Funkcialaj Ekvacioj, **28** (1985), 39–46

On the Periodic Problem for the Equation $x''(t)+g(x(t))=f(t)$

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§ 0. Introduction

In this note we propose to prove the existence of periodic solutions for second order equations of the type

$$x''(t)+g(x(t))=f(t) \tag{0.1}$$

where $f(t)$ is a 2π-periodic fuction and $g(x)$ is a monotone continuous function, such that $|g(x)|\leqq\gamma|x|+C$, $x\in\boldsymbol{R}$, $C=\text{const.}>0$, $\gamma=\text{const.}$, $0\leq\gamma<1$ under conditions upon $g(x)$ and $f(t)$ analogous to those of Landesman and Lazer [6].

Our result is based on a technique introduced in the basic work of Brezis and Nirenberg [4] for attacking nonlinear partial differential equations, in particular, boundary value problems for semilinear equations. First we consider the equation

$$\varepsilon x_{\varepsilon,1}+x_\varepsilon''+g(x_\varepsilon)=f \tag{0.1$_\varepsilon$}$$

where $x_{\varepsilon,1}=(2\pi)^{-1}\int_0^{2\pi}x_\varepsilon(t)dt$ and ε is a small positive number.

Next we establish bounds for x_ε independent of ε. With the aid of standard truncation of $g(x)$ and Schauder fixed point theorem we solve the equation (0.1$_\varepsilon$). Finally we carry out a limit process as $\varepsilon\to0$ and using the Minty trick we obtain a solution of (0.1).

We note that periodic solutions of the equation (0.1) were treated in the work of Fučik and Lovicar [5] by a different method. Reissig [9] prove an analogous result to ours but our assumptions on $f(t)$ and definition of solution is more general.

§ 1. Notations and main result

We consider second order equation of the type

$$x''(t)+g(x(t))=f(t) \tag{1.1}$$

where $f(t)\in L^\infty(\boldsymbol{R})$ is 2π-periodic function of t.

Definition. *By a solution of* (1.1) *we mean a 2π-perodic function $x(t) \in L^\infty(\boldsymbol{R})$ such that $x''(t) \in L(\boldsymbol{R})$ and* (1.1) *holds a.e. on* $\boldsymbol{R}$.

We note that if $x(t) \in L^\infty(\boldsymbol{R})$ is a 2π-perodic function then $x(t)$ can be expanded in a Fourier series $x(t) = \sum_k c_k \exp(ikt)$, $c_k = \bar{c}_{-k}$, such that $\sum_k |c_k|^2 < \infty$ because $x(t) \in L^2(0, 2\pi)$ and if $x''(t)$ is the generalised derivative then $x''(t) = -\sum k^2 c_k \exp(ikt)$ (see [3]). Then we can consider the operator L: $L^2(0, 2\pi) \to L^2(0, 2\pi)$, $Lx(t) = x''(t)$ because $x''(t) \in L^\infty(\boldsymbol{R})$ by the Definition. Let N be the kernel of the operator L in $L^2(0, 2\pi)$. It is clear that N is reduced to constant functions and if P is the projection on N in $L^2(0, 2\pi)$ then $Px(t)$ is the average of $x(t)$ over $[0, 2\pi]$:

$$Px(t) = (2\pi)^{-1} \int_0^{2\pi} x(t)\,dt.$$

We assume that $g(x)$: $\boldsymbol{R} \to \boldsymbol{R}$ is a monotone increasing continuous function and

$$\text{(G)} \qquad |g(x)| \leq \gamma |x| + C, \qquad x \in \boldsymbol{R},$$

$$\text{(F)} \qquad g(-\infty) < Pf(t) < g(+\infty),$$

where $C = \text{const.} > 0$, $\gamma = \text{const.}$, $0 \leq \gamma < 1$.

Theorem. *The equation* (1.1) *admits at least one periodic solution $x(t)$ under assumptions* (G), (F).

It is easy to see that condition (F) is a necessary condition. If $g(-\infty) = -\infty$ and $g(+\infty) = +\infty$ the condition (F) shows that (1.1) is solvable if $f(t) \in L^\infty(\boldsymbol{R})$ is a 2π-perodic function. The statement of the theorem is valid for $\gamma < 1$. For example the equation $x''(t) + x(t) = f(t)$ does not admit solution for every 2π-periodic function $f(t) \in L^\infty(\boldsymbol{R})$ because -1 is an eigenvalue of L.

Without loss of generality we assume $g(0) = 0$. Otherwise we may subtract $g(0)$ from both sides of (1.1) and consider an equation which satisfies the same hypotheses (G) and (F). In the proof of Theorem we need the following inequality of Wirtinger (see [2]).

Proposition (Writinger's inequality). *If $u(t)$ is a 2π-periodic function and $\int_0^{2\pi} u(t)\,dt = 0$ then*

$$\text{(1.2)} \qquad \int_0^{2\pi} (u(t))^2 dt \leq \int_0^{2\pi} (u'(t))^2 dt.$$

Equality in (1.2) *holds if* $u(t) = c_1 \cos t + c_2 \sin t$.

An easy proof of Wirtinger's inequality is based on Fourier representations.

§ 2. Proof of the main result

N is a closed subspace of $L^2(0, 2\pi)$ and we have a decomposition $L^2(0, 2\pi)= N\oplus N^{\perp}$. Then each $u \in L^2(0, 2\pi)$ can be uniquely decomposed as $u=u_1+u_2$ where $u_1 \in N$ is a constant and $u_2 \in N^{\perp}$. We consider the approximate equation

$$\varepsilon x_{\varepsilon,1}+x_\varepsilon''+g(x_\varepsilon)=f. \tag{2.1}$$

Step 1. Estimates for x_ε.

There exists a function $v(t) \in N^{\perp}$ and a constant w such that $f(t)=g(w)+v''$ by the assumption (F). Thus

$$\varepsilon x_{\varepsilon,1}+(x_\varepsilon-v)''+g(x_\varepsilon)-g(w)=0.$$

Taking the $L^2(0, 2\pi)$ scalar product with $x_\varepsilon-v$ and using periodicity of $x_\varepsilon-v$ we obtain

$$\varepsilon\int_0^{2\pi} x_{\varepsilon,1}(x_\varepsilon-v)dt+\int_0^{2\pi}(g(x)-g(w))(x_\varepsilon-v)dt=\int_0^{2\pi}|(x-v)'|^2dt$$

$(x-v)'$ is 2π-periodic function and $\int_0^{2\pi}(x-v)'dt=0$. We have by Wirtinger's inequality

$$2\pi\varepsilon|x_{\varepsilon,1}|^2+\int_0^{2\pi}(g(x)-g(w))(x-v)dt\le\int_0^{2\pi}|(x-v)''|^2dt$$

or

$$2\pi\varepsilon|x_{\varepsilon,1}|^2+\int_0^{2\pi}(g(x_\varepsilon)-g(w))(x_\varepsilon-v)dt\le\|g(w)g(x_\varepsilon)-\varepsilon x_{\varepsilon,1}\|^2. \tag{2.2}$$

We denote by $(\cdot, \cdot)$, $\|\cdot\|$ and $|\cdot|$ the scalar product, the norm in $L^2(0, 2\pi)$ and usual norm in $\boldsymbol{R}$. By (G):

$$\begin{aligned}(g(x_\varepsilon)-g(w))(x_\varepsilon-v)&=(g(x_\varepsilon)-g(w))(x_\varepsilon-w+w-v)\\ &\ge|g(x_\varepsilon)-g(w)||x_\varepsilon-w|-|g(x_\varepsilon)-g(w)||w-v|\\ &\ge|g(x_\varepsilon)-g(w)|(\gamma^{-1}|g(x_\varepsilon)|-C\gamma^{-1}-|w|)-|g(x_\varepsilon)-g(w)||w-v|.\end{aligned}$$

We shall denote by $C_j, j=1, 2, \cdots$ various positive constants in further estimates. By the previous estimates

$$\int_0^{2\pi}(g(x_\varepsilon)-g(w))(x_\varepsilon-v)dt\ge\gamma^{-1}\|g(x_\varepsilon)\|^2-C_1\|g(x_\varepsilon)\|-C_2. \tag{2.3}$$

We shall use the elementary inequality $(a+b)^2\le(1+\delta)a^2+(1+\delta^{-1})b^2$, $a, b \in \boldsymbol{R}$, $\delta>0$ for estimation of the right hand side of (2.2). Let $\delta>0$ be such that

$\gamma<(1+\delta)^{-1}<1$. We have

$$\|g(w)-g(x_\varepsilon)-x_{\varepsilon,1}\|^2\leq(1+\delta)\|g(x_\varepsilon)\|^2+C_3\varepsilon^2|x_{\varepsilon,1}|^2+C_4. \tag{2.4}$$

Let us take ε sufficiently small such that $2\pi\varepsilon-C_3\varepsilon^2>k\varepsilon$, where k is a positive number less than π and $k_1=\gamma^{-1}-(1+\delta)>0$. Then

$$k\varepsilon|x_{\varepsilon,1}|^2+k_1\|g(x_\varepsilon)\|^2-C_1\|g(x_\varepsilon)\|-C_5\leq 0. \tag{2.5}$$

By (2.5) $\sqrt{\varepsilon}\,|x_{\varepsilon,1}|\leq C_6$ and $\|g(x_\varepsilon)\|\leq C_6$ independently of ε. Then from (2.1)

$$\|x_\varepsilon''\|\leq C_7. \tag{2.6}$$

Using Wirtinger's inequality and (2.6) we have

$$\|x_\varepsilon'\|\leq C_7 \quad\text{and}\quad \|x_{\varepsilon,2}\|\leq C_7. \tag{2.7}$$

Now we shall derive $L^1(0, 2\pi)$ estimate for x_ε. Let's fix a number r such that $0<r<\min(g(+\infty)-Pf, Pf-g(-\infty))$. Hence for $h: |h|<r$ we have $g(-\infty)<Pf+h<g(+\infty)$ and then there exists a constant w_h such that $f+h=g(w_h)+v''$ and $|w_h|\leq C_8$ independently of h. We have

$$\varepsilon x_{\varepsilon,1}+(x_\varepsilon-v)''+g(x_\varepsilon)-g(w_h)=-h.$$

Taking the L^2 scalar product with $x_\varepsilon-v$ and working as before we conclude that $\int_0^{2\pi} x_\varepsilon\, h dt\leq C_9$ where C_9 is independent of ε and h for all $h: |h|<r$. Then

$$\|x_\varepsilon\|_{L^1}\leq C_{10}. \tag{2.8}$$

By (2.8)

$$|x_{\varepsilon,1}|\leq C_{11}, \tag{2.9}$$

because $x_{\varepsilon,1}=(2\pi)^{-1}\int_0^{2\pi} x_\varepsilon(t)dt$. Hence

$$\|x_\varepsilon\|\leq C_{12} \tag{2.10}$$

because $x_\varepsilon=x_{\varepsilon,1}+x_{\varepsilon,2}$ and $\|x_\varepsilon\|^2\leq|x_{\varepsilon,1}|^2+\|x_{\varepsilon,2}\|^2$. Using the Sobolev's inequality, (2.6), (2.7) and (2.10) we obtain

$$\|x_\varepsilon\|_{L^\infty}\leq C_{13}. \tag{2.11}$$

Step 2. Solvability of the equation (2.1).

Let $\underline{M}$ and $\overline{M}$ are two numbers such that

$$g(-\infty)<g(\underline{M})<Pf<g(\overline{M})<g(+\infty).$$

We can assume that C_{13} of (2.10) satisfies

$$C_{13}=M>\max(|\underline{M}|, |\overline{M}|).$$

Let $\tilde{g}(x)$ be a bounded monotone and continuous function which coincides with $g(x)$ for $|x|\leq M$:

$$\tilde{g}(x)=\begin{cases} g(x), & |x|\leq M, \\ g(M), & x>M, \\ g(-M), & x<-M. \end{cases}$$

We consider the equation

$$\varepsilon x_1+x''+\tilde{g}(x)=f \tag{2.12}$$

which solution we shall denote by $\tilde{x}_\varepsilon$.

Since $g(\underline{M})=g(\overline{M})$ and $g(\overline{M})=g(\overline{M})$ assumptions (G), (F) hold for g:

$$g(-\infty)<\tilde{g}(-\infty)\leq\tilde{g}(\underline{M})<Pf<\tilde{g}(\overline{M})\leq\tilde{g}(+\infty)<g(+\infty)$$

and

$$|\tilde{g}(x)|\leq|g(x)|\leq\gamma|x|+C.$$

Therefore the solution $\tilde{x}_\varepsilon$ of (2.12) satisfies the same estimates (2.10), (2.11) as x_ε. Consequently if the solution $\tilde{x}_\varepsilon$ exists then $\tilde{x}_\varepsilon$ is also a solution of (2.1) because $\tilde{g}(x)$ coincides with $g(x)$ for $|x|\leq M=C_{13}$. It is clear that (3.12) is equivalent to the system

$$\varepsilon x_1+\tilde{g}_1(x_1+x_2)=f_1, \tag{2.13}$$

$$x_2''+\tilde{g}_2(x_1+x_2)=f_2. \tag{2.14}$$

For a fixed $x_2\in N^{\perp}$ there exists an unique solution $x_1=\varphi(x_2)$ of the equation (2.13). Indeed, the function $G_{x_2}(x_1)=g_1(x_1+x_2)=(2\pi)^{-1}\cdot\int_0^{2\pi}\tilde{g}(x_1+x_2(t))dt$ is a monotone continuous function in x_1 because $g(x)$ is monotone. Hence (2.12) can be writen as $\varepsilon x_1+G_{x_2}(x_1)=f_1$ and the last equation has an unique solution for each fixed x_2.

The mapping $\varphi\colon N^{\perp}\to N$ is continuous in $L^2(0, 2\pi)$. Indeed let $x_{n,2}\to x_2$ in $N^{\perp}$. We have

$$\varepsilon x_{n,1}+\tilde{g}_1(x_{n,1}+x_{n,2})=f_1,$$
$$\varepsilon x_1+\tilde{g}_1(x_1+x_2)=f_1$$

and so

$$\varepsilon\|x_{n,1}-x_1\|^2+(\tilde{g}_1(x_{n,1}+x_{n,2})-\tilde{g}_1(x_1+x_2),\ x_{n,1}-x_1)=0.$$

We have by the monotonicity of g:

$$\varepsilon\|x_{n,1}-x_1\|^2 \leq (\tilde{g}_2(\tilde{x}_n)-\tilde{g}_2(x), x_{n,2}-x_2)$$
$$\leq (\|\tilde{g}_2(x_n)\|+\|\tilde{g}_2(x)\|)\,\|x_{n,2}-x_2\| \leq 2K\|x_{n,2}-x_2\|$$

because $|\tilde{g}_2(x)| \leq |\tilde{g}(x)| \leq K$, where $K=\max(|g(M)|, |g(-M)|)$ (see the definition of $\tilde{g}$). Hence $x_{n,1}\to x_1$ in $L^2(0, 2\pi)$.

To solve (2.14) we have to find a solution x_2 of

$$x_2''+\tilde{g}_2(x_1+\varphi(x_2))=f_2 \quad \text{or} \quad x_2=L^{-1}(f_2-\tilde{g}_2(x_2+\varphi(x_2))).$$

We shall prove that the operator $T\colon N^\perp\to N^\perp$,

$$Tx_2=L^{-1}(f_2-g_2(x_2+\varphi(x_2)))$$

has a fixed point. The operator $L^{-1}\colon N^\perp\to N^\perp$ is a compact operator and $\|L^{-1}\|_{L(N^\perp, N^\perp)}=1$. Then T is a continuous and compact operator on N because $\varphi\colon N^\perp\to N$ is continuous and $L^{-1}\colon N^\perp\to N^\perp$ is a compact operator. For any $x_2\in N^\perp$: $\|Tx_2\|=\|L^{-1}(f_2-\tilde{g}_2(x_2+\varphi(x_2)))\|\leq\|f_2\|+\|\tilde{g}_2(x_2+\varphi(x_2))\|\leq\|f_2\|+(2\pi)^{1/2}K=R$, because g_2 is a bounded function. Now we can apply the Schauder fixed point theorem to the operator T on the ball $B=\{x_2\in N^\perp\colon \|x_2\|\leq R\}$. $T\colon B\to B$ is compact operator and then there exists $x_2\in B$ such that $Tx_2=x_2$. Consequently the system (2.13), (2.14) is solvable and so the equation (2.1) as it is mentioned above.

Step 3. Convergence.

We will study the passage to the limit in (2.1) as $\varepsilon\to 0$. For a suitable sequence $\varepsilon_n\to 0$ we may assert in view of the bounds (2.6), (2.10) that

$$x_{\varepsilon_n}\longrightarrow x \qquad \text{weak in } L^2(0, 2\pi),$$
$$x''_{\varepsilon_n}\longrightarrow x'' \qquad \text{weak in } L^2(0, 2\pi).$$

Now we shall use the monotonicity of g and the compactness of L^{-1} on $N^\perp$. We have $(x''_{\varepsilon_n}, x_{\varepsilon_n})\to(x'', x)$ because $(x''_{\varepsilon_n}, x_{\varepsilon_n})=(x''_{\varepsilon_n}, x_{\varepsilon_n,2})\to(x'', x_2)=(x'', x)$ and the sequence $\{x_{\varepsilon_n,2}\}$ is compact in $L^2(0, 2\pi)$ by compactness of L^{-1} on $N^\perp$. By the monotonicity of g for arbitrary function $\psi\in L^2(0, 2\pi)$ we have

$$\int_0^{2\pi}(g(x_\varepsilon)-g(\psi))(x_\varepsilon-\psi)\,dt\geq 0$$

or

$$\int_0^{2\pi}(f-x''_\varepsilon-\varepsilon x_{\varepsilon,1}-g(\psi))(x_\varepsilon-\psi)\,dt\geq 0.$$

Passing to the limit as $\varepsilon_n\to 0$ we obtain

$$\int_0^{2\pi}(f-x''-g(\psi))(x-\psi)dt\geq 0.$$

Now we use the Minty trick: for $\chi\in L^2(0, 2\pi)$ and t positive set $\psi=x-t\chi$. After dividing by t we find

$$\int_0^{2\pi}(f-x''-g(x-t\chi)\chi)\,dt\geq 0.$$

Letting $t\to 0$ and using the fact that χ is an arbitrary function in $L^2(0, 2\pi)$ we may conclude that $x''+g(x)=f$ a.e. on $\boldsymbol{R}$. Theorem is completely proved.

We note that Amann and Mancini [6, Lemma 2.1] give an abstract result about the mehtod applied in the proof of Theorem. In this work the approximative equation (2.1) is chosen as in [4]. It is possible to choose as an approximate equation $\varepsilon x_\varepsilon+x_\varepsilon''+g(x_\varepsilon)=f$ also.

By the method applied to prove the Theorem we can prove an analogous existence theorem to system of the type

$$x''(t)+G(X(t))=F(t)$$

where $F(t)=(f_1(t), f_2(t), \cdots, f_n(t))\in(L^\infty(\boldsymbol{R}))^n$ is 2π-periodic mapping, $G\colon \boldsymbol{R}^n\to\boldsymbol{R}^n$ is continuous mapping such that: $0\leq\gamma<1$,

$$0\leq(G(X)-G(Y), X-Y)_n\leq\|X-Y\|_n^2, \tag{2.15}$$

where $X, Y\in\boldsymbol{R}^n$, $\|X\|_n\geq r$, $\|Y\|_n\geq r$ (here $(\cdot,\cdot)_n$ and $\|\cdot\|_n$ denote the scalar product and norm in $\boldsymbol{R}^n$). Then there exists 2π-periodic mapping $X(t)=(x_1(t), x_2(t), \cdots, x_n(t))$ such that $X(t)\in(L^\infty(\boldsymbol{R}))^n$ and $X''(t)\in(L^\infty(\boldsymbol{R}))^n$ and the system is satisfied for a.e. t. If $F(t)\in(L^2(\boldsymbol{R}))^n$ analogous result follows from the result of Mawhin [7, Theorem 2]- then $X(t)\in(L^2(\boldsymbol{R}))^n$ such that $X''(t)\in(L^2(\boldsymbol{R}))^n$. To continuations of this results for semilinear equations in Hilbert spaces is devoted our next work.

The author wishes to thank Professor N. Popivanov for his helpful discussions and comments.

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(Ricevita la 22-an de aŭgusto, 1983)