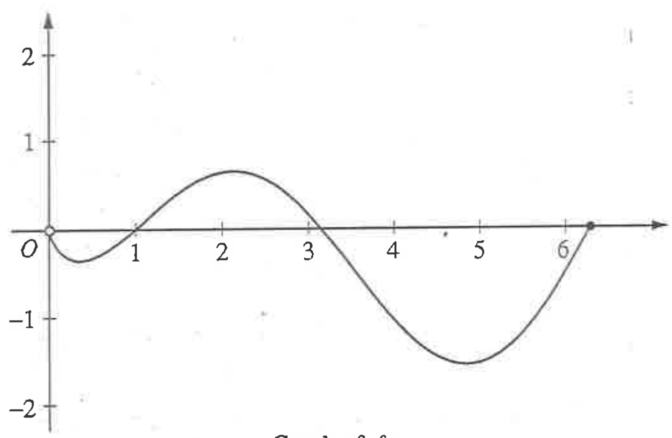
CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



Graph of f

b) g is increasing on $(0, \pi)$ since $g'(x) \geq 0$ when $0 < x \leq \pi$.

c) g has an absolute minimum at $x = 2\pi$ since $g'(x) < 0$ when $x < 2\pi$ and $g'(2\pi) = 0$.

4. Let f be the function given by $f(x) = (\ln x)(\sin x)$. The figure above shows the graph of f for $0 < x \leq 2\pi$.

   The function g is defined by $g(x) = \int_1^x f(t)\,dt$ for $0 < x \leq 2\pi$. ← When $0 < x < 1$ $g(x) = -\int_x^1 f(t)dt$.

   (a) Find $g(1)$ and $g'(1)$. $g(1) = \int_1^1 f(t)dt = 0$; $g'(x) = f(x) \rightarrow g'(1) = f(1) = (\ln 1)(\sin 1) = 0$

   (b) On what intervals, if any, is g increasing? Justify your answer.

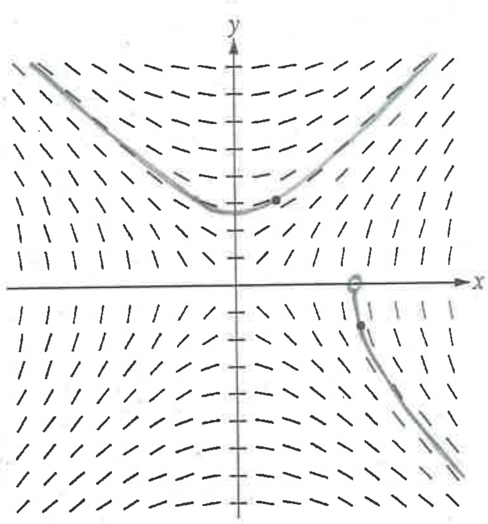
   (c) For $0 < x \leq 2\pi$, find the value of x at which g has an absolute minimum. Justify your answer.

   (d) For $0 < x < 2\pi$, is there a value of x at which the graph of g is tangent to the x-axis? Explain why or why not. Yes, g is tangent to the x-axis when $x = 1$ since $g(1) = 0$ and $g'(1) = f(1) = 0$.

5. Consider the differential equation $\frac{dy}{dx} = \frac{x}{y}$, where $y \neq 0$.

   (a) The slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(3, -1)$, and sketch the solution curve that passes through the point $(1, 2)$.

   (<u>Note:</u> The points $(3, -1)$ and $(1, 2)$ are indicated in the figure.)



b) $\frac{dy}{dx} = \frac{1}{2}$

$y - 2 = \frac{1}{2}(x - 1)$

$\therefore y = \frac{1}{2}x + \frac{3}{2}$

c) $\int y\,dy = \int x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$

$\rightarrow y^2 = x^2 + C$

$(-1)^2 = 3^2 + C \rightarrow C = -8$

$\therefore y = -\sqrt{x^2 - 8}, \ x > \sqrt{8}$

   (b) Write an equation for the line tangent to the solution curve that passes through the point $(1, 2)$.

   (c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(3) = -1$, and state its domain.

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6. Let $g(x) = xe^{-x} + be^{-x}$, where b is a positive constant. $xe^{-x} + be^{-x} = \frac{x+b}{e^x}$

   (a) Find $\lim_{x \to \infty} g(x)$. $\lim_{x\to\infty} \frac{x+b}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$ (use l'Hospital's Rule since $\frac{\infty + b}{e^{\infty}} = \frac{\infty}{\infty}$ is an indeterminate form.)

   (b) For what positive value of b does g have an absolute maximum at $x = \frac{2}{3}$? Justify your answer.

   (c) Find all values of b, if any, for which the graph of g has a point of inflection on the interval $0 < x < \infty$. Justify your answer.

b) $g'(x) = \frac{e^x - (x+b)e^x}{e^{2x}} = \frac{1 - x - b}{e^x}$

$= 0$ when $1 - x - b = 0$

At $x = \frac{2}{3}$, $b = 1 - \frac{2}{3} = \frac{1}{3}$

$\xleftarrow{\ +\ }\ \frac{2}{3}\ \xrightarrow{\ -\ }$

→ local max @ $x = \frac{2}{3}$

since there are no other local max or min points this must be an absolute max.

c) $g''(x) = \frac{x + b - 2}{e^x}$

$= 0$ when $x + b - 2 = 0$

$b = 2 - x$

since $b, x > 0$ it follows that $0 < b < 2$