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Global Solutions of Reaction-Diffusion Equations

By

Tomasz DLOTKO

(Silesian University, Poland)

Introduction

There is an increasing interest in long time studying of reaction-diffusion equations and their generalizations (e.g. [6], [1], [2], [3], [11]). For such considerations it is necessary to have global (for all $t \geqslant 0$) in time estimates of solutions and their different norms. We give here such global bounds for the Sobolev and Hölder norms of solutions and their spatial derivatives, then we are able to show global existence of solutions. Our estimates are obtained using the J. Moser method [9], as developed recently N. Alikakos [1], [2].

Summary. The essence of this reasoning is as follows; if we assume for the coefficients a_{ij}, b_i, f of (10) (§2) some independent of time smoothness conditions, then two global in time a priori estimates

$$\|u(t, \cdot)\|_{L^2(\Omega)} \leqslant \text{const.}, \quad \|u_t(t, \cdot)\|_{L^2(\Omega)} \leqslant \text{const.} \tag{0}$$

are necessary and sufficient for global estimates ((22)) of Hölder norms of the solution u as well as its existence. Estimates (0) are satisfied in particular if f, b_i are dominated (compare (a) in Theorem 1, (d) in Theorem 2) by the first positive eigenvalue of the linear problem

$$\begin{cases} -\sum_{i,j} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial v}{\partial x_j}\right) = \lambda v, \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Preliminaries. $\boldsymbol{R}^+ = [0, \infty)$, $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\Omega)$, $|\Omega|$-means the Lebesgue measure of Ω, $\nabla u(t, x)$ denotes $(u = u(t, x),\ x \in \boldsymbol{R}^n)$ the vector $(\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$.

By the Steklov average of the function $u(t, x)$ we mean

$$u_h(t, x) := \frac{1}{h}\int_t^{t+h} u(z, x)dz, \quad h > 0.$$

We use here the usual notation for Sobolev spaces; L^p, H_0^1, $W^{2,p}$ (see [6], [7], [12]) and for the Hölder spaces ([7]; C instead of H); $C^{j+\nu}(\Omega)$, $C^{1+\alpha/2, 2+\alpha}(\bar{D})$. Also the symbols (e.g.) $C^1_{\text{loc}}(\boldsymbol{R}^+; L^\infty(\Omega))$ are used in the usual sense [12].

Throughout the work non-specified integrals are taken over $\Omega \subset \boldsymbol{R}^n$, all sums are taken from 1 to n.

We need the following;

Lemma 1. *Let* $x \in C^1([0, \infty))$ *and* $y \in C^0([0, \infty))$, *let* $\alpha, \gamma \geqslant 0$, $\beta, \lambda > 0$ *and* $\eta \in [0, 1)$ *be real constants. If*

$$0 \leqslant \lambda x(t) \leqslant y(t) \qquad for \quad t \geqslant 0,$$
$$x'(t) \leqslant \alpha - \beta y(t) + \gamma y^{\eta}(t)$$

then $0 \leqslant x(t) \leqslant \max\{x(0); r_0/\lambda\}$, $t \geqslant 0$, *where* r_0 *denotes the unique positive root of the equation*; $\alpha - \beta r + \gamma r^{\eta} = 0$.

Proof. Clearly for $y(t) > r_0$ the right side of the differential inequality is negative, hence x is decreasing. So if $x(0) > r_0/\lambda$, then x decreases until reaching r_0/λ (possibly at $t = \infty$), but if $x(0) \leqslant r_0/\lambda$ then x could never be grater than r_0/λ.

Remark 1. Considering a little more general differential inequality

$$x'(t) \leqslant \alpha(t) - \beta y(t) + \gamma(t) y^{\eta}(t),$$

where α, γ are non-negative continuous functions, $\alpha(t), \gamma(t) \to 0$ as $t \to \infty$, one can verify that (under assumptions of Lemma 1) $\limsup_{t\to\infty} x(t) \leqslant 0$ or equivalently $\lim_{t\to\infty} x(t) = 0$.

We need also the following lemma concerning weak derivatives:

Lemma 2. *Assume* $u \in H^1(Q) \cap L^\infty(Q)$, $v \in H_0^1(Q) \cap L^\infty(Q)$. *Then* $uv \in H^1(Q)$, *and (for the Sobolev derivatives)*

$$(uv)_{x_i} = u_{x_i} v + u v_{x_i}.$$

We left an easy proof. As a consequence of Lemma 2, for the powers of $u \in H_0^1(Q) \cap L^\infty(Q)$ (for Sobolev derivatives)

$$(u^k)_{x_i} = k u_{x_i} u^{k-1}.$$

The following lemma concerns the t-derivatives:

Lemma 3. *When* $u \in Y = \{v;\ v, v_t \in L^2([0, T]; L^\infty(\Omega))\}$, *then*

(i) *u is equal (a.e.) to a function (denoted further by u) in* $C^0([0, T]; L^\infty(\Omega))$,

(ii) *for all* $p \in N$ *holds* ($\cdot_t$*-distributional derivative with values in* $L^\infty(\Omega)$): *for all* $0 \leqslant T_1 \leqslant T_2 \leqslant T$

$$\langle u(T_2, \cdot), u^p(T_2, \cdot)\rangle - \langle u(T_1, \cdot), u^p(T_1, \cdot)\rangle$$
$$= (p+1) \int_{T_1}^{T_2} \langle u_t(t, \cdot), u^p(t, \cdot)\rangle dt.$$

The proof is an easy consequence of Chapt. IV of [5].

Note that Lemma 3 remains valid (for distributional derivatives with values in $L^2(\Omega)$) also for

$$u \in \tilde{Y} = \{v;\ v \in C^0([0, T];\ L^\infty(\Omega)),\ v_t \in L^2([0, T];\ L^2(\Omega))\}.$$

§ 1. A priori L^∞ bounds

Let $x \in \Omega \subset \boldsymbol{R}^n$, Ω a bounded domain with $\partial\Omega \in C^1$, $D = \boldsymbol{R}^+ \times \Omega$. Consider the following problem;

$$(1)\quad u_t = \sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(t, x, u, \nabla u)) + \sum_i b_i(t, x, u, \nabla u)u_{x_i} + f(t, x, u, \nabla u)$$

with the initial-boundary conditions

$$(2)\qquad u = 0 \quad \text{on} \quad \partial\Omega, \quad u(0, x) = u_0(x).$$

The main part of (1) is assumed to be elliptic

$$\exists\, a_0 > 0 \quad \sum_{i,j} a_{ij}(t, x, u, \nabla u)u_{x_i} \geqslant a_0 \sum_i u_{x_i}^2,$$

uniformly in $(t, x, u, \nabla u) \in \boldsymbol{R}^+ \times \bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n =: A$. Moreover

$$|a_{ij}(t, x, u, p)| \leqslant |a(t, x)|\,|u| + |a_1(t, x)|\,|p|,$$

a, $a_1 \in L^\infty(D)$. By a weak solution of (1)–(2) we understand the function $u \in L^2_{\text{loc}}(\boldsymbol{R}^+;\ H^1_0(\Omega))$, $u_t \in L^2_{\text{loc}}(\boldsymbol{R}^+;\ L^2(\Omega))$, satisfying for almost all $t \in \boldsymbol{R}^+$ and all $\eta \in H^1_0(\Omega)$

$$(3)\qquad \langle u_t, \eta\rangle = -\int \sum_{i,j} a_{ij}\eta_{x_i}dx + \int \sum_i b_i u_{x_i}\eta dx + \int f\eta dx.$$

Note (see [12]) that u is equivalent to a function in $C^0_{\text{loc}}(\boldsymbol{R}^+;\ L^2(\Omega))$, hence it is reasonable to take u_0 from $L^2(\Omega)$. This definition has rather formal character here. Its sources are in the results of J. L. Lions; for existence questions see [7].

The following theorem is connected with [11]. Its proof is given along the lines as proposed in Theorem 3.1 of [1];

Theorem 1. *Assume there exists a weak solution u of (1)–(2) belonging to $L^2_{\text{loc}}(\boldsymbol{R}^+;\ L^\infty(\Omega) \cap H^1_0(\Omega))$ with $u_t \in L^2_{\text{loc}}(\boldsymbol{R}^+;\ L^\infty(\Omega))$, and let $|b_i| \leqslant B$, f locally bounded,*

$$\exists\, C, D > 0 \qquad uf(t, x, u, \nabla u) \leqslant Cu^2 + D \quad \textit{in the set } A.$$

Anyone of the conditions:

(a) *$f(t, x, 0, p) = 0$ ($p \in \boldsymbol{R}^n$), there exists $\delta > 0$ such that $\partial f/\partial u \leqslant \lambda(a_0 - \delta)$*

(λ *as in* (4)), $b_i=0$, $i=1,\cdots,n$,

(b) $\|u\|_{L^2(\Omega)}\to 0$ *when* $t\to\infty$, *then gives* $\|u\|_{L^p(\Omega)}\to 0$, $p\in[1,\infty)$, *when* $t\to\infty$. *If we assume only*

(c) *the* $L^2(\Omega)$ *norm of* u *is bounded globally in* t, *then the same is true for its* L^∞ *norm.*

Proof. We show first that (a)$\Rightarrow$(b). Take the test function in (3) equal to u (this belongs to $L^\infty\cap H_0^1$) to obtain

$$\begin{aligned}\langle u_t, u\rangle &= -\int\sum_{i,j} a_{ij}(t,x,u,\nabla u)u_{x_i}dx\\ &\quad+\int[f(t,x,u,\nabla u)-f(t,x,0,\nabla u)]u\,dx\\ &\leqslant -a_0\int\sum_i(u_{x_i})^2\,dx+\int\frac{\partial f}{\partial u}(t,x,\tilde{u},\nabla u)u^2dx.\end{aligned}$$

With the use of (a), Poincaré inequality;

$$\forall\, v\in H_0^1(\Omega)\;\lambda\|v\|^2_{L^2(\Omega)}\leqslant\|v\|^2_{H_0^1(\Omega)}\equiv\|\nabla v\|^2_{L^2(\Omega)},\tag{4}$$

$\lambda=\lambda(n,\Omega)>0$, and Lemma 1.2, Chapt. III of [12] we get:

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)}\leqslant -2\delta\lambda\|u\|^2_{L^2(\Omega)},$$

or equivalently

$$\|u(t,\cdot)\|^2_{L^2(\Omega)}\leqslant\|u_0\|^2_{L^2(\Omega)}\exp(-2\delta\lambda t).$$

Moreover now $C=\lambda(a_0-\delta)$, $D=0$.

Now we show that (c) implies the $L^\infty(\Omega)$ boundedness. Take as the test function in (3) u^{2^k-1} (this belongs to $H_0^1\cap L^\infty$ as follows from Lemma 2), transform the components in the following way:

$$\begin{aligned}&-\int\sum_{i,j}a_{ij}(t,x,u,\nabla u)(u^{2^k-1})_{x_i}dx\\ &=-(2^k-1)\int u^{2k-2}\sum_{i,j}a_{ij}u_{x_i}dx\leqslant -a_0(2^k-1)\int u^{2^k-2}\cdot\sum_i u^2_{x_i}\,dx\\ &=-a_0(2^k-1)2^{2-2k}\int\sum_i[(n^{2^{k-1}})_{x_i}]^2dx.\end{aligned}$$

To estimate the next component

$$\int\sum_i b_iu_{x_i}u^{2^k-1}dx=2^{1-k}\int\sum_i b_i(u^{2^{k-1}})_{x_i}u^{2^{k-1}}dx$$

we must use Hölder inequality and the consequence (compare [1], p. 209) of the Nirenberg-Gagliardo interpolation inequality ([7], Th. 2.2, Chapt. II, §2-for functions vanishing at $\partial\Omega$):

$$\forall\ w \in H_0^1(\Omega) \quad \|w\|_{L^2(\Omega)} \leqslant \beta(n, \Omega)\, \|\nabla w\|_{L^2(\Omega)}^{\theta} \|w\|_{L^1(\Omega)}^{1-\theta},$$

where $\theta = n/n+2$. This, with the use of Young's inequality (see [7]) with $m = 1/\theta$ is transformable into: for every $w \in H_0^1(\Omega)$

$$\|w\|_{L^2(\Omega)}^2 \leqslant \varepsilon \|\nabla w\|_{L^2(\Omega)}^2 + C_\varepsilon \|w\|_{L^1(\Omega)}^2, \tag{5}$$

where ε is an arbitrary positive number, $C_\varepsilon = \text{const.}\ \varepsilon^{-n/2}$. We thus have

$$\begin{aligned}
&\int \sum_i b_i u_{x_i} u^{2^k-1} dx \\
&\leqslant Bn\, 2^{1-k} \Big(\int \sum_i [(u^{2^{k-1}})_{x_i}]^2\, dx \Big)^{1/2} \cdot \Big(\int u^{2^k} dx \Big)^{1/2} \\
&\leqslant Bn\, 2^{1-k} \|\nabla u^{2^{k-1}}\|_{L^2(\Omega)} \cdot [\varepsilon \|\nabla u^{2^{k-1}}\|_{L^2(\Omega)}^2 + C_\varepsilon \|u^{2^{k-1}}\|_{L^1(\Omega)}^2]^{1/2} \\
&\leqslant Bn\sqrt{\varepsilon}\, 2^{1-k} \|\nabla u^{2^{k-1}}\|_{L^2(\Omega)}^2 + Bn\sqrt{C_\varepsilon}\, 2^{1-k} \|\nabla u^{2^{k-1}}\|_{L^2(\Omega)} \|u^{2^{k-1}}\|_{L^1(\Omega)}.
\end{aligned}$$

We also have:

$$\begin{aligned}
&\int f(t, x, u, \nabla u) u^{2^k-1} dx \leqslant C \int u^{2^k} dx + D \int u^{2^k-2} dx \\
&\quad \leqslant (C+D) \int u^{2^k} dx + D \int u^{2^{k-1}} dx \\
&\quad \leqslant (C+D)\, [\varepsilon' \|\nabla u^{2^{k-1}}\|_{L^2(\Omega)}^2 + C_{\varepsilon'} \|u^{2^{k-1}}\|_{L^1(\Omega)}^2] + D \int u^{2^{k-1}} dx.
\end{aligned}$$

Denoting $v := u^{2^{k-1}}$, gathering estimates, we obtain:

$$\begin{aligned}
(6)\quad 2^k \langle u_t, u^{2^k-1} \rangle &\leqslant \|\nabla v\|_{L^2(\Omega)}^2 [-a_0(2^k-1)2^{2-k} + 2Bn\sqrt{\varepsilon} + (C+D)\varepsilon' 2^k] \\
&\quad + \|\nabla v\|_{L^2(\Omega)} \|v\|_{L^1(\Omega)} 2Bn\sqrt{C_\varepsilon} + [2^k D \|v\|_{L^1(\Omega)} + 2^k (C+D) C_{\varepsilon'} \|v\|_{L^1(\Omega)}^2].
\end{aligned}$$

Choose ε and $\varepsilon'(k) = \varepsilon_k'$ such that

$$2Bn\sqrt{\varepsilon} = a_0/2, \quad (C+D)\varepsilon_k' 2^k = a_0/2$$

(hence the first bracket is less than or equal to $-a_0$), denote the inductive bound of $\|v\|_{L^2(\Omega)}^2$ by $m_k := \sup_{t \geqslant 0} \|u(t, \cdot)\|_{L^{2^k}(\Omega)}^{2^k}$, integrate over any interval $[T_1, T_2]$ and use Lemma 1 to get the integral inequality

$$\int u^{2^k}(T_2, x) dx - \int u^{2^k}(T_1, x) dx \leqslant \int_{T_1}^{T_2} (\text{right side of (6)})\, dt.$$

Since $u \in C_{\text{loc}}^0(\boldsymbol{R}^+; L^\infty(\Omega))$, then the same reasoning as in the proof of Lemma 1 shows that its theses are preserved:

$$\int u^{2^k}(t, x)dx$$

$$\leqslant \max\left\{\int u_0^{2^k}(x)dx, \frac{1}{\lambda a_0}[Bnm_{k-1}\sqrt{C_\varepsilon/a_0}+(2^k(C+D)C_{\varepsilon_k'}m_{k-1}^2+2^kDm_{k-1}\right.$$

$$\left.+(Bnm_{k-1}\sqrt{C_\varepsilon/a_0})^2)^{1/2}]^2\right\}.$$

Noting that ε is independent of k and $\varepsilon_k'=a_0/2^{k+1}(C+D)$, we have $C_\varepsilon=\text{const.}$, $C_{\varepsilon_k'}=\text{const. } 2^{(k+1)n/2}$, and the estimate of $\int u^{2^k}dx$ may be rewritten (with new constants independent of k) as: $k=2, 3,\cdots$

$$\int u^{2^k}(t, x)dx \tag{7}$$

$$\leqslant \max\left\{\int u_0^{2^k}(x)dx,\ m_{k-1}^2(\tilde{B}+\tilde{C}2^{(k+1)n/2+k})+m_{k-1}2^k\tilde{D}\right\}.$$

Since $k\leqslant(k+1)n$, taking the 2^k roots of both sides of (7) and supremum at the left side, we obtain (the 2^k root is an increasing function):

$$m_k^{2^{-k}} \leqslant \max\{\|u_0\|_{L^{2^k}(\Omega)},\ [m_{k-1}^2(\tilde{B}+\tilde{C}2^{(k+1)3n/2})+m_{k-1}2^k\tilde{D}]^{2^{-k}}\}. \tag{8}$$

Since $u_0\in L^\infty(\Omega)$, then

$$\|u_0\|_{L^{2^k}(\Omega)} \leqslant c\|u_0\|_{L^\infty(\Omega)} \leqslant c_1\|u_0\|_{L^2(\Omega)};\quad k=1, 2,\cdots,\quad c=\max\{1, |\Omega|\}$$

(if $\|u_0\|_{L^\infty}\neq 0$ then also $\|u_0\|_{L^2}>0$). We must show that the sequence $m_k^{2^{-k}}$ which is convergent to $\sup_{t\geqslant 0}\|u(t,\cdot)\|_{L^\infty(\Omega)}$ (see [13] p. 34), is bounded.

Take $x_1=\max\{\sup_{t\geqslant 0}\|u(t,\cdot)\|_{L^2(\Omega)}, 1\}$ and increase the constant $\tilde{C}$ to $\tilde{C}'=\max\{c_1^4, 1, \tilde{B}, \tilde{C}, \tilde{D}\}$. Clearly $m_k^{2^{-k}}$ is dominated by the solution x_k of the recurrence, $k=2, 3,\cdots$

$$x_k = \max\{c_1\|u_0\|_{L^{2^k}(\Omega)},\ [x_{k-1}^{2^k}(\tilde{B}+\tilde{C}'2^{(k+1)3n/2})+x_{k-1}^{2^{k-1}}2^k\tilde{D}]^{2^{-k}}\}.$$

It is easy to verify (since $x_1\geqslant 1$, $\tilde{C}'\geqslant 1$) that both $x_k\geqslant 1$, $k=1, 2,\cdots$. Now the bracket [ ] will be dominated by

$$x_{k-1}^{2^k}3\tilde{C}'2^{(k+1)3n/2},$$

and clearly $x_k\leqslant x_k'$, $k\in N$, where $x_1'=x_1$ and

$$x_k' = \max\{c_1\|u_0\|_{L^2(\Omega)},\ x_{k-1}'[3\tilde{C}'2^{(k+1)3n/2}]^{2^{-k}}\}.$$

Since $\tilde{C}'\geqslant c_1^4$, $x_1'\geqslant\|u_0\|_{L^2(\Omega)}$, then for $k=2$ the maximum is attained on the second term. But the sequence $\{x'_k\}$ is increasing ($\tilde{C}' \geqslant 1$), hence the maximum is always attained on second term

$$x'_k = x'_{k-1}[3\tilde{C}'2^{(k+1)3n/2}]^{2^{-k}}$$

or by induction

$$\text{(9)} \qquad \sup_{t\geqslant 0} \|u(t,\cdot)\|_{L^\infty(\Omega)} \leqslant x'_\infty = \max\{\sup_{t\geqslant 0}\|u(t,\cdot)\|_{L^2(\Omega)}, 1\}3\tilde{C}'(2P)^{3n/2}$$

where $P := \prod_{k=1}^{\infty}(2^k)^{2^{-k}} < \infty$. Hence we have the L^∞ bound.

There are two possibilities to show that (b) gives $L^p(\Omega)$ convergence. Since (through just shown) (b) implies the $L^\infty(\Omega)$ boundedness (by M), we can use the estimate ($p \geqslant 2$)

$$\|u(t,\cdot)\|_{L^p(\Omega)} \leqslant M^{(p-2)/p}\left(\int u^2 dx\right)^{1/p} \longrightarrow 0, \quad t \longrightarrow \infty;$$

and then for $p \in [1, 2)$ Hölder inequality. Another possibility is to combine inductively (6) with Remark 1.

§ 2. Estimates of Hölder norms

We are now able to estimate first order spatial derivatives of solutions, however not for such a general form of the equation as (1). Consider the problem ($i, j = 1, \cdots, n$)

$$\text{(10)} \qquad u_t = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_i b_i(x)\frac{\partial u}{\partial x_i} + f(t, x, u, \nabla u)$$

$$\text{(11)} \qquad u = 0 \quad \text{on} \quad \partial\Omega, \quad u(0, x) = u_0(x) \in C^2(\bar{\Omega}).$$

The main part $a_{ij} \in C^1(\bar{\Omega})$, $i, j = 1, \cdots, n$, is assumed to be elliptic (constant $a_0 > 0$), $b_i \in C^0(\bar{\Omega})$, $|b_i| \leqslant B$, and for this paragraph we assume that the solution u of (10), (11) is globally bounded in time; $|u| \leqslant M$. The continuous function f is assumed to satisfy in the set $K := \boldsymbol{R}^+ \times \bar{\Omega} \times [-M, M] \times \boldsymbol{R}^n$ a uniform Lipschitz condition with respect to t (constant N_1) and ∇u;

$$|f(t, x, u, u\nabla_1) - f(t, x, u, \nabla u_2)| \leqslant N_3 \sum_i \left|\frac{\partial(u_1 - u_2)}{\partial x_i}\right|,$$

to be differentiable in u; $\partial f/\partial u \leqslant N_2$, and to be globally bounded in K; $|f| \leqslant M_1$. These assumptions gives in particular (compare [4] p. 202) uniqueness of the classical solution of (10), (11). In this paragraph we do not want to use too sharp smoothness assumptions for u; let the classical solution u exists and belongs to $C^{1,2}_{\text{loc}}(\bar{D})$ (classical solution with bounded derivatives u_{x_i}, u_t, $u_{x_i x_j}$ in any compact subset of $\bar{D}$). We are able to formulate

Theorem 2. *If one of the following conditions holds*:

(d) $\exists\, \delta>0\, (N_1+N_2)/\lambda+(N_3+B)\sqrt{n/\lambda} \leqslant a_0-\delta$ (λ *from* (4)),

(e) $\|u_t\|_{L^2(\Omega)}$ *is bounded independently of time, then* u_t *is bounded in both* $L^p(\Omega)$, $p\in[1, \infty)$, *independently of time.*

Proof. Take the Steklov average of both sides of (10), differentiate the result with respect to t and multiply in $L^2(\Omega)$ by $u_{ht}^{2^k-1}$, $k=1, 2,\cdots$, to get

$$2^{-k}\frac{d}{dt}\int u_{ht}^{2^k}\,dx = \int\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u_{ht}}{\partial x_j}\right)u_{ht}^{2^k-1}dx$$
$$+\int\sum_i b_i(x)\frac{\partial u_{ht}}{\partial x_i}u_{ht}^{2^k-1}dx+\int f_{ht}u_{ht}^{2^k-1}dx.$$

For the first right side component we have

$$\int\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_{ht}}{\partial x_j}\right)u_{ht}^{2^k-1}dx$$
$$=-\int\sum_{i,j}a_{ij}\frac{\partial u_{ht}}{\partial x_j}\frac{\partial}{\partial x_i}(u_{ht}^{2^k-1})dx$$
$$\leqslant -a_0(2^k-1)2^{2-2k}\int\sum_i\left[\frac{\partial}{\partial x_i}(u_{ht}^{2^{k-1}})\right]^2dx,$$

because $u_{ht}=0$ on $\partial\Omega$. For the second one

$$\int\sum_i b_i(x)\frac{\partial u_{ht}}{\partial x_i}u_{ht}^{2^k-1}dx$$
$$=2^{1-k}\int\sum_i b_i(x)(u_{ht}^{2^{k-1}})\cdot(u_{ht}^{2^{k-1}})_{x_i}dx$$
$$\leqslant\left(\int\sum_i B^2[(u_{ht}^{2^{k-1}})_{x_i}]^2dx\, n\int u_{ht}^{2^k}dx\right)^{1/2}.$$

With the use of conditions assumed for f:

$$(13)\qquad \int f(t, x, u, \nabla u)_{ht}u_{ht}^{2^k-1}dx$$
$$=\int u_{ht}^{2^k-1}\cdot\frac{1}{h}[f(t+h, x, u(t+h, x), \nabla u(t+h, x))$$
$$-f(t, x, u(t, x), \nabla u(t, x))]dx$$
$$\leqslant N_1\int|u_{ht}|^{2^k-1}dx+N_2\int u_{ht}^{2^k}dx+N_3\int\sum_i\left|\frac{\partial u_{ht}}{\partial x_i}\right||u_{ht}|^{2^k-1}dx$$

$$\leqslant (N_1+N_2)\int u_{ht}^{2^k}dx + N_1\int u_{ht}^{2^{k-1}}dx$$

$$+ N_3\left[\int\sum_i (u_{htx_i}u_{ht}^{2^{k-1}-1})^2 dxn\int u_{ht}^{2^k}dx\right]^{1/2}.$$

For $k=1$ the last expression is estimated with the use of Poincaré and Cauchy inequalities:

$$\int f_{ht}u_{ht}dx$$

$$\leqslant [\lambda^{-1}(N_1+N_2+\varepsilon_0 N_1/2)+N_3\sqrt{n/\lambda}]\cdot\|\nabla u_{ht}\|_{L^2(\Omega)}^2 + (N_1/2\varepsilon_0)|\Omega|.$$

When $\varepsilon_0 N_1/\lambda<\delta$, then from (12) and the above estimates it follows that

$$\frac{d}{dt}\|u_{ht}\|_{L^2(\Omega)}^2 \leqslant -\delta\|\nabla u_{ht}\|_{L^2(\Omega)}^2 + (N_1/\varepsilon_0)|\Omega|, \tag{14}$$

so using Lemma 1

$$\|u_{ht}(t,\cdot)\|_{L^2(\Omega)}^2 \leqslant \max\left\{\|u(0,\cdot)_{ht}\|_{L^2(\Omega)}^2, \frac{N_1}{\lambda\varepsilon_0\delta}|\Omega|\right\}.$$

This through the limit passage $h\searrow 0$ shows that (d)⇒(e) (compare the comment after formula (16)).

For $k=2, 3,\cdots$ it follows from the previous estimates with the use of (5), that ($v:=u_{ht}^{2^{k-1}}$):

$$\int[\sum_i b_i u_{htx_i}+f_{ht}]u_{ht}^{2^k-1}dx$$

$$\leqslant \|\nabla v\|_{L^2(\Omega)}^2[\varepsilon(N_1+N_2)+2^{1-k}n(N_3+B)\varepsilon']$$

$$+ \|\nabla v\|_{L^2(\Omega)}2^{1-k}n(N_3+B)C_{\varepsilon'}\|v\|_{L^1(\Omega)}$$

$$+ (N_1+N_2)C_\varepsilon\|v\|_{L^1(\Omega)}^2+N_1\|v\|_{L^1(\Omega)}.$$

Choosing $\varepsilon(k)=\varepsilon_k$ and ε' such that the bracket after $\|\nabla v\|_{L^2(\Omega)}^2$ is less than $a_0 2^{-k}$ we arrive at the estimate

$$\frac{d}{dt}\|v\|_{L^2(\Omega)}^2 \leqslant [-a_0(2^k-1)2^{2-k}+a_0]\|\nabla v\|_{L^2(\Omega)}^2 \tag{15}$$

$$+ 2n(N_3+B)C_{\varepsilon'}\|\nabla v\|_{L^2(\Omega)}\|v\|_{L^1(\Omega)}$$

$$+ 2^k[(N_1+N_2)C_\varepsilon\|v\|_{L^1(\Omega)}^2+N_1\|v\|_{L^1(\Omega)}]$$

$$\leqslant -a_0\|\nabla v\|_{L^2(\Omega)}^2 + 2n(N_3+B)C_{\varepsilon'}m_{k-1}\|\nabla v\|_{L^2(\Omega)}$$

$$+ 2^k[(N_1+N_2)C_\varepsilon m_{k-1}^2 + N_1 m_{k-1}],$$

where we denote the (inductive) bound by $m_{k-1} := \sup_{\boldsymbol{R}^+} \|v\|_{L^1(\Omega)}$. We then get from Lemma 1 a global in time estimate (for explicit v);

$$(16) \quad \int u_{ht}^{2^k}(t, x)dx$$
$$\leqslant \max\left\{\int u_{ht}^{2^k}(0, x)dx, \frac{1}{\lambda}(\text{root } (\|\nabla v\|_{L^2(\Omega)}^2) \text{ of the right side of (15))}\right\}.$$

After the limit passage $h\searrow 0$ this estimate remains valid for $\int u_t^{2^k}dx$. Note only that, in the presence of our smoothness assumptions for u, $u_t(0, x)$ can be found through the limit passage $t\searrow 0$ from the equation (10), then its $L^{2^k}(\Omega)$ norm will be estimated by $W^{2,2^k}(\Omega)$ or $C^2(\bar{\Omega})$ norm of u_0. However we have only the L^{2^k} estimates, $k=2, 3,\cdots$, and this in the scale of L^p spaces is equivalent to global estimates of u_t in both $L^p(\Omega)$. Our proof is over.

The result of Theorem 2 will be used to obtain global in time estimates of certain norms of the solution u. Consider our parabolic equation with arbitrarily fixed $t>0$ (this moment a parameter) as the elliptic problem:

$$(17) \quad \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) + \sum_i b_i(x) \frac{\partial u}{\partial x_i} = [-f(t, x, u, \nabla u) + u_t] = \tilde{f}(t, x)$$
$$u = 0 \quad \text{on} \quad \partial\Omega \quad (\text{now let } \partial\Omega \in C^2),$$

where $\tilde{f}$ is bounded in both $L^p(\Omega)$ independently of t ($|f| \leqslant M_1$). For this kind of problem the solution (which coincide with $u(t, x)$) belongs to $W^{2,p}(\Omega)$ (compare [8, p. 161], [10, 2.5]) and satisfies the estimate

$$(18) \quad \|u(t, \cdot)\|_{W^{2,p}(\Omega)} \leqslant \text{const.} \quad (\|\tilde{f}\|_{L^p(\Omega)} + \|u(t, \cdot)\|_{L^1(\Omega)})$$
$$\leqslant \text{const.} \quad (\|\tilde{f}\|_{L^p(\Omega)} + M|\Omega|).$$

It follows from (18) with $p>n+1$ and Sobolev Imbedding Theorem ([6], [10], [12])

$$W^{k,p}(\Omega) \hookrightarrow C^{j+\nu}(\bar{\Omega}), \quad 0 \leqslant \nu \leqslant k - n/p - j,$$

that $(p>n+1)$

$$(19) \quad \|u(t, \cdot)\|_{C^{1+\nu}(\bar{\Omega})} \leqslant \text{const.} \|u(t, \cdot)\|_{W^{2,p}(\Omega)},$$

where $1-n/p=\nu>1/(n+1)$. This together with (18) means that $\nabla u \in L^\infty(\boldsymbol{R}^+; C^\nu(\bar{\Omega}))$ with global in t bound for $\|\nabla u\|_{C^\nu(\bar{\Omega})}$. Moreover as a result of Theorem 2 for every $p\geqslant 1$, $u_t \in L^\infty(\boldsymbol{R}^+; L^p(\Omega))$. Fix an arbitrary $T>0$, then for $p=p_0>2n+2$ and the full gradient of u:

$$(u_t, \nabla u) \in L^\infty(\boldsymbol{R}^+; L^{p_0}(\Omega)) \subset L^{p_0}(D_\tau^{\tau+T}),$$

where $D_\tau^{\tau+T} = D \cap \{(t, x);\ \tau \leqslant t \leqslant \tau + T\}$, τ-arbitrary. Again through the Sobolev Theorem, actually in $\boldsymbol{R}^{n+1}$

(20) $$u \in C^{\mu,\mu}(D_\tau^{\tau+T}), \quad \mu = 1/2,$$

with the norm bounded independently of τ and T. It follows from Lemma 3.1, Chapt. II, §3 of [7], (19) and (20) that ∇u satisfies in $\bar{D}$ global in time Hölder condition with respect to t, with exponent $\delta = \mu\nu/(1+\nu)$ $(\delta < \nu/2)$. This together with (19) gives our fundamental estimate:

(21) $$\nabla u \in C^{\delta/2,\delta}(\bar{D}),$$

where the Hölder norm of ∇u is estimated globally in time.

§ 3. Global existence of solutions

The estimate (21) will be used to show global existence of the solution of (10)–(11). This will be done by a standard method as indicated below. Additionally to the conditions assumed in §2 let a_{ij} belongs to $C^{1+\alpha}(\bar{\Omega})$, $i, j = 1, \cdots, n$, $b_i \in C^\alpha(\bar{\Omega})$, and f satisfies a uniform Hölder condition in the set K (exponent α) with respect to x. It then follows from Theorem 9, p. 205 of [4];

Proposition 1. *Let $u_0 \in C^{2+\alpha}(\bar{\Omega})$, $\partial\Omega \in C^{2+\alpha}$ and let the compatibility conditions $u_0 = 0$ on $\partial\Omega$,*

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u_0}{\partial x_j}\right) + f(0, x, 0, \nabla u_0) = 0 \quad on \quad \partial\Omega$$

be satisfied. Then under assumptions (c) *in Theorem* 1 *and* (e) *in Theorem* 2 *there exists a unique solution u of* (10), (11). *It is globally bounded in $C^{1+\gamma/2,2+\gamma}(\bar{D})$ with $\gamma = \min\{\alpha, \delta\}$.*

Proof. Existence of such a solution in any finite cylinder $[0, H] \times \Omega$ follows from the results of A. Friedman [4]. Note only that as a consequence of (21) and our smoothness assumptions the composite function $f(t, x, u, \nabla u)$ belongs to $C^{\gamma/2,\gamma}(\bar{D})$. Denoting $f(t, x, u, \nabla u) =: g(t, x)$ and adopting for a moment Friedman's notation ([4] p. 63) it thus follows from the Schauder interior estimates ([4] p. 64) that in any bounded cylinder $D_\tau^{\tau+T}$ ($\tau \geqslant 0$ arbitrary, $T > 0$ fixed)

$$|u|_{2+\gamma} \leqslant K(|u|_0 + |d^2 g|_\gamma),$$

where K is independent of τ. Next since our boundary data and the lateral surface are independent of time and the Schauder boundary estimates has local character ([4] p. 121), it follows that

(22) $$\|u\|_{C^{1+\gamma/2,2+\gamma}(\bar{D})} \leqslant \text{const.}.$$

This completes the proof.

Remark 2. Nobody could expect global a priori estimates when the conditions of the type (a), (d) are disturbed in both set A. Let $w_1(x)\not\equiv 0$ be the eigenfunction corresponding to the first positive eigenvalue λ of the linear problem

$$Lw = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right) = -\lambda w,\ w|_{\partial\Omega} = 0\,.$$

Then clearly $u(t, x):=w_1(x)\exp(\varepsilon t)$, $\varepsilon>0$, is an unbounded solution of

$$\begin{cases} u_t = Lu + (\lambda+\varepsilon)u, \\ u(0, x) = w_1(x), \quad u|_{\partial\Omega} = 0. \end{cases}$$

Remark 3. There are two main consequences of global a priori estimates (22) important for long time studies of solutions of (10), (11).

1) The trajectories are compact in $C^2(\bar{\Omega})$ phase space — this is needed in the LaSalle invariance principle ([6] p. 91) type arguments,

2) Through the Ascoli-Arzela theorem one can extract a sequence of times $\{t_n\}_{n\in\boldsymbol{N}}$, $t_n\nearrow\infty$, such that $u(t_n, x)\to v(x)$ in $C^2(\bar{\Omega})$, $u_t(t_n, x)\to\omega(x)$ in $C^0(\bar{\Omega})$, $n\to\infty$, and if $f(t_n, x,\cdot,\cdot)\to g(x,\cdot,\cdot)$, then

$$\omega(x) = \sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) + \sum_i b_i\frac{\partial v}{\partial x_i} + g(x, v, \nabla v),$$

$v=0$ on $\partial\Omega$. So the limit function v is a solution of an elliptic type problem.

This however is not usually enough for existence of $\lim_{t\to\infty} u(t, x)$; for example any periodic, non-constant trajectorie of (10) (with f periodic) satisfies 2) but has no such a limit.

Note also that the estimate (18) ensures through Rellich's type theorem (c.f. [12], Chapt. II, §1) compactness of trajectories in $W^{1,q}(\Omega)$; $q\geqslant 1$ — this together with (16) (weak compactness of u_t in $L^p(\Omega)$) remains to justify the passage with t to infinity for parabolic problems in weak formulation.

Remark 4. We have restricted our considerations here to the first boundary problem, but a parallel reasoning (see Summary) is possible for other types of boundary conditions; compare [3] for the analogon of our Theorem 2.

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nuna adreso:
Institute of Mathematics
Silesian University
40-007 Katowice
Bankowa 14, Poland

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