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On a System of Difference Equations of the Form $y(x-1)=f(x, y(x))$ with $\lim_{x\to\infty} \frac{\partial f}{\partial y}(x, 0)=I$

By

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§1. Introduction

We consider a system of nonlinear difference equations of the form

$$\text{(E)} \qquad y(x-1)=f(x, y(x)),$$

where x is a complex variable, y is a complex n-dimensional vector with components $\{y_j\}$, and $f(x, y)$ is an n-dimensional vector whose components are holomorphic and bounded functions of x and y for

$$(1.1) \qquad |x|>R_0, \quad |\arg x|<\frac{\pi}{2}, \quad \|y\|\equiv\max_{j=1}^{n}|y_j|<\delta_0,$$

where R_0 is sufficiently large and δ_0 is sufficiently small. For a row vector k with components $\{k_j\}$ non-negative integers and a column vector z with components $\{z_j\}$, the symbol z^k stands for the scalar expression $z_1^{k_1}\cdot z_1^{k_2}\cdots z_n^{k_n}$ and $|k|$ denotes the length $k_1+k_2+\cdots+k_n$. We have an expansion

$$(1.2) \qquad f(x, y)=f_0(x)+A(x)y+\sum_{|k|\geqq 2} f_k(x)y^k$$

which is uniformly convergent for (1.1). Here the coefficients $f_0(x), f_k(x), |k|\geqq 2$ are n-dimensional vectors and $A(x)$ is an n by n matrix with components holomorphic and bounded for $|x|>R_0, |\arg x|<\frac{\pi}{2}$. We assume that the components of these coefficients admit asymptotic expression in powers of x^{-1} as x tends to infinity through the sector $|\arg x|<\frac{\pi}{2}$.

The matrix A_0 which is defined by

$$(1.3) \qquad A_0=\lim_{\substack{x\to\infty \\ |\arg x|<\pi/2}} A(x)$$

plays an important role in the study of the behavior of local solutions. The case of a single nonlinear difference equation has been studied by several authors, for example, J. Horn [4], S. Tanaka [7] and K. Takano [5]. W. A. Harris Jr. and Y. Sibuya [2], [3] have treated the case of a system of nonlinear difference equations under the hypothesis that the matrix $A_0 - I$ is nonsingular, I being the unit matrix of order n. K. Takano [6] has developed his previous paper [5] to the case when one and only one of the eigenvalues of the matrix A_0 is equal to unity and the others are neither zero nor unity in absolute value.

In our paper we discuss the case when the matrix A_0 is equal to the unit matrix:

$$A_0 = I. \tag{1.4}$$

Our results are summerized in the following three theorems. Throughout this paper we use ε for an arbitrary pre-assigned positive number, R and δ for sufficiently large and small positive numbers respectively.

Theorem 1. *If there exists a formal power series solution*

$$\sum_{j=1}^{\infty} p_j x^{-j} \tag{1.5}$$

for equation (E), *where* p_j, $j=1, 2, \cdots$, *are n-dimensional constant vectors, then there exists an actual solution* $\phi_0(x)$ *of equation* (E) *holomorphic in a domain of the form*

$$|x| > R, \quad |\arg x| < \frac{\pi}{2} - \varepsilon \tag{1.6}$$

and asymptotically developable into the formal solution as x tends to infinity through the sector (1.6).

Remark 1. As can be easily verified, a sufficient condition for equation (E) to possess a formal solution of the form (1.5) is that the matrix which is defined by

$$\Lambda = \lim_{\substack{x \to \infty \\ |\arg x| < \pi/2}} \frac{d}{dx^{-1}} A(x) \tag{1.7}$$

has no eigenvalues equal to a positive integer and moreover

$$f_0(x) = O(x^{-2}), \; f_k(x) = O(x^{-1}), \; |k| = 2.$$

By using the particular solution $\phi_0(x)$ obtained in Theorem 1, we apply the transformation

$$y(x) = \phi_0(x) + z(x), \tag{1.8}$$

then the transformed equation possesses the identically zero solution as a particular solution. Hence the system is reduced to an equation of the form (E) with $f_0(x)\equiv 0$. But the sector $|\arg x|<\frac{\pi}{2}$ must be replaced by a slightly narrower one:

$$|\arg x|<\frac{\pi}{2}-\varepsilon.$$

Theorem 2. *We consider the system of linear difference equations*

$$(E_1)\qquad y(x-1)=A(x)y(x),$$

which consists of the linear part of (E) with $f_0(x)\equiv 0$. Let Λ be of Jordan's canonical form:

$$\Lambda=\begin{pmatrix} \lambda_1 & & & & \\ \varepsilon_1 & \lambda_2 & & & \\ & \varepsilon_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \cdot & \lambda_{n-1} \\ & & & & \varepsilon_{n-1}\lambda_n \end{pmatrix}$$

with

$$(1.9)\qquad \mathrm{Re}\,\lambda_1\leqq \mathrm{Re}\,\lambda_2\leqq\cdots\leqq \mathrm{Re}\,\lambda_n.$$

By applying an appropriate transformation of the form

$$(1.10)\qquad y(x)=(I+P(x))u(x),$$

we can transform (E_1) into the system of equations of the form

$$(E_1')\qquad u(x-1)=(I+\Lambda x^{-1}+B(x))u(x),$$

with the following properties:

(i) *$P(x)$ is an n by n matrix function which is holomorphic in a domain of the form* (1.6) *and asymptotically developable into a power series in x^{-1} as x tends to infinity through the sector $|\arg x|<\pi/2-\varepsilon$,*

(ii) *$B(x)$ is a lower triangular matrix whose components $b_{rs}(x)$ are monomials in x^{-1} of the form $b_{rs}(x)=b_{rs}x^{-\lambda_r+\lambda_s-1}$ such that if $b_{rs}\neq 0$, then the quantity $\lambda_r-\lambda_s$ must be equal to a positive integer.*

Remark 2. The assumption that the matrix Λ is of Jordan's canonical form with inequality (1.9) can be always satisfied by carrying out, if necessary, a linear transformation with constant coefficients. By applying transformation (1.10) to equation (E) with $f_0(x)\equiv 0$, the matrix $A(x)$ is reduced to the matrix of the form $I+$

$\Lambda x^{-1}+B(x)$. The vectors corresponding to $f_k(x)$, $|k|\geqq 2$, are holomorphic and bounded functions of x having the same asymptotic property as before.

Theorem 3 (*Main Theorem*). *Let Λ and $B(x)$ be the same as in Theorem 2. Then we consider the system of nonlinear difference equations*

$$(\mathrm{E}_2) \qquad y(x-1)=(I+\Lambda x^{-1}+B(x))y(x)+\sum_{|k|\geqq 2} f_k(x)y(x)^k,$$

where the power series in the right hand side is again uniformly convergent for a domain of the form (1.1). *Assume that all the eigenvalues of Λ have positive real part and they satisfy inequalities*

$$(1.11) \qquad 0<\mathrm{Re}\,\lambda_1\leqq \mathrm{Re}\,\lambda_2\leqq\cdots\leqq \mathrm{Re}\,\lambda_n.$$

Assume moreover that

$$(1.12) \qquad f_k(x)=O(x^{-2}), \qquad |k|\geqq 2.$$

Then there exists a transformation $y(x)=\phi(x, u(x))$ of the form

$$(1.13) \qquad y(x)=u(x)+\sum_{|k|\geqq 2} p_k(x)u(x)^k$$

such that equation (E_2) *is transformed into the system of the form*

$$(\mathrm{E}_2') \qquad \begin{aligned} &u_1(x-1)=(1+\lambda_1 x^{-1})u_1(x),\\ &u_r(x-1)=(1+\lambda_r x^{-1})u_r(x)+\varepsilon_{r-1}x^{-1}u_{r-1}(x)\\ &\qquad +\sum_{j=1}^{r-1} b_{rj}x^{-\lambda_r+\lambda_j-1}u_j(x)\\ &\qquad +\sum_{k\in S_r} c_{rk}x^{-\lambda_r+k_1\lambda_1+\cdots+k_{r-1}\lambda_{r-1}-1}u_1(x)^{k_1}\cdots u_{r-1}(x)^{k_{r-1}},\\ &\qquad\qquad r=2, 3, \cdots, n,\end{aligned}$$

where

(i) *$p_k(x)$, $|k|\geqq 2$, are n-dimensional vector functions holomorphic and bounded in a domain of the form* (1.6) *and asymptotically developable into power series in x^{-1} as x tends to infinity through the sector $|\arg x|<\dfrac{\pi}{2}-\varepsilon$,*

(ii) *S_r is a finite set of row vectors $k=(k_1, \cdots, k_n)$ such that $k_r=k_{r+1}=\cdots=k_n=0$ and $\lambda_r-k_1\lambda_1-\cdots-k_{r-1}\lambda_{r-1}$ are positive integers,*

(iii) *c_{rk}, ($k\in S_r$, $r=2, 3, \cdots, n$) are constants,*

(iv) *the power series in u appearing in the right hand side of* (1.13)

$$(1.14) \qquad u+\sum_{|k|\geqq 2} p_k(x)u^k$$

is uniformly convergent for

(1.15)
$$|x|>R, \quad |\arg x|<\frac{\pi}{2}-\varepsilon, \quad \left|\arg\frac{x}{\lambda_j}\right|<\frac{\pi}{2}-\varepsilon,$$
$$j=1, 2, \cdots, n, \qquad \|u\|<\delta,$$

so that its sum $\phi(x, u)$ *represents there a holomorphic and bounded function of* (x, u).

Remark 3. If we substitute a general solution $u_0(x)$ of reduced equation (E_2') for u in (1.14), the resulting expression

$$\phi_0(x)+(I+P(x))\phi(x, u_0(x))$$

represents a local solution of equation (E) provided that the values of $(x, u_0(x))$ satisfy inequalities (1.15).

Remark 4. If, instead of assuming condition (1.11) as in Theorem 3, we assume a milder condition

$$\mathrm{Re}\,\lambda_1\leqq\cdots\leqq\mathrm{Re}\,\lambda_t\leqq 0<\mathrm{Re}\,\lambda_{t+1}\leqq\cdots\leqq\mathrm{Re}\,\lambda_n,$$

then we have a particular solution of the form

$$y_j(x)=\sum_{|k|\geqq 2} p_k^j(x)u_{t+1}(x)^{k_{t+1}}\cdots u_n(x)^{k_n}, \qquad j=1, 2, \cdots, t,$$
$$y_j(x)=u_j(x)+\sum_{|k|\geqq 2} p_k^j(x)u_{t+1}(x)^{k_{t+1}}\cdots u_n(x)^{k_n}, \qquad j=t+1, t+2, \cdots, n,$$

where the functions $u_j(x)$, $j=t+1, t+2, \cdots, n$, must satisfy the equations

$$u_{t+1}(x-1)=(1+\lambda_{t+1}x^{-1})u_{t+1}(x),$$
$$u_r(x-1)=(1+\lambda_r x^{-1})u_r(x)+\varepsilon_{r-1}x^{-1}u_{r-1}(x)+\sum_{j=t+1}^{r-1} b_{rj}x^{-\lambda_r+\lambda_j-1}u_j(x)$$
$$+\sum_{k\in S_r'} c_{rk}x^{-\lambda_r+k_{t+1}\lambda_{t+1}+\cdots+k_{r-1}\lambda_{r-1}-1}u_{t+1}(x)^{k_{t+1}}\cdots u_n(x)^{k_n},$$
$$r=t+2, t+3, \cdots, n,$$

and S_r' denote a subset of the finite set of row vectors $k=(k_1, \cdots, k_n)$ such that

(i) $k_1=k_2=\cdots=k_t=k_r=k_{r+1}=\cdots=k_n=0$,

(ii) $\lambda_r-\sum_{j=t+1}^{r-1} k_j\lambda_j$ is a positive integer.

§ 2. Proof of Theorem 1

In order to prove Theorem 1 (in this section) and in order to prove convergence of the formal transformation appearing in Theorem 3 (in section 5), we want to apply a fixed-point theorem in the function space.

Fixed-Point Theorem.

Let D be a domain in C^m, C being the complex plane, F be a family of holomorphic functions $\phi(x)\colon D\to C^n$, and T be a mapping from F into F. We assume the following:

(i) F is convex,

(ii) F is closed in the sense of uniform convergence on compact sets,

(iii) If a sequence $\{\phi_j(x)\}_{j=1}^{\infty}$ is convergent on each compact subset of the domain D, then the same is true for the sequence $\{T\phi_j(x)\}_{j=1}^{\infty}$,

(iv) TF is locally uniformly bounded in the domain D.

Under these assumptions there exists a fixed point $\tilde{\phi}(x)$ such that the equation $T\tilde{\phi}(x)=\tilde{\phi}(x)$ holds.

To prove Theorem 1, let N be any large positive integer and let us put

$$\phi_N(x)=\sum_{j=1}^{N-1} p_j x^{-j}. \tag{2.1}$$

We apply for equation (E) a transformation such that

$$y(x)=\eta(x)+\phi_N(x). \tag{2.2}$$

Then equation (E) is transformed into an equation of the form

$$\eta(x-1)=f_N(x,\eta(x)), \tag{E$'$}$$

where

$$f_N(x,\eta)=f(x,\eta+\phi_N(x))-\phi_N(x-1). \tag{2.3}$$

It is easily verified that

$$\|f_N(x,\eta)\|\leqq\alpha_N|x|^{-N-1}+(1+\alpha|x|^{-1})\|\eta\|, \tag{2.4}$$

$$\|f_N(x,\eta)-f_N(x,\eta')\|\leqq(1+\alpha|x|^{-1})\|\eta-\eta'\|, \tag{2.5}$$

for

$$|x|>R'_N,\quad |\arg x|<\frac{\pi}{2},\quad \|\eta\|<\delta'_N,\quad \|\eta'\|<\delta'_N,$$

by choosing positive numbers α, α_N, R'_N and δ'_N appropriately.

We denote by F the family of the functions $\eta(x)$ which are holomorphic in a domain of the form

$$D(R_N)\equiv\left\{x;\,|x|>R_N,\,|\arg x|<\frac{\pi}{2}-\varepsilon\right\}, \tag{2.6}$$

and satisfy there

$$\|\eta(x)\|\leqq K_N|x|^{-N}, \tag{2.7}$$

where R_N and K_N are to be determined in a suitable way.

Let κ be a positive number such that $0<\kappa<\cos\left(\frac{\pi}{2}-\varepsilon\right)$. As can be easily verified, the following inequalities

$$(2.8)\qquad |1+x^{-1}|^{-j}<1-j\kappa|x|^{-1}, \qquad (1\leqq j\leqq N)$$

hold in the domain $D(R''_N)$, by choosing R''_N sufficiently large.

The positive integer N and positive numbers K_N and R_N are to be determined so that

$$(2.9)\qquad N>\frac{\alpha}{\kappa}, \quad K_N>\frac{\alpha_N}{N\kappa-\alpha}, \quad R_N>\max\left\{\left(\frac{K_N}{\delta'_N}\right)^{1/N}, R'_N, R''_N\right\}.$$

We define a mapping T on F by

$$(2.10)\qquad T\eta(x)=f_N(x+1, \eta(x+1)).$$

In order to prove that the mapping T is well defined, we show that inequalities

$$|x+1|>R'_N, \quad |\arg x|<\frac{\pi}{2}, \quad \|\eta(x+1)\|<\delta'_N$$

hold for $x\in D(R_N)$ and $\eta(x)\in F$. It is evident that if an x is in $D(R_N)$, then the $x+1$ is also in $D(R_N)$. Therefore $\eta(x+1)$ is well defined. By using the inequality $R_N>R'_N$ in (2.9), we have $|x+1|>R'_N$. Now we have

$$\begin{aligned}\|\eta(x+1)\|&\leqq K_N|x+1|^{-N} && \text{by (2.7)}\\ &\leqq K_N R_N^{-N}<\delta'_N && \text{by (2.9).}\end{aligned}$$

Thus, it has been shown that the mapping T is well defined.

In addition, following inequalities

$$\begin{aligned}&\|f_N(x+1, \eta(x+1))\|\\ &\leqq\alpha_N|x+1|^{-N-1}+(1+\alpha|x+1|^{-1})\|\eta(x+1)\| && \text{by (2.4)}\\ &\leqq(\alpha_N+\alpha K_N)|x+1|^{-N-1}+K_N|x+1|^{-N} && \text{by (2.7)}\\ &\leqq(\alpha_N+\alpha K_N)|x|^{-N-1}+K_N|x|^{-N}(1-N\kappa|x|^{-1}) && \text{by (2.8)}\\ &\leqq K_N|x|^{-N} && \text{by (2.9)}\end{aligned}$$

derive that T maps each function belonging to F into F.

It is evident that we can apply Fixed-Point Theorem to the present case. Hence there is a solution $\eta_0(x)$ for equation (E′) holomorphic in $D(R_N)$ with $\|\eta_0(x)\|=O(x^{-N})$.

In order to prove the uniqueness of a solution, assume that there exists another solution $\eta_1(x)$ for equation (E′) holomorphic in $D(R_N)$ with $\|\eta_1(x)\|=O(x^{-N})$. We have in $D(R_N)$

$$\begin{aligned}&\|\eta_0(x)-\eta_1(x)\|\\&\quad=\|f_N(x+1,\eta_0(x+1))-f_N(x+1,\eta_1(x+1))\|\\&\quad\leqq(1+\alpha|x|^{-1})\|\eta_0(x+1)-\eta_1(x+1)\| \qquad \text{by (2.5)}\\&\quad\leqq K(1+\alpha|x|^{-1})|x|^{-N} \qquad (K\text{: a constant }>0).\end{aligned}$$

Let us set

$$\sigma(x)=\|\eta_0(x)-\eta_1(x)\|\,|x|^N, \qquad x\in D(R_N)$$

then we can easily verify that $\sigma(x)$ is bounded and satisfies

$$\sigma(x)\leqq(1-(N\kappa-\alpha)|x|^{-1})\sigma(x+1).$$

Hence we have

$$\sigma(x)\leqq\prod_{j=0}^{m}\left(1-\frac{N\kappa-\alpha}{|x|+j}\right)K' \qquad (K'\text{: a constant }>0),$$

for an arbitrary positive integer m. By virtue of the relation

$$\lim_{m\to\infty}\prod_{j=0}^{m}\left(1-\frac{N\kappa-\alpha}{|x|+j}\right)=0,$$

we have $\sigma(x)\equiv0$; that is, $\eta_0(x)\equiv\eta_1(x)$.

In order to prove the asymptotic developability of a solution, let us denote $\tilde{\eta}_N(x)$ the unique solution given above. The function $y_N(x)$ given by

$$y_N(x)=\tilde{\eta}_N(x)+\phi_N(x)$$

is a solution of equation (E) holomorphic in $D(R_N)$.

For our purpose, it is sufficient to prove that this function does not depend on N. To do this, let M be any integer larger than N. By the same reasoning as above, the expression

$$y_M(x)=\tilde{\eta}_M(x)+\phi_M(x)$$

is also a solution of equation (E) holomorphic in $D(R_M)$. Obviously the function

$$y_M(x)-\phi_N(x)\equiv\tilde{\eta}_M(x)+\sum_{j=N}^{M-1}p_jx^{-j}$$

is a holomorphic solution for equation (E′) in $D(R_M)$ and satisfies the order condition

$$\|y_M(x)-\phi_N(x)\|=O(x^{-N}).$$

By the uniqueness of a solution, we must have the identity

$$\tilde{\eta}_M(x)+\sum_{j=N}^{M-1} p_j x^{-j} \equiv \tilde{\eta}_N(x),$$

in the common part of $D(R_N)$ and $D(R_M)$. Consequently we get

$$y_M(x) \equiv y_N(x).$$

Thus the proof of Theorem 1 is accomplished.

§ 3. Proof of Theorem 2

By applying transformation (1.10) to equation (E_1), we have

$$(3.1) \qquad (I+P(x-1))u(x-1)=(I+\Lambda x^{-1}+A(x))(I+P(x))u(x).$$

Let $B(x)$ be an n by n matrix function satisfying

$$(3.2) \qquad (I+\Lambda x^{-1}+A(x))(I+P(x))=(I+P(x-1))(I+\Lambda x^{-1}+B(x)).$$

Namely,

$$(3.3) \qquad P(x-1)(I+\Lambda x^{-1}+B(x))=(I+\Lambda x^{-1}+A(x))P(x)+A(x)-B(x).$$

Our aim is to determine $P(x)$ so that $B(x)$ may take a form as simple as possible. Let us substitute formal power series

$$(3.4) \qquad \sum_{j=2}^{\infty} A_j x^{-j}, \quad \sum_{j=0}^{\infty} B_j x^{-j}, \quad \sum_{j=1}^{\infty} P_j x^{-j}, \quad \sum_{j=1}^{\infty} \left\{ \sum_{r=0}^{j-1} \binom{j-1}{r} P_{j-r} \right\} x^{-j},$$

for $A(x)$, $B(x)$, $P(x)$ and $P(x-1)$ in equation (3.3) respectively. By equating coefficients of like powers of the both sides, we have following recursive relations:

$$B_0=0, \qquad B_1=0,$$
$$P_1+P_1\Lambda-\Lambda P_1=A_2-B_2,$$
$$\cdots\cdots$$
$$(3.5) \qquad mP_m+P_m\Lambda-\Lambda P_m=A_{m+1}-B_{m+1}+M_m,$$

where M_m is a polynomial in matrices $A_2,\cdots, A_m$, $B_2,\cdots, B_m$, $P_1,\cdots, P_{m-1}$ and Λ. Comparing the (r, s) elements of both sides of equation (3.5), we get

$$(3.6) \qquad (m-\lambda_r+\lambda_s)P_m^{rs}=A_{m+1}^{rs}-B_{m+1}^{rs}+M_m^{rs}+\varepsilon_{r-1}P_m^{r-1,s}-\varepsilon_s P_m^{r,s+1}.$$

Now we introduce the linear order "$<$" in the set of the indices of an n by n matrix such that:

$(r', s') < (r, s)$ means that (i) $s' > s$, otherwise
(ii) $s' = s$ and $r' < r$.

Under this linear order condition, the element $P_m^{r-1,s}$ and $P_m^{r,s+1}$ are considered as known quantities. Now we put

$$\text{(3.7)} \qquad \begin{aligned} &B_{m+1}^{rs} = 0, \qquad \text{if } \lambda_r - \lambda_s \neq m, \\ &B_{m+1}^{rs} = A_{m+1}^{rs} + M_m^{rs} + \varepsilon_{r-1} P_m^{r-1,s} - \varepsilon_s P_m^{r,s+1}, \qquad \text{if } \lambda_r - \lambda_s = m. \end{aligned}$$

By considering assumption (1.9), if $\lambda_r - \lambda_s$ is a positive integer m, then r must be larger than s. Thus, it has been verified that the matrix $B(x)$ is of the form stated at (i) in Theorem 2.

In order to prove holomorphy of $P(x)$, we use Theorem 1. We take the s-th column of both sides of equation (3.3):

$$(1 + \lambda_s x^{-1}) p_s(x-1) + \varepsilon_s x^{-1} p_{s+1}(x-1) + \sum_{j=s+1}^{n} b_{js}(x) p_j(x-1)$$
$$= (I + \Lambda x^{-1} + A(x)) p_s(x) + d_s(x),$$

where $p_j(x)$ and $d_j(x)$, $j = 1, 2, \cdots, n$, are the j-th column vectors of matrices $P(x)$ and $A(x) - B(x)$ respectively. Rewriting the above equation, we see that $p_s(x)$ satisfies an equation of the form

$$\text{(3.8)} \qquad p_s(x-1) = \tilde{f}_0(x) + (I + \tilde{\Lambda} x^{-1} + \tilde{A}(x)) p(x).$$

By using Theorem 1, we can conclude that the statement with respect to the matrix $P(x)$ in Theorem 2 holds.

§4. Construction of formal transformation in Theorem 3

Instead of constructing formal transformation (1.13) directly, it is convenient to apply transformations of the following form

$$y(x) = u^1(x) + \sum_{|k|=2} q_k(x) u^1(x)^k,$$
$$u^1(x) = u^2(x) + \sum_{|k|=3} q_k(x) u^2(x)^k,$$
$$\cdots\cdots$$
$$u^{m-1}(x) = u^m(x) + \sum_{|k|=m+1} q_k(x) u^m(x)^k,$$
$$\cdots\cdots$$

successively. Let

$$y(x) = u^m(x) + \sum_{|k| \geq 2} Q_k^m(x) u^m(x)^k$$

be the composite of the first m transformations above. If we make m tend to infinity, then we have a desired formal transformation. So we want to stress a transformation of the form

$$y(x)=u(x)+\sum_{|k|=N} q_k(x)u(x)^k, \tag{4.1}$$

with $q_k(x)=O(x^{-1})$.

By applying transformation (4.1) to equation (E_2), we have

$$\begin{aligned} u(x-1)=&(I+\Lambda x^{-1}+B(x))(u(x)+\sum_{|k|=N} q_k(x)u(x)^k) \\ &+\sum_{|k|\geqq 2} f_k(x)(u(x)+\cdots)^k \\ &-\sum_{|k|=N} q_k(x-1)((I+\Lambda x^{-1}+B(x))u(x)+\cdots)^k+\cdots. \end{aligned} \tag{4.2}$$

Rewriting the right hand side, we have

$$u(x-1)=(I+\Lambda x^{-1}+B(x))u(x)+\sum_{|k|\geqq 2} g_k(x)u(x)^k. \tag{4.3}$$

It is evident that

$$g_k(x)=f_k(x), \qquad \text{for } |k|<N. \tag{4.4}$$

The term $g_k(x)u(x)^k$ with $|k|=N$ is picked up only in the expression

$$\begin{aligned} &(I+\Lambda x^{-1}+B(x))q_k(x)u(x)^k+f_k(x)u(x)^k \\ &\quad -\sum_{|h|=N} q_h(x-1)((I+\Lambda x^{-1}+B(x))u(x))^k. \end{aligned}$$

Considering that the matrix $I+\Lambda x^{-1}+B(x)$ is of lower triangular form, we introduce in the set of n-dimensional row vectors with length N, a linear order "$<$" such that:

$$\begin{aligned} h<k \text{ means that } h_i<k_i \qquad &\text{for some } i,\ 1\leqq i\leqq n-1, \quad \text{and} \\ h_j=k_j \qquad &\text{for all } j<i. \end{aligned}$$

Then the $g_k(x)$'s with $|k|=N$ are written as follows:

$$\begin{aligned} g_k(x)=&(I+\Lambda x^{-1}+B(x))q_k(x)+f_k(x)-q_k(x-1)\prod_{j=1}^{n}(1+\lambda_j x^{-1})^{k_j} \\ &-\sum_{\substack{|h|=N \\ h<k}} q_h(x-1)c_{kh}(x), \end{aligned} \tag{4.5}$$

where $c_{kh}(x)$ are polynomials in x^{-1} with $c_{kh}(x)=O(x^{-1})$. Namely we have

$$\prod_{j=1}^{n}(1+\lambda_j x^{-1})^{k_j} q_k(x-1)$$
(4.6)
$$=(I+\Lambda x^{-1}+B(x))q_k(x)+f_k(x)-\sum_{\substack{|h|=N\\ h<k}} q_h(x-1)c_{kh}(x)-g_k(x).$$

Let us substitute formal power series

$$(4.7)\qquad \sum_{j=1}^{\infty} q_j x^{-j},\quad \sum_{j=1}^{\infty}\left\{\sum_{r=0}^{j-1}\binom{j-1}{r}q_{j-r}\right\}x^{-j},\quad \sum_{j=2}^{\infty} v_j x^{-j},\quad \sum_{j=0}^{\infty} g_j x^{-j}$$

for $q_k(x)$, $q_k(x-1)$, $f_k(x)-\sum_{\substack{|h|=N\\ h<k}} q_h(x-1)c_{kh}(x)$, $g_k(x)$ in equation (4.6) respectively. By equating coefficients of like powers of the both sides, we have following recursive relations:

$$g_0=0, \qquad g_1=0$$
$$((1+k\lambda)I-\Lambda)q_1=v_2-g_2,$$
$$\cdots\cdots$$
$$(4.8)\qquad ((m+k\lambda)I-\Lambda)q_m=v_{m+1}-g_{m+1}+w_m,$$

with a simplified notation

$$(4.9)\qquad k\lambda=k_1\lambda_1+\cdots+k_n\lambda_n,$$

where w_m is a linear combination of $q_1,\cdots,q_{m-1}$ with coefficients n by n matrices.

Taking the r-th elements of both sides of equation (4.8), we get

$$(4.10)\qquad (m-\lambda_r+k\lambda)q_m^r=v_{m+1}^r-g_{m+1}^r+w_m^r+\varepsilon_{r-1}q_m^{r-1}.$$

Since $q_m^{r'}$, $r'<r$, are already determined, we put

$$(4.11)\qquad \begin{aligned} &g_{m+1}^r=0, && \text{for } \lambda_r-k\lambda\neq m,\\ &g_{m+1}^r=v_{m+1}^r+w_m^r+\varepsilon_{r-1}q_m^{r-1}, && \text{for } \lambda_r-k\lambda=m.\end{aligned}$$

Thus equation (4.6) has a formal power series solution. Moreover equation (4.6) can be rewritten of the form

$$q_k(x-1)=\tilde{f}_0(x)+(I+\tilde{\Lambda}x^{-1}+\tilde{B}(x))q_k(x).$$

Hence by applying Theorem 1, it is shown that the functions $q_k(x)$ have the asymptotic property. As the function $p_k(x)$ appearing in the composite transformation (1.13) is a polynomial in $q_h(x)$, with $|h|\leqq|k|$, we have the statement (i) in Theorem 3.

By assumption (1.11), the relation $\lambda_r-k\lambda=m$, (m: an integer>0) derives that $k_r=k_{r+1}=\cdots=k_n=0$. Hence, if the r-th element $g_k^r(x)$ of the vector $g_k(x)$ is not identically zero, then it must take the form as follows:

$$g_k^r(x) = c_{rk} x^{-\lambda_r + k_1\lambda_1 + \cdots + k_{r-1}\lambda_{r-1} - 1}. \tag{4.12}$$

Hence, if we apply to equation (E_2) the composite transformation (1.13), then by virtue of equation (4.4), (4.12), we have equation (E_2') stated in Theorem 3 in section 1.

§ 5. Convergence of formal transformation in Theorem 3

We will write equation (E_2') simply as follows:

$$u(x-1) = G(x, u(x)). \tag{5.1}$$

Solving this equation conversely with respect to $u(x)$, we have

$$\begin{aligned} &u(x) = H(x, u(x-1)); \qquad \text{or} \\ &u_j(x) = H_j(x, u(x-1)), \qquad j = 1, 2, \cdots, n. \end{aligned} \tag{5.2}$$

The proof of convergence of (1.14) is based on the Fixed-Point Theorem. Let N be any large positive integer and let us set

$$\phi_N(x, u) = u + \sum_{\substack{k\mu < N \\ |k| \geqq 2}} p_k(x) u^k, \tag{5.3}$$

with simplified notations

$$\begin{aligned} &k\mu = k_1\mu_1 + \cdots + k_n\mu_n, \\ &\mu_j = \mathrm{Re}\lambda_j, \qquad j = 1, 2, \cdots, n. \end{aligned} \tag{5.4}$$

Let $u(x)$ be a holomorphic and bounded solution of reduced equation (E_2'). We apply for equation (E_2) the transformation

$$y(x) = \eta(x) + \phi_N(x, u(x)). \tag{5.5}$$

Then we have

$$\eta(x-1) = f_N(x, u(x), \eta(x)), \tag{5.6}$$

where

$$\begin{aligned} f_N(x, u, \eta) = &(I + \Lambda x^{-1} + B(x))(\eta + \phi_N(x,u)) \\ &+ \sum_{|k| \geqq 2} f_k(x)(\eta + \phi_N(x, u))^k - \phi_N(x-1, G(x, u)). \end{aligned} \tag{5.7}$$

It is easily verified that

$$\|f_N(x, u, \eta)\| \leqq \alpha_N |x|^{-1} \|u\|_\mu^N + (1 + \alpha |x|^{-1}) \|\eta\|, \tag{5.8}$$

(5.9) $$\|f_N(x, u, \eta)-f_N(x, u, \eta')\|\leqq(1+\alpha|x|^{-1})\|\eta-\eta'\|,$$

for

$$|x|>R'_N, \quad |\arg x|<\frac{\pi}{2}-\varepsilon, \quad \|u\|_\mu\equiv\max_{j=1}^{n}|u_j|^{1/\mu_j}<\delta'_N,$$
$$\|\eta\|<\Delta'_N, \qquad \|\eta'\|<\Delta'_N,$$

by choosing positive numbers α, α_N, R'_N, δ'_N and Δ'_N appropriately.

Let F be the family of functions $\phi(x, u)$ which are holomorphic in a domain $D_\lambda(R_N, \delta_N)$ of the form

(5.10) $$D_\lambda(R_N, \delta_N)\equiv\Big\{(x, u); \quad |x|>R_N, \quad |\arg x|<\frac{\pi}{2}-\varepsilon, \quad \left|\arg\frac{x}{\lambda_j}\right|<\frac{\pi}{2}-\varepsilon, \quad 1\leqq j\leqq n, \quad \|u\|_\mu<\delta_N\Big\},$$

and satisfy there

(5.11) $$\|\phi(x, u)\|\leqq K_N[\max\{|x|^{-1}, \|u\|_\mu\}]^N,$$

where R_N, K_N, δ_N are to be determined in a suitable way.

Let κ be a positive number defined in section 2, and λ be a complex number. As can be easily verified, the following inequalities

(5.12) $$|1+\lambda(x+1)^{-1}|^{-j}\leqq1-j\kappa|\lambda||x|^{-1}, \quad j=1, 2,\cdots, N$$

hold for

$$|x|>R''_N, \quad |\arg x|<\frac{\pi}{2}-\varepsilon, \left|\arg\frac{x}{\lambda}\right|<\frac{\pi}{2}-\varepsilon,$$

by choosing R''_N sufficiently large. We can assume that

(5.13) $$\sum_{r=1}^{n-1}|\varepsilon_r| + \sum_{r,s=1}^{n}|b_{rs}|+\sum_{r=2}^{n}\sum_{k\in S_r}|c_{rk}|<\kappa\cdot\min_{r=1}^{n}|\lambda_r|,$$

if necessary by carrying out a linear transformation with constant coefficients. Let L, ρ and β be positive numbers such that

(5.14) $$\sum_{r=1}^{n-1}|\varepsilon_r|+\sum_{r,s=1}^{n}|b_{rs}|+\sum_{r=2}^{n}\sum_{k\in S_r}|c_{rk}|<L<\kappa\cdot\min_{r=1}^{n}|\lambda_r|,$$
$$0<\rho<\mu_n^{-1}\Big(\kappa\cdot\min_{r=1}^{n}|\lambda_r|-L\Big),$$
$$0<\beta<\min\{\kappa, \rho\}.$$

It is easily verified that

$$(5.15)\qquad \begin{aligned} &(1-(\kappa|\lambda_r|-L)|x|^{-1})^{1/\mu_r}<1-\rho|x|^{-1}, \qquad r=1,2,\cdots,n\\ &(1-\min\{\kappa,\rho\}|x|^{-1})^N<1-N\beta|x|^{-1}, \end{aligned}$$

for $|x|>R''_N$, if R''_N is sufficiently large.

Now we take a positive integer N and positive numbers K_N, δ_N and R_N so that

$$(5.16)\qquad \begin{aligned} &N>\frac{\alpha}{\beta}, \quad K_N>\frac{\alpha_N}{N\beta-\alpha}, \quad \delta_N<\min\{\delta'_N, (\varDelta'_N K_N^{-1})^{1/N}\},\\ &R_N>\max\{\delta_N^{-1}, R'_N, R''_N\}. \end{aligned}$$

We define a mapping T on F by

$$(5.17)\qquad T\phi(x,u)=f_N(x+1, H(x+1,u), \phi(x+1, H(x+1,u))).$$

First, we will show that

$$(5.18)\qquad \|H(x+1,u)\|_\mu<\delta'_N,$$

$$(5.19)\qquad \|\phi(x+1, H(x+1,u))\|<\varDelta'_N,$$

for (x,u) in $D_\lambda(R_N,\delta_N)$. We have

$$\begin{aligned} &|H_1(x+1,u)|\\ &\quad =|1+\lambda_1(x+1)^{-1}|^{-1}|u_1|\\ &\quad \leqq(1-\kappa|\lambda_1||x|^{-1})[\max\{|x|^{-1}, \|u\|_\mu\}]^{\mu_1} \qquad \text{by (5.12)}\\ &\quad \leqq(1-(\kappa|\lambda_1|-L)|x|^{-1})[\max\{|x|^{-1}, \|u\|_\mu\}]^{\mu_1}. \end{aligned}$$

Assuming that

$$\begin{aligned} &|H_j(x+1,u)| \qquad (\equiv|H_j|)\\ &\quad \leqq(1-(\kappa|\lambda_j|-L)|x|^{-1})[\max\{|x|^{-1}, \|u\|_\mu\}]^{\mu_j}, \qquad j<r, \end{aligned}$$

we have

$$\begin{aligned} &|H_r(x+1,u)|\\ &\quad \leqq|1+\lambda_r(x+1)^{-1}|^{-1}[|u_r|+|\varepsilon_{r-1}||x|^{-1}|H_{r-1}|\\ &\qquad +\sum_{j=1}^{r-1}|b_{rj}||x|^{-\mu_r+\mu_j-1}|H_j|\\ &\qquad +\sum_{k\in S_r}|c_{rk}||x|^{-\mu_r+k\mu-1}|H_1|^{k_1}\cdots|H_{r-1}|^{k_{r-1}}],\\ &\quad \leqq(1-\kappa|\lambda_r||x|^{-1})(1+L|x|^{-1})[\max\{|x|^{-1}, \|u\|_\mu\}]^{\mu_r}, \qquad \text{by (5.14)} \end{aligned}$$

$$\leqq (1-(\kappa|\lambda_r|-L)|x|^{-1})[\max\{|x|^{-1}, \|u\|_\mu\}]^{\mu r}.$$

By induction, we thus have

$$(5.20)\qquad \begin{aligned} \|H(x+1, u)\|_\mu &\leqq (1-\rho|x|^{-1})\cdot\max\{|x|^{-1}, \|u\|_\mu\}, &&\text{by (5.15)}\\ &< \max\{R_N^{-1}, \delta_N\} = \delta_N < \delta_N', &&\text{by (5.16)}. \end{aligned}$$

Inequality (5.18) is thus proved. Then we have

$$\begin{aligned} &\|\phi(x+1, H(x+1, u))\| \\ &\quad \leqq K_N[\max\{|x+1|^{-1}, \|H(x+1, u)\|_\mu\}]^N &&\text{by (5.11)}\\ &\quad \leqq K_N[\max\{R_N^{-1}, \delta_N\}]^N &&\text{by (5.18)}\\ &\quad \leqq K_N\delta_N^N < \Delta_N' &&\text{by (5.16)}. \end{aligned}$$

Thus the mapping T is well defined.

In order to prove that $TF\subset F$, we must show that

$$(5.21)\qquad \|T\phi(x, u)\| \leqq K_N[\max\{|x|^{-1}, \|u\|_\mu\}]^N.$$

We have

$$\begin{aligned} &\|f_N(x+1, H(x+1, u), \phi(x+1, H(x+1, u)))\| \\ &\quad \leqq \alpha_N|x+1|^{-1}\|H(x+1, u)\|_\mu^N + (1+\alpha|x+1|^{-1})\|\phi(x+1, H(x+1, u))\| \\ &&&\text{by (5.8)}\\ &\quad \leqq \alpha_N|x|^{-1}[\max\{|x|^{-1}, \|u\|_\mu\}]^N + \alpha|x|^{-1}\|\phi(x+1, H(x+1, u))\| \\ &\qquad + \|\phi(x+1, H(x+1, u))\| &&\text{by (5.20)}\\ &\quad \leqq (\alpha_N+\alpha K_N)|x|^{-1}[\max\{|x|^{-1}, \|u\|_\mu\}]^N \\ &\qquad + K_N[\max\{|x|^{-1}(1-\kappa|x|^{-1}), (1-\rho|x|^{-1})\max\{|x|^{-1}, \|u\|_\mu\}\}]^N \\ &&&\text{by (5.20)}\\ &\quad \leqq (\alpha_N+\alpha K_N)|x|^{-1}[\max\{|x|^{-1}, \|u\|_\mu\}]^N \\ &\qquad + K_N(1-N\beta|x|^{-1})[\max\{|x|^{-1}, \|u\|_\mu\}]^N &&\text{by (5.15)}\\ &\quad \leqq K_N[\max\{|x|^{-1}, \|u\|_\mu\}]^N. \end{aligned}$$

Thus inequality (5.21) is verified. By virtue of Fixed-Point Theorem, there exists a function $\phi_0(x, u)$ corresponding to a fixed point, which is holomorphic in $D_\lambda(R_N, \delta_N)$ and satisfies

$$(5.22)\qquad \phi_0(x, u) = f_N(x+1, H(x+1, u), \phi_0(x+1, H(x+1, u))),$$

and

$$(5.23)\qquad \|\phi_0(x, u)\| \leqq K_N[\max\{|x|^{-1}, \|u\|_\mu\}]^N.$$

In order to prove the uniqueness of our solution, suppose that there is another solution $\phi_1(x, u)$. By using $u(x)$ with

$$\|u(x)\|_\mu < \delta_N$$

we have

$$\begin{aligned} &\|\phi_0(x, u(x)) - \phi_1(x, u(x))\| \\ &\quad \leqq (1+\alpha|x|^{-1})\|\phi_0(x+1, u(x+1)) - \phi_1(x+1, u(x+1))\| && \text{by (5.9)} \\ &\quad \leqq (1+\alpha|x|^{-1})K[\max\{|x|^{-1}, \|H(x+1, u(x))\|_\mu\}]^N && (K\text{: a constant}>0) \\ &\quad \leqq (1+\alpha|x|^{-1})K[\max\{|x|^{-1}, \|u(x)\|_\mu\}]^N && \text{by (5.20).} \end{aligned}$$

Let us set

$$\sigma(x) = \|\phi_0(x, u(x)) - \phi_1(x, u(x))\| [\max\{|x|^{-1}, \|u(x)\|_\mu\}]^{-N},$$

for $(x, u(x))$ in $D_\lambda(R_N, \delta_N)$. We can easily verify that $\sigma(x)$ is bounded and satisfies

$$\sigma(x) \leqq (1-(N\beta-\alpha)|x|^{-1})\sigma(x+1).$$

Hence we get $\sigma(x) \equiv 0$ as in section 2. Namely

$$\phi_0(x, u(x)) \equiv \phi_1(x, u(x)).$$

As the function $u(x)$ is chosen arbitrarily, so long as it is a holomorphic solution of (E_2') with $\|u(x)\|_\mu < \delta_N$, we conclude that

$$\phi_0(x, u) \equiv \phi_1(x, u) \qquad \text{in } D_\lambda(R_N, \delta_N).$$

We denote $P_N(x, u)$ instead of $\phi_0(x, u)$. We can prove as in section 2 that the sum $P_N(x, u) + \phi_N(x, u)$ is independent of N. This proves the uniform convergence of the formal transformation in Theorem 3.

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