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On the Non-Local Problem with a Functional for Parabolic Equation

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§ 1. Introduction.

Consider a linear equation of parabolic type

$$(1)\qquad Lu=\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_ix_j}+\sum_{i=1}^{n}b_i(x,t)u_{x_i}+c(x,t)u-u_t=f(x,t)$$

in $D=\Omega\times(0, T]$, where Ω is a bounded domain in R_n. We denote by Γ the lateral surface of D, i.e., $\Gamma=\partial\Omega\times[0, T]$. In this paper we investigate the following non-local problem: given functions f, ϕ and ψ defined on D, Γ and $\bar{\Omega}$ respectively, find a solution of (1) satisfying the conditions

$$(2)\qquad u(x,t)=\phi(x,t)\qquad \text{on } \Gamma$$

and

$$(3)\qquad u(x,0)+F(x,u(\cdot,\cdot))=\Psi(x)\qquad \text{on } \Omega,$$

where F is a mapping on $\bar{\Omega}\times C(\bar{D})$ having the following property: for every $x\in\Omega$ and $u\in C(\bar{D})$ there exists a point

$$(\tilde{x},\tilde{t})\in\bar{D}\quad\text{with}\quad \tilde{t}>0\quad\text{such that}\quad |F(x,u(\cdot,\cdot))|\leq|u(\tilde{x},\tilde{t})|.$$

In section 2 we establish the uniqueness of the solution of the problem (1), (2) and (3) and give an apriori bound for the solution in terms of f, ϕ and Ψ. In Theorem 3 of section 3 we establish the existence of a solution of the non-local problem, with

$$F(x,u(\cdot,\cdot))=\int_D u(y,\tau)d\mu^x(y,\tau).$$

The results are then applied to derive the existence and uniqueness of a solution of the non-local problem in an infinite strip. In particular we establish an integral representation of a solution of the non-local problem in $R_n\times(0, T]$ and give a construction of a solution with $\Psi\in L^p$. Most of the theorems of this paper extend the results of Chabrowski [4], where the non-local problem, with the condition (3) replaced by

$$(3') \qquad u(x,0)+\sum_{i=1}^{N}\beta_i(x)u(x,T_i)=\Psi(x) \qquad \text{on } \Omega,$$

was investigated. Finally we point out that a certain class of non-local problems was studied by Bicadze and Samarskii [2]. Subsequently their results were extended by Kerefov [6] and Vabishchevich [8]. In particular the latter authors investigated the non-local problem (1), (2) and (3′) with $N=1$.

§ 2. Uniqueness and a priori bounds.

Throughout this section we make the following assumption

(A) The coefficients a_{ij}, $b_i(i,j=1,\cdots,n)$ and c are continuous on D and $c(x,t)\leq 0$ on D. Furthermore for every vector $\xi\in R_n$ and all $(x,t)\in D$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j>0.$$

By $C^{2,1}(D)$ we denote the set of functions u continuous on D with their derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i\partial x_j$ $(i,j=1,\cdots,n)$ and $\partial u/\partial t$ (at $t=T$ the derivative is understood as the lefthand derivative).

Lemma 1. *Suppose that the mapping F has the following property*

(B) *for every $x\in\Omega$ and every $u\in C(\bar{D})$ there exists a point $(\tilde{x},\tilde{t})\in\bar{D}$ with $\tilde{t}>0$ such that*

$$|F(x,u(\cdot,\cdot))|\leq|u(\tilde{x},\tilde{t})|.$$

Then the problem (1), (2) *and* (3) *has at most one solution in $C^{2,1}(D)\cap C(\bar{D})$.*

Proof. Let $u\in C^{2,1}(D)\cap C(\bar{D})$ be a solution of the homogeneous problem

$$Lu=0 \qquad \text{in } D,$$
$$u=0 \qquad \text{on } \Gamma$$

and

$$u(x,0)+F(x,u(\cdot,\cdot))=0 \qquad \text{on } \Omega.$$

Suppose that $u\not\equiv 0$. We may assume that u takes on a negative value at certain point of $\bar{D}$. By the strong maximum principle (Friedman [5], Chapter 2) there exists a point $x_0\in\Omega$ such that $u(x_0,0)=\min_{\bar{D}}u<0$. By the property (B) there exists a point $(x_1,T_1)\in\bar{D}$ with $T_1>0$ such that

$$|u(x_0,0)|\leq|F(x_0,u(\cdot,\cdot))|\leq|u(x_1,T_1)|.$$

It is clear that $x_1 \in \Omega$. If $u(x_1, T_1) \leq 0$ we get a contradiction. Hence $u(x_1, T_1) > 0$ and consequently there exists a point $x_2 \in \Omega$ such that $u(x_2, 0) = \max_{\bar{D}} u > 0$. Again by the property (B) we can find a point $(x_3, T_3) \in \bar{D}$ with T_3 such that

$$u(x_2, 0) = |F(x_2, u(\cdot, \cdot))| \leq |u(x_3, T_3)|.$$

It is obvious that $x_3 \in \Omega$. If $u(x_3, T_3) \geq 0$ we get a contradiction, hence $u(x_3, T_3) < 0$. Now we must distinguish two cases

$$|u(x_0, 0)| < u(x_2, 0) \quad \text{and} \quad u(x_2, 0) \leq |u(x_0, 0)|.$$

In the first case we have

$$|u(x_0, 0)| < u(x_2, 0) \leq |u(x_3, T_3)|.$$

Since both values $u(x_0, 0)$ and $u(x_3, T_3)$ are negative u attains its negative minimum at (x_3, T_3) and we arrive at a contradiction. Similarly in the second case

$$u(x_2, 0) \leq |u(x_0, 0)| \leq |u(x_1, T_1)| = u(x_1, T_1)$$

and u takes on a positive maximum at (x_1, T_1) and again we arrive at a contradiction.

Lemma 2. *Suppose that the mapping F has the following properties.*

(C_1) $-1 \leq F(x, 1)$ *for every* $x \in \Omega$.

(C_2) *For every point* $x_0 \in \Omega$ *such that* $F(x_0, 1) > -1$, $F(x_0, \cdot)$ *is decreasing and* $F(x_0, l) = F(x_0, 1)l$ *for every constant* l.

(C_3) *For every point* $x_0 \in \Omega$ *such that* $F(x_0, 1) = -1$ *and every* $u \in C(\bar{D})$ *there exists a point* $(x_1, t_1) \in \bar{D}$ *with* $t_1 > 0$ *such that*

$$-F(x_0, u(\cdot, \cdot)) \leq u(x_1, T_1).$$

Let $u \in C^{2,1}(D) \cap C(\bar{D})$. *If* $Lu \geq 0$ (≤ 0) *in* D, $u(x, t) \leq 0$ (≥ 0) *on* Γ *and* $u(x, 0) + F(x, u(\cdot, \cdot)) \leq 0$ (≥ 0) *on* Ω, *then* $u(x, t) \leq 0$ (≥ 0) *on* $\bar{D}$.

Proof. It suffices to prove the first part of the theorem. We may assume that there exists $x_0 \in \Omega$ such that $u(x_0, 0) = \max_{\bar{D}} u > 0$. Now we distinguish two cases: $F(x_0, 1) > -1$ and $F(x_0, 1) = -1$. In the first case it follows from (C_2) that

$$u(x_0, 0) + F(x_0, 1)u(x_0, 0) \leq u(x_0, 0) + F(x_0, u) \leq 0$$

and consequently $u(x_0, 0) \leq 0$ and we get a contradiction. In the second case there exists a point $(x_1, t_1) \in \bar{D}$ such that

$$u(x_0, 0) \leq -F(x_0, u) \leq u(x_1, t_1)$$

and u takes on a positive maximum at $(x_1, t_1) \in D$ and again we arrive at a contradiction.

From Lemma 2 we deduce the following a priori bound for a solution of the problem (1), (2) and (3).

Lemma 3. *Let $c(x, t) \leq -d$ in D, where d is a positive constant and let the assumptions* (C_1), (C_2) *and* (C_3) *of Lemma* 2 *hold and moreover suppose that*

(C_4) *for every $x \in \Omega$, $F(x, u)$ is linear in $u \in C(\bar{D})$ and furthermore for every $0 < \beta < d$ there exists a positive constant $\gamma(\beta) < 1$ such that*

$$-F(x, e^{-\beta t}) \leq \gamma(\beta) \qquad \text{for all } x \in \Omega.$$

If $u \in C^{2,1}(D) \cap C(\bar{D})$ is a solution of the problem (1), (2) *and* (3) *then for every $\beta < d$*

$$|u(x, t)| \leq (d-\beta)^{-1} e^{\beta T} \sup_D |f(x, t)| + e^{\beta T} \sup_\Gamma |\phi(x, t)| + (1-\gamma(\beta))^{-1} \sup_\Omega |\Psi(x)|$$

on $\bar{D}$.

Proof. Let $u(x, t) = v(x, t) e^{-\beta t}$, where $0 < \beta < d$. Then v satisfies the equation

$$L_1 v = \sum_{i,j=1}^n a_{ij}(x, t) v_{x_i x_j} + \sum_{i=1}^n b_i(x, t) v_{x_i} + (c(x, t) + \beta) v - v_t = e^{\beta t} f(x, t)$$

in D with $c(x, t) + \beta < \beta - d < 0$ on D and the conditions

$$v(x, t) = \phi(x, t) e^{\beta t} \qquad \text{on } \Gamma,$$

and

$$v(x, 0) + F(x, v e^{-\beta t}) = \Psi(x) \qquad \text{on } \Omega.$$

We may assume that

$$M = e^{\beta t} \sup_D |f(x, t)| < \infty, \qquad M_1 = e^{\beta T} \sup_\Gamma |\phi(x, t)| < \infty$$

and $M_2 = \sup_D |\Psi(x)| < \infty$ since otherwise there is nothing to prove. Put

$$w = v - \frac{M}{d-\beta} - M_1 - \frac{M_2}{1-\gamma}.$$

Then

$$L_1 w = f e^{\beta t} - (c+\beta) \frac{M}{d-\beta} - (c+\beta) M_1 - (c+\beta) \frac{M_2}{1-\gamma} \geq 0$$

in D,

$$w(x, t) \leq 0 \qquad \text{on } \Gamma$$

and

$$w(x, 0)+F(x, e^{-\beta t}w)=\Psi-\frac{M}{d-\beta}-M_1-\frac{M_2}{1-\gamma}-F\left(x, e^{-\beta t}\frac{M}{1-\beta}\right)$$
$$-F(x, e^{-\beta t}M_1)-F\left(x, \frac{M_2}{1-\gamma}e^{-\beta t}\right)$$
$$\leq M_2\left(1-\frac{1}{1-\gamma}+\frac{\gamma}{1-\gamma}\right)+(\gamma-1)M_1+(\gamma-1)\frac{M}{1-\beta}\leq 0$$

on Ω. Lemma 2 implies that $w\leq 0$ on $\bar{D}$. Similarly we can establish the inequality

$$v(x, t)\geq -\frac{M}{d-\beta}-M_1-\frac{M_2}{1-\gamma}$$

considering the auxiliary function

$$z(x, t)=v(x, t)+\frac{M}{d-\beta}+M_1+\frac{M_2}{1-\gamma}.$$

From the proof of Lemma 2 the following result is obtained.

Lemma 4. *Suppose that the mapping F has the following properties.*
(D_1) $-1<F(x, 1)$ *for every* $x\in\Omega$.
(D_2) *For every* $x\in\Omega$ $F(x, u)$ *is decreasing in* u *in* $C(\bar{D})$ *and* $F(x, l)=lF(x, 1)$ *for every constant* l.
Let $u\in C^{2,1}(D)\cap C(\bar{D})$. *If* $Lu\geq 0$ (≤ 0) *in* D, $u(x, t)\leq 0$ (≥ 0) *on* Γ *and* $u(x, 0)+F(x, u(\cdot, \cdot))\leq 0$ (≥ 0) *on* Ω, *then* $u(x, t)\leq 0$ (≥ 0) *on* $\bar{D}$.

Using Lemma 4 one can establish the following version of Lemma 3 with the assumption $c(x, t)\leq -d$ in D omitted.

Lemma 5. *Let* $-(1/T+1)\leq F(x, 1)$ *on* Ω *and assume that for every* $x\in\Omega$ $F(x, u)$ *is linear in* $u\in C(\bar{D})$ *and that* $F(x, u)$ *is decreasing in* u. *If* $u\in C^{2,1}(D)\cap C(\bar{D})$ *is a solution of the problem* (1), (2) *and* (3) *then*

$$|u(x, t)|\leq (T+1)\sup_D|f(x, t)|+\sup_\Gamma|\phi(x, t)|+\frac{T+1}{T}\sup_\Omega|\Psi(x)|$$

for all $(x, t)\in\bar{D}$.

To prove the last a priori bound we use Lemma 4 with the auxiliary functions

$$v(x, t)=u(x, t)\pm\left[(t+1)\sup_D|f(x, t)|+\sup_\Gamma|\phi(x, t)|+\frac{T+1}{T}\sup_\Omega|\Psi(x)|\right].$$

§ 3. Existence of solution in a bounded cylinder.

For the existence we shall need the following assumptions

(A_1) There exist positive constants λ_0 and λ_1 such that

$$\lambda_0 |\xi| \leq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leq \lambda_1 |\xi|^2$$

for all $(x, t) \in D$, $a_{ij}=a_{ji}$ $(i, j=1, \cdots, n)$.

(A_2) The coefficients a_{ij}, b_i $(i, j=1, \cdots, n)$, c and f are Hölder continuous in D (exponent α).

Moreover we assume that $\partial\Omega \in C^{2+\alpha}$. Under these assumptions the Green function $G(x, t; y, \tau)$ $(x, y \in \Omega, \tau<t)$ for the operator L exists (see Friedman [5], p. 81–85).

Let $D_r=\Omega\times[r, T]$, where $r>0$. In this section we assume that the mapping F is given by the formula

$$F(x, u)=\int_D u(y, \tau)d\mu^x(y, \tau), \tag{4}$$

where $\{\mu^x\}$ $(x \in \bar{\Omega})$ is a family of signed Borel measures on $\bar{D}$ with compact supports in $\bar{D}_r$.

Theorem 1. *Let the assumptions* (A_1) *and* (A_2) *hold. Assume that* $c(x, t)\leq 0$ *in* D *and that a family of signed Borel measures* $\{\mu^x\}$ $(x \in \bar{\Omega})$ *has the following properties*

(i) $|\mu^x|\leq 1$ *for all* $x \in \bar{\Omega}$ *and for every* $u \in C(\bar{D})$ *the integral* $\int_D u(y, \tau)d\mu^x(y, \tau)$ *is continuous on* $\bar{\Omega}$.

(ii) $\int_D \left[\int_\Omega G(y, \tau; z, 0)\phi(z)dz\right] d\mu^x(y, \tau)=0,$

and

$$\int_D \left[\int_0^\tau G(y, \tau; z, \delta)f(z, \delta)dzd\delta\right] d\mu^x(y, \tau)=0$$

on $\partial\Omega$ *for every* $\phi \in L^2(\Omega)$ *and* $f \in C(\bar{D})$ *respectively.*

If $\Psi \in C(\bar{\Omega})$ *and* $\Psi(x)=0$ *on* $\partial\Omega$, *then the problem* (1), (2) *and* (3) *with* $\phi\equiv 0$ *has a unique solution in* $C^{2,1}(D)\cap C(\bar{D})$.

Proof. It is obvious that the mapping F satisfies the condition (B) of Lemma 1. We try to find a solution in the form

$$u(x, t)=\int_\Omega G(x, t; y, 0)u(y, 0)dy-\int_0^t\int_\Omega G(x, t; y, \tau)f(y, \tau)dyd\tau, \tag{5}$$

where $u(\cdot, 0) \in C(\bar{\Omega})$ is to be determined. The condition (3) (with F given by (4)) leads to the integral equation

$$(6)\qquad u(x,0)+\int_D\left[\int_\Omega G(y,\tau;z,0)u(z,0)dz\right]d\mu^x(y,\tau)$$
$$=\Psi(x)+\int_D\left[\int_0^\tau d\delta\int_\Omega G(y,\tau;z,\delta)f(z,\delta)\right]d\mu^x(y,\tau).$$

It is easy to show that the linear mapping of $L^2(\Omega)$ into $L^2(\Omega)$ given by

$$T\phi(x)=\int_D\left[\int_\Omega G(y,\tau;z,0)\phi(z)dz\right]d\mu^x(y,\tau)$$

is compact. The homogeneous equation

$$\phi(x)+\int_D\left[\int_\Omega G(y,\tau;z,0)\phi(z)dz\right]d\mu^x(y,\tau)=0$$

has only the trivial solution in $L^2(\Omega)$. Indeed if ϕ is a soultion in $L^2(\Omega)$ of the above equation, then by the assumptions (i) and (ii) $\phi\in C(\bar{\Omega})$ and $\phi(x)=0$ on $\partial\Omega$. Hence the integral

$$u(x,t)=\int_\Omega G(x,t;y,0)\phi(y)dy$$

is a solution in $C^{2,1}(D)\cap C(\bar{D})$ of the homogeneous problem (1), (2) and (3). By Lemma 1 $u\equiv 0$ and consequently $\phi\equiv 0$. By the Fredholm theory of compact operators the equation (6) has a unique solution $u(\cdot,0)$ in $L^2(\Omega)$. It follows from the assumption (i) and (ii) that the integral

$$\int_D\left[\int_0^\tau d\delta\int_\Omega G(y,\tau;z,\delta)f(z,\delta)dz\right]d\mu^x(y,\tau)$$

is continuous on Ω and vanishes on $\partial\Omega$, hence $u(\cdot,0)\in C(\bar{\Omega})$ and $u(y,0)=0$ on $\partial\Omega$. Thus the formula (4) gives a solution in $C^{2,1}(D)\cap C(\bar{D})$ to the problem (1), (2) and (3).

We briefly mention here that Lemma 1 and the method used in the proof of Theorem 1 lead to the existence and the uniqueness of a solution to the non-local problem with the condition (3) replaced by

$$(7)\qquad u(x,0)+g\left(\int_D u(y,\tau)d\mu^x(y,\tau)\right)=\Psi(x),$$

where g is a Lipschitz continuous function on $(-\infty,\infty)$.

Theorem 2. *Suppose that the assumptions of Theorem* 1 *hold. Let g be a function on $(-\infty,\infty)$ such that $g(0)=0$ and that*

$$|g(u_1)-g(u_2)|\leq A|u_1-u_2|$$

for all u_1 *and* u_2 *in* $(-\infty, \infty)$, *where* $A<1$ *is a positive constant. If* $\Psi \in C(\bar{\Omega})$ *and* $\Psi(x)=0$ *on* $\partial\Omega$, *then the problem* (1), (2) *and* (7) (*with* $\phi\equiv 0$) *has a unique solution in* $C^{2,1}(D)\cap C(\bar{D})$.

Proof. We try to find a solution in the form (4), where $u(\cdot, 0) \in C(\bar{\Omega})$ is to be determined. The condition (6) leads to the equation

$$u(x, 0)+g\left(\int_D\left[\int_\Omega G(y, \tau; z, 0)u(z, 0)dz\right]d\mu^x(y, \tau)\right.$$
$$\left.-\int_D\left[\int_0^\tau\int_\Omega G(y, \tau; z, \delta)f(z, \delta)dzd\delta\right]d\mu^x(y, \tau)\right)=\Psi(x)$$

on Ω. Define the mapping T: $C(\bar{\Omega})\to C(\bar{\Omega})$ by the formula

$$v(x, 0)=Tu=-g\left(\int_D\left[\int_\Omega G(y, \tau; z, 0)u(z, 0)dz\right]d\mu^x(y, \tau)\right.$$
$$\left.-\int_D\left[\int_0^\tau\int_\Omega G(y, \tau; z, \delta)f(z, \delta)dzd\delta\right]d\mu^x(y, \tau)\right)+\Psi(x),$$

where $C(\bar{\Omega})$ is equipped with the supremum norm. It is easy to see that T is a contraction mapping. Consequently by the Banach fixed point theorem there exists a unique solution $u(\cdot, 0) \in C(\bar{\Omega})$. Since $g(0)=0$ it follows from the assumption (ii) that $u(\cdot, 0)=0$ on $\partial\Omega$ and the formula (5) gives a solution to the problem (1), (2) and (7).

Theorem 2 continues to hold if $A=1$ provided $c(x, t)\leq -d$ on D, where d is a positive constant. Indeed using the transformation $u(x, t)=e^{-\beta t}v(x, t)$, where $0<\beta<d$, we reduce this case to the case with a Lipschitz constant less than 1.

In the next two theorems (Theorems 3 and 4 below) we assume that the mapping F is given by the following formula

$$F(x, u(\cdot, \cdot)) = -\int_D u(y, \tau)d\mu^x(y, \tau), \tag{8}$$

where $\{\mu^x\}$ $(x \in \Omega)$ is a family of non-negative Borel measures on $\bar{D}$ with compact supports in D_r, $r>0$. The proofs are based on a priori bounds given in Lemma 3 and 4.

Theorem 3. *Let the assumptions* (A_1) *and* (A_2) *hold, and let* $c(x, t)\leq -d$ *on* D *where* d *is a positive constant. Suppose that* $\{\mu^x\}$ $(x \in \bar{\Omega})$ *is a family of non-negative Borel measures on D satisfying the following conditions*

(i) *for every* $u \in C(\bar{D})$, $\int_D u(y, \tau)d\mu^x(y, \tau)$ *is a continuous function on* $\bar{\Omega}$ *and* $\mu^x(D)\leq 1$ *for all* $x \in \bar{\Omega}$,

(ii)
$$\int_D \left[\int_\Omega G(y, \tau; z, 0)\phi(z)dz\right] d\mu^x(y, \tau)=0$$

and

$$\int_D \left[\int_0^\tau G(y, \tau; z, \delta)f(z, \delta)dzd\delta\right] d\mu^x(y, \tau)=0$$

on $\partial\Omega$ for every $\phi \in L^2(\Omega)$ and $f \in C(\bar{D})$ respectively.

Let ϕ and Ψ be continuous functions on Γ and $\bar{\Omega}$ respectively and moreover assume that there exists a continuous extension Φ of ϕ into $\bar{D}$ such that

$$\Psi(x)-\Phi(x, 0)+\int_D \Phi(y, \tau)d\mu^x(y, \tau)=0 \qquad on\ \partial\Omega.$$

Then the problem (1), (2) *and* (3) *has a unique solution n $C^{2,1}(D)\cap C(\bar{D})$.*

Proof. We first assume that $\Phi\equiv 0$ on $\bar{D}$, hence $\Psi(x)=0$ on $\partial\Omega$. We try to find a solution in the form (5), where $u(\cdot, 0) \in C(\bar{\Omega})$ is to be determined. The condition (3) leads to the Fredholm integral equation of the first kind

$$u(x, 0)-\int_D \left[\int_\Omega G(y, \tau; z, 0)v(z)dz\right] d\mu^x(y, \tau)$$
$$=\Psi(x)-\int_D \left[\int_0^\tau\int_\Omega G(y, \tau; z, \delta)f(z, \delta)dzd\delta\right] d\mu^x(y, \tau)$$

on Ω. By the same argument as in Theorem 1 we show that the above equation has a unique solution u in $L^2(\Omega)$. If follows from (i) and (ii) that $u(\cdot, 0) \in C(\bar{\Omega})$ and $u(y, \tau)=0$ on $\partial\Omega$ and the formula (5) gives a solution in this case.

Suppose next that $\phi\not\equiv 0$, but assume that the extension Φ belongs to $\bar{C}^{2+\alpha}(D)$. Introducing $v=u-\Phi$ we immediately obtain, by the previous result the existence of a solution v to $Lv=f-L\Phi$, which vanishes on Γ and satisfies the condition

$$v(x, 0)-\int_D v(y, \tau)d\mu^x(y, \tau)=\Psi(x)-\Phi(x, 0)+\int_D \Phi(y, \tau)d\mu^x(y, \tau)$$

on Ω. Finally we consider the general case, where Φ is only assumed to be continuous. By Theorem 2 in Friedman [5] p. 60 and the Weierstrass approximation theorem there exists a sequence of polynomials Φ_m on $\bar{D}$ which approximates Φ uniformly on $\bar{D}$. Now we define a sequence of functions $\{\Psi_m\}$ on $\partial\Omega$ by the formula

$$\Psi_m(x)=\Phi_m(x, 0)-\int_D \Phi_m(y, \tau)d\mu^x(y, \tau)$$

$m=1, 2, \cdots$. Since $\lim_{m\to\infty} \Psi_m(x)=\Psi(x)$ uniformly on $\partial\Omega$, one can construct a sequence of functions $\{\tilde{\Psi}_m\}$ in $C(\bar{\Omega})$ such that $\lim_{m\to\infty} \tilde{\Psi}_m(x)=\Psi(x)$ uniformly on $\bar{\Omega}$ and $\tilde{\Psi}_m(x)=\Psi_m(x)$ on $\partial\Omega$ for all m. By what we have already proved there exist solutions to the problem

$$Lu_m=f(x, t) \quad \text{in } D$$
$$u_m(x, t)=\Phi_m(x, t) \quad \text{on } \Gamma,$$

and

$$u_m(x, 0)-\int_D u_m(y, \tau)d\mu^x(y, \tau)=\tilde{\Psi}_m(x) \quad \text{on } \Omega.$$

By Lemma 3 the sequence $\{u_m\}$ is uniformly convergent on $\bar{D}$ to a function $u \in C(\bar{D})$. It is clear that u satisfies the conditions (2) and (4). Using Friedman-Schauder interior estimates (Friedman [5], Theorem 5 p. 64) one can easily prove that u satisfies the equation (1).

Remark. In the above proof we followed the argument used in the proof of Theorem 9 in Friedman [5] (p. 70–71). For the definition of the space $\bar{C}^{2+\alpha}(D)$ see Friedman [5] (p. 61–62).

We conclude this section with result which readily follows from Lemma 4 and the argument given in the proof of Theorem 3.

Theorem 4. *Let the hypothesis* (A_1) *and* (A_2) *hold and let* $c(x, t)\leq 0$ *in D. Assume that* $\{\mu^x\}$ $(x \in \bar{\Omega})$ *is a family of non-negative Borel measures on D satisfying the condition* (ii) *of Theorem* 1 *and*

(i″) *For every* $u \in C(\bar{D})$, $\int_D u(y, \tau)d\mu^x(y, \tau)$ *is a continuous function on* $\bar{\Omega}$ *and* $\mu^x(D)\leq(1/1+T)$ *for all* $x \in \bar{\Omega}$.

Let ϕ *and* Ψ *be continuous functions on* Γ *and* $\bar{\Omega}$ *respectively and finally assume that there exists a continuous extension* Φ *of* ϕ *into* $\bar{D}$ *such that*

$$\Psi(x)-\Phi(x, 0)+\int_D \Phi(y, \tau)d\mu^x(y, \tau)=0$$

on $\partial\Omega$. *Then the problem* (1), (2) *and* (3) *has a unique solution in* $C^{2,1}(D)\cap C(\bar{D})$.

To illustrate the results of this section consider the following example. Let $T_i \in (0, T]$ $(i=1, \cdots, N)$ and put

$$d\mu^x=\sum_{i=1}^{N} \beta_i(x)d\delta_{(x,T_i)},$$

where $\beta_i \in C(\bar{\Omega})$ $(i=1, \cdots, N)$, $0 \leq \sum_{i=1}^{N} \beta_i(x) \leq 1$ on $\bar{\Omega}$ and $\delta_{(x,T_i)}$ denotes the Dirac measure concentrated at (x, T_i). Since $G(x, t; y, \tau)=0$ for $x \in \partial\Omega$, $y \in \Omega$ and $t>\tau$, the assumptions (i) and (ii) of Theorem 3 are obviously satisfied. Theorem 3 yields the existence of a unique solution in $C^{2,1}(D) \cap C(\bar{D})$ to the problem (1), (2) and (3) provided

$$\Psi(x)=\phi(x, 0)-\sum_{i=1}^{N} \beta_i(x)\phi(x, T_i) \qquad \text{on } \partial\Omega.$$

Here the condition (3) takes the form

$$u(x, 0)-\sum_{i=1}^{N} \beta_i(x)u(x, T_i)=\Psi(x) \qquad \text{on } \Omega. \tag{9}$$

The finite sum in (8) can be replaced by an infinite series, i.e., the condition (8) becomes

$$u(x, 0)-\sum_{i=1}^{\infty} \beta_i(x)u(x, T_i)=\Psi(x) \qquad \text{on } \Omega, \tag{9'}$$

where $\sum_{i=1}^{\infty} \beta_i(x)$ converges uniformly on $\bar{\Omega}$ and $\inf_i T_i>0$. The non-local problem (1), (2) and (9) (or (9′)) has been studied in [4].

§ 4. Existence of solution in $\mathbf{R}_n \times (0, \mathbf{T}]$.

Throughout this section we make the following assumptions.

(B_1) There exist positive constants λ_0 and λ_1 such that for every vector $\xi \in R_n$

$$\lambda_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leq \lambda_1|\xi|^2$$

for all $(x, t) \in R_n \times [0, T]$, $a_{ij}=a_{ji}$ $(i, j=1, \cdots, n)$.

(B_2) The coefficients a_{ij}, b_i $(i, j=1, \cdots, n)$ and c are bounded and Hölder continuous on $R_n \times [0, T]$ and moreover $c(x, t) \leq -d$, where d is a positive constant.

Let $H(x, \delta)=\prod_{i=1}^{n} \cosh \delta x_i$. It is clear that there exist positive constants γ and δ_0 such that

$$LH \leq -\gamma H \qquad \text{in } R_n \times [0, T]$$

for all $0<\delta \leq \delta_0$.

(B_3) Let $\{\mu^x\}$ $(x \in R_n)$ be a family of non-negative Borel measures on $R_n \times [0, T]$ with compact supports in $R_n \times [r, T]$, where $r>0$ such that

$$\int_0^T \int_{R_n} \exp(\delta_0 \sum_{i=1}^{n} |y_i|) d\mu^x(y, \tau) \leq 1$$

for all $x \in R_n$.

We shall say that a function u defined on $R_n \times [0, T]$ belongs to $E(R_n \times [0, T])$ if there exist positive constants M and $\delta < \delta_0$ such that

$$|u(x, t)| \leq M \exp (\delta \sum_{i=1}^{n} |x_i|)$$

for all $(x, t) \in R_n \times [0, T]$.

We shall say that a function v defined on R_n belongs to $E(R_n)$ if there exist positive constants M and $\delta < \delta_0$ such that

$$|v(x)| \leq M \exp (\delta \sum_{i=1}^{n} |x_i|)$$

for all $x \in R_n$.

In the following theorem we establish the existence of a solution of the equation (1) satisfying the condition

$$u(x, 0) - \int_0^T \int_{R_n} u(y, \tau) d\mu^x(y, \tau) = \Psi(x) \qquad \text{on } R_n. \tag{10}$$

For an increasing sequence $\{P_m\}$ of positive numbers tending to infinity we put

$$D_m = (|x| < P_m) \times (0, T] \quad \text{and} \quad \Gamma_m = (|x| = P_m) \times [0, T].$$

Theorem 5. *Assume that there exists a sequence of cylinders D_m (described above) such that the restriction $\{\mu^x\}$ $(x \in \bar{D}_m)$ to the cylinder D_m satisfies the conditions* (i) *and* (ii) *of Theorem* 3 *for every m. If $f \in E(R_n \times [0, T])$ is an Hölder continuous function on compact subsets of $R_n \times [0, T]$ and $\Psi \in C(R_n) \cap E(R_n)$, then there exists a unique solution in $C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T]) \cap E(R_n \times [0, T])$ of the problem* (1), (10).

Proof. The proof is similar to that of Theorem III in [1] (see also Krzyżański [7]). It is clear that there exist positive constants M and $\delta < \delta_0$ such that

$$|f(x, t)| \leq M \exp (\delta \sum_{i=1}^{n} |x_i|) \quad \text{and} \quad |\Psi(x)| \leq M \exp (\delta \sum_{i=1}^{n} |x_i|)$$

on $R_n \times [0, T]$ and R_n respectively. Let $\phi(x, t)$ be a continuous function on $R_n \times [0, T]$ such that

$$|\phi(x, t)| \leq M \exp (\delta \sum_{i=1}^{n} |x_i|) \qquad \text{on } R_n \times [0, T],$$

$$\phi(x, 0) = \Psi(x) \text{ on } R_n \text{ and } \phi(x, t) = 0 \text{ on } R_n \times [r, T].$$

By Theorem 3, there exists for every m a unique solution in $C^{2,1}(D_m) \cap C(\bar{D}_m)$ of the problem

$$Lu_m = f \qquad \text{in } D_m$$
$$u_m(x, t) = \phi(x, t) \qquad \text{on } \Gamma_m$$

and

$$u_m(x, 0) - \int_{D_m} u_m(y, \tau) d\mu^x(y, \tau) = \Psi(x) \qquad \text{for } |x| < P_m.$$

Now we extend u_m into the strip $R_n \times [0, T]$ by defining $u_m(x, t) = \phi(x, t)$ on $R_n \times [0, T] - \bar{D}_m$ $(m = 1, 2, \cdots)$. Put

$$u_m(x, t) = v_m(x, t) H(x, \delta) \qquad m = 1, 2, \cdots.$$

The function v_m satisfies the equation

$$\sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 v_m}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \left(b_i(x, t) + \frac{2}{H(x, \delta)} \sum_{j=1}^{n} a_{ij}(x, t) \frac{\partial H}{\partial x_j} \right) \frac{\partial v_m}{\partial x_i} + (H(x, \delta)^{-1} LH) v_m - \frac{\partial v_m}{\partial t} = H(x, \delta)^{-1} f(x, t) \tag{11}$$

in D_m and the following conditions

$$v_m(x, t) = \phi(x, t) H(x, \delta)^{-1} \qquad \text{on } \Gamma_m$$

and

$$v_m(x, t) - \int_{D_m} H(x, \delta)^{-1} v_m(y, \tau) H(y, \tau) d\mu^x(y, \tau) = H(x, \delta)^{-1} \Psi(x)$$

for $|x| < P_m$. It follows from Lemma 3 that $|v(x, t)| \le M_1$ in $R_n \times [0, T]$ for all m, where M_1 is a positive constant independent of m. Now let $\delta < \delta_1 < \delta_0$ and put

$$u_p(x, t) = \bar{v}_p(x, t) H(x, \delta_1)$$

and

$$u_{pq}(x, t) = u_p(x, t) - u_q(x, t) = H(x, \delta_1)[\bar{v}_p(x, t) - \bar{v}_q(x, t)] = H(x, \delta_1) \bar{v}_{pq}(x, t)$$

for $p < q$. The function $\bar{v}_{pq}$ satisfies the homogeneous equation (11) with $H(x, \delta)$ replaced by $H(x, \delta_1)$ and moreover

$$\bar{v}_{pq}(x, t) = H(x, \delta_1)^{-1} [\phi(x, t) - u_p(x, t)] \qquad \text{on } \Gamma_p$$

and

$$\bar{v}_{pq}(x, 0) - \int_{D_m} \bar{v}_{pq}(y, \tau) H(y, \delta_1) H(x, \delta_1)^{-1} d\mu^x(y, \tau) = 0$$

for $|x|<P_p$. Since $\lim_{|x|\to\infty} H(x, \delta)H(x, \delta_1)^{-1}=0$, it follows from Lemma 3 that the sequence $\{u_m\}$ satisfies the uniform Cauchy condition on every compact subset of $R_n\times[0, T]$. Put $\lim_{p\to\infty} \bar{v}_p(x, t)=v(x, t)$ and $u(x, t)=v(x, t)H(x, \delta_1)$ on $R_n\times[0, T]$. It is obvious that $u\in E(R_n\times[0, T])$ is continuous on $R_n\times[0, T]$ and satisfies (10) by the Lebesgue Dominated Convergence Theorem. The fact that u satisfies the equation (1) follows from the Friedman-Schauder interior estimates.

To establish the uniqueness, let $u\in C^{2,1}(R_n\times(0, T])\cap C(R_n\times[0, T])\cap E(R_n\times[0, T])$ be a solution of the problem (1) and (10) (with $f\equiv 0$ and $\Psi\equiv 0$). There exist positive constants M and $\delta<\delta_0$ such that $|u(x, t)|\leq MH(x, \delta)$ on $R_n\times[0, T]$. Now choose $\delta<\delta_1<\delta_0$ and put $u(x, t)=v(x, t)H(x, \delta_1)$. Given $\epsilon>0$ we can find a positive number R such that $|v(x, t)|<\epsilon$ for $(x, t)\in(|x|\geq R)\times[0, T]$. Since v satisfies the homogeneous equation (11) with $H(x, \delta)$ replaced by $H(x, \delta_1)$, it follows from Lemma 3 that

$$|v(x, t)|\leq \varepsilon e^{\beta T} \qquad \text{on } (|x|\leq R)\times[0, T],$$

where $0<\beta<d$ and the uniqueness easily follows.

Remark. Let

$$d\mu^x=\sum_{i=1}^{N}\beta_i(x)d\delta_{(x,T_i)},$$

where $T_i\in(0, T]$ $(i=1, \cdots, N)$, $\beta_i\in C(R_n)$ $(i=1, \cdots, N)$ and $0\leq\sum_{i=1}^{N}\beta_i(x)\leq 1$ on R_n. By virtue of properties of the Green function in a cylinder the assumptions (i) and (ii) of Theorem 3 are trivially satisfied. Theorem 5 yields the existence and uniqueness of a solution of (1) satisfying

$$u(x, 0)-\sum_{i=1}^{N}\beta_i(x)u(x, T_i)=\Psi(x) \qquad \text{on } R_n.$$

It follows from the proof of Theorem 5 that the assumption (B_3) is irrelevant because in this case the approximating sequence v_m satisfies the condition

$$v_m(x, 0)-\sum_{i=1}^{N}\beta_i(x)v_m(x, T_i)=\Psi(x)H(x, \delta)^{-1} \qquad \text{for } |x|<R_m.$$

If f and Ψ are bounded functions the assumption (B_3) can be replaced by weaker condition

(B_3') Let $\{\mu^x\}$ $(x\in R_n)$ be a family of signed Borel measures with compact supports in $R_n\times[r, T]$, $r>0$, such that

$$|\mu^x|(R_n\times[0, T])\leq 1$$

for all $x \in R_n$ and assume that $\int_0^T \int_{R_n} v(y, \tau) d\mu^x(y, \tau)$ is continuous on R_n for every continuous bounded function v on $R_n \times [0, T]$.

Let $C_b(R_n \times [0, T])$ denote the space of continuous and bounded functions on $R_n \times [0, T]$ equipped with supremum norm.

Theorem 6. *Suppose that the assumptions* (B_1), (B_2) *and* (B_3') *hold. If f is a bounded function on $R_n \times [0, T]$ and Ψ is a continuous bounded function on R_n, then the problem* (1), (10) *has a bounded solution in* $C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T])$.

Proof. Introduce the transformation $u(x, t) = v(x, t)e^{-\beta t}$, where $0 < \beta < d$, then

$$L_1 v = \sum_{i,j=1}^{n} a_{ij}(x, t) v_{x_i x_j} + \sum_{i=1}^{n} b_i(x, t) v_{x_i} + (c(x, t) + \beta) v - v_t = f(x, t) e^{\beta t}$$

on $R_n \times (0, T]$. Let $\Gamma(x, t; y, \tau)$, where $x, y \in R_n$, $0 \le \tau < t \le T$, be the fundamental solution for the operator L_1. Since $c(x, t) + \beta < 0$ on $R_n \times [0, T]$

$$\int_{R_n} \Gamma(x, t; y, \tau) dy \le 1$$

for all $x, y \in R_n$ and $0 \le \tau < t \le T$ (see Friedman [5]). Consider the mapping S: $C_b(R_n \times [0, T]) \to C_b(R_n \times [0, T])$ given by the following formula

$$v(x, t) = Sw(x, t) = \int_{R_n} \Gamma(x, t; y, 0) \int_0^T \int_{R_n} e^{-\beta\tau} w(z, \tau) d\mu^y(z, \tau) dy$$
$$+ \int_{R_n} \Gamma(x, t; y, 0) \Psi(y) dy - \int_0^t \int_{R_n} \Gamma(x, t; y, \tau) f(y, \tau) dy d\tau.$$

It is clear that if $w \in C_b(R_n \times [0, T])$ then Sw satisfies the equation $L_1 v = f$ in $R_n \times (0, T]$ and

$$v(x, 0) = \int_0^T \int_{R_n} e^{-\beta\tau} w(y, \tau) d\mu^x(y, \tau) + \Psi(x) \qquad \text{on } R_n.$$

Since supp $\mu^x \subset R_n \times [r, T]$ for all $x \in R_n$, S is a contraction and the result follows from the Banach fixed point theorem.

We are unable to prove the uniqueness of a solution under the hypothesis of Theorem 6. In Theorem 7 below we establish the uniqueness of a solution in the case where the measure satisfies the condition

(B_3'') $$d\mu^x(y, \tau) = \beta(x) d\delta_x(y) d\nu^x(\tau),$$

where $0 \le \beta(x) \le 1$, $\beta \in C(R_n)$, δ_x denotes the Dirac measure on R_n concentrated at $x \in R_n$ and $\{\nu^x\}$ ($x \in R_n$) is a family of non-negative Borel measures on $[0, T]$ such that supp $\nu^x \subset [r, T]$, $r > 0$ and $\nu^x([0, T]) \le 1$ for all $x \in R_n$. Moreover we assume that the integral $\int_0^T v(x, \tau)d\nu^x(\tau)$ is continuous on R_n for every continuous bounded function v on $R_n \times [0, T]$.

Theorem 7. *Suppose that the hypothesis* (B_1), (B_2) *and* (B_3'') *hold. If f is a bounded function on $R_n \times [0, T]$ and Hölder continuous on every compact subset of $R_n \times [0, T]$ and Ψ is a continuous bounded function on R_n, then the problem* (1), (10) *has a unique bounded solution in* $C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T])$.

The existence of a solution follows from Theorem 6. The uniqueness can be proved using the method given in the proof of Theorem 5.

We note here that under the assumptions of Theorem 6 one can establish the existence of a solution of (1) satisfying the condition

$$u(x, 0) - g\left(\int_0^T \int_{R_n} u(y, \tau)d\mu^x(y, \tau)\right) = \Psi(x)$$

on R_n, where g is a Lipschitz function with a constant less than 1 and $g(0)=0$.

§ 5. Integral representation of solutions.

Throughout this section we make the assumptions (B_1), (B_2) and (B_3).

In the sequel we shall need the following lemma, the proof of which is routine.

Lemma 6. *Let f and Ψ be bounded functions on $R_n \times [0, T]$ and R_n respectively. If u is a solution in $E(R_n \times [0, T]) \cap C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T])$ of the problem* (1), (10) *then for every* $0 < \beta < d$

$$|u(x, t)| \le (d-\beta)^{-1} e^{\beta T} \sup_{R_n \times [0,T]} |f(x, t)| + (1 - e^{-\beta r})^{-1} \sup_{R_n} |\Psi(x)|$$

on $R_n \times [0, T]$.

Theorem 8. *Suppose that the hypothesis of Theorem 5 hold. Let Ψ be a bounded and continuous function on R_n and f a bounded function on $R_n \times [0, T]$ and Hölder continuous on every compact subset of $R_n \times [0, T]$. Then the solution of the problem* (1), (10) *is given by the formula*

$$u(x, t) = \int_{R_n} M(x, t, y)[\Psi(y) + F(y)]dy \tag{12}$$

for all $(x, t) \in R_n \times (0, T]$, *where*

$$F(x) = -\int_0^T \int_{R_n} \left[\int_0^\tau \int_{R_n} \Gamma(y, \tau; z, \delta) f(z, \delta)dzd\delta\right] d\mu^x(y, \tau)$$

and the kernel function $M(x, t, y)$ has the following properties

$$M(x, t, \cdot) \in L^p(R_n)$$

for all $(x, t) \in R_n \times (0, T]$ *and* $1 \leq p < \infty$, $M(x, t, y)$ *as a function of* (x, t) *satisfies the equation* $LM=0$ *in* $R_n \times (0, T]$ *for almost all* $y \in R_n$ *and moreover* M *satisfies the equation*

$$(13) \qquad M(x, t, w) = -\int_{R_n} \Gamma(x, t; y, 0)\left[\int_0^T \int_{R_n} M(z, \tau, w) d\mu^y(z, \tau)\right] dy + \Gamma(x, t; w, 0)$$

for all $(x, t) \in R_n \times (0, T]$ *and almost all* $w \in R_n$.

Proof. We assume initially that $f \equiv 0$. Let Ψ be a continuous and bounded function in $L^p(R_n)$. By Lemma 6 the unique solution of the problem (1), (10) in $C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T]) \cap E(R_n \times (0, T])$ is bounded on $R_n \times [0, T]$. First we prove that for each $\delta > 0$ there exists a positive constant $C(\delta)$ such that

$$(14) \qquad |u(x, t)| \leq C(\delta) \|\Psi\|_{L^p(R_n)}$$

on $R_n \times [\delta, T]$. To prove (14) we first assume that $\mu^x(R_n \times [0, T]) \leq \beta_0$ on R_n, where $0 < \beta_0 < 1$. Consider the Cauchy problem for the homogeneous equation (1) with the initial condition

$$z(x, 0) = \int_0^T \int_{R_n} u(y, \tau) d\mu^x(y, \tau) + \Psi(x).$$

The unique solution z in $C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T]) \cap E(R_n \times [0, T])$ is given by

$$z(x, t) = \int_{R_n} \Gamma(x, t; y, 0) \int_0^T \int_{R_n} u(z, \tau) d\mu^y(z, \tau) dy + \int_{R_n} \Gamma(x, t; y, 0) \Psi(y) dy$$

for all $(x, t) \in R_n \times (0, T]$ (Friedman [5], p. 26). Since u is a solution of the same problem we obtain

$$(15) \qquad u(x, t) = \int_{R_n} \Gamma(x, t; y, 0) \int_0^T \int_{R_n} u(z, \tau) d\mu^y(z, \tau) dy + \int_{R_n} \Gamma(x, t; y, 0) \Psi(y) dy$$

for all $(x, t) \in R_n \times (0, T]$. Now it is well known that

$$(16) \qquad 0 < \Gamma(x, t; y, 0) \leq C_1 t^{-n/2} \exp\left(-\mathscr{H}\frac{|x-y|^2}{t}\right)$$

for all $(x, t) \in R_n \times (0, T]$ and $y \in R_n$, where C_1 and $\mathscr{H}$ are positive constants (Friedman [5], p. 24). Applying the Hölder inequality we derive from (15) and (16) that

$$(17) \qquad \sup_{R_n \times [r, T]} |u(x, t)| \leq C_1 (1-\beta_0)^{-1} r^{-n/2p} \left(\int_{R_n} e^{-q\mathscr{H}|x|^2} dx\right)^{1/q} \left(\int_{R_n} |\Psi(x)|^p dx\right)^{1/p},$$

where $1/p+1/q=1$ (with obvious modification if $p=1$). Using again the representation (15) and the estimates (16) and (17) we obtain

$$|u(x, t)|\leq[\beta_0(1-\beta_0)^{-1}C_1C_2+C_1C_3t^{-n/p}]\|\Psi\|_{L^p(R_n)} \tag{18}$$

for all $(x, t)\in R_n\times(0, T]$, where

$$C_2=r^{-n/2p}\Big(\int_{R_n}e^{-\varkappa q|x|^2}dx\Big)^{1/q}\quad\text{and}\quad C_3=\Big(\int_{R_n}e^{-q\varkappa|x|^2}dx\Big)^{1/q},$$

and the estimate (14) easily follows. By (14) the mapping $\Psi\to u(x, t)$ defines a linear functional on $C_b(R_n)\cap L^p(R_n)$ continuous on L^p-norm. Consequently the representation (12) follows from the Riesz representation theorem of a linear continuous functional on $L^p(R_n)$. To derive (13) observe that by (12) and (15) we have for every continuous bounded function Ψ

$$\int_{R_n}M(x, t, w)\Psi(w)dw=\int_{R_n}\Gamma(x, t; y, 0)\Big[\int_0^T\int_{R_n}M(z, \tau, w)\Psi(w)dz\Big]d\mu^y(z, \tau)dy$$
$$+\int_{R_n}\Gamma(x, t; w, 0)\Psi(w)dw.$$

Consequently if we fix $(x, t)\in R_n\times(0, T]$, applying Fubini's theorem, we obtain the identity (13) for almost all $w\in R_n$. Now choose $w\in R_n$ such that the integral

$$\int_{R_n}\Gamma(x, T; y, 0)\Big[\int_0^T\int_{R_n}M(z, \tau, w)d\mu^y(z, \tau)\Big]dy$$

is finite. Then by Theorem 1 in Watson [8] the integral

$$\int_{R_n}\Gamma(x, t; y, 0)\Big[\int_0^T\int_{R_n}M(z, \tau, w)d\mu^y(z, \tau)\Big]dy$$

is finite for all $(x, t)\in R_n\times(0, T]$ and represents a solution of the equation $Lv=0$ in $R_n\times(0, T]$ and the last assertion of the theorem follows. The general case follows by means of the following transformations

$$u(x, t)=e^{-\beta t}v(x, t),\qquad\text{where } 0<\beta<d$$

and

$$z(x, t)=u(x, t)+\int_0^t d\tau\int_{R_n}\Gamma(x, t; y, \tau)f(y, \tau)dy.$$

Similarly in the case of a bounded cylinder one can prove

Theorem 9. *Suppose the assumptions of Theorem* 1 *hold. Let u be a solution in* $C^{2,1}(D)\cap C(\bar{D})$ *of the problem* (1), (2) *and* (3) *with* $\phi\equiv0$. *Then*

$$u(x, t)=\int_{\Omega} m(x, t, y)[\Psi(y)+F(y)]dy$$

for all $(x, t) \in D$, *where*

$$F(x)=-\int_{D}\left[\int_{0}^{\tau}\int_{\Omega} G(y, \tau; z, \delta) f(z, \delta) dz d\delta\right] d\mu^{x}(y, \tau) \qquad \text{on } \Omega,$$

and the kernel function $m(x, t, y)$ *has the following properties*:

$$m(x, t, \cdot) \in L^{p}(\Omega) \quad 1 \leq p < \infty \qquad \text{for all } (x, t) \in D,$$

$m(x, t, y)$ *as a function of* (x, t) *satisfies the equation* $Lm=0$ *for almost all* $y \in \Omega$. *Moreover*

$$m(x, t, w)=-\int_{\Omega} G(x, t; y, 0)\left[\int_{D} m(z, \tau, w) d\mu^{y}(z, \tau)\right] dy+G(x, t; w, 0)$$

for all $(x, t) \in D$ *and almost all* $y \in \Omega$.

§ 6. Some generalizations of non-local problem.

It follows from Theorem 8 (the inequality (14)) that the problem (1), (10) can be solved for $\Psi \in L^{p}(R_{n})$ $1 \leq p < \infty$, but this requires a new formulation of the condition (10).

We shall say that a function $u(x, t)$ defined on $R_{n} \times (0, T]$ has a parabolic limit at $y \in R_{n}$ if there exists a number b such that for all $\gamma > 0$, we have

$$\lim_{\substack{(x,t)\to(y,0)\\ |x-y|<\gamma\sqrt{t}}} u(x, t)=b.$$

We express this briefly by writing $p-\lim_{(x,t)\to(y,0)} u(x, t)=b$. (See Chabrowski [3], p. 257).

Let $\Psi \in L^{p}(R_{n})$. We shall say that a function u belonging to $C^{2,1}(R_{n} \times (0, T])$ is a solution of the problem (1), (10) if it satisfies the equation (1) in $R_{n} \times (0, T]$ and

$$p-\lim_{(x,t)\to(y,0)} u(x, t)=\int_{0}^{T}\int_{R_{n}} u(z, \tau) d\mu^{y}(z, \tau)+\Psi(x) \tag{19}$$

for almost all $y \in R_{n}$.

Theorem 10. *Suppose the hypothesis of Theorem* 5 *hold. If* $\Psi \in L^{p}(R_{n})$, $1 \leq p < \infty$ *and* f *is a bounded function on* $R_{n} \times [0, T]$ *and Hölder continuous on every compact subset of* $R_{n} \times [0, T]$, *then there exists a solution of the problem* (1), (10).

Proof. Let $\{\Psi_{r}\}$ be a sequence of function in $C(R_{n})$ with compact supports which converges in L^{p} to Ψ. By Theorem 5 and Lemma 6 there exists a unique bounded solution u_r in $C^{2,1}(R_n\times(0, T])\cap C(R_n\times[0, T])$ to the problem

$$Lu_r=f \qquad \text{in } R_n\times(0, T],$$
$$u_r(x, 0)-\int_0^T\int_{R_n} u_r(y, \tau)d\mu^x(y, \tau)=\Psi_r(x) \qquad \text{on } R_n.$$

It follows from (14) that

$$|u_r(x, t)-u_s(x, t)|\leq C(\delta)\|\Psi_r-\Psi_s\|_{L^p}$$

for all $(x, t)\in R_n\times[\delta, T]$. Hence u_r converges uniformly on $R_n\times[\delta, T]$ for every $\delta>0$ to a continuous function $u(x, t)$ on $R_n\times(0, T]$. As in the proof of Theorem 8 it is easy to establish the representation

$$u_r(x, t)=\int_{R_n}\Gamma(x, t; y, 0)\int_0^T\int_{R_n} u(z, \tau)d\mu^y(z, \tau)+\int_{R_n}\Gamma(x, t; y, 0)\Psi(y)dy$$
$$-\int_0^t d\tau\int_{R_n}\Gamma(x, t; y, \tau)f(y, \tau)dy$$

for all $(x, t)\in R_n\times(0, T]$, $r=1, 2, \cdots$. Letting $r\to\infty$ we obtain

$$u(x, t)=\int_{R_n}\Gamma(x, t; y, 0)\int_0^T\int_{R_n} u(z, \tau)d\mu^y(z, \tau)+\int_{R_n}\Gamma(x, t; y, 0)\Psi(y)dy$$
$$-\int_0^t d\tau\int_{R_n}\Gamma(x, t; y, \tau)f(y, \tau)dy$$

on $R_n\times(0, T]$. Since u is bounded on $R_n\times[\delta, T]$ for every $\delta>0$ and supp $\mu^y\subset R_n\times[r, T]$ for all $y\in R_n$ it is easy to see that u satisfies the equation (1) in $R_n\times(0, T]$. It follows from Theorem 3.1 in Chabrowski [3] that

$$p-\lim_{(x,t)\to(y,0)} u(x, t)=\int_0^T\int_{R_n} u(z, \tau)d\mu^y(z, \tau)+\Psi(y)$$

for almost all $y\in R_n$.

Finally consider the non-local problem with the condition (10) replaced by

$$u(x, 0)-\beta(x)\int_0^T\int_{R_n} u(y, \tau)d\mu(y, \tau)=\Psi(x) \qquad \text{on } R_n, \tag{20}$$

where μ is a non-negative Borel measure on $R_n\times[0, T]$ such that supp $\mu\subset R_n\times[r, T]$, $r>0$, and

$$\int_0^T\int_{R_n}\exp\left(\delta_0\sum_{i=1}^n|y_i|\right)d\mu(y, \tau)\leq 1.$$

If $\beta\in C(R_n)$, $0\leq\beta(x)\leq 1$ on R_n and has a compact support in R_n, then there exists an increasing sequence of cylinders $\{D_m\}$ satisfying the assumptions (i) and (ii) of

Theorem 3 and such that $\bigcup_{m\geq 1} D_m=R_n\times(0, T]$. Consequently Theorem 5 guarantees the existence of a solution of the problem (1), (10) provided $\Psi\in C(R_n)\cap E(R_n)$ and $f\in E(R_n\times[0, T])$ and is Hölder continuous on every compact subsət of $R_n\times[0, T]$.

Lemma 7. *Suppose that the hypotheses* (B_1), (B_2) *and* (B_3) *hold. Assume that* $\beta_i\in C_0(R_n)$, $0\leq\beta_i(x)\leq 1$ *on* R_n $(i=1, 2)$. *Assume further that f is a bounded function on* $R_n\times[0, T]$ *and Hölder continuous on every compact subset of* $R_n\times[0, T]$ *and that* Ψ *is a continuous and bounded function on* R_n. *If* u_i $(i=1, 2)$ *are solutions in* $C^{2,1}(R_n\times(0, T])\cap C(R_n\times[0, T])\cap E(R_n\times[0, T])$ *of the problem* (1), (20) *with* $\beta=\beta_i$ $(i=1, 2)$ *then*

$$|u_1(x, t)-u_2(x, t)|\leq K_1\sup_{R_n\times[r,T]}|u_1(x, t)|\sup_{R_n}|\beta_1(x)-\beta_2(x)| \tag{21}$$

for all $(x, t)\in R_n\times[0, T]$, *where* K_1 *is a positive constant independent of* u_1 *and* u_2, *and moreover for every* $\delta>0$ *there exists a positive constant* K_2 *independent of* u_1 *and* u_2 *such that*

$$|u_1(x, t)-u_2(x, t)|\leq K_2(\delta)\sup_{R_n\times[r,T]}|u(x, t)|\,\|\beta_1-\beta_2\|_{L^p} \tag{22}$$

for all $(x, t)\in R_n\times[\delta, T]$, $1\leq p<\infty$.

Proof. The proof of the estimates (21) and (22) is similar to that of the estimate (14). Therefore we only sketch the proof of the estimate (22). As in Lemma 6 we establish the representation

$$u_i(x, t)=\int_{R_n}\Gamma(x, t; y, 0)\beta(y)\int_0^T\int_{R_n}u_i(z, \tau)d\mu(z, \tau)dy$$
$$+\int_{R_n}\Gamma(x, t; y, 0)\Psi(y)dy-\int_0^t d\tau\int_{R_n}\Gamma(x, t; y, \tau)f(y, \tau)dy$$

on $R_n\times(0, T]$. Hence

$$u_1(x, t)-u_2(x, t)=\int_{R_n}\Gamma(x, t; y, 0)[\beta_1(y)-\beta_2(y)]\int_0^T\int_{R_n}u_1(z, \tau)d\mu(z, \tau)dy \tag{23}$$
$$+\int_{R_n}\Gamma(x, t; y, 0)\beta_2(y)\int_0^T\int_{R_n}[u_1(z, \tau)-u_2(z, \tau)]d\mu(z, \tau)dy$$

on $R_n\times(0, T]$. Assume first that $0\leq\beta_1(x)\leq\delta<1$ $(i=1, 2)$, where γ is a constant. Since $\int_{R_n}\Gamma(x, t; y, 0)dy\leq 1$ on $R_n\times(0, T]$ the identity (23) implies

$$\sup_{R_n\times[r,T]}|u_1(x, t)-u_2(x, t)|\leq(1-\gamma)^{-1}\sup_{R_n\times[r,T]}|u_1(x, t)|\sup_{R_n}|\beta_1(x)-\beta_2(x)|.$$

Applying again the identity (23) we derive the estimate (21). In the general case we use the transformation $u(x, t)=e^{-\gamma t}v(x, t)$, where $0<\gamma<d$.

Let $$\tilde{C}(R_n)=\{u;\, u\in C(R_n) \text{ and } \lim_{|x|\to\infty} u(x)=0\}.$$

Theorem 11. *Let the hypothesis* (B_1), (B_2) *and* (B_3) *hold and let* $\beta\in\tilde{C}(R_n)$ *and* $0\leq\beta(x)\leq 1$ *on* R_n. *Assume that f is a bounded function on* R_n *and Hölder continuous on every compact subset of* $R_n\times[0, T]$ *and that* Ψ *is a continuous bounded function on* R_n. *Then there exists a unique bounded solution in* $C^{2,1}(R_n\times(0, T])\cap C(R_n\times[0, T])$ *of the problem* (1), (20).

Approximating β by a sequence of functions in $C_0(R_n)$ the above result easily follows from the estimate (21) and the Friedman-Schauder interior estimates.

Theorem 12. *Let the hypothesis* (B_1), (B_2) *and* (B_3) *hold. If f is a bounded function on* $R_n\times[0, T]$ *and Hölder continuous on every compact subset of* $R_n\times[0, T]$, $\Psi\in L^p(R_n)$, $\beta\in L^p(R_n)$ $1\leq p<\infty$ *and* $0\leq\beta(x)\leq 1$ *on* R_n, *then there exists a solution of the problem* (1), (20), *where the condition* (20) *is understood in the sense of the parabolic limit.*

Proof. Let $\{\Psi_m\}$ be a sequence of functions in $C_b(R_n)\cap L^p(R_n)$ and $\{\beta_m\}$ be a sequence of functions in $C_o(R_n)$ converging in L^p to Ψ and β respectively. For every m there exists a unique solution u_m in $C^{2,1}(R_n\times(0, T])\cap C(R_n\times[0, T])$ of the problem

$$Lu_m=f \qquad \text{in } R_n\times(0, T],$$
$$u_m(x, 0)-\beta_m(x)\int_0^T\int_{R_n} u(y, \tau)d\mu(y, \tau)=\Psi(x) \qquad \text{on } R_n.$$

By the estimate (14) the sequence $\{u_m\}$ is uniformly bounded on every strip $R_n\times[\delta, T]$, $\delta>0$. Let $q>s$, it is obvious that

$$\begin{aligned} u_q(x, t)-u_s(x, t)=&\int_{R_n}\Gamma(x, t; y, 0)[\beta_q(y)-\beta_s(y)]\int_0^T\int_{R_n} u_q(z, \tau)d\mu(z, \tau)dy\\ &+\int_{R_n}\Gamma(x, t; y, 0)\beta_q(y)\int_0^T\int_{R_n}[u_q(z, \tau)-u_s(z, \tau)]d\mu(z, \tau)dy\\ &+\int_{R_n}\Gamma(x, t; y, 0)[\Psi_q(y)-\Psi_s(y)]dy \end{aligned}$$

on $R_n\times(0, T]$. A sin the proof of Lemma 7 one can easily show that for every $\delta>0$ there is a positive constant $C(\delta)$ such that

$$\sup_{R_n\times[\delta,T]}|u_q(x, t)-u_s(x, t)|\leq C(\delta)[\|\beta_q-\beta_s\|_{L^p}\sup_{R_n\times[\tau,T]}|u_q(x, t)|+\|\Psi_q-\Psi_s\|_{L^p}].$$

The result easily follows from the Friedman-Schauder interior estimates.

References

[1] Besala, P., On solutions of Fourier's first problem for a systems of non-linear parabolic equations in an unbounded domain, Ann. Polon. Math., **13** (1963), 247–265.

[2] Bicadze, A. V. and Samarskii, A. A., On some simple generalizations of linear elliptic boundary problems, Dokl. Akad. Nauk. SSSR, **185** (1969), 398–400.

[3] Chabrowski, J., Representation theorems for parabolic systems, J. Austral. Math. Soc., (Series A), **32** (1982), 246–288.

[4] ——, On non-local problems for parabolic equations, Nagoya Math. J., **93** (1984), 109–131.

[5] Friedman, A., *Partial differential equations of parabolic type,* Prentice-Hall, Englewood Cliffs, N. J., 1964.

[6] Kerefov, A. A., Non-local boundary value problems for parabolic equation, Differencial'nye Uravnenija, **15** (1979), 52–55.

[7] Krzyzański, M., Sur les solutions de l'équation linéaire du type parabolique déterminées par les conditions initiales, Annales de la Societé Polonaise de Mathématique, **18** (1945), 145–156, and note complémentaire, ibidem, **20** (1947), 7–9.

[8] Vabishchevich, P. N., Non-local parabolic problems and the inverse heat-conduction problem, Differencial'nye Uravnenija, **17** (1981), 761–765.

[9] Watson, N. A., Uniqueness and representation theorems for parabolic equations, J. London Math. Soc., (2), **8** (1974), 311–321.

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