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Gronwall's Inequality for Systems of Multiple Volterra Integral Equations

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1. Introduction.

In 1919 T. H. Gronwall [1] made use of a lemma which, in a generalized form, is a basic tool in the theory of ordinary differential equations. The following generalization of the original lemma is known as Gronwall's Lemma [2].

Gronwall's Lemma. *Suppose $u(t)$, $\phi(t)$, and $k(t)$ are real valued continuous functions for $t \in [a, b]$. Suppose $k(t)$ is nonnegative on $[a, b]$ and*

$$u(t) \leq \phi(t) + \int_a^t k(\tau) u(\tau) d\tau, \qquad t \in [a, b].$$

Then

$$u(t) \leq \phi(t) + \int_a^t k(\tau) \exp\left[\int_\tau^t k(s) ds\right] \phi(\tau) d\tau, \qquad t \in [a, b]. \tag{1}$$

If $\phi'(t)$ is continuous the estimate (1) may be given in the form (see e.g., [3])

$$u(t) \leq \exp\left[\int_a^t k(\tau) d\tau\right] \phi(a) + \int_a^t \exp\left[\int_\tau^t k(s) ds\right] \phi'(\tau) d\tau. \tag{2}$$

The importance of the inequalities (1) and (2), of course, lies in the fact that the right hand side of each is independent of $u(t)$.

This useful lemma has been generalized in many ways and these generalizations have a variety of applications. The purpose of this paper is to establish a generalization of the lemma for a class of systems of multiple Volterra integral equations. We will also see that both estimates, (1) and (2), have natural analogues for these systems.

We will introduce a fundamental solution for the Volterra equation under consideration as the solution of a related Volterra equation. This fundamental solution, which is a generalization of the fundamental matrix in ordinary differential equations, is the generalization of the function $\exp\left[\int_\tau^x k(s) ds\right]$ appearing in (1) and (2).

A generalization of Gronwall's Lemma for a scalar equation in one independent variable has been given in terms of a resolvent kernel [4] while others have been obtained for equations in several variables in terms of the Riemann function [5], [6], [7]. The fundamental solution introduced here is related to both the resolvent and the Riemann function. We will investigate these relations and show that the result established here may be used to obtain several generalizations given by other authors.

The main result is given in Section 2 and is established by applying a theorem valid in partially ordered Banach spaces. Section 3 is devoted to a discussion of the relationships between the result given here and generalizations obtained by other authors.

Although we do not consider them here, our generalization has several applications. In particular the author has used this result to obtain stability results for systems of multiple Volterra equations. These stability results will appear elsewhere.

2. Main Result.

Unless specified otherwise we will adopt the following notation for the remainder of the paper. Let R denote the reals. Let $x=(x_1, x_2, \cdots, x_n) \in R^n$, $x \leq y$ iff $x_i \leq y_i$ for $1 \leq i \leq n$, and for $a, b \in R^n$ then $[a, b]=\{x \mid x \in R^n,\ a \leq x \leq b\}$. Let $|\cdot|$ be an arbitrary vector norm on R^n and let $\|M\|=\sup_{|x|=1}|Mx|$ be the norm of an $m \times n$ matrix M. Let α_k denote any combination of the integers $\{1, 2, \cdots, n\}$ taken k at a time. We will assume that the elements in any combination $\alpha_k=\{i_1, i_2, \cdots, i_k\}$ have been ordered (i.e., $i_1<i_2<\cdots<i_k$). Given the combination α_k we let $\alpha_k'=\{1, 2, \cdots, n\}-\alpha_k$.

Let $\alpha_k=\{i_1, i_2, \cdots, i_k\}$. For $x \in R^n$ we define $x_{\alpha_k}=(x_{i_1}, x_{i_2}, \cdots, x_{i_k})$. We denote the multiple integral symbol $\int_{a_{i_1}}^{x_{i_1}} \int_{a_{i_2}}^{x_{i_2}} \cdots \int_{a_{i_k}}^{x_{i_k}}$ by $\int_{a_{\alpha_k}}^{x_{\alpha_k}}$ and the sequence of differentials $dr_{i_k} dr_{i_{k-1}} \cdots dr_{i_1}$ by dr_{α_k}. If $g: R^n \to R^m$, we define the partial derivative $g_{x_{i_1} x_{i_2} \cdots x_{i_k}}=g_{x_{\alpha_k}}$. For $x, y \in R^n$ we introduce the following: let

$$w_i(x, y ; \alpha_k)=\begin{cases} x_i & \text{if } i \notin \alpha_k \\ y_i & \text{if } i \in \alpha_k \end{cases}$$

and $w(x, y ; \alpha_k)=(w_1(x, y ; \alpha_k), w_2(x, y ; \alpha_k), \cdots, w_n(x, y ; \alpha_k))$.

Let u, ϕ: $[a, b] \to R^m$ and for each α_k, $1 \leq k \leq n$, let $K_{\alpha_k}(x, r_{\alpha_k})$ be an $m \times m$ matrix function for $x \in [a, b]$, $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$. We then consider the linear system of m Volterra integral equations in n independent variables of the form

$$u(x)=\phi(x)+\sum_{\substack{\alpha_k \\ 1 \leq k \leq n}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) u(w(x, r ; \alpha_k)) dr_{\alpha_k}. \tag{3}$$

It may be shown that if $\phi(x)$ and the matrix functions $K_{\alpha_k}(x, r_{\alpha_k})$ are continuous then Equation (3) has a unique continuous solution on $[a, b]$. Also, we observe that there are 2^n-1 integrals appearing in Equation (3). We simplify the notation further by using $\sum$ to mean $\sum\limits_{\substack{\alpha_k \\ 1\le k\le n}}$.

We now define the fundamental solution for Equation (3).

Definition. Suppose the matrix function $A(x;\xi)$ satisfies the equation

$$A(x;\xi)=I+\sum \int_{\xi_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k})A(w(x,r;\alpha_k);\xi)dr_{\alpha_k} \tag{4}$$

for $a\le\xi\le x\le b$. Then $A(x;\xi)$ is called a *fundamental solution* for Equation (3).

If the functions $K_{\alpha_k}(x, r_{\alpha_k})$ are continuous it may be shown that $A(x;\xi)$ exists, is unique, and is continuous for $a\le\xi\le x\le b$. The following theorem, which may be verified directly, gives the solution of Equation (3) in terms of the fundamental solution. The two forms for the solution of Equation (3) given in this theorem allow us to establish analogues for inequalities (1) and (2).

Theorem 1. *Suppose $\phi(x)$ is continuous on $[a, b]$ and each matrix function $K_{\alpha_k}(x, r_{\alpha_k})$ is continuous for $x\in[a, b]$, $a_{\alpha_k}\le r_{\alpha_k}\le x_{\alpha_k}\le b_{\alpha_k}$.*

i) *If each $A_{\xi_{\alpha_k}}(x;\xi)$ is continuous for $a\le\xi\le x\le b$ then the unique continuous solution of Equation* (3) *on $[a, b]$ is*

$$u(x)=\phi(x)+\sum(-1)^k\int_{a_{\alpha_k}}^{x_{\alpha_k}} A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\phi(w(x,r;\alpha_k))dr_{\alpha_k}. \tag{5}$$

ii) *If each $\phi_{x_{\alpha_k}}$ is continuous on $[a, b]$ then the unique continuous solution of Equation* (3) *is*

$$u(x)=A(x;a)\phi(a)+\sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}. \tag{6}$$

Our main result is based on Theorem 1 and the following theorem which may be found in [8].

Theorem 2. *Suppose F is a complete metric space and is partially ordered in such a way, that if an increasing sequence $(g_n)\subset F$ has the limit g_0, then $g_n<g_0$ for all n. Suppose T is an order preserving ($f_1<f_2\Rightarrow Tf_1<Tf_2$) contraction on F and f_0 is the unique fixed point of T. Then $f\in F$ and $f<Tf$ implies $f<f_0$.*

For any real λ consider the space $C_\lambda[a, b]$ consisting of the set of continuous functions $g\colon [a, b]\to R^m$ normed by $\|g\|_\lambda=\sup\limits_{x\in[a,b]}\left\{|g(x)|\exp\left[-\lambda\left(\sum\limits_{i=1}^{n} x_i\right)\right]\right\}$. It may be shown that $C_\lambda[a, b]$ is a Banach space. Let $K\subset C_\lambda[a, b]$ be the positive cone of functions such that $\phi(x)=(\phi_1(x), \phi_2(x), \cdots, \phi_m(x))\in K$ iff $\phi_i(x)\ge 0$, $1\le i\le m$. We consider the partial order on $C_\lambda[a, b]$ defined such that $g_1 < g_2$ iff $g_2 - g_1 \in K$. We notice that this partial order has the property that if (g_n) is an increasing sequence in $C_\lambda[a, b]$ converging to g_0, then $g_n < g_0$ for all n.

We now have the following generalization of Gronwall's Lemma.

Theorem 3. *Suppose $\phi(x)$ is continuous on $[a, b]$ and for each α_k with $1 \leq k \leq n$ the $m \times m$ matrix function $K_{\alpha_k}(x, r_{\alpha_k})$ is continuous and has nonnegative elements. If $u\colon [a, b] \to R^m$ is continuous and*

$$u(x) < \phi(x) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) u(w(x, r; \alpha_k)) dr_{\alpha_k}$$

for $x \in [a, b]$ then $u(x) < v(x)$ where $v(x)$ is the unique continuous solution of Equation (3) on $[a, b]$. If in addition each $A_{\xi_{\alpha_k}}(x; \xi)$ is continuous on $a \leq \xi \leq x \leq b$ then

$$u(x) < \phi(x) + \sum (-1)^k \int_{a_{\alpha_k}}^{x_{\alpha_k}} A_{r_{\alpha_k}}(x; w(x, r; \alpha_k)) \phi(w(x, r; \alpha_k)) dr_{\alpha_k}. \tag{7}$$

Or if instead we make the additional assumption that each $\phi_{x_{\alpha_k}}(x)$ is continuous on $[a, b]$ then

$$u(x) < A(x; a)\phi(a) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x; w(a, r; \alpha_k)) \phi_{r_{\alpha_k}}(w(a, r; \alpha_k)) dr_{\alpha_k}. \tag{8}$$

Proof. The continuity of each $K_{\alpha_k}(x, r_{\alpha_k})$ on the compact domain implies there is a constant $M > 0$ such that $\|K_{\alpha_k}(x, r_{\alpha_k})\| \leq M$ for each α_k. Choose λ_0 so that $\lambda_0 > 1$ and $\dfrac{M(2^n - 1)}{\lambda_0} < 1$. Define T on $C_{\lambda_0}[a, b]$ such that for $g \in C_{\lambda_0}[a, b]$ then

$$(Tg)(x) = \phi(x) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) g(w(x, r; \alpha_k)) dr_{\alpha_k}.$$

It follows from the continuity of the functions $\phi(x)$, $g(x)$, and $K_{\alpha_k}(x, r_{\alpha_k})$ that $(Tg)(x)$ is continuous on $[a, b]$.

Take $g_1, g_2 \in C_{\lambda_0}[a, b]$. Then we have

$$\begin{aligned}
&|(Tg_1)(x) - (Tg_2)(x)| \\
&\leq \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} M\, |g_1(w(x, r; \alpha_k)) - g_2(w(x, r; \alpha_k))| \\
&\qquad \cdot \exp\left[-\lambda_0\left(\sum_{i=1}^{n} w_i(x, r; \alpha_k)\right)\right] \exp\left[\lambda_0\left(\sum_{i=1}^{n} w_i(x, r; \alpha_k)\right)\right] dr_{\alpha_k} \\
&\leq M\, \|g_1 - g_2\|_{\lambda_0} \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \exp\left[\lambda_0\left(\sum_{i=1}^{n} w_i(x, r; \alpha_k)\right)\right] dr_{\alpha_k}
\end{aligned}$$

$$\leq M\,\|g_1-g_2\|_{\lambda_0}\sum\frac{1}{\lambda_0^k}\exp\left[\lambda_0\Big(\sum_{i=1}^n x_i\Big)\right]$$

$$\leq\frac{M(2^n-1)}{\lambda_0}\|g_1-g_2\|_{\lambda_0}\exp\left[\lambda_0\Big(\sum_{i=1}^n x_i\Big)\right].$$

Therefore $\|Tg_1-Tg_2\|_{\lambda_0}\leq\dfrac{M(2^n-1)}{\lambda_0}\|g_1-g_2\|_{\lambda_0}$ and T is a contraction on $C_{\lambda_0}[a,b]$.

Now suppose $g_1, g_2\in C_{\lambda_0}[a, b]$ such that $g_1<g_2$. Then since the elements in each $K_{\alpha_k}(x, r_{\alpha_k})$ are nonnegative we see that $K_{\alpha_k}(x, r_{\alpha_k})g_1(w(x, r;\alpha_k))<K_{\alpha_k}(x, r_{\alpha_k})g_2(w(x, r;\alpha_k))$ for each α_k. Thus $Tg_1<Tg_2$ and T is order preserving. The result now follows directly from Theorems 1 and 2. ∎

We note that our method of proof depends on the proper choice of norm on the space of continuous functions. For further discussion of this idea and related topics, the reader is referred to a paper by Chu and Diaz [9].

In the case when $m=n=1$ and $K_1(x, r)=k(r)$ (x, r real) we see the $A(x;r)=\exp\left[\int_r^x k(s)ds\right]$ and estimates (7) and (8) reduce to estimates (1) and (2) respectively. The form of the estimates (7) and (8) suggest that the fundamental solution defined above is the natural generalization of the exponential function appearing in Gronwall's Lemma; in fact the fundamental solution defined here is a generalization of the fundamental matrix in the theory of ordinary differential equations.

3. Generalizations Related to Theorem 3.

We now turn our attention to a discussion of several generalizations of Gronwall's Lemma and show how some of these may be obtained from Theorem 3.

Chu and Metcalf [4] have given the following extension of Gronwall's Lemma for scalar integral equations in one variable ($x, \xi, r\in R$): Suppose the functions u and ϕ are continuous on [0, 1] and $K(x,\xi)$ is continuous and nonnegative on $0\leq\xi\leq x\leq 1$. If

$$u(x)\leq\phi(x)+\int_0^x K(x, r)u(r)dr, \qquad 0\leq x\leq 1,$$

then

$$u(x)\leq\phi(x)+\int_0^x H(x, r)\phi(r)dr, \qquad 0\leq x\leq 1,$$

where $H(x,\xi)=\sum_{i=1}^{\infty}K_i(x,\xi)$, $0\leq\xi\leq x\leq 1$, is the resolvent kernel, and $K_i (i=1, 2,\cdots)$ are the iterated kernels of K.

It may be shown [10] that the resolvent kernel $H(x, r)$ satisfies the integral equation

$$H(x,\xi)=K(x,\xi)+\int_\xi^x K(x,r)H(r,\xi)dr.$$

Under the assumption that $A_\xi(x;\xi)$ is continuous for $0\leq\xi\leq x\leq 1$ we see from Equation (4) that

$$A_\xi(x;\xi)=-K(x,\xi)+\int_\xi^x K(x,r)A_\xi(r;\xi)dr$$

in this case. Hence, by uniqueness, we see that $A_\xi(x;\xi)=-H(x,\xi)$. Using this fact and Theorem 3 in the form of inequality (7) (with $m=n=1$ and $a=0$) we obtain the result given by Chu and Metcalf.

Conlan and Diaz [11] have used the following generalization of Gronwall's Lemma to study existence and uniqueness for an n-th order hyperbolic partial differential equation: If γ, M, and L are nonnegative constants, if in the region $0\leq x\leq b$ ($b\in R^n$, $b>0$, $b<\infty$) the real valued function $u(x)$ is continuous and nonnegative, and if

$$u(x)\leq\gamma+L\sum_{\substack{\alpha_k\\ 1\leq k\leq n-1}}\int_{0_{\alpha_k}}^{x_{\alpha_k}} u(w(x,r;\alpha_k))dr_{\alpha_k}+M\int_0^x u(r)dr$$

for $x\in[0,b]$, then $u(x)\leq\gamma K$ for $x\in[0,b]$ where K is a constant depending on L, M, and b. We see that under the assumptions made by Conlan and Diaz, Theorem 3 may be applied directly and inequality (8) implies $u(x)\leq A(x;0)\gamma$. Their result then follows from the continuity of $A(x;0)$ on $[0,b]$.

Other generalizations have been given when $u(x)$ satisfies the following special inequality ($K_{\alpha_k}\equiv 0$, $1\leq\alpha_k\leq n-1$, $K_{\alpha_n}(x,r)\equiv K(r)$)

$$u(x)<\phi(x)+\int_a^x K(r)u(r)dr,\qquad (a,x,r\in R^n). \tag{9}$$

In order to establish the connection between these generalizations and that given in Theorem 3 we introduce the matrix function $\bar{A}(x;\xi)$ satisfying the equation

$$\bar{A}(x;\xi)=I+(-1)^n\int_\xi^x \bar{A}(r;\xi)K(r)dr,\qquad a\leq x\leq\xi\leq b. \tag{10}$$

If $K(x)$ is continuous on $[a,b]$, which will be assumed here, it may be shown that Equation (10) has a unique solution continuous in x and ξ. When $n=1$ we see that $\bar{A}(x;\xi)$ is the transpose of a fundamental matrix for the adjoint system. We also note that if $m=1$ then $\bar{A}(x;\xi)$ is the so called Riemann function [12] for the hyperbolic equation $u_x(x)=K(x)u(x)$. If $A(x;\xi)$ is the fundamental solution for the equation associated with inequality (9) then it is possible to establish the following reciprocity relation:

$$A(x\,;\,\xi)=\bar{A}(\xi\,;\,x),\qquad a\leq\xi\leq x\leq b. \tag{11}$$

We will now use Equation (11) to obtain the special form of the estimate (7) when $u(x)$ satisfies inequality (9). Using Equation (11), Equation (10), and the continuity of $\bar{A}(x\,;\,\xi)$ in its first variable we see that for each α_k with $1\leq k\leq n-1$

$$A_{\xi_{\alpha_k}}(x\,;\,\xi)=(-1)^n\int_{x_{\alpha_k'}}^{\xi_{\alpha_k'}}\bar{A}(w(r,\xi\,;\,\alpha_k)\,;\,x)K(w(r,\xi\,;\,\alpha_k))dr_{\alpha_k'} \tag{12}$$

and

$$A_\xi(x\,;\,\xi)=(-1)^n\bar{A}(\xi\,;\,x)K(\xi) \tag{13}$$

for $a\leq x\leq\xi\leq b$. It follows from Equation (12) that $A_{\xi_{\alpha_k}}(x\,;\,w(x,\xi\,;\,\alpha_k))=0$ for each α_k with $1\leq k\leq n-1$. Using this fact, Equation (13), and Equation (11) again we see that (7) now becomes

$$u(x)<\phi(x)+\int_a^x A(x\,;\,r)K(r)\phi(r)dr. \tag{14}$$

Snow [5], [6] has used a different method to obtain two generalizations of Gronwall's Lemma when $n=2$ and $u(x)$ satisfies an inequality of the form (9). Since the result given in [5] follows from the one given in [6] we show that the main result in [6] follows from Theorem 3 above. Changing notation so that x, y, a, b, r, $s\in R$ we have the following statement of Snow's Theorem: Suppose D is a domain in R^2 and u, ϕ: $D\to R^m$ are continuous on D. Suppose $K(x,y)$ is a continuous symmetric $m\times m$ matrix function having non-negative elements on D. Let $P_0(a,b)$ and $P(x,y)$ be points in D such that $(a,b)\leq(x,y)$ and let G be the rectangle having the line joining P_0P as its diagonal. Suppose the matrix $V(r,s\,;\,x,y)$ satisfies the characteristic value problem

$$V_{rs}(r,s\,;\,x,y)=K(r,s)V(r,s\,;\,x,y),\qquad V(x,s\,;\,x,y)=V(r,y\,;\,x,y)=I. \tag{15}$$

Let D^+ be the connected subdomain of D containing P and on which $V(r,s\,;\,x,y)$ has nonnegative elements. If $G\subset D^+$ and $u(x,y)$ satisfies

$$u(x,y)<\phi(x,y)+\int_a^x\int_b^y K(r,s)u(r,s)dsdr \tag{16}$$

then

$$u(x,y)<\phi(x,y)+\int_a^x\int_b^y V^T(r,s\,;\,x,y)K(r,s)\phi(r,s)dsdr$$

where V^T is the transpose of V.

Let $A(x,y\,;\,\xi,\eta)$ be the fundamental solution for the equation associated with

(16). If we assume $G \subset D$ we see that under Snow's hypotheses Theorem 3, where the estimate (7) now takes the form given in (14), implies

$$u(x, y) < \phi(x, y) + \int_a^x \int_b^y A(x, y; r, s) K(r, s) \phi(r, s) ds dr. \tag{17}$$

Integrating the equation in problem (15) and using the characteristic data we have

$$V(\xi, \eta; x, y) = I + \int_\xi^x \int_\eta^y K(r, s) V(r, s; x, y) ds dr, \qquad (\xi, \eta) \leq (x, y). \tag{18}$$

Comparing Equation (18) with the Equation (10) for $\bar{A}$, using Equation (11), and using the symmetry of $K(x, y)$ we see that $\bar{A}(\xi, \eta; x, y) = V^T(\xi, \eta; x, y) = A(x, y; \xi, \eta)$ and hence we obtain the estimate obtained by Snow.

We point out that provided $G \subset D$ we obtain the estimate (17) with no assumption that $A(x, y; \xi, \eta)$ have nonnegative elements on a subregion $D^+ \supset G$. In fact, under the hypothesis that $K(x, y)$ has nonnegative elements it follows from Theorem 3 that $A(x, y; \xi, \eta)$ has nonnegative elements for all $(\xi, \eta) \in G$. We also note that the estimate (17) is valid without the symmetry assumption on the matrix $K(x, y)$.

More recently Young [7] has generalized Snow's method for a scalar inequality of the form (9) in n independent variables. Returning to the notation used earlier we have the following statement of Young's extension: Let Ω be an open set in R^n and let $a, x \in \Omega$ such that $a < x$. Suppose $u(x)$, $\phi(x)$, and $k(x) \geq 0$ are real valued and continuous on Ω. Let $V(\xi; x)$ be the solution of the characteristic value problem

$$(-1)^n v_\xi(\xi; x) = k(\xi) v(\xi; x), \quad v(\xi; x) = 1 \qquad \text{for } \xi_i = x_i, \ 1 \leq i \leq n. \tag{19}$$

Let Ω^+ be the connected subdomain of Ω containing x such that $v(\xi; x) \geq 0$ for all $\xi \in \Omega^+$. If $[a, x] \subset \Omega^+$ and

$$u(x) \leq \phi(x) + \int_a^x k(r) u(r) dr \tag{20}$$

then

$$u(x) \leq \phi(x) + \int_a^x \phi(r) k(r) v(r; x) dr.$$

Suppose $[a, x] \subset \Omega$. It follows from problem (19) that $v(\xi; x)$ is the solution of the equation

$$v(\xi; x) = 1 + (-1)^n \int_x^\xi k(r) v(r; x) dr. \tag{21}$$

Comparing (21) with Equation (10) and making use of Equation (11) we see that

$v(\xi\,;x)=\bar{A}(\xi\,;x)=A(x\,;\xi)$, $\xi\leq x$. Using this in (14) with $m=1$ and $K(x)\equiv k(x)$ we obtain Young's conclusion.

Walter [13] has also given a generalization for a scalar inequality of the form (9) in n independent variables. Walter's concluding estimate is given in terms of a function $h^*(x,\xi)$ defined as a series of functions determined from an iteration procedure. It may be shown that the function $h^*(x,\xi)$ and the fundamental solution $A(x\,;\xi)$ for this case are identical and hence our result is consistent with the result given in [13].

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