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Asymptotic Behavior of the Shock Curve and the Entropy Solution to the Scalar Conservation Law with Periodic Initial Data

By

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1. Introduction

We consider the scalar conservation law in a single space variable

$$u_t + f(u)_x = 0, \qquad t > 0,\ x \in \boldsymbol{R}, \tag{1.1}$$

with initial data

$$u(x, 0) = u_0(x), \qquad x \in \boldsymbol{R}. \tag{1.2}$$

We assume that f is C^2 and uniformly convex: $f''(u) \geq \varepsilon > 0$, $u \in \boldsymbol{R}$. To select the physically meaningful solution we impose the entropy condition

$$\frac{u(x+a, t) - u(x, t)}{a} \leq E/t, \qquad a > 0,\ t > 0, \tag{1.3}$$

where $E > 0$ is a constant independent of x, t, and a.

It is well known that for arbitrary bounded measurable initial data there exists uniquely a global weak solution to (1.1), (1.2) and (1.3), which is called the entropy solution. Generally, this solution develops discontinuities in a finite time even though $u_0(x)$ is smooth, so such a solution is called the shock wave (see Smoller [4]). The purpose of this paper is to give the asymptotic formula for the shock wave for large time in the case where the initial data u_0 is periodic.

Lax [1] gives an explicit formula for the entropy solution for all time (see also Lax [2]). Put $f'(u) = a(u)$ and $a^{-1}(u) = b(u)$. We define the conjugate function $g(z)$ by

$$g(z) = zb(z) - f(b(z)).$$

Let

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(1.4) $$G(x, y, t) = \int_0^y u_0(z)\,dz + tg\left(\frac{x-y}{t}\right).$$

Then for fixed $t > 0$, for all but a countable set of x the minimum of (1.4) is taken at exactly one point $y = y(x, t)$. The entropy solution is given by

(1.5) $$u(x, t) = b\left(\frac{x - y(x, t)}{t}\right).$$

For fixed $t > 0$, we call the point $x = x(t)$ a shock point if the solution $u(\cdot, t)$ has a jump discontinuity across $x(t)$. The entropy condition (1.3) requires that at the shock point it holds $u(x(t) - 0, t) > u(x(t) + 0, t)$. Also we call the locus of $x(t)$ a shock curve.

Let

$$t^* = \inf\{t > 0 \mid \exists x = x(t) \text{ such that } u(x(t) - 0, t) > u(x(t) + 0, t)\}.$$

In Section 2, using Lax's formula (1.5) we shall give a practical method of determining the location of the shock point $x(t)$ for all $t > t^*$ and also the behavior of the solution at the point of discontinuity.

The method in Section 2 can be applied to the case where the initial data u_0 is periodic. In Section 3, we can find the shock curve and the entropy solution in the form of asymptotic expansions for large time. We shall give an explicit method of constructing these asymptotic expansions. Suppose that $u_0(x)$ is periodic with period p, and has mean value zero, that is, $\int_0^p u_0(x)\,dx = 0$. Let c be the point where $U(x) = \int_0^x u_0(y)\,dy$ takes its minimum. Then we can also prove that to construct these asymptotic expansions for large time, it is sufficient to assume the regularity of the initial data $u_0(x)$ only in a neighborhood of $x = c$. Our result is a more precise description of the asymptotic behavior of the entropy solution than Lax's saw tooth wave [1].

2. A method of determining the shock point $x(t)$

We may assume without loss of generality that $f(0) = f'(0) = 0$, since in (1.1) we may replace $f(u)$ by $f(u) - f(0) - f'(0)u$ and $u(x, t)$ by $u(x + f'(0)t, t)$. Now let the initial data $u_0(x)$ be a bounded C^2 function such that

1. There exists a unique point m such that $u_0'(x) > 0$ for $x < m$, $u_0'(m) = 0$ and $u_0'(x) < 0$ for $x > m$,
2. There exists a unique point $\tau > m$ such that $(a \circ u_0)''(x) < 0$ for $m \le x < \tau$, $(a \circ u_0)''(\tau) = 0$, and $(a \circ u_0)''(x) > 0$ for $\tau < x$.

Put

$$t^* = -\{(a \circ u_0)'(\tau)\}^{-1}.$$

Note that since $a(u)$ is an increasing function,

$$u_0(y) - b\left(\frac{x-y}{t}\right) = 0$$

is equivalent to

$$a \circ u_0(y) - \frac{x-y}{t} = 0.$$

If $0 < t \leq t^*$ then for each $x \in \boldsymbol{R}$ the two graphs $z = b\left(\frac{x-y}{t}\right)$ and $z = u_0(y)$ intersect at a unique point $y = y_0(x, t)$.

If $t^* < t$ then there exist two functions $\beta_1(t), \beta_2(t) \in C^\infty((t^*, \infty))$ such that $\beta_1(t) < \tau < \beta_2(t)$ and $-1/t = (a \circ u_0)'(\beta_1(t)) = (a \circ u_0)'(\beta_2(t))$. Let $\alpha_i(t) = (a \circ u_0)(\beta_i(t))t + \beta_i(t)$, $i = 1, 2$; see Figure 1.

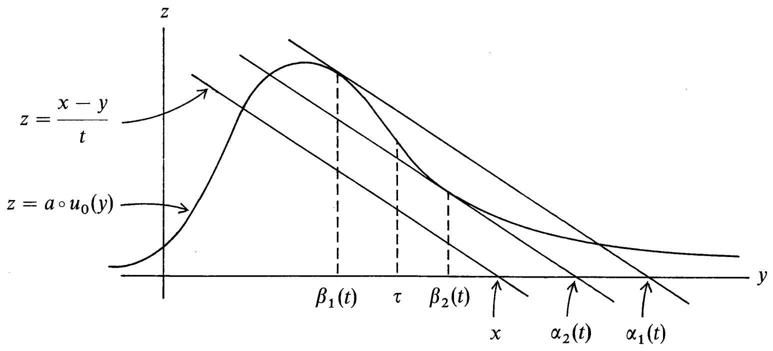

Figure 1.

Then we can easily check that $\alpha_1(t) > \alpha_2(t)$ and

$$\text{(2.1)} \qquad \lim_{t \to t^*+0} \alpha_i(t) = \tau - \frac{(a \circ u_0)(\tau)}{(a \circ u_0)'(\tau)}, \qquad i = 1, 2.$$

Now we observe that for each $x \in (-\infty, \alpha_2(t)]$ the two graphs $z = b\left(\frac{x-y}{t}\right)$, $z = u_0(y)$ intersect at a unique point $y = y_1(x, t)$, for each $x \in [\alpha_1(t), \infty)$ at a unique point $y = y_2(x, t)$ and for each $x \in (\alpha_2(t), \alpha_1(t))$ at three distinct points $y = y_1(x, t), \tilde{y}(x, t), y_2(x, t)$ $(y_1(x, t) < \tilde{y}(x, t) < y_2(x, t))$. Note that for fixed $t > t^*$, $y_1(x, t)$ is a continuous function of $x \in (-\infty, \alpha_1(t))$ and $y_2(x, t)$ is a continuous function of $x \in (\alpha_2(t), +\infty)$. For $x \in (\alpha_2(t), \alpha_1(t))$ let D_1 and D_2 be the left and right regions surrounded by the two graphs respectively; see Figure 2.

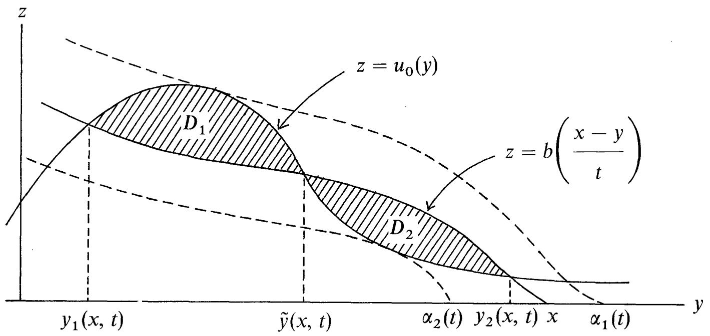

Figure 2.

We denote the area of D_i by $|D_i|$ $(i = 1, 2)$.

Theorem 2.1. (1) *If* $0 < t \leq t^*$ *then* $u(x, t) = u_0(y_0(x, t))$.
(2) *Let* $t^* < t$ *and* $x = x(t) \in (\alpha_2(t), \alpha_1(t))$ *be the point where* $|D_1| = |D_2|$ *holds. Then we have*

(a) *If* $x \in (-\infty, x(t))$ *then* $u(x, t) = u_0(y_1(x, t))$.
(b) *If* $x \in (x(t), +\infty)$ *then* $u(x, t) = u_0(y_2(x, t))$.
(c) $x(t)$ *is the shock point where* $u(x(t) - 0, t) > u(x(t) + 0, t)$ *and* $u_0(y_1(x(t), t)) = u(x(t) - 0, t)$, $u_0(y_2(x(t), t)) = u(x(t) + 0, t)$.

(3) *The limit of the shock curve* $x^* = \lim_{t \to t^* + 0} x(t)$ *is given by*

$$x^* = \tau - \frac{(a \circ u_0)(\tau)}{(a \circ u_0)'(\tau)}.$$

Proof. We make use of Lax's formula (1.5). Since $y(x, t)$ minimizes $G(x, y, t)$, we have

$$\frac{\partial}{\partial y} G(x, y(x, t), t) = u_0(y(x, t)) - b\left(\frac{x - y(x, t)}{t}\right) = 0. \tag{2.2}$$

Here we have used the equality

$$g'(z) = b(z) + zb'(z) - f'(b(z))b'(z) = b(z).$$

Let us consider the case where the two graphs $z = b\left(\frac{x - y}{t}\right)$ and $z = u_0(y)$ intersect at a unique point $y = \xi(x, t)$. Then we can easily observe that

$\frac{\partial^2}{\partial y^2} G(x, \xi, t) > 0$ and therefore $\xi = y(x, t)$. Now let us consider the case where the two graphs intersect at three points, that is, $t^* < t$ and $x \in (\alpha_2(t), \alpha_1(t))$. Then $y_1(x, t)$ and $y_2(x, t)$ are the points where $G(x, y, t)$ takes its relative minimum. To select the point which minimizes $G(x, y, t)$ we rewrite (1.4) as

$$G(x, y, t) = \int_0^y u_0(z)\,dz + \int_y^x b\left(\frac{x-z}{t}\right)dz.$$

Now let $x \in (\alpha_2(t), x(t))$. Then from Figure 2 we can easily observe that $|D_1| > |D_2|$, which implies $G(x, y_1(x, t), t) < G(x, y_2(x, t), t)$. Hence we have $y(x, t) = y_1(x, t)$. Similarly, for $x \in (x(t), \alpha_1(t))$ we have $y(x, t) = y_2(x, t)$. Therefore, combining with (1.5) and (2.2) we obtain (1) and (2).
(3) is immediately obtained from $\alpha_1(t) > x(t) > \alpha_2(t)$ and (2.1).

Q.E.D.

Remark. Whitham obtained a result similar to the above theorem in [5, Secs. 2.8, 2.9], but his proof seems to be incomplete.

3. Construction of the asymptotic expansions

We shall apply the method in Section 2 to the case where the initial data $u_0(x)$ is periodic with period p, and also give the asymptotic formulae for the shock curve $x(t)$ and the entropy solution $u(x, t)$.

From the uniqueness of the entropy solution, it follows that for all $t > 0$, $u(\cdot, t)$ is a periodic function of period p. Hence it is enough to give the asymptotic formulae over one period. Without loss of generality we may assume that $f(0) = f'(0) = 0$ and $\frac{1}{p}\int_0^p u_0(y)\,dy = 0$, since in (1.1) we may replace $f(u)$ by $f(u+m) - f(m) - f'(m)u$ and $u(x, t)$ by $u(x + f'(m)t, t) - m$ where $m = \frac{1}{p}\int_0^p u_0(y)\,dy$.

Suppose that $U(y) = \int_0^y u_0(z)\,dz$, also periodic, has a unique minimum in each period. Let

$$\min_{0 \le y < p} U(y) = U(c),$$

and let $u_0(y)$ be C^2 in a neighborhood of $y = c$.
Then $u_0(c) = 0$ and $u_0(y)$ is strictly increasing in a neighborhood of $y = c$.

Theorem 3.1 (Main result). *Let $N \geq 1$, and assume that*

1. *The bounded measurable initial data* $u_0(y)$ *of period* p *is* C^{N+1} *in a neighborhood of* $y = c$,
2. $u_0'(c) > 0$,
3. $f(u)$ *is* $C^{N+2}(\boldsymbol{R})$.

Then the shock curve $x(t)$ *and the solution* $u(x, t)$ *admit the following asymptotic expansions as* $t \to +\infty$:

$$x(t) = c + \frac{p}{2} + \sum_{k=1}^{N-1} c_k t^{-k} + O(t^{-N}),$$

$$u(x, t) = \begin{cases} \dfrac{x - c}{f''(0)} t^{-1} + \displaystyle\sum_{k=2}^{N} A_k(x) t^{-k} + O(t^{-N-1}) & c \le x < x(t) \\ \dfrac{x - c - p}{f''(0)} t^{-1} + \displaystyle\sum_{k=2}^{N} A_k(x - p) t^{-k} + O(t^{-N-1}) & x(t) < x \le c + p. \end{cases}$$

The coefficients c_k $(k \ge 1)$, $A_k(x)$ $(k \ge 2)$ *can be determined successively by solving certain recursive equations.*
In particular,

$$c_1 = \frac{f^{(3)} p^2}{24 (f^{(2)})^2}, \quad c_2 = \frac{-p^2}{12 (f^{(2)})^2 u_0^{(1)}} \left(\frac{f^{(3)}}{f^{(2)}} + \frac{u_0^{(2)}}{2 (u_0^{(1)})^2} \right),$$

$$A_2(x) = \frac{-f^{(3)} (x - c)^2}{2 (f^{(2)})^3} - \frac{x - c}{(f^{(2)})^2 u_0^{(1)}},$$

where

$$f^{(i)} = f^{(i)}(0), \quad u_0^{(i)} = u_0^{(i)}(c).$$

Remark. When the periodic initial data u_0 is bounded and measurable Lax [1] showed that $u(x, t)$ tends to a saw tooth function $v(x, t)$ as $t \to +\infty$:

$$v(x, t) = \begin{cases} \dfrac{x - c}{f''(0)} t^{-1} & c \le x < c + \dfrac{p}{2} \\ \dfrac{x - c - p}{f''(0)} t^{-1} & c + \dfrac{p}{2} < x \le c + p, \end{cases}$$

which corresponds to the first term of each asymptotic expansion of Theorem 3.1 (see also Yoshikawa [6]).

As before, let $y(x, t)$ be the point where $G(x, y, t)$ takes its minimum.

Lemma 3.2. *We have*

$$c - y(x, t) \longrightarrow 0 \quad \text{or } c + p - y(x, t) \longrightarrow 0 \text{ as } t \longrightarrow +\infty,$$

uniformly in a.e. $x \in [c, c + p]$.

We shall prove this lemma in Appendix.

Let t be sufficiently large. For each $x \in [c, c+p]$ the two graphs $z = u_0(y)$ and $z = b\left(\frac{x-y}{t}\right)$ restricted on $y \in [c, c+p]$ intersect at more than two points. From among these points of intersection choose the point whose y-coordinate $y_1(x, t)$ is nearest to c and the point whose y-coordinate $y_2(x, t)$ is nearest to $c + p$; see Figure 3.

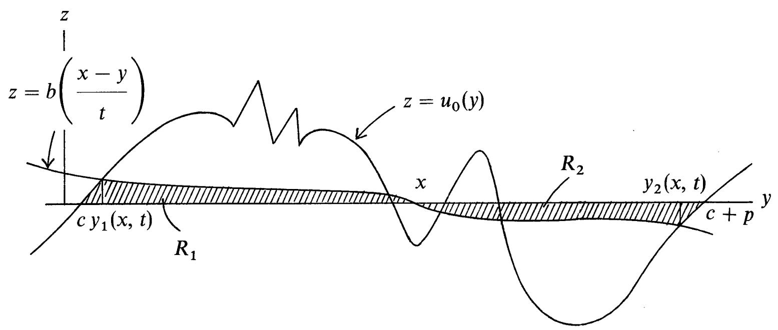

Figure 3.

Let R_1 be the region surrounded by $z = u_0(y)$ $(c \leq y \leq y_1(x, t))$, $z = b\left(\frac{x-y}{t}\right)$ $(y_1(x, t) \leq y \leq x)$ and y-axis $(c \leq y \leq x)$ and let R_2 be the region surrounded by $z = b\left(\frac{x-y}{t}\right)$ $(x \leq y \leq y_2(x, t))$, $z = u_0(y)$ $(y_2(x, t) \leq y \leq c+p)$ and y-axis $(x \leq y \leq c+p)$. We denote the area of R_i by $|R_i|$ $(i = 1, 2)$.

Proposition 3.3. *Let t be large enough and $x = x(t) \in [c, c+p]$ be the point where $|R_1| = |R_2|$ holds. Then we have*

(1) *If $x \in [c, x(t))$ then $u(x, t) = u_0(y_1(x, t))$,*

(2) *If $x \in (x(t), c+p]$ then $u(x, t) = u_0(y_2(x, t))$,*

(3) *$x(t)$ is the shock point where $u(x(t)-0, t) > u(x(t)+0, t)$ and $u_0(y_1(x(t), t)) = u(x(t)-0, t)$, $u_0(y_2(x(t), t)) = u(x(t)+0, t)$.*

Proof. From Lemma 3.2 it follows $y(x, t) = y_1(x, t)$ or $y(x, t) = y_2(x, t)$ for sufficiently large t. Now let $x \in [c, x(t))$. From Figure 3 we can easily observe that $|R_1| < |R_2|$, which implies

$$\int_c^{y_1(x,t)} u_0(y)\,dy + \int_{y_1(x,t)}^{y_2(x,t)} b\left(\frac{x-y}{t}\right) dy + \int_{y_2(x,t)}^{c+p} u_0(y)\,dy < 0.$$

Combining with $\int_c^{c+p} u_0(y)\,dy = 0$ yields

$$\int_{y_1(x,t)}^{y_2(x,t)} \left\{u_0(y) - b\left(\frac{x-y}{t}\right)\right\} dy > 0.$$

Then applying the argument as in the proof of Theorem 2.1, we obtain (1). The proof of (2) and (3) is almost parallel to that of (1). Q.E.D.

From Proposition 3.3 we conclude that for sufficiently large t the shock curve $x(t)$ satisfies

$$\int_{y_2(x(t),t)-p}^{y_1(x(t),t)} u_0(y)\,dy + \int_{y_1(x(t),t)}^{y_2(x(t),t)} b\left(\frac{x(t)-y}{t}\right) dy = 0. \tag{3.1}$$

Proof of Theorem 3.1. Put

$$y_1(t) = y_1(x(t), t),\ y_2(t) = y_2(x(t), t) \quad \text{and} \quad \varepsilon = t^{-1}.$$

We seek the asymptotic expansions of $x(t)$, $y_1(t)$ and $y_2(t)$ in the following forms:

$$\begin{aligned} &x^{N-1}(t) = x^{N-1}(\varepsilon) = c_0 + \sum_{k=1}^{N-1} c_k \varepsilon^k, \\ &y_1^N(t) = y_1^N(\varepsilon) = c + \sum_{k=1}^{N} a_k \varepsilon^k, \quad y_2^N(t) = y_2^N(\varepsilon) = c + p + \sum_{k=1}^{N} b_k \varepsilon^k. \end{aligned} \tag{3.2}$$

In the first step we shall give an explicit method of determining the coefficients in each expansion of (3.2). The second step is devoted to prove that each expansion of (3.2) actually approximates $x(t)$, $y_1(t)$ and $y_2(t)$ respectively as $t \to +\infty$. Finally, in the third step we shall give an asymptotic expansion of $u(x, t)$.

(First Step) From Figure 3

$$b\left(\frac{x(t)-y_1(t)}{t}\right) = u_0(y_1(t)), \quad b\left(\frac{x(t)-y_2(t)}{t}\right) = u_0(y_2(t)).$$

Then

$$\frac{x(t)-y_1(t)}{t} = a \circ u_0(y_1(t)), \quad \frac{x(t)-y_2(t)}{t} = a \circ u_0(y_2(t)). \tag{3.3}$$

Moreover from (3.1) it follows

$$\int_{y_2(t)-p}^{y_1(t)} u_0(y)\,dy + tg\left(\frac{x(t)-y_1(t)}{t}\right) - tg\left(\frac{x(t)-y_2(t)}{t}\right) = 0,$$

so that by (3.3)

$$\int_{y_2(t)-p}^{y_1(t)} u_0(y)\,dy + tg\circ a\circ u_0(y_1(t)) - tg\circ a\circ u_0(y_2(t)) = 0. \tag{3.4}$$

Hence we obtain a system of equations for $x(\varepsilon) = x(t)$, $y_1(\varepsilon) = y_1(t)$, and $y_2(\varepsilon) = y_2(t)$:

$$\varepsilon(x(\varepsilon) - y_1(\varepsilon)) = a\circ u_0(y_1(\varepsilon)), \tag{3.5}$$

$$\varepsilon(x(\varepsilon) - y_2(\varepsilon)) = a\circ u_0(y_2(\varepsilon)), \tag{3.6}$$

$$\varepsilon\int_{y_2(\varepsilon)-p}^{y_1(\varepsilon)} u_0(y)\,dy + g\circ a\circ u_0(y_1(\varepsilon)) - g\circ a\circ u_0(y_2(\varepsilon)) = 0. \tag{3.7}$$

Lemma 3.4. *For sufficiently large t, $x(t)$ is differentiable and* (3.4) *is equivalent to*

$$\frac{dx(t)}{dt} = \frac{f(u_0(y_1(t)) - f(u_0(y_2(t)))}{u_0(y_1(t)) - u_0(y_2(t))}. \tag{3.8}$$

We shall prove this lemma in Appendix.

By virtue of this lemma we may replace (3.7) by

$$-\varepsilon^2\frac{dx(\varepsilon)}{d\varepsilon} = \frac{f(u_0(y_1(\varepsilon)) - f(u_0(y_2(\varepsilon)))}{u_0(y_1(\varepsilon)) - u_0(y_2(\varepsilon))}. \tag{3.7$'$}$$

From the assumptions we write

$$\begin{aligned} u_0(y) &= \sum_{k=1}^{N} u_k(y-c)^k + O((y-c)^{N+1}) \qquad (y \longrightarrow c),\\ f(u) &= \sum_{k=2}^{N+1} f_k u^k + O(u^{N+2}) \qquad (u \longrightarrow 0), \end{aligned} \tag{3.9}$$

where

$$u_k = \frac{u_0^{(k)}(c)}{k!},\ u_1 > 0 \quad\text{and}\quad f_k = \frac{f^{(k)}(0)}{k!}. \tag{3.10}$$

Now expanding $a(u)$, $u_0(y)$, $f(u)$ in (3.5), (3.6) and (3.7′) by (3.9) we substitute (3.2) into them. Putting all the coefficients in each term of the same power of ε to be zero, we obtain a recurrent system of equations for c_0, c_k, a_k and $b_k (k \geq 1)$.
From (3.5) and (3.6) we obtain

$$\varepsilon(c_0 - c) + \sum_{k=1}^{N-1} (c_k - a_k)\varepsilon^{k+1} = \sum_{k=2}^{N+1} kf_k\{\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} a_m\varepsilon^m)^l\}^{k-1} + O(\varepsilon^{N+1}), \tag{3.11}$$

$$\varepsilon(c_0 - c - p) + \sum_{k=1}^{N-1} (c_k - b_k)\varepsilon^{k+1} = \sum_{k=2}^{N+1} kf_k\{\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} b_m\varepsilon^m)^l\}^{k-1} + O(\varepsilon^{N+1}). \tag{3.12}$$

Putting the terms of order ε to be zero we have

$$c_0 - c = 2f_2u_1a_1, \tag{3.13}$$

$$c_0 - c - p = 2f_2u_1b_1, \tag{3.14}$$

which imply $a_1 \neq b_1$.
Now substituting (3.2) into (3.7′) and taking $a_1 \neq b_1$ into account we obtain

$$\begin{aligned} -\varepsilon^2 \sum_{k=1}^{N-1} kc_k\varepsilon^{k-1} &= f_2\{\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} a_m\varepsilon^m)^l + \sum_{l=1}^{N} u_l(\sum_{m=1}^{N} b_m\varepsilon^m)^l\} \\ &+ f_3\{[\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} a_m\varepsilon^m)^l]^2 + \sum_{l=1}^{N} u_l(\sum_{m=1}^{N} a_m\varepsilon^m)^l \cdot \sum_{l=1}^{N} u_l(\sum_{m=1}^{N} b_m\varepsilon^m)^l \\ &+ [\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} b_m\varepsilon^m)^l]^2\} + \cdots\cdots\cdots \\ &+ f_{N+1}\sum_{k=0}^{N} [\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} a_m\varepsilon^m)^l]^{N-k} [\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} b_m\varepsilon^m)^l]^k \\ &+ O(\varepsilon^{N+1}). \end{aligned} \tag{3.15}$$

Putting the terms of order ε to be zero, we have

$$f_2(u_1a_1 + u_1b_1) = 0.$$

Combining with (3.13) and (3.14) we obtain

$$c_0 = c + \frac{p}{2},\ a_1 = \frac{p}{4f_2u_1}, \quad \text{and} \quad b_1 = \frac{-p}{4f_2u_1}.$$

Now putting the terms of order ε^2 in (3.11), (3.12) and (3.15) to be zero, we have

$$c_1 - a_1 = 2f_2(u_1 a_2 + u_2 a_1^2) + 3f_3 u_1^2 a_1^2,$$
$$c_1 - b_1 = 2f_2(u_1 b_2 + u_2 b_1^2) + 3f_3 u_1^2 b_1^2,$$
$$-c_1 \quad = f_2(u_1 a_2 + u_2 a_1^2 + u_1 b_2 + u_2 b_1^2) + f_3(u_1^2 a_1^2 + u_1^2 a_1 b_1 + u_1^2 b_1^2).$$

Therefore we obtain

$$c_1 = \frac{f_3 p^2}{16 f_2^2}, \quad a_2 = \frac{-p}{16 f_2^2 u_1}\left(\frac{f_3 p}{f_2} + \frac{2}{u_1} + \frac{u_2 p}{u_1^2}\right),$$

and

$$b_2 = \frac{-p}{16 f_2^2 u_1}\left(\frac{f_3 p}{f_2} - \frac{2}{u_1} + \frac{u_2 p}{u_1^2}\right).$$

Similarly, putting the terms of order ε^3 to be zero, we obtain

$$c_2 = \frac{-p^2}{16 f_2^2 u_1}\left(\frac{f_3}{f_2} + \frac{u_2}{3u_1^2}\right).$$

The same argument can also be applied to determine a_k, b_k and $c_k (k \geq 2)$ inductively. Equating to zero each term of order $\varepsilon^n (n \leq N)$ we obtain

$$c_{n-1} - a_{n-1} \quad = 2f_2 u_1 a_n + P_n(f_2,\ldots,f_{n+1}, u_1,\ldots,u_n, a_1,\ldots,a_{n-1}),$$
$$c_{n-1} - b_{n-1} \quad = 2f_2 u_1 b_n + Q_n(f_2,\ldots,f_{n+1}, u_1,\ldots,u_n, b_1,\ldots,b_{n-1}),$$
$$-(n-1)c_{n-1} = f_2(u_1 a_n + u_1 b_n)$$
$$+ R_n(f_2,\ldots,f_{n+1}, u_1,\ldots,u_n, a_1,\ldots,a_{n-1}, b_1,\ldots,b_{n-1}),$$

where P_n, Q_n and R_n are the polynomials of their arguments. Therefore we can determine a_n, b_n and c_{n-1} from a_k and b_k $(k \leq n-1)$.

(Second Step) In this step we shall prove

$$\text{(3.16)} \qquad \begin{aligned} &x(\varepsilon) - x^{N-1}(\varepsilon) = O(\varepsilon^N), \quad y_1(\varepsilon) - y_1^N(\varepsilon) = O(\varepsilon^{N+1}), \\ &y_2(\varepsilon) - y_2^N(\varepsilon) = O(\varepsilon^{N+1}). \end{aligned}$$

From (3.11) and (3.12) it follows

$$\varepsilon(x^{N-1}(\varepsilon) - y_1^N(\varepsilon)) = a \circ u_0(y_1^N(\varepsilon)) + O(\varepsilon^{N+1})$$

and

$$\varepsilon(x^{N-1}(\varepsilon) - y_2^N(\varepsilon)) = a \circ u_0(y_2^N(\varepsilon)) + O(\varepsilon^{N+1}).$$

From Lemma 3.4 and (3.15) it can be easily seen that

$$\varepsilon\int_{y_2^N(\varepsilon)-p}^{y_1^N(\varepsilon)} u_0(y)\,dy + g\circ a\circ u_0(y_1^N(\varepsilon)) - g\circ a\circ u_0(y_2^N(\varepsilon)) = O(\varepsilon^{N+2}).$$

Hence combining with (3.5) (3.6) and (3.7) we have

$$\begin{aligned}(3.17)\qquad & a\circ u_0(y_1^N(\varepsilon)) - a\circ u_0(y_1(\varepsilon)) - \varepsilon(x^{N-1}(\varepsilon) - x(\varepsilon)) + \varepsilon(y_1^N(\varepsilon) - y_1(\varepsilon)) \\ &= O(\varepsilon^{N+1})\end{aligned}$$

$$\begin{aligned}(3.18)\qquad & a\circ u_0(y_2^N(\varepsilon)) - a\circ u_0(y_2(\varepsilon)) - \varepsilon(x^{N-1}(\varepsilon) - x(\varepsilon)) + \varepsilon(y_2^N(\varepsilon) - y_2(\varepsilon)) \\ &= O(\varepsilon^{N+1})\end{aligned}$$

and

$$\begin{aligned}(3.19)\qquad & \varepsilon\left(\int_{y_1(\varepsilon)}^{y_1^N(\varepsilon)} u_0(y)\,dy + \int_{y_2^N(\varepsilon)-p}^{y_2(\varepsilon)-p} u_0(y)\,dy\right) + g\circ a\circ u_0(y_1^N(\varepsilon)) \\ & - g\circ a\circ u_0(y_1(\varepsilon)) - \{g\circ a\circ u_0(y_2^N(\varepsilon)) - g\circ a\circ u_0(y_2(\varepsilon))\} = O(\varepsilon^{N+2}).\end{aligned}$$

Hereafter we denote the open interval $(\min(a, b), \max(a, b))$ by $I(a, b)$. By the mean value theorem there exist $\xi_i^N = \xi_i^N(\varepsilon)\in I(y_i^N(\varepsilon), y_i(\varepsilon))$, $i = 1, 2$ such that

$$a\circ u_0(y_i^N(\varepsilon)) - a\circ u_0(y_i(\varepsilon)) = (a\circ u_0)'(\xi_i^N)(y_i^N(\varepsilon) - y_i(\varepsilon)), \qquad i = 1, 2,$$

there exist $\eta_1^N = \eta_1^N(\varepsilon)\in I(y_1^N(\varepsilon), y_1(\varepsilon))$, $\eta_2^N = \eta_2^N(\varepsilon)\in I(y_2(\varepsilon) - p, y_2^N(\varepsilon) - p)$ such that

$$\int_{y_1(\varepsilon)}^{y_1^N(\varepsilon)} u_0(y)\,dy = u_0(\eta_1^N)(y_1^N(\varepsilon) - y_1(\varepsilon)),$$

$$\int_{y_2(\varepsilon)-p}^{y_2^N(\varepsilon)-p} u_0(y)\,dy = u_0(\eta_2^N)(y_2^N(\varepsilon) - y_2(\varepsilon)),$$

and there exist $v_i^N = v_i^N(\varepsilon)\in I(a\circ u_0(y_i^N(\varepsilon)), a\circ u_0(y_i(\varepsilon))$, $i = 1, 2$ such that

$$\begin{aligned} & g\circ a\circ u_0(y_i^N(\varepsilon)) - g\circ a\circ u_0(y_i(\varepsilon)) \\ &\qquad = b(v_i^N)(a\circ u_0(y_i^N(\varepsilon)) - a\circ u_0(y_i(\varepsilon))) \\ &\qquad = b(v_i^N)(a\circ u_0)'(\xi_i^N)(y_i^N(\varepsilon) - y_i(\varepsilon)),\ i = 1, 2.\end{aligned}$$

Now substituting them into (3.17), (3.18) and (3.19) we have the following linearized system

$$A\begin{pmatrix} y_1^N(\varepsilon) - y_1(\varepsilon) \\ y_2^N(\varepsilon) - y_2(\varepsilon) \\ x^{N-1}(\varepsilon) - x(\varepsilon)\end{pmatrix} = \begin{pmatrix} O(\varepsilon^{N+1}) \\ O(\varepsilon^{N+1}) \\ O(\varepsilon^{N+2})\end{pmatrix},$$

where

$$A = (a_{ij})_{1 \le i \le 3, 1 \le j \le 3}$$
$$= \begin{pmatrix} (a \circ u_0)'(\xi_1^N) + \varepsilon & 0 & -\varepsilon \\ 0 & (a \circ u_0)'(\xi_2^N) + \varepsilon & -\varepsilon \\ \varepsilon u_0(\eta_1^N) + b(v_1^N)(a \circ u_0)'(\xi_1^N) & -\varepsilon u_0(\eta_2^N) - b(v_2^N)(a \circ u_0)'(\xi_2^N) & 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} y_1^N(\varepsilon) - y_1(\varepsilon) \\ y_2^N(\varepsilon) - y_2(\varepsilon) \\ x^{N-1}(\varepsilon) - x(\varepsilon) \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} O(\varepsilon^{N+1}) \\ O(\varepsilon^{N+1}) \\ O(\varepsilon^{N+2}) \end{pmatrix}, \tag{3.20}$$

where $\det A$, the determinant of the matrix A, is given by

$$\begin{aligned} \det A = {} & \varepsilon (a \circ u_0)'(\xi_1^N)(a \circ u_0)'(\xi_2^N)(b(v_1^N) - b(v_2^N)) \\ & + \varepsilon^2 \{ b(v_1^N)(a \circ u_0)'(\xi_1^N) + (a \circ u_0)'(\xi_2^N) u_0(\eta_1^N) \\ & - b(v_2^N)(a \circ u_0)'(\xi_2^N) - (a \circ u_0)'(\xi_1^N) u_0(\eta_2^N) \} \\ & + \varepsilon^3 (u_0(\eta_1^N) - u_0(\eta_2^N)), \end{aligned} \tag{3.21}$$

and b_{ij}, the cofactor of a_{ij}, is given by

$$\begin{aligned} b_{11} &= -\varepsilon \{ \varepsilon u_0(\eta_2^N) + b(v_2^N)(a \circ u_0)'(\xi_2^N) \}, \\ b_{12} &= \varepsilon \{ \varepsilon u_0(\eta_2^N) + b(v_2^N)(a \circ u_0)'(\xi_2^N) \}, \quad b_{13} = \varepsilon \{ (a \circ u_0)'(\xi_2^N) + \varepsilon \} \\ b_{21} &= -\varepsilon \{ \varepsilon u_0(\eta_1^N) + b(v_1^N)(a \circ u_0)'(\xi_1^N) \}, \\ b_{22} &= \varepsilon \{ \varepsilon u_0(\eta_1^N) + b(v_1^N)(a \circ u_0)'(\xi_1^N) \}, \quad b_{23} = \varepsilon \{ (a \circ u_0)'(\xi_1^N) + \varepsilon \}, \\ b_{31} &= -\{ (a \circ u_0)'(\xi_2^N) + \varepsilon \} \{ \varepsilon u_0(\eta_1^N) + b(v_1^N)(a \circ u_0)'(\xi_1^N) \}, \\ b_{32} &= \{ (a \circ u_0)'(\xi_1^N) + \varepsilon \} \{ \varepsilon u_0(\eta_2^N) + b(v_2^N)(a \circ u_0)'(\xi_2^N) \}, \\ b_{33} &= \{ (a \circ u_0)'(\xi_1^N) + \varepsilon \} \{ (a \circ u_0)'(\xi_2^N) + \varepsilon \}. \end{aligned}$$

Now we shall prove

$$\det A \ge C\varepsilon^2 \qquad \text{for some constant } C > 0. \tag{3.22}$$

Since

$$b(v_i^N) = \int_0^1 b(a \circ u_0(y_i(\varepsilon)) + \{ a \circ u_0(y_i^N(\varepsilon)) - a \circ u_0(y_i(\varepsilon)) \} s)\, ds, \qquad i = 1, 2,$$

we have

$$b(v_1^N) - b(v_2^N) = \int_0^1 [\{a \circ u_0(y_1^N(\varepsilon)) - a \circ u_0(y_2^N(\varepsilon))\} s$$
$$+ \{a \circ u_0(y_1(\varepsilon)) - a \circ u_0(y_2(\varepsilon))\}(1-s)] b'(\theta)\, ds,$$
$$\theta \in I(a \circ u_0(y_1(\varepsilon)) + \{a \circ u_0(y_1^N(\varepsilon)) - a \circ u_0(y_1(\varepsilon))\} s,$$
$$a \circ u_0(y_2(\varepsilon)) + \{a \circ u_0(y_2^N(\varepsilon)) - a \circ u_0(y_2(\varepsilon))\} s).$$

Now we observe that the integrand is positive and

$$b'(\theta) = \frac{1}{f''(b(\theta))} > c > 0$$

for some constant c, so taking $a_1 \neq b_1$ into account gives

$$b(v_1^N) - b(v_2^N) \geq \frac{c}{2}\{a \circ u_0(y_1^N(\varepsilon)) - a \circ u_0(y_2^N(\varepsilon)) + a \circ u_0(y_1(\varepsilon)) - a \circ u_0(y_2(\varepsilon))\}$$
$$> \frac{c}{2}(a \circ u_0(y_1^N(\varepsilon)) - a \circ u_0(y_2^N(\varepsilon)))$$
$$= \frac{c}{2}[\sum_{k=2}^{N+1} k f_k \{(\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} a_m \varepsilon^m)^l\}^{k-1}$$
$$- \sum_{k=2}^{N+1} k f_k \{(\sum_{l=1}^{N} u_l(\sum_{m=1}^{N} b_m \varepsilon^m)^l\}^{k-1}] + O(\varepsilon^{N+1}) \geq c'\varepsilon$$

for some constant $c' > 0$.
Since $(a \circ u_0)'(\xi_i^N) > c > 0$, $i = 1, 2$, for sufficiently small $\varepsilon > 0$ we obtain

$$\varepsilon (a \circ u_0)'(\xi_1^N)(a \circ u_0)'(\xi_2^N)(b(v_1^N) - b(v_2^N)) \geq C\varepsilon^2.$$

Moreover, noting that the second and third terms of the right hand side of (3.21) are positive we obtain (3.22).
Since $v_i^N \in I(a \circ u_0(y_i^N(\varepsilon)), a \circ u_0(y_i(\varepsilon)))$ and $b(0) = 0$ imply $b(v_i^N) = O(\varepsilon)$, it can be easily seen that

$$b_{ij} = O(\varepsilon^2),\ i, j = 1, 2,\ b_{i3} = O(\varepsilon),\ i = 1, 2,\ b_{3j} = O(\varepsilon),\ j = 1, 2,$$
$$b_{33} = O(1). \tag{3.23}$$

Therefore, combining (3.20) with (3.22) and (3.23) we obtain

$$\begin{pmatrix} y_1^N(\varepsilon) - y_1(\varepsilon) \\ y_2^N(\varepsilon) - y_2(\varepsilon) \\ x^{N-1}(\varepsilon) - x(\varepsilon) \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} O(\varepsilon^2) & O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon^2) & O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) & O(1) \end{pmatrix}\begin{pmatrix} O(\varepsilon^{N+1}) \\ O(\varepsilon^{N+1}) \\ O(\varepsilon^{N+2}) \end{pmatrix}$$

$$= \frac{1}{\det A} \begin{pmatrix} O(\varepsilon^{N+3}) \\ O(\varepsilon^{N+3}) \\ O(\varepsilon^{N+2}) \end{pmatrix} = \begin{pmatrix} O(\varepsilon^{N+1}) \\ O(\varepsilon^{N+1}) \\ O(\varepsilon^{N}) \end{pmatrix}.$$

This proves (3.16).

(Third Step) Consider the case $c \leq x < x(t)$. From Proposition 3.3 and Figure 3 it follows

(3.24) $$u(x, t) = u_0(y_1(x, t)) = b\left(\frac{x - y_1(x, t)}{t}\right).$$

Then

(3.25) $$\varepsilon(x - y_1(x, \varepsilon)) = a \circ u_0(y_1(x, \varepsilon)).$$

We seek the asymptotic expansion of $y_1(x, \varepsilon)$ of the form

$$y_1^N(x, \varepsilon) = c + \sum_{k=1}^{N} \alpha_k \varepsilon^k.$$

Substituting it into (3.25) we obtain

$$\varepsilon(x - c) - \sum_{k=1}^{N} \alpha_k \varepsilon^{k+1} = \sum_{k=2}^{N+1} k f_k \{(\sum_{l=1}^{N} u_l (\sum_{m=1}^{N} \alpha_m \varepsilon^m)^l\}^{k-1} + O(\varepsilon^{N+1}).$$

As before, putting the terms of order $\varepsilon^k (k \geq 1)$ to be zero gives

$$\alpha_1 = \frac{x - c}{2 f_2 u_1}, \ \alpha_2 = \left(-\frac{u_2}{u_1} - \frac{3 f_3 u_1}{2 f_2}\right)\left(\frac{x - c}{2 f_2 u_1}\right)^2 - \frac{x - c}{(2 f_2 u_1)^2}, \text{ and so on.}$$

Also applying the mean value theorem to

$$a \circ u_0(y_1^N(x, \varepsilon)) - a \circ u_0(y_1(x, \varepsilon)) + \varepsilon(y_1^N(x, \varepsilon) - y_1(x, \varepsilon)) = O(\varepsilon^{N+1})$$

leads to

$$y_1^N(x, \varepsilon) - y_1(x, \varepsilon) = O(\varepsilon^{N+1}).$$

Therefore from (3.24) it follows that

$$\begin{aligned} u(x, t) &= u_0(y_1(x, \varepsilon)) = u_0(y_1^N(x, \varepsilon)) + O(\varepsilon^{N+1}) \\ &= \sum_{k=1}^{N} u_k (\sum_{l=1}^{N} \alpha_l \varepsilon^l)^k + O(\varepsilon^{N+1}) = \sum_{k=1}^{N} A_k \varepsilon^k + O(\varepsilon^{N+1}), \end{aligned}$$

where

$$A_1 = u_1 \alpha_1 = \frac{x-c}{2f_2},$$

$$A_2 = u_1 \alpha_2 + u_2 \alpha_1^2 = -\frac{3 f_3 u_1^2}{2f_2}\left(\frac{x-c}{2f_2 u_1}\right)^2 - u_1 \frac{x-c}{(2f_2 u_1)^2}, \text{ and so on.}$$

The same argument can also be applied to the case $x(t) < x \le c + p$. Finally, combining with (3.10) we obtain the result. Q.E.D.

The similar argument can be applied to the case where the initial data $u_0(y)$ is degenerate at $y = c$.

Corollary. *Let $\sigma \ge 3$ be an odd number and let $N \ge \sigma$. Assume that*
1. *the bounded measurable initial data $u_0(y)$ of period p is C^{N+1} in a neighborhood of $y = c$,*
2. $u_0^{(n)}(c) = 0$ $1 \le n \le \sigma - 1$, $u_0^{(\sigma)}(c) > 0$,
3. *$f(u)$ is $C^{N'+2}(\boldsymbol{R})$, where N' is the smallest integer such that $N' + 1 \ge \dfrac{N+1}{\sigma}$.*

Then the shock curve $x(t)$ and the solution $u(x, t)$ admit the following asymptotic expansions as $t \to +\infty$:

$$x(t) = c + \frac{p}{2} + \sum_{k=2}^{N-\sigma} c_k t^{-k/\sigma} + O(t^{-(N-\sigma+1)/\sigma}),$$

$$u(x, t) = \begin{cases} \dfrac{x-c}{f''(0)} t^{-1} + \displaystyle\sum_{k=1}^{N-\sigma} A_k(x) t^{-1-k/\sigma} + O(t^{-(N+1)/\sigma}) & c \le x < x(t) \\ \dfrac{x-c-p}{f''(0)} t^{-1} + \displaystyle\sum_{k=1}^{N-\sigma} A_k(x-p) t^{-1-k/\sigma} + O(t^{-(N+1)/\sigma}) & x(t) < x \le c+p. \end{cases}$$

The coefficients c_k, $A_k(x)$ $(k \ge 1)$ can be determined successively by solving certain recursive equations.
In particular,

$$c_2 = \frac{-1}{(\sigma+2)(\sigma+1)} \frac{u_0^{(\sigma+1)}}{u_0^{(\sigma)}} \left(\frac{2 f^{(2)} u_0^{(\sigma)}}{\sigma!\, p}\right)^{-2/\sigma},$$

$$A_1(x) = \frac{-1}{f^{(2)}} \left(\frac{\sigma!\,(x-c)}{f^{(2)} u_0^{(\sigma)}}\right)^{1/\sigma}.$$

A. Appendix

Proof of Lemma 3.2. Assume that there exist a constant $\varepsilon > 0$ and sequences $\{t_j\}_1^\infty$; $t_j \to +\infty (j \to \infty)$, $\{x_j\}_1^\infty$; $x_j \in [c, c+p]$ such that $|c - y(x_j, t_j)| > \varepsilon$ and

$|c + p - y(x_j, t_j)| > \varepsilon$ for sufficiently large j. Then from the definition of c it follows

$$\int_0^{y(x_j,t_j)} u_0(z)\,dz - \int_0^c u_0(z)\,dz > \delta \tag{A.1}$$

for some constant $\delta > 0$.

On the other hand, since $tg\left(\frac{p}{t}\right) \to 0$ as $t \to +\infty$, we have

$$0 < t_j g\left(\frac{x_j - y(x_j, t_j)}{t_j}\right) < \delta \text{ and } 0 < t_j g\left(\frac{x_j - c}{t_j}\right) < \delta \text{ for sufficiently large } j.$$

Therefore, combining with (A.1) we have

$$\int_0^{y(x_j,t_j)} u_0(z)\,dz + t_j g\left(\frac{x_j - y(x_j, t_j)}{t_j}\right) > \int_0^c u_0(z)\,dz + t_j g\left(\frac{x_j - c}{t_j}\right).$$

This contradicts the minimizing property of $y(x_j, t_j)$. Q.E.D.

Proof of Lemma 3.4. The differentiability of $y_1(x, t)$, $y_2(x, t)$ and $x(t)$ is easily proved by applying the implicit function theorem to (2.2) and (3.1) (see Schaeffer [3]). Now we shall prove (3.8). Differentiating (3.4) gives

$$\begin{aligned} u_0(y_1(t))\frac{dy_1(t)}{dt} &+ a \circ u_0(y_1(t))u_0(y_1(t)) - f(u_0(y_1(t))) \\ &+ tu_0(y_1(t))(a \circ u_0)'(y_1(t))\frac{dy_1(t)}{dt} - \Bigg(u_0(y_2(t))\frac{dy_2(t)}{dt} \\ &+ a \circ u_0(y_2(t))u_0(y_2(t)) - f(u_0(y_2(t)) \\ &+ tu_0(y_2(t))(a \circ u_0)'(y_2(t))\frac{dy_2(t)}{dt}\Bigg) = 0. \end{aligned}$$

Here we have used the equality

$$g \circ a(u) = a(u)u - f(u).$$

Then combining with (3.3) and

$$(a \circ u_0)'(y_1(t))\frac{dy_i(t)}{dt} = \frac{-(x(t) - y_i(t))}{t^2} + \frac{1}{t}\left(\frac{dx(t)}{dt} - \frac{dy_i(t)}{dt}\right), \qquad i = 1, 2,$$

we obtain (3.8).

Conversely, (3.8) implies

$$\int_{y_2(t)-p}^{y_1(t)} u_0(y)\,dy + tg \circ a \circ u_0(y_1(t)) - tg \circ a \circ u_0(y_2(t)) = C,$$

where C is a constant independent of t. Then applying the mean value theorem we have

$$\int_{y_2(t)-p}^{y_1(t)} u_0(y)\,dy + t(a\circ u_0(y_1(t)) - a\circ u_0(y_2(t)))\,b(\xi(t)) = C,$$

$$\xi(t)\in I\,(a\circ u_0(y_1(t)),\ a\circ u_0(y_2(t))).$$

From (3.3) it follows

$$a\circ u_0(y_1(t)) - a\circ u_0(y_2(t)) = \frac{1}{t}\{y_2(t) - y_1(t)\}$$

so that

$$\int_{y_2(t)-p}^{y_1(t)} u_0(y)\,dy + \{(y_2(t)) - (y_1(t))\}\,b(\xi(t)) = C.$$

Hence, by letting $t\to+\infty$ we have $y_1(t)\to c$, $y_2(t)\to c+p$ and $\xi(t)\to 0$. Therefore we obtain $C=0$. Q.E.D.

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References

[1] Lax, P.D., Hyperbolic systems of conservation laws, II. Comm. Pure Appl. Math., **10** (1957), 537–566.

[2] Lax, P.D., *Hyperbolic systems of conservation laws and the mathematical theory of shock waves,* Conf. Board Math. Sci., 11, SIAM, 1973.

[3] Schaeffer, D.G., A regularity theorem for conservation laws, Adv. in Math., **11** (1973), 368–386.

[4] Smoller, J., *Shock waves and reaction-diffusion equations,* Springer Verlag, 1983.

[5] Whitham, G.B., *Linear and nonlinear waves,* Wiley-Interscience, 1974.

[6] Yoshikawa, A., *Introduction to systems of non-linear conservation laws,* Sophia Kokyuroku in Math., 21, Sophia Univ., 1985.

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