NORMALIZED FLOW AND SPERNER THEORY OF COXETER GROUPS

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Abstract. In this paper, we give an overview on investigations into the Sperner property of posets and particularly the posets induced by applying natural orders to Coxeter groups. We first explore an elementary proof of Sperner's original theorem using Hall's matching condition and discuss the limitations of this technique. We will then discuss the development of the stronger normalized matching condition and normalized flow property and their applications to the problem of Sperner, and finally review recent applications of these techniques to the natural orders of Coxeter groups.

Contents

1. Introduction: Sperner's Theorem

Sperner's theorem is an elementary theorem in extremal set theory proven by Emanuel Sperner in 1928. Its statement is as follows:

Theorem 1.1. *(Sperner's Theorem [1]) Let S be a collection of subsets of $\{1, 2, ...n\}$ such that for all $A, B \in S$, we have that $A \subseteq B$ if and only if $A = B$. Then,*

$$|S| \leqslant \binom{n}{\lfloor n/2 \rfloor} \tag{1.2}$$

As it pertains to the study of the Sperner property of general ranked posets, it will be instructive to consider a proof of Sperner's theorem using a consideration of the n-element Boolean algebra $\mathscr{B}_n$ as a poset.

Theorem 1.3. *(Hall's Matching Theorem [2]) Let $G = (A \sqcup B, E)$ be a bipartite graph. For any $W \subset A$, let $N(W) \subset B$ be the set of vertices in B connected by edges to vertices in W. That is, $b \in N(W)$ if and only if there exists $a \in W$ such that*

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$(a,b) \in E$. *Then, there exists an injective function* $f : A \to B$ *with* $(a, f(a)) \in E$ *for all* $a \in A$ *if and only if for all* $W \subseteq A$,

$$|W| \leqslant |N(W)| \tag{1.4}$$

The inequality (1.4) is known as Hall's Matching Condition. To prove Sperner's theorem we will need also to define posets and the Boolean Algebra.

Definitions 1.5. A *poset* is a pair $(S, \leqslant)$ where S is a set and $\leqslant$ is a reflexive, transitive, and antisymmetric binary relation on S. That is, for all $a, b, c \in S$,

i. $a \leqslant b \wedge b \leqslant a \iff a = b$
ii. $a \leqslant b \wedge b \leqslant c \implies a \leqslant c$

Given a poset $(S, \leqslant)$ and $x, y \in S$, we write $x < y$ to say $x \leqslant y$ and $x \neq y$. We say x is covered by y, or $x \lessdot y$, if $x < y$ and there is no $z \in S$ such that $x < z < y$. We call $W \subset S$ a *chain* if $\leqslant$ is total on W, which means that for every $x, y \in W$, either $x \leqslant y$ or $y \leqslant x$. Alternatively, W is called an *antichain* if there exist no $x, y \in W$ with $x < y$.

A poset $(S, \leqslant_S)$ is a *ranked poset* if there exists a rank function $r : S \to \mathbb{N}$ such that

i. $x \leqslant_S y \implies r(x) \leqslant r(y)$
ii. $x \lessdot_S y \implies r(y) = r(x) + 1$

The pre-image of a singleton set under the rank function r is called a *rank*.

The Boolean Algebra $\mathscr{B}_n$ is the poset $(\mathcal{P}([n]), \subseteq)$ where $\mathcal{P}([n])$ is the power set of $\{1, 2, ..., n\} = [n]$. It is a ranked poset with the natural rank function $r(W) = |W|$. A rank $r^{-1}(k)$ is just the set of k-element subsets of $[n]$, of which there are $\binom{n}{k}$. Denoting $R_k = r^{-1}(k)$, it follows that for all k,

$$|R_k| = |R_{n-k}| \tag{1.6}$$

and that

$$|R_0| \leqslant |R_1| \leqslant ... \leqslant |R_{\lfloor n/2 \rfloor}| \geqslant |R_{\lfloor n/2 \rfloor + 1}| \geqslant ... \geqslant |R_{n-1}| \geqslant |R_n| \tag{1.7}$$

These properties of $\mathscr{B}_n$ are referred to as *rank-symmetry* (1.6) and *rank-unimodality* (1.7). We will use this consideration of $\mathscr{B}_n$ as a ranked poset in our proof of Sperner's theorem, which can be equivalently stated as follows:

Theorem 1.8. *(Sperner's Theorem) Let* $S \subseteq \mathscr{B}_n$ *be an antichain. Then,*

$$|S| \leqslant \binom{n}{\lfloor n/2 \rfloor}$$

Proof. Let $R_0, R_1, ..., R_n$ be the ranks of $\mathscr{B}_n$. Take $k \in \{1, ..., \lfloor n/2 \rfloor - 1\}$. Let $G_k = (R_k \sqcup R_{k+1}, E)$ where $E = \{(X, Y) \in R_k \times R_{k+1} \mid X \subseteq Y\}$. Take $S \in R_k$. Considered as a vertex in G_k, S must have degree $n - k$: there is one edge from S for every subset of $[n]$ size $k+1$ containing it, and the only way to get such a subset is to add to S one of the $n - k$ elements of $[n]$ that it does not already contain. Similarly, every vertex in R_{k+1} has degree $k + 1$, where each edge from $T \in R_{k+1}$ corresponds to taking out one of T's elements, of which there are $k + 1$.

Take some $W \subset R_k$. Since every vertex in W has degree $n - k$, there are $|W|(n - k)$ edges between W and $N(W)$. Since every vertex in $N(W)$ has degree $(k + 1)$, there are $|N(W)|(k + 1)$ edges between $N(W)$ and R_k. This means that there are $|N(W)|(k+1)-|W|(n-k)$ edges between $N(W)$ and $R_k \backslash W$. By the initial choice of k, $k+1 \leqslant (n-k)$, so we have that the number of edges between $N(W)$ and $R_k \backslash W$ is less than or equal to $(n-k)(|N(W)| - |W|)$. Since this number certainly cannot be negative, it follows that $|N(W)| \geqslant |W|$, so Hall's matching condition applies and there exists matching $f_k : R_k \to R_{k+1}$. It follows then by definition of G_k that $A \subseteq f_k(A)$ for every $A \in R_k$.

An analogous process can be used to construct matchings $f_k : R_{k+1} \to R_k$ for $k \in \{\lfloor n/2 \rfloor, ..., n-1\}$ such that $f_k(A) \subseteq A$. It follows then that for every element $A \in \mathscr{B}_n$, if A is not already of rank $\lfloor n/2 \rfloor$ itself, there exists a composition of matchings that takes A to some element of rank $\lfloor n/2 \rfloor$: if A is of rank k, it follows that

$$R_{\lfloor n/2 \rfloor} \ni f(A) = \begin{cases} (f_{\lfloor n/2 \rfloor - 1} \circ ... \circ f_{k+1} \circ f_k)(A) & k < \lfloor n/2 \rfloor \\ (f_{\lfloor n/2 \rfloor} \circ ... \circ f_{k-2} \circ f_{k-1})(A) & k > \lfloor n/2 \rfloor \\ A & k = \lfloor n/2 \rfloor \end{cases}$$

By the construction of the matchings, for every $A \in \mathscr{B}_n$, $A \subseteq f(A)$ or $f(A) \subseteq A$. Furthermore, by injectivity of the matchings and since each matching follows the edges between ranks of $\mathscr{B}_n$ induced by inclusion of sets, $f(A) = f(B) \implies A \subseteq B$ or $B \subseteq A$. Since f is surjective onto $R_{\lfloor n/2 \rfloor}$ (each element of that rank is its own image), f determines a partition of $\mathscr{B}_n$ into $|R_{\lfloor n/2 \rfloor}| = \binom{n}{\lfloor n/2 \rfloor}$ parts, each of which is a chain. The result then follows immediately from this partition: suppose there is a set $A \subset \mathscr{B}_n$ with $|A| > \binom{n}{\lfloor n/2 \rfloor}$. Then, by the pigeonhole principle, there must be $x, y \in A$ both in the same part. Since this part is a chain, $x \subseteq y$ or $y \subseteq x$, and since $x, y \in A$, A cannot be an antichain.

□

As becomes clear in the later part of the proof, the bound on $|S|$ given in (1.2) emerges as the greatest size of all ranks in $\mathscr{B}_n$. Since any rank in a ranked poset is itself an antichain, it follows that any upper bound on the size of antichains in a ranked poset must be at least the size of the largest rank. Sperner's theorem shows that $\mathscr{B}_n$ has the special property that the largest rank is itself an antichain of maximum size. This property is aptly called the Sperner property. For the same reason as there is interest in Sperner's theorem being proven for $\mathscr{B}_n$, so too is there combinatorial interest in proving other ranked posets are Sperner. Particularly when a poset has a simple and meaningful rank function, it can be much easier to count the members of the largest rank (e.g. the number of subsets size $\lfloor n/2 \rfloor$) than to otherwise derive the maximal size of any antichain. It is not generally the case that a ranked poset is Sperner; see figure 1 for an example.

There are two useful generalizations of the Sperner property in the k-Sperner and strong Sperner properties.

Definition 1.9. A finite ranked poset $(S, \leqslant)$ is k-Sperner for some $k \in \mathbb{N}$ if for every choice of k antichains $A_1, ..., A_k$

$$\left| \bigcup_{i=1}^{k} A_i \right| \leqslant \left| \bigcup_{i=1}^{k} R_{j_i} \right|$$

where $R_{j_1}, ..., R_{j_k}$ are the k ranks of greatest cardinality in S.

Definition 1.10. A finite ranked poset $(S, \leqslant)$ is strong Sperner if it is k-Sperner for every k.

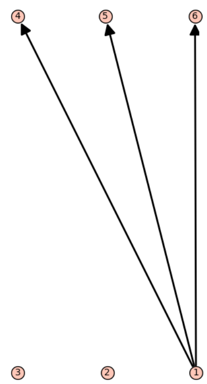


FIGURE 1. The Hasse Diagram of a non-Sperner ranked poset. The ranks are $\{1,2,3\}$ and $\{4,5,6\}$, but the largest antichain is $\{2,3,4,5,6\}$.

A poset is Sperner iff it is 1-Sperner; it follows that a strong Sperner poset is also Sperner. A result of Erdős shows that $\mathscr{B}_n$ is strong Sperner, which in turn implies Sperner's theorem.

Theorem 1.11. *(Erdős [3]) $\mathscr{B}_n$ is strong Sperner.*

Both Erdős's proof and the proof given above of Sperner's theorem are based heavily on the fact that vertices in the same rank have the same degree and can thus be considered analogous to one another. This is a symmetry particular to $\mathscr{B}_n$ that typically does not apply to other ranked posets, nor even to other Sperner or strong Sperner posets. Approaches such as these are therefore quite limited as to the scope of posets they can show to be Sperner.

2. Normalized Flow and the Product Theorem

The proof we gave of Sperner's theorem hinged on showing that the bipartite graphs between ranks in the Hasse diagram of $\mathscr{B}_n$ had matchings and the fact that $\mathscr{B}_n$ is rank-unimodal. In general it follows that any rank-unimodal poset with matchings between ranks is Sperner. However, showing such a matching directly can be difficult for more complex posets and there is no obvious way of inductively combining matchings. In what follows, we will discuss the "normalized flow property," a particular form of which is a strengthening of the Sperner property that facilitates powerful techniques of showing a poset is Sperner through decomposition.

Definition 2.1. A bipartite graph $(A \sqcup B, E)$ with $|A| \leqslant |B|$ is said to satisfy the *normalized matching condition* if for all $X \subset A$,

$$\frac{|X|}{|A|} \leqslant \frac{|N(X)|}{|B|}$$

This inequality is equivalent to $|X| \leqslant \frac{|A|}{|B|}|N(X)|$, so with $|A| \leqslant |B|$ this is a stronger inequality than Hall's matching condition, $|X| \leqslant |N(X)|$. Harper showed further that the normalized matching condition is closely related to the existence of flows in the graph.

Definition 2.2. Let $G = (V = (A \sqcup B), E)$ be a bipartite graph with a non-negative real-valued weight function $\nu : V \to \mathbb{R}_{\geqslant 0}$ considered as a measure; that is, for $X \subseteq V$

$$\nu(X) := \sum_{x \in X} \nu(x)$$

Then a *normalized flow* on G is a function $f : E \to \mathbb{R}_{\geqslant 0}$ with the conditions that for all $a \in A$,

$$\sum_{b \in N(a)} f(a, b) = \frac{\nu(a)}{\nu(A)}$$

and similarly for all $b \in B$,

$$\sum_{a \in N(b)} f(a, b) = \frac{\nu(b)}{\nu(B)}$$

These two notions are closely related by Ford-Fulkerson's max-flow min-cut theorem.

Theorem 2.3. *(Max-flow Min-cut Theorem [4]) The maximum flow in a network is equal to the minimum capacity across all cuts in the network.*

Which has a pertinent corollary first mentioned by Harper [5] without proof:

Corollary 2.4. *A bipartite graph $G = (V = A \sqcup B, E)$ with vertex capacity ν has a flow of capacity $\nu(A)$ iff $\nu(X) \leqslant \nu(N(X))$ for all $X \subseteq A$.*

Proof. Construct a flow network $G' = (V', E', \omega)$ with $V' = \{s, t\} \cup V$ and $E' = E \cup (\{s\} \times A) \cup (B \times \{t\})$, and with

$$\omega(x, y) = \begin{cases} \nu(y) & x = s \\ \nu(x) & y = t \\ \infty & \text{otherwise} \end{cases}$$

Since the flow along any edge from s to $a \in A$ is determined by the amount of flow coming out of a, and similarly the flow along any edge to t from $b \in B$ is determined by the flow into b, there is a natural bijection between flows on G and flows on G' that matches the amount of flow on each edge from A to B. It follows then that the maximum flow capacity on G' is the maximum flow capacity on G. Any set of edges spanning a cut of finite capacity on G' is determined by a set $X \subseteq A$ and is of the form

$$C(X) = \{(s, x) \mid x \in A \backslash X\} \cup \{(y, t) \mid y \in N(X)\}$$

Suppose $C(\varnothing) = \{s\} \times A$ is the minimum cut with capacity $c(C(\varnothing))$. This is equivalent to saying that for all $X \subseteq A$, $c(C(X)) \geqslant c(C(\varnothing))$. By construction of $C(X)$ and the definition of cut capacity, it follows that

$$c(C(X)) = c(C(\varnothing)) + \sum_{y \in N(X)} \nu(y) - \sum_{x \in X} \nu(x) = c(C(\varnothing)) + \nu(N(X)) - \nu(X)$$

Therefore $C(\varnothing)$ is the minimal cut iff $\nu(N(X)) \geqslant \nu(X)$ for all $X \subseteq A$. □

Corollary 2.5. *A bipartite graph $G = (A \sqcup B, E)$ with vertex capacity $\nu \equiv 1$ has a normalized flow iff it satisfies the normalized matching condition.*

Proof. Consider G' as the same graph as G but with the alternate vertex capacity function

$$\nu'(x) = \begin{cases} \frac{1}{|A|} & x \in A \\ \frac{1}{|B|} & x \in B \end{cases}$$

Again think of ν' as a measure in the sense that $\nu'(X) = \frac{|X|}{|A|}$, $\nu'(Y) = \frac{|Y|}{|B|}$ for $X \subseteq A, Y \subseteq B$. By Corollary 2.4, G' has a flow of capacity $\nu'(A) = 1$ if and only if $\nu'(X) \leqslant \nu'(N(X))$ for every $X \subseteq A$. By definition of ν', this condition is equivalent to the normalized matching condition on G. Furthermore, a flow on G' having capacity $\nu'(A) = 1$ is equivalent to a normalized flow on G with capacity function $\nu \equiv 1$: since the maximum flow on G' has capacity $\nu'(A)$, each vertex in A must have maximum flow going through it, as must the vertices in B since $\nu'(B) = \frac{|B|}{|B|} = 1 = \nu'(A)$. It follows then that the edge flows in the maximum flow on G' follow this condition for all $a \in A$:

$$\sum_{b \in N(a)} f(a,b) = \nu'(a) = \frac{1}{|A|} = \frac{\nu(a)}{\nu(A)}$$

and similarly for all $b \in B$,

$$\sum_{a \in N(b)} f(a,b) = \nu'(b) = \frac{1}{|B|} = \frac{\nu(b)}{\nu(B)}$$

which are precisely the conditions defining normalized flow on G. □

The weight function $\nu \equiv 1$ is of particular interest to the problem of determining whether a poset is Sperner, as will be shown later. These definitions of normalized flow and of the normalized matching condition are applied to posets in much the same way as we applied Halls' matching condition in the proof of Sperner's theorem by applying that property to the subgraph of adjacent ranks in the Hasse diagram.

Definition 2.6. A finite ranked and weighted poset $(P, \leqslant, \nu)$ is said to have the *normalized flow property* (NFP) if every pair of adjacent ranks R_k, R_{k+1} considered as a bipartite graph has a normalized flow with respect to ν.

This brings us to the crucial result in the study of normalized flow on posets, namely the Product Theorem.

Definition 2.7. Let $(P, \leqslant_P), (Q, \leqslant_Q)$ be ranked, weighted posets with rank functions r_P, r_Q and weight functions ν_P, ν_Q. Define the *product* of P and Q, $P \times Q$, as a poset $(P \times Q, \leqslant)$ with $\leqslant$ defined as

$$(p,q) \leqslant (p',q') \iff (p \leqslant_P p') \wedge (q \leqslant_Q q')$$

with a weight function $\nu : P \times Q \to \mathbb{R}_{\geqslant 0}$ defined as

$$\nu(p,q) = \nu_P(p)\nu_Q(q)$$

Finally, the rank function for $P \times Q$ is given by $r \equiv r_P + r_Q$.

Definition 2.8. Let $(P, \leqslant, \nu)$ be a ranked weighted poset with ranks $R_1, ..., R_n$ sorted from lowest to highest rank. P is called *log-concave* or *2-positive* if for every k,

$$\nu(R_k)\nu(R_{k+2}) \leqslant \nu(R_{k+1})^2$$

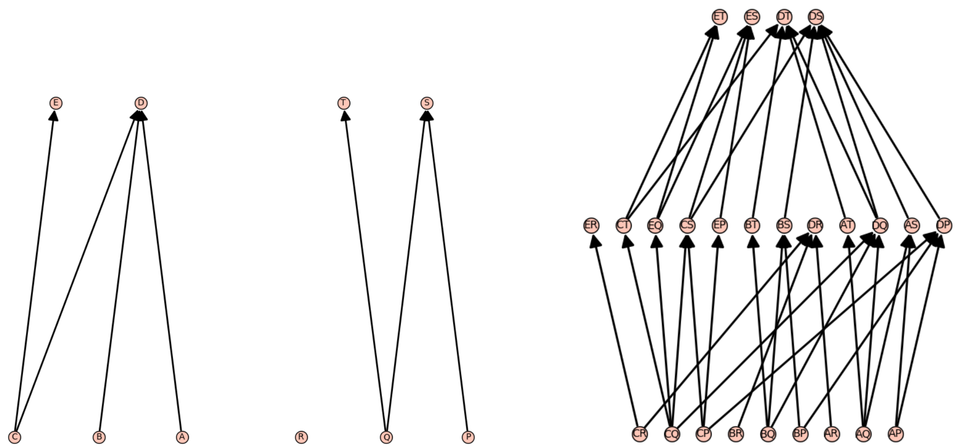


FIGURE 2. The product of Sperner posets is not necessarily Sperner. Both $\{A, B, C, D, E\}$ (left) and $\{P, Q, R, S, T\}$ (middle) are strong Sperner under their respective partial orders, but their product (right) is not Sperner: its largest rank is R_1 with $|R_1| = 12$, but $(R_1 \backslash \{(D, R)\}) \cup \{(A, R), (B, R)\}$ is an antichain size 13.

Theorem 2.9. *(Product Theorem) Let P, Q be two ranked, weighted, log-concave posets with the normalized flow property. Then, $P \times Q$ is a ranked, weighted, log-concave poset with the normalized flow property.*

Proof. See Harper [5]. □

3. Applications of NFP to Sperner Theory

The Product Theorem yields a way in which NFP posets can be combined and thus considered in an inductive way which does not otherwise apply to Sperner posets: it is not generally true that the product of Sperner or even strong Sperner posets is Sperner (see Figure 2). The crux of this consideration is that, as Harper proved, the Sperner property follows from a particular form of NFP.

Theorem 3.1. *(Generalized Sperner's Theorem) Let $(P, \leqslant, \nu)$ be a ranked weighted poset with NFP. Define a Sperner weight of order k as a function $\omega_k : P \to \mathbb{R}_{\geqslant 0}$ with the following properties:*

(i) $\omega_k(x) \leqslant \nu(x)$ for all $x \in P$.
(ii) For every chain $C \subset P$,

$$\sum_{x \in C} \frac{\omega_k(x)}{\nu(x)} \leqslant k$$

And as usual, ω_k is considered as a measure with $\omega_k(S) = \sum_{s \in S} \omega_k(s)$. Then, the supremum of $\omega_k(P)$ over all k-order Sperner weights ω_k is less than or equal to the weight of the k largest ranks in P. That is,

$$\sup_{\omega_k} \ \omega_k(P) \leqslant \max_{|I|=k} \sum_{i \in I} \nu(R_i) \tag{3.2}$$

Proof. See Harper [5]. □

Corollary 3.3. *Let $(P, \leqslant, \nu)$ be a ranked poset with weight function $\nu \equiv 1$. If P has NFP, then P is strong Sperner.*

Proof. Take some $k \in \mathbb{N}$. Let $A_1, ..., A_k \subset P$ be antichains. Define the function

$$\omega_k(x) = \begin{cases} 1 & x \in \bigcup_{i=1}^{k} A_i \\ 0 & \text{otherwise} \end{cases}$$

This ω_k is a k-order Sperner weight: $\omega_k(x) \leqslant 1 = \nu(x)$ for all x. Any chain C can only intersect any antichain A at most once, so it follows that

$$\sum_{x \in C} \frac{\omega_k(x)}{\nu(x)} = \sum_{x \in C} \omega_k(x) = |C \cap (A_1 \cup ... \cup A_k)| \leqslant k$$

Since ω_k is a Sperner weight and P has NFP, the bound provided by the generalized Sperner theorem applies, which implies

$$\max_{|I|=k} \sum_{i \in I} |R_i| \geqslant \omega_k(P) = |A_1 \cup ... \cup A_k|$$

That is, the size of the k biggest ranks is at least the size of the choice of antichains $A_1, ..., A_k$. Since this was a general choice of antichains, P is k-Sperner by definition. Since this was a general choice of k, P is strong Sperner by definition. □

This result combines quite powerfully with the Product Theorem since it takes $\nu \equiv 1$; since the weight function of the product poset is the product of its factor posets' weight functions, a product of two log-concave NFP posets with $\nu \equiv 1$ will itself be a log-concave NFP poset with $\nu \equiv 1$ and will therefore be strong Sperner. This means that if one knows two posets are Sperner particularly because it has NFP with $\nu \equiv 1$, posets of that type can be inductively combined to study more complex classes of poset. Two of the simplest posets that are deployed in this sort of study are *chains* and *claws*. A chain poset is essentially how we described them before: a poset whose order is total, i.e. a poset $(\{x_1, ..., x_n\}, \leqslant)$ with $x_1 \leqslant x_2 \leqslant ... \leqslant x_n$. Denote such a chain poset as $\mathscr{C}_n$. A claw poset, on the other hand, is a poset of form $(\{x_1, ..., x_n\}, \leqslant\}$ where $x_1 \leqslant x_i$ for all $i \geqslant 2$, and all other x_i, x_j are incomparable. Each of these can be used as factors for more complex posets.

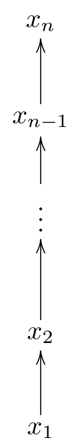
Example 3.4. To demonstrate the power of the Product Theorem, we here employ it to write a much more succinct proof of the fact that $\mathscr{B}_n$ is strong Sperner.

Proof. Represent $\mathscr{C}_2$ as $(\{0, 1\}, \leqslant)$ with $0 \leqslant 1$. There is an isomorphism of posets $f : \mathscr{C}_2^n \to \mathscr{B}_n$ defined by $f(a_1, ..., a_n) = \{i \in [n] \mid a_i = 1\}$. Since $\mathscr{B}_n$ is isomorphic to a product of chains, it has normalized flow with $\nu \equiv 1$ and is strong Sperner. □

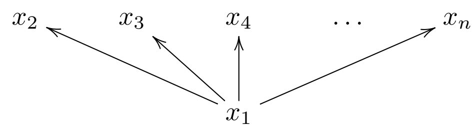
Example 3.5. Define the *lattice of factors* $\mathscr{L}(n)$ for $n \in \mathbb{N}$ as a poset $(S, \leqslant)$ where $S = \{m \in \mathbb{N} \mid m|n\}$ and $a \leqslant b \iff a|b$. The lattice of factors $\mathscr{L}(n)$ is strong Sperner for all $n \in \mathbb{N}$.

Proof. The number n has unique prime factorization $n = \prod_{i=1}^{k} p_i^{a_i}$. For every $x, y \in \mathscr{L}(n)$, x and y their own prime decompositions $x = \prod_{i=1}^{k} p_i^{x_i}$, $y = \prod_{i=1}^{k} p_i^{y_i}$ and from this follows the fact $x \leqslant y$ if and only if $x_i \leqslant y_i$ for all i. Represent each $\mathscr{C}_n$ as having the set $\{1, ..., n\}$ and the usual $\leqslant$ relation of the natural numbers. There is then a natural isomorphism of posets $f : \mathscr{L}(n) \to \bigtimes_{i=1}^{k} \mathscr{C}_{a_i}$ defined by

$$f\left(p_1^{x_1} p_2^{x_2} ... p_k^{x_k}\right) = (x_1, x_2, ..., x_k)$$



(A) Chain poset $\mathscr{C}_n$.



(B) Claw poset $\mathcal{C}_n$.

Since $\mathscr{L}(n)$ is isomorphic to a product of chains, it has NFP with $\nu \equiv 1$ and is strong Sperner. □

This ability to combine NFP posets inductively proves extremely powerful, and it is because of this capacity for inductive combination that Harper writes "if the analogue of Sperner's theorem were to be proven for any infinite family of posets, *it would almost have to follow from* NFP" [6]. In the following, we will review a recent result that has used NFP decomposition to prove the Sperner property for posets that have not been proven Sperner in any other way.

4. The Prefix Order on Coxeter Groups

Definition 4.1. A *Coxeter group* is a group defined by the presentation
$$\langle r_1, ..., r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$$
where $m_{ii} = 1$ and $m_{ij} \geqslant 2$ for $i \neq j$.

For finite Coxeter groups, this definition generalizes the notion of the symmetry group of a regular polytope, and every finite coxeter group is precisely the symmetry group of some regular polytope. There are four infinite classes of Coxeter group, denoted A_n, B_n, D_n, and $I_2(n)$. The groups $A_n \cong S_n$ are the symmetries of the n-simplices, the groups B_n are the symmetry groups of the n-cubes, the D_n groups are those of the n-demicubes, and the $I_2(n) \cong D_{2n}$ are the symmetry groups of regular polygons. Apart from these 4 classes there are only finitely many finite Coxeter groups, called the exceptional groups.

Since each symmetry group is generated by its reflections, there is a natural consideration of a "length" of each element in terms of the reflections, particularly how many reflections it takes to express that element as a product. Furthermore, there is a notion of comparability where two symmetries can be closely related by reflections, each being the product of the other with a reflection. We formalize these notions below:

Definition 4.2. Let G be a Coxeter group with $T \subset G$ the set of reflections (elements conjugate to simple reflections). The *absolute length* of an element $x \in G$, denoted $l_T(x)$, is defined as the least number of reflections in T it takes to express x as a product.

With this notion of absolute length, the relationship of two symmetries being one reflection away from each other can be further specified into a partial order, where the shorter of the two symmetries is less than the longer.

Definition 4.3. Let G be a Coxeter group with reflection T. The *prefix order* or equivalently the *absolute order*, denoted $\leqslant$, is defined as follows:

$$a \leqslant b \iff l_T(a^{-1}b) + l_T(a) = l_T(b)$$

It follows from this definition that $a \leqslant b$ iff there is a shortest possible representation of b as a product of reflections that contains a representation of a as a prefix. That is, $a \leqslant b$ iff there exists reflections $r_1, ..., r_{l_T(b)}$ such that $b = r_1 r_2 ... r_{l_T(b)}$ and $a = r_1 r_2 ... r_{l_T(a)}$.

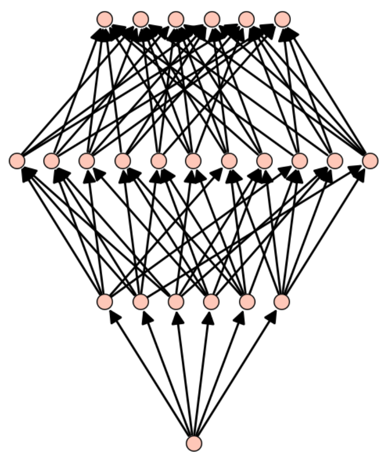


FIGURE 4. A Hasse diagram for Coxeter Group A_3 under the absolute order.

The Hasse diagrams given by this order on Coxeter groups typically lack much of the symmetry that was used to show that $\mathscr{B}_n$ was Sperner and strong Sperner. In particular, the rank graphs are often not biregular; this can be seen in Figure 4 where in rank 1 the rightmost vertex has more outward edges than the vertex to its left. There is also no single element of highest rank in A_3, meaning it cannot be a product of chains. In fact, the arguments used to show that these orders are Sperner do not use a direct isomorphism at all, but rather use an embedding of NFP constructions as *spanning subposets*.

Definition 4.4. Let $P = (S, \leqslant)$ be a ranked poset with rank function $r : S \to \mathbb{N}$. A poset $P' = (S, \preceq)$ defined on the same vertex set S is a *spanning subposet* if $x \preceq y \implies x \leqslant y$ and if r is still a rank function on P'.

Lemma 4.5. *Let $P = (S, \leqslant)$ be a ranked poset and let $P' = (S, \preceq)$ be a spanning subposet of P. Then, if P' has NFP with respect to some ν, then P also has NFP with respect to that same ν.*

The following proof elaborates on an idea first mentioned by Gaetz and Gao [9].

Proof. Let R_k, R_{k+1} be adjacent ranks in P. Since P' has the same ranks as P, these ranks have a normalized flow with respect to $\nu \equiv 1$ along the edges determined by $\preceq$. Since $a \preceq b \implies a \leqslant b$, all of the edges between R_k, R_{k+1} as ranks of P' are also there between them as ranks of P. There is therefore a normalized flow between R_k, R_{k+1} as ranks of P constructed by taking the flow between them as ranks of P' and assigning zero flow to all extra edges introduced by expanding to P. Since any two ranks of P have normalized flow, P has NFP by definition. □

This lemma underlies the final step to the following theorems:

Theorem 4.6. [8] *Coxeter groups of classes A_n and B_n have NFP with respect to $\nu \equiv 1$ and are strong Sperner.*

Theorem 4.7. [9] *Coxeter groups of class $I_2(n)$ have normalized flow with respect to $\nu \equiv 1$ and are strong Sperner.*

Gaetz and Gao also verified by computer search that all of the exceptional group types have NFP as well. The arguments for the non-exceptional cases all operated on the same principle of embedding products of claws $\mathcal{C}_n$ (with aptly chosen n values) as spanning subposets into the Coxeter poset.

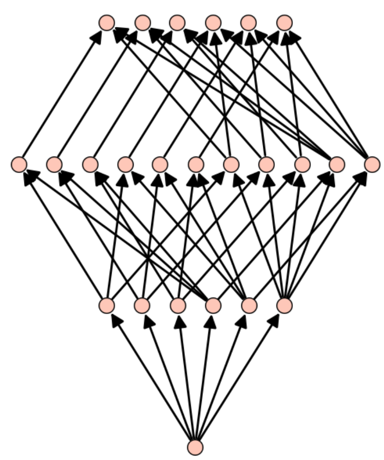


FIGURE 5. The Hasse diagram of $\mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4$. This has NFP by the Product Theorem and is a spanning subposet of A_3 (see Figure 4).

Whether or not the D_n groups are Sperner and/or have NFP is still open. Gaetz and Gao verified that D_n has NFP for $n \leqslant 8$ by computer search and conjectured the D_n groups do have NFP in general, but showed that there is product of claws isomorphic to a spanning subposet of D_n.

5. Conclusion

In the preceding we have attempted to show the power that NFP and the product theorem have in trying to answer these combinatorial questions. There are still many problems yet to be approached with this technique; we lay out a few possible routes based on the preceding discussion.

Question 5.1. *Do the D_n groups have NFP? Is there some product of chains and claws that embeds as a spanning subposet?*

Question 5.2. *For what other kinds of groups does the notion of generator word length and prefix order apply? Do the posets induced by those orders have NFP?*

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