If π is cuspidal, then just like in the ramified case for GL_1, the L-factor carries no information.

Proof. If π is cuspidal, there is not much to show. We have:

$$Z(W, \chi, s) = \int_{F^\times} \phi_W(x)\chi^{-1}(x)|x|^{2s-1}\, d^\times x$$

Since $\phi_W \in \mathcal{K}(\pi) = C_c^\infty(F^\times)$, the integral breaks into a sum of finitely many terms based on the valuation of x, so $Z(W, \chi, s) \in \mathbf{C}[q^{-2s}, q^{2s}]$.

If $\pi = \pi_{\mu_1,\mu_2}$ with $\mu_1/\mu_2 \neq 1, |\cdot|, |\cdot|^{-1}$ then we have:

$$\phi_W(x) = |x|^{1/2}\left(\mu_1(x)\Phi_1(x) + \mu_2(x)\Phi_2(x)\right)$$

with $\Phi_j \in C_c^\infty(F)$ and $\mu_1, \mu_2 \colon F^\times \to \mathbf{C}^\times$ characters. Then we have:

$$Z(W, \chi, s) = \zeta\left(\Phi_1, \chi^{-1}\mu_1, 2s - \frac{1}{2}\right) + \zeta\left(\Phi_2, \chi^{-1}\mu_2, 2s - \frac{1}{2}\right)$$

We will write $z = 2s - \frac{1}{2}$ here and in the future. By Theorem 1(B), this is contained in:

$$L(\chi^{-1}\mu_1, z)\mathbf{C}[q^{-2s}, q^{2s}] + L(\chi^{-1}\mu_2, z)\mathbf{C}[q^{-2s}, q^{2s}] = L(\chi^{-1}\mu_1, z)L(\chi^{-1}\mu_2, z)\mathbf{C}[q^{-2s}, q^{2s}]$$

The claimed equality holds since $\mu_1 \neq \mu_2$ implies the two GL_1 L-functions have different poles (as one can see by looking at the defining formula). For the cases when $\mu_1/\mu_2 = 1, |\cdot|, |\cdot|^{-1}$, similar arguments hold but with minor variations due to the precise characterization of the Kirillov models $\mathcal{K}(\pi)$. See [Go, pp.145–147] for details. □

Now, we give a computation of the γ factors in the case of principal series. Paralleling the GL_1 theory for ramified characters, the case of cuspidal representations is more subtle and involves an analogue of the Gauss sum. See [BH, Section 25] for details.

Theorem 9. If $\pi = \pi_{\mu_1,\mu_2}$, then:

$$\gamma_\pi(\chi, s) = \gamma\left(\chi^{-1}\mu_1, 2s - \frac{1}{2}\right) \cdot \gamma\left(\chi^{-1}\mu_2, 2s - \frac{1}{2}\right)$$

Proof. Choose W from Lemma 5 and let $\phi = \phi_W$. We have the following claims:

Claim 10. There exists some $\Phi \in C_c^\infty(F)$ which extends to $\Phi \in \mathcal{B}_{\mu_1,\mu_2}$ such that

$$\Phi(w^{-1}\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)) = \Phi(y)$$

and the Fourier transform $\widehat{\Phi} \in C_c^\infty(F^\times)$ satisfies $Z(W, \chi, s) = \zeta(\Phi, \chi^{-1}\mu_2, z)$. Here $\mathcal{B}_{\mu_1,\mu_2}$ is the space of locally constant functions $\varphi : G_F \to \mathbf{C}$ satisfying

$$\varphi\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \cdot g\big) = \mu_1(a)\mu_2(b)|a/b|^{1/2}\varphi(g).$$

Proof of Claim 10: Set $\widehat{\Phi}(x) = \mu_2^{-1}(x)|x|^{-1/2}\phi_W(x) \in C_c^\infty(F^\times)$. By Fourier inversion,

$$\Phi(-y) = \int \hat{\Phi}(x)\psi(xy)dx \in C_c^\infty(F).$$

By the Fourier transform [Go, Equation (148)], it follows that $\Phi \in \mathcal{B}_{\mu_1,\mu_2}$ and $\Phi(w^{-1}\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)) = \Phi(y)$. This proves the claim. ■

Claim 11. Choose Φ as in the proof of Claim 10. Define Φ_w by $g \mapsto \Phi(gw)$ so its restriction $\Phi_w \in C_c^\infty(F)$ satisfies $x \mapsto \Phi\big(w^{-1}\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)w\big)$. Then, with $z = 2s - \frac{1}{2}$,

$$Z\Big(\widetilde{W}, \chi^{-1}\omega_\pi, 1-s\Big) = \zeta\Big(\widehat{\Phi}_w, \chi\mu_1^{-1}, 1-z\Big).$$

Proof of Claim 11: By a direct substitution of the definitions of Z, ζ and Φ,

$$\begin{aligned} Z\Big(\widetilde{W}, \chi^{-1}\omega_\pi, 1-s\Big) &= \int_{F^\times} \phi_{\widetilde{W}}(x)\chi\omega_\pi^{-1}(x)|x|^{1-2s}\, d^\times x \\ &= \int_{F^\times} \widehat{\Phi}(x)\mu_2(x)|x|^{1/2} \cdot \chi(x)\mu_1^{-1}(x)\mu_2(x)^{-1/2} \cdot |x|^{1-z}|x|^{-1/2}\, d^\times x \\ &= \int_{F^\times} \widehat{\Phi}(x)\chi\mu_1^{-1}(x)|x|^{1-z}\, d^\times x \\ &= \zeta\Big(\widehat{\Phi}_w, \chi\mu_1^{-1}, 1-z\Big). \end{aligned}$$

This proves the claim. ■

Claim 12. Choose Φ as in the proof of Claim 10. Then

$$\zeta\big(\Phi_w, \chi^{-1}\mu_1, z\big) = (\mu_1\chi^{-1})(-1)\zeta\big(\Phi, \chi\mu_2^{-1}, 1-z\big)$$

Proof of Claim 12: By Claim 11,

$$\Phi_w(y) = \Phi\Big(w^{-1}\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)w\Big) = \Phi\Big(\left(\begin{smallmatrix} 1 & -1/y \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/y & 0 \\ 0 & y \end{smallmatrix}\right)w\left(\begin{smallmatrix} 1 & -1/y \\ 0 & 1 \end{smallmatrix}\right)\Big).$$

As $\Phi \in \mathcal{B}_{\mu_1,\mu_2}$ by Claim 10, the above expression is equal to

$$\mu^{-1}(y)|y|^{-1}\Phi\Big(w\left(\begin{smallmatrix} 1 & -1/y \\ 0 & 1 \end{smallmatrix}\right)\Big) = \omega_\pi(-1)\mu^{-1}(y)|y|^{-1}\Phi(-1/y).$$

Substituting this formula into the integral $\zeta(\Phi_w, \chi^{-1}\mu_1, z)$, it follows by the change of variables $-1/y \mapsto y$ that

$$\begin{aligned} \zeta\big(\Phi_w, \chi^{-1}\mu_1, z\big) &= \int_{F^\times} \Phi_w(y)\chi^{-1}\mu_1(y)|y|^z d^\times y \\ &= \omega_\pi(-1)\int_{F^\times} \Phi(-1/y)\chi^{-1}\mu_2(y)|y|^{z-1}d^\times y \\ &= \chi^{-1}\mu_1(-1)\int_{F^\times} \Phi(y)\chi\mu_2^{-1}(y)|y|^{1-z}d^\times y \\ &= \chi^{-1}\mu_1(-1)\zeta(\Phi, \chi\mu_2^{-1}, 1-z). \end{aligned}$$

This proves the claim. ■

Now, we have, by repeatedly applying the claims as well as the GL_1 theorems:

$$\begin{aligned}(\mu_1\chi^{-1})(-1)\gamma(\mu_2\chi^{-1}, z)Z(W, \chi, s) &= (\mu_1\chi^{-1})(-1)\gamma(\mu_2\chi^{-1}, z)\zeta\left(\widehat{\Phi}, \chi^{-1}\mu_2, z\right)\\ &= (\mu_1\chi^{-1})(-1)\zeta\left(\Phi, \chi\mu_2^{-1}, 1-z\right)\\ &= \zeta\left(\Phi_w, \chi^{-1}\mu_1, z\right)\\ &= \gamma(x\mu_1^{-1}, 1-z)\zeta(\widehat{\Phi_w}, \chi\mu_1^{-1}, 1-z)\\ &= \gamma(\chi\mu_1^{-1}, 1-z)Z\left(\widetilde{W}, \chi^{-1}\omega_\pi, 1-s\right)\end{aligned}$$

By Theorem 6, the righthand side equals

$$= \gamma(\chi\mu_1^{-1}, 1-z)\gamma_\pi(\chi, s)Z(W, \chi, s).$$

Our choice of W from Lemma 5 satisfies $Z(W, \chi, s) \not\equiv 0$ so we may divide this term from both sides to deduce that

$$\gamma_\pi(\chi, s) = \mu_1\chi_1^{-1}(-1)\frac{\gamma(\mu_2\chi^{-1}, z)}{\gamma(\mu_1^{-1}\chi, 1-z)}.$$

Applying Theorem 1(A) to the denominator proves the theorem. □

Theorem 13. Define $\epsilon_\pi(\chi, s) = \gamma_\pi(\chi, s)\frac{L_\pi(\chi,s)}{L_\pi(\chi^{-1}\omega_\pi,1-s)}$. Then we have the functional equation:

$$\epsilon_\pi(\chi, s)\epsilon_\pi(\chi^{-1}\omega_\pi, 1-s) = \omega_\pi(-1)$$

and $\epsilon_\pi(\chi, s) = aq^{bs}$ for some $a \in \mathbf{C}^\times$ and $b \in \mathbf{Z}$.

Finally, in the principal series case, we can compute the ϵ factors. For the cuspidal case, the ϵ factor equals the γ factor (as the L-functions defining ϵ_π are trivial) so we again refer the reader to [BH, Section 25] for details.

Theorem 14. If $\pi = \pi_{\mu_1,\mu_2}$ with $\mu_1/\mu_2 \neq |\cdot|, |\cdot|^{-1}$, then we have:

$$\epsilon_\pi(\chi, s) = \epsilon(\chi^{-1}\mu_1, 2s - \tfrac{1}{2}) \cdot \epsilon(\chi^{-1}\mu_2, 2s - \tfrac{1}{2})$$

where the ϵ factors on the right are the ϵ factors from the GL_1 theory.

Proof. This follows from Theorems 9 and 13 as well as the relationship between ϵ factors and γ factors in the GL_1 case. For the cases when $\mu_1/\mu_2 =\neq |\cdot|, |\cdot|^{-1}$, see [Go, pages 1.49–1.52]. □

Now, we will prove Theorem 13:

Proof. The equation $\epsilon_\pi(\chi, s)\epsilon_\pi(\chi^{-1}\omega_\pi, 1-s) = \omega_\pi(-1)$ follows directly from the definition and the functional equation for γ given in Theorem 6.

Now, again by Theorem 6, we have:

$$\frac{Z(W, \chi, s)}{L_\pi(\chi, s)}\epsilon_\pi(\chi, s) = \frac{Z\left(\widetilde{W}, \omega_\pi\chi^{-1}, 1-s\right)}{L_\pi(\chi^{-1}\omega_\pi, 1-s)}$$

By Theorem 8, the right-hand side is in $\mathbf{C}[q^{-2s}, q^{2s}]$, so it is entire in s. Now, choose W such that $Z(W, \chi, s) = L_\pi(\chi, s)$. This implies that $\epsilon_\pi(\chi, s) \in \mathbf{C}[q^{-2s}, q^{2s}]$ as well. Similarly, we see that $\epsilon_\pi(\chi^{-1}\omega_\pi, 1-s) \in \mathbf{C}[q^{-2s}, q^{2s}]$. Since their product is $\omega_\pi(-1)$, we have:

$$\epsilon_\pi(\chi, s) \in \mathbf{C}[q^{-2s}, q^{2s}]^\times$$

Thus, $\epsilon_\pi(\chi, s) = aq^{bs}$ for some $a \in \mathbf{C}^\times$ and $b \in \mathbf{Z}$. □

References

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